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In the Name of God

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* Full Professor

** Associate Professor

We would like to acknowledge the help of Narjes khatoon Zohorian in the preparation of this issue.

Letter from the Editor in Chief

I would like to welcome you to the Iranian Journal of Numerical Analysis and Optimization (IJNAO). This journal is published biannually and supported by the Faculty of Mathematical Sciences at the Ferdowsi University of Mashhad. Faculty of Mathematical Sciences with three centers of excellence and three research centers is well-known in mathematical communities in Iran.
The main aim of the journal is to facilitate discussions and collaborations between specialists in applied mathematics, especially in the fields of numerical analysis and optimization, in the region and worldwide.
Our vision is that scholars from different applied mathematical research disciplines, pool their insight, knowledge and efforts by communicating via this international journal.
In order to assure high quality of the journal, each article is reviewed by subject-qualified referees.
Our expectations for IJNAO are as high as any well-known applied mathematical journal in the world. We trust that by publishing quality research and creative work, the possibility of more collaborations between researchers would be provided. We invite all applied mathematicians especially in the fields of numerical analysis and optimization to join us by submitting their original work to the Iranian Journal of Numerical Analysis and Optimization.

Mohammad Hadi Farahi

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Analysing panel flutter in supersonic flow by Hopf bifurcation

Z. Monfared* and Z. Dadi

Abstract

This paper is devoted to the study of a partial differential equation (PDE) governing panel motion in supersonic flow. This PDE can be transformed to an ODE by means of a Galerkin method. Here by using a criterion which is closely related to the Routh-Hurwitz criterion, we investigate the mentioned transformed ODE from Hopf bifurcation point of view. In fact we obtain a region for existence of simple Hopf bifurcation for it. With the aid of Matlab and Hopf bifurcation tool, flutter and limit cycle oscillations of panel are verified. Moreover, Hopf bifurcation theory is used to analyse the flutter speed of the system.

Keywords: Panel flutter; Limit cycle; Hopf bifurcation; Routh-Hurwitz criterion; Vibrations.

1 Introduction

Aerodynamics is a branch of dynamics concerned with studying the motion of air, particularly when it interacts with a solid object such as an Aircraft structure.

On the other hand, flow-induced structural vibration is one of the most technical problems affecting the reliability, cost and safety of aircraft structures. The vibration caused by a fluid flowing around a body is known as flow-induced vibration. Flow-induced vibrations best describe the interaction that occurs between the fluid's dynamic forces and a structure's inertial, damping, and elastic forces. The study of flow-induced vibrations has rapidly developed in aeronautical and nonaeronautical engineering. In aeronautics, flow-induced vibration is often referred to as flutter. Flutter is the instability of aeronautics structures under unsteady aerodynamic loadings. Panel flutter

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is a phenomenon of self-exciting vibrations of skin panels of flight vehicle at high flight speeds resulting from the interaction between an elastic structure and the flow around the structure. Such vibrations typically have high amplitude and cause fatigue damage of skin panels. This flutter phenomenon was first observed during World War II; however, formal studies did not appear until the 1950s. Supersonic panel flutter is a key design consideration for some high-speed aerospace vehicles like spacecrafts and missiles, [14, 17, 20]. Moreover, for nonlinear systems flutter is usually interpreted as a limit cycle oscillation (LCO). If a limit cycle is created the system will oscillate forever.

Aeroelastic flutter is a catastrophic structural failure, which needs to be avoided within the flight envelope of an aircraft structure. Hence, engineering researchers have paid much attention to studying the flutter and limit cycle motions of thin panels in recent years (see [2, 12, 14, 15, 16, 17, 19, 20]).

Furthermore, Hopf bifurcation theory can be utilized as an important tool for the determination of the flutter and limit cycle vibrations of panels. In addition, Hopf bifurcation tool can be used to analyze the flutter speed of the system. Hence, with the use of thin panels in shuttles and large space stations, nonlinear dynamics, bifurcations, and the chaos of thin panels have become more and more important. In the past decade, researchers have made a number of studies into nonlinear oscillations, bifurcations, and the chaos of thin panels. Holmes [5] studied flow-induced oscillations and bifurcations of thin panels and gave a finite-dimensional analysis. Then based on the analysis in [5], Holmes and Marsden [7] considered an infinite-dimensional analysis for flow-induced oscillations and pitchfork and fold bifurcations of thin panels. Holmes [6] then simplified this problem to a two-degrees-of-freedom nonlinear system and used center manifolds and the theory of normal forms to study the degenerate bifurcations. Yang and Sethna [18] used an averaging method to study the local and global bifurcations in parametrically excited, nearly square plates. From the von Karman equation, they simplified this system to a parametrically excited two-degrees-of-freedom nonlinear oscillators and analyzed the global behaviour of the averaged equations. Based on the studies in [18], Feng and Sethna [3] made use of the global perturbation method developed by Kovacic and Wiggins [8] to study further the global bifurcations and chaotic dynamics of a thin panel under parametric excitation, and obtained the conditions in which Silnikov-type homoclinic orbits and chaos can occur. Zhang et al. [19] investigated both the local and global bifurcations of a simply supported at the fore-edge, rectangular thin plate subjected to transversal and in-plane excitations simultaneously.

In this paper, a problem of flow-induced oscillations, that of panel flutter is considered. In fact here we investigate a partial differential equation which describes panel motion and obtain a region for the existence of a special type of Hopf bifurcation for it. This type of Hopf bifurcation occurs where a pair of complex conjugate eigenvalues of the Jacobian matrix passes through the imaginary axis while all other eigenvalues have negative real parts. Furthermore, the existence of this type of Hopf bifurcation leads to flutter and limit

cycle motions of the panel which can cause failure of the structure. To the best of our knowledge, it is the first time that such a region for the existence of periodic solutions and Hopf bifurcation is being investigated. Moreover, by means of Matlab and the fourth and fifth-order Runge-Kutta (RK-45) method we do some numerical simulations. These simulations present our theoretical results and flutter and limit cycle oscillations of thin panel. Moreover, the flutter speed is obtained by using Hopf bifurcation tool.

2 Preliminaries

In this section, we state some mathematical concepts and basic results.

2.1 A criterion based on the Routh-Hurwitz criterion for the existence of simple Hopf bifurcation

Important criterion that gives necessary and sufficient conditions for all of the roots of the characteristic polynomial to lie in the left half of the complex plane is known as the Routh-Hurwitz Criterion. This criterion is stated in the next theorem, see [16].

Theorem 2.1. *(Routh-Hurwitz Criterion).Consider a polynomial of the form*

$$a_k z^k + a_{k-1} z^{k-1} + a_{k-2} z^{k-2} + \ \ldots \ + a_0,$$

the roots of this polynomial lie in the open left half-plane if and only if all the leading principal minors of the $k \times k$ matrix

$$Q = \begin{pmatrix} a_1 & a_0 & \cdots & 0 \\ a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{2k-1} & a_{2k-2} & \cdots & a_k \end{pmatrix},$$

are positive and $a_k > 0$; we assume that $a_j = 0$ if $j < 0$ or $j > k$.

In dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden qualitative or topological change in its behaviour. In general, at a bifurcation point, the local stability properties of equilibria, periodic orbits or other invariant sets change. Moreover, in many applications we are concerned with a special type of Hopf bifurcations, where a pair of complex conjugate eigenvalues of the Jacobian matrix passes through the imaginary axis while all other eigenvalues have negative real parts. These are called simple Hopf

bifurcations in [10], in order to distinguish them from the Hopf bifurcations with some other eigenvalues on the right half plane. Now consider the system

$$\dot{X} = f(X, \mu), \qquad X \in \mathbb{R}^n,\ \mu \in \mathbb{R}. \tag{1}$$

We use the notation in [4, 13] and mention the following theorem which states the sufficient conditions for existence of Hopf bifurcation.

Theorem 2.2. *Suppose that system* (1)*, has an equilibrium* (x_0, μ_0) *at which the following properties are satisfied:*

(SH1) $D_x f_{\mu_0}(x_0)$ *has a simple pair of pure imaginary eigenvalues and other eigenvalues have negative real parts. Therefore, there exists a smooth curve of equilibria* $(x(\mu), \mu)$ *with* $x(\mu_0) = x_0$*. The eigenvalues* $\lambda(\mu), \bar{\lambda}(\mu)$ *of* $D_x f_{\mu_0}(x(\mu))$ *which are imaginary at* $\mu = \mu_0$ *vary smoothly with* μ*. Furthermore, if,*

(SH2) $\frac{d}{d\mu}(Re\lambda(\mu))\Big|_{\mu=\mu_0} = d \neq 0,$

then Hopf bifurcation will occur.

Proof. See [4]. □

Even though numerical computation of eigenvalues is feasible, it is ideal to have a criterion stated in terms of the coefficient of the characteristic polynomials rather than the traditional Hopf bifurcation criterion which is based on the property of eigenvalues. Specially for higher dimensional systems with many parameters, this criterion will be more convenient. See [10].
We denote the characteristic polynomial of the Jacobian matrix $J(\mu)$ of (1) as:

$$P(\lambda; \mu) = det(\lambda I_n - J(\mu)) = p_0(\mu) + p_1(\mu)\lambda + ... + p_n(\mu)\lambda^n,$$

where every $p_i(\mu)$ is a smooth function of μ, and $p_n(\mu) = 1$. And we consider the case $p_0(\mu) > 0$, because there is not any nonnegative real root. Let

$$L_n(\mu) = \begin{pmatrix} p_1(\mu) & p_0(\mu) & \cdots & 0 \\ p_3(\mu) & p_2(\mu) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{2n-1}(\mu) & p_{2n-2}(\mu) & \cdots & p_n(\mu) \end{pmatrix},$$

where $p_i(\mu) = 0$ if $i < 0$ or $i > n$. Moreover, when $p_0(\mu) > 0$ by the R-H criterion the polynomial $P(\lambda; \mu)$ of λ has all roots with negative real parts if and only if the following n principal sub determinants of $L_n(\mu)$ are positive:

- $D_1(\mu) = det(L_1(\mu)) = p_1(\mu) > 0$
- $D_2(\mu) = det(L_2(\mu)) = det \begin{pmatrix} p_1(\mu) & p_0(\mu) \\ p_3(\mu) & p_2(\mu) \end{pmatrix} > 0$

  $\vdots$

- $D_n(\mu) = det(L_n(\mu)) > 0$. Since $D_n(\mu) = p_n(\mu)D_{n-1}(\mu)$ and in our case $p_n(\mu) = 1$ the R-H criterion conditions can be stated as

- $p_0(\mu) > 0, \quad D_1 > 0, \quad D_2 > 0, \quad ... \ , \quad D_{n-1} > 0.$

Theorem 2.3. *Assume there is a smooth curve of equilibria $(x(\mu), \mu)$ with $x(\mu_0) = x_0$ for (1). Then conditions (SH1) and (SH2) for a simple Hopf bifurcation are equivalent to the following conditions on the coefficients of the characteristic polynomial $P(\lambda; \mu)$:*

(1) $p_0(\mu_0) > 0, \quad D_1(\mu_0) > 0, \quad ... \ , \quad D_{n-2}(\mu_0) > 0, \quad D_{n-1}(\mu_0) = 0$

(2) $\frac{dD_{n-1}(\mu_0)}{d\mu} \neq 0.$

Proof. see [10]. □

2.2 Galërkin method

Suppose we wish to solve the following boundary value problem of partial differential equations over the interval $a \leq z \leq b$,

$$L[y(z,t)] + f(z,t) = 0,$$

$$y(a,t) = z_a, y(b,t) = z_b.$$

A Galërkin method is used to approximate the problem by a sequence of finite dimensional problems. In other words, we consider the problem as a flow defined on a space V and then choose the finite dimensional subspace $V_N \subset V$ of dimensional N and project our problem onto V_N. Reducing the problem to a finite dimensional vector subspace allows us to numerically compute u_N (the solution of PDE) as a finite linear combination of the basis vectors in V_N.

Now, let $\{\varphi_j\}_{j=1}^N$ be an orthonormal basis of the finite dimensional subspace V_N that satisfy the boundary conditions of the problem. Therefore, we can write

$$u_N(z,t) = \sum_{j=1}^{N} a_j(t)\varphi_j.$$

Through the use of orthogonal functions the error function E_N, representing the difference between the exact and approximate solution, is minimized such that

$$\int_a^b E_N(z,t).\varphi_j \ dz = 0, \quad \forall j = 1, 2, ..., N,$$

where,

$$E_N(z,t) = L[u_N(z,t)] + f(z,t).$$

Each term in above equation gives an ODE in time for the N coefficients $\{a_j(t)\}_{j=1}^{N}$ and these must be solved numerically. For more information see [5, 7].

2.3 Dynamic pressure

In fluid dynamics, dynamic pressure (indicated with q, or Q, and sometimes called velocity pressure) is the quantity of air measured by most airspeed instruments and defined by

$$q = \frac{1}{2}\rho v^2,$$

where ρ and v are density and velocity of the flow respectively. See [1], Section 3.5.

3 Formulation of the problem

Consider a supersonic stream of fluid passes above a thin plate with the length 1, fixed at the edges $z = 0$ and $z = 1$ .The panel is simultaneously subjected to an in-plane tensile load Γ. The fluid velocity is characterized in terms of the dynamic pressure q, see Figure 1. Using nondimensional quantities, and assuming that the panel bends in a cylindrical mode (so that $w(z, y, t) = w(z, t)$ is independent of y), the following nonlinear partial-differential equation, which is essentially a one dimensional version of the von Karman equations is considered for a thin plate:

$$\begin{aligned} &w_{tt} + \alpha w_{tzzzz} + \sqrt{q}\delta w_t - \{\Gamma + k\int_0^1 w_z^2 dz + \sigma \int_0^1 w_z.w_{tz} dz\}w_{zz} \\ &+w_{zzzz} + qw_z = 0, \end{aligned} \tag{2}$$

see [4, 7]. Here $w = w(z, t)$ is the transverse displacement of the panel, $\alpha, \sigma \geq 0$ are (linear) viscoelastic damping parameters associated with the panel, $\delta > 0$ represents fluid damping and $k > 0$ is a measure of the nonlinear axial (membrance) restoring forces generated in the panel due to transverse displacement. Moreover, all these parameters are assumed to be fixed, except q which can vary.
Now, consider equation (2) with the following simply supported boundary conditions

$$w(0, t) = w_{zz}(0, t) = w(1, t) = w_{zz}(1, t) = 0. \tag{3}$$

A plate subjected to a compressive in-plane load with fluid flow over its surface may undergo complex motions resulting in dynamic instabilities (flutter) and associated limit cycle motions.

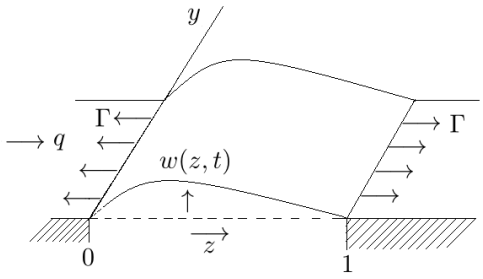


Figure 1: The panel flutter problem

4 Bifurcation analysis

It does not seem possible to solve the equation (2) explicitly. As it is mentioned in [7] by using a Galërkin method the partial differential equation (2) together with the boundary conditions (3) can be transformed to a ordinary differential equation. In fact in [7] because of the simply supported boundary condition (3), the following family of orthogonal basis

$$\{\varphi_j(z)\}_{j=1}^{N} = \{sin(j\pi z)\}_{j=1}^{N}$$

is chosen. Then by writing , $w(z,t) = \sum_{j=1}^{N} a_j(t)\varphi_j(z)$ and applying the Galërkin procedure for $N = 2$ (two modes) to the governing Equation (2) and using the orthonormality of the bases and the relationships

$$\int_0^1 w_j''''.w_s \; dz = \int_0^1 w_j''.w_s'' \; dz,$$

$$\int_0^1 w_j''.w_s \; dz = -\int_0^1 w_j'.w_s' \; dz,$$

the following ODE in the time dependent amplitude coefficients $a_j(t)$ is obtained.

$$\dot{x} = A_q x + f(x), \qquad x \in \mathbb{R}^4, q \in \mathbb{R}, \tag{4}$$

where

$$x = \begin{pmatrix} a_1 \\ a_2 \\ \dot{a_1} \\ \dot{a_2} \end{pmatrix},$$

$$A_q = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \pi^2(\Gamma - \pi^2) & 8q/3 & -(\alpha\pi^4 + \sqrt{q}\delta) & 0 \\ -8q/3 & 4\pi^2(\Gamma - 4\pi^2) & 0 & -(16\alpha\pi^4 + \sqrt{q}\delta) \end{pmatrix},$$

and

$$f = \begin{pmatrix} 0 \\ 0 \\ -f_1 \\ -f_2 \end{pmatrix},$$

in which

$$f_1 = \tfrac{\pi^4}{2}\{k(a_1^2 + 4a_2^2) + \sigma(a_1\dot{a_1} + 4a_2\dot{a_2})\}a_1,$$

$$f_2 = 2\pi^4\{k(a_1^2 + 4a_2^2) + \sigma(a_1\dot{a_1} + 4a_2\dot{a_2})\}a_2.$$

Moreover, $a_i = x_i (i = 1, 2)$ are the amplitudes of normal two modes.

Now we investigate Hopf bifurcations of system (2) for the trivial equilibrium position $x = 0$ or $w(z, t) \equiv 0$ by using a criterion which is closely related to the Routh-Hurwitz (R-H) criterion to obtain a region of simple Hopf bifurcation parameters. For this purpose, we mention the sufficient conditions for existence of simple Hopf bifurcation of system (4) in the following theorem.

Theorem 4.1. *Suppose that for $q = q_0$ and the trivial equilibrium position $x = 0$ of (4), the following relations satisfy:*

1) $p_0(q_0) > 0$

2) $p_1(q_0) > 0$

3) $det \begin{pmatrix} p_1(q_0) & p_0(q_0) \\ p_3(q_0) & p_2(q_0) \end{pmatrix} > 0$

4) $D_3(q_0) = det \begin{pmatrix} p_1(q_0) & p_0(q_0) & 0 \\ p_3(q_0) & p_2(q_0) & p_1(q_0) \\ 0 & 1 & p_3(q_0) \end{pmatrix} = 0$

5) $\dfrac{dD_3(q_0)}{dq} \neq 0,$

where

$$p_0(q_0) = 4\pi^2(\Gamma - \pi^2)(\Gamma - 4\pi^2) + \frac{64q_0^2}{9},$$
$$p_1(q_0) = -\pi^2\{(\Gamma - \pi^2)(16\alpha\pi^4 + \sqrt{q_0}\delta) + 4(\Gamma - 4\pi^2)(\alpha\pi^4 + \sqrt{q_0}\delta)\},$$
$$p_2(q_0) = (\alpha\pi^4 + \sqrt{q_0}\delta)(16\alpha\pi^4 + \sqrt{\pi}\delta) - \pi^2(5\Gamma - 17\pi^2),$$
$$p_3(q_0) = 17\alpha\pi^4 + 2\sqrt{q_0}\delta.$$

Then q_0 is a simple Hopf bifurcation value for system (4) at the trivial equilibrium position $x = 0$.

Proof. By computing the characteristic polynomial of (4) and using theorem 2.3., the above assertion can be proved. □

The panel transverse displacement and velocity for two modes ($N = 2$) can be evaluated by the following equations

$$w(z,t) = \sum_{j=1}^{2} a_j(t) sin(j\pi z), \tag{5}$$

$$w_t(z,t) = \sum_{j=1}^{2} \dot{a_j}(t) sin(j\pi z). \tag{6}$$

Therefore, by the above theorem we can find a region for existence of simple Hopf bifurcation for equation (4) and therefore for equation (2).

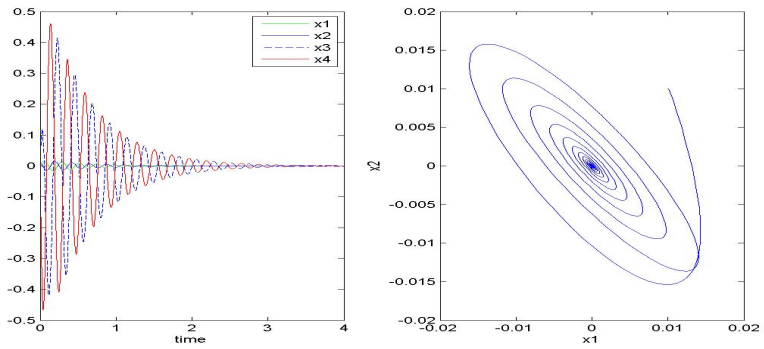
5 Numerical simulation

Here numerical simulations are carried out to support our theoretical results and show panel flutter.

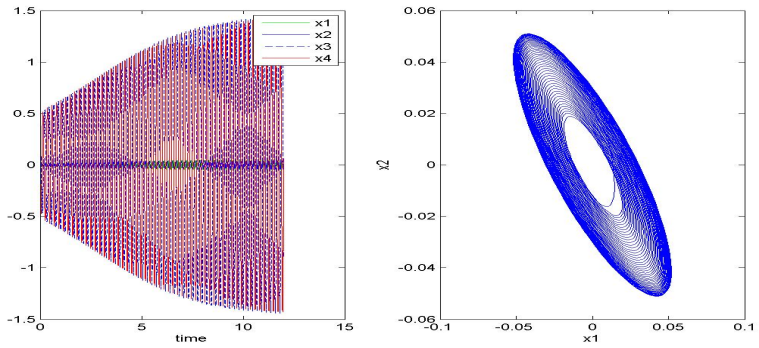
Example. Consider system (4), by the aid of software Auto for $\delta = 1, \Gamma = 2, \alpha = 0$ and $q_0 \simeq 323.24789021$ we can show that this system undergoes Hopf bifurcation at the equilibrium $x = (x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$. Moreover, these values of parameters by [4, 5, 7] are physically meaningful. In addition, for the mentioned values of parameters the characteristic polynomial of the Jacobian matrix of (4) at x for q_0 is :

$$P(\lambda; q_0) \simeq 857953.77463 + 27998.10001\lambda + 1880.50639\lambda^2 + 35.95819\lambda^3 + \lambda^4.$$

So



(a)



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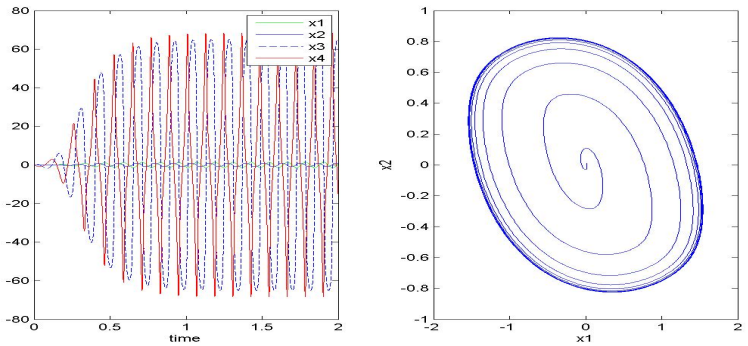
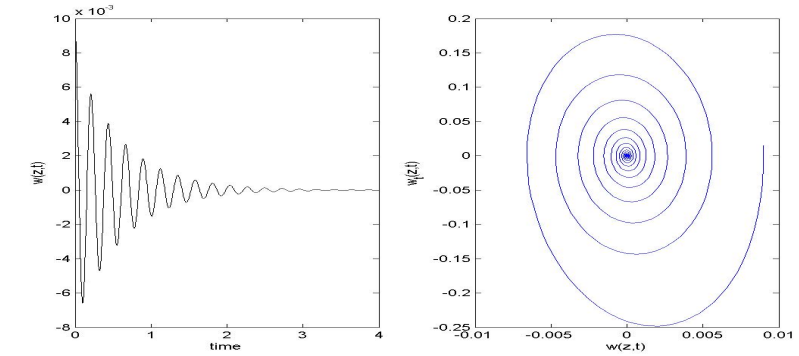
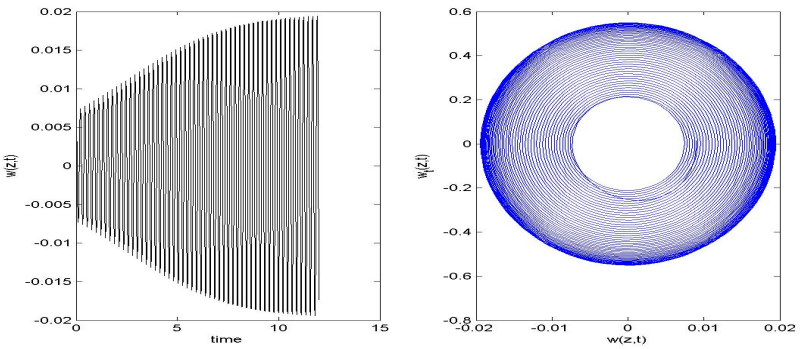


(c)

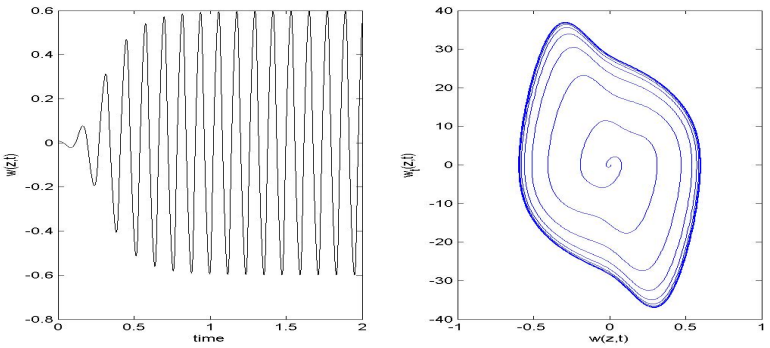
Figure 2: The related trajectories in the phase space $x_1 - x_2$ and time history for different values of q. Initial condition is very close to the equilibrium $x = 0$: (a)$q = 300$; (b) $q = 327$; (c) $q = 1000$



(a)



(b)



(c)

Figure 3: Phase Portrait in the space $w - w_t$ and time History of the panel transverse displacement for different values of q : (a)$q = 300$; (b) $q = 327$; (c) $q = 1000$

$$p_0(q_0) \simeq 857953.774634701 > 0,$$

$$D_1(q_0) = p_1(q_0) \simeq 27998.1000129 > 0,$$

$$D_2(q_0) = det \begin{pmatrix} p_1(q_0) & p_0(q_0) \\ p_3(q_0) & p_2(q_0) \end{pmatrix} \simeq 21800139.664474831 > 0,$$

$$D_3(q_0) = det \begin{pmatrix} p_1(q_0) & p_0(q_0) & 0 \\ p_3(q_0) & p_2(q_0) & p_1(q_0) \\ 0 & 1 & p_3(q_0) \end{pmatrix} = 0,$$

and

$$\frac{dD_3(q_0)}{d\omega} \simeq -4937513.3537057098 \neq 0.$$

Therefore, by Theorem 4.1., $q_0 \simeq 323.24789021876$ is a simple Hopf bifurcation value for (4) at x. Furthermore, for q_0 the Jacobian matrix A_q at x has a pair of pure imaginary eigenvalues $\lambda_1, \lambda_2 \simeq \pm 27.90392896678841 89i$, and a couple of complex conjugate eigenvalues with negative real part $Re\lambda_3 = Re\lambda_4 \simeq -17.979095923287$.

We solved the equation (4) by means of the fourth and fifth-order Runge-Kutta (RK-45) method. Then by numerical simulations we showed the occurrence of Hopf bifurcation. The related trajectories in the phase space $x_1 - x_2$ (the space of the amplitudes of two modes) and the time history responses for different values of parameter q are presented in Figure 2.

Moreover, by equations (5) and (6), the panel transverse displacement ($w(z,t)$) and velocity ($w_t(z,t)$) for two modes can be determined. So, the related trajectories in the phase space $w - w_t$ and the time history responses of the panel transverse displacement for different values of parameter q are illustrated in Figure 3.

As it is illustrated in Figures 2 and 3, the equilibrium point $x = 0$ or $w(z,t) \equiv 0$ is a stable focus when the dynamic pressure of flow is less than q_0. In this case by passing the time the amplitude of panel vibrations will vanish. While for $q > q_0$ the equilibrium point $x = 0$ or $w(z,t) \equiv 0$ turns out to be an unstable focus surrounded by a stable limit cycle. That means that by passing the time, the amplitude of panel oscillations will increase and finally panel will vibrate with a fixed period for ever. Furthermore, in this case for large values of the dynamic pressure q, panel vibrations can have high amplitude and cause catastrophic failure of the structure. This increase of amplitude will be faster for larger values of q.

In addition, since the Hopf bifurcation occurs for the dynamic pressure $q_0 \simeq 323.24789021876\ Pa$, due to Section 2.3., by knowing the density of flow (air) the flutter speed can be obtained. For example at sea level and at 15 $^\circ C$, air has a density of approximately 1.225 $\frac{kg}{m^3}$. Hence, for this density of air the flutter speed is approximately 22.97284609 in the absence of linear

viscoelastic damping parameter α. In Figure 3 our simulations show flutter and limit cycle oscillations (LCO) of the panel without considering the viscoelastic damping parameter α.

6 Conclusion

In this paper, we extended the previous results in [5, 6, 7, 8, 17, 18, 19] to study the vibrations of a thin panel fixed at two edges. Indeed we found a region for existence of simple Hopf bifurcation for a partial differential equation governing panel motion. Because, the existence of simple Hopf bifurcation can lead to flutter and limit cycle oscillations of the panel. Numerical simulations were carried out by using the fourth and fifth-order Runge-Kutta method, to support our analytical results. In fact by simulations and Hopf bifurcation theory, we showed the occurrence of flutter and limit cycle motions of thin panel. Then Hopf bifurcation tool was used to calculate the flutter speed of the system. Moreover, numerical simulations presented vibrations of thin panel can have high amplitude which cause damage of panel.

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Hopf bifurcation in a general n-neuron ring network with n time delays

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Abstract

In this paper, we consider a general ring network consisting of n neurons and n time delays. By analyzing the associated characteristic equation, a classification according to n is presented. It is investigated that Hopf bifurcation occurs when the sum of the n delays passes through a critical value. In fact, a family of periodic solutions bifurcate from the origin, while the zero solution loses its asymptotically stability. To illustrate our theoretical results, numerical simulation is given.

Keywords: Ring network; Stability; Periodic solution; Hopf bifurcation; Time delay.

1 Introduction

Since Hopfield constructed a simplified neural network (NN) model [7, 15], the dynamic behaviors (such as stability, periodic oscillatory, limit cycles, bifurcation and chaos) of continuous-time neural networks have received much attention due to their applications in optimization, signal processing, image processing, solving nonlinear algebraic equations, pattern recognition, associative memories and so on (see, [3, 10, 11, 18] and references therein). It is well known that time delays exist in the signal transmission, thus Marcus and Westervelt proposed an NN model with delays, based on the Hopfield NN model [12]. The time delays are regarded as parameters; consequently, periodic solutions often appear as solutions of delay differential equations (DDEs) through Hopf bifurcation. For example, existence of periodic solutions, in a special type of DDEs, has been discussed in [9]. In [8, 16, 17], the authors used Hopf bifurcation theory to study some kinds of neural networks.

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A ring network is a network topology in which each node connects to exactly two other nodes, forming a single continuous pathway for signals through each node a ring. Data travel from node to node, with each node along the way handling every packet.

Ring networks have been found in a variety of neural structures such as cerebellum [5], and even in chemistry and electrical engineering. In the field of neural networks, rings are studied to gain insight into the mechanisms underlying the behavior of recurrent networks. In [16], Wang and Han investigated the continuous-time bidirectional ring network model. Many results have been reported in the literature on the dynamics of ring neural networks (see [2, 18]). The local and global stability of ring networks with delays have been discussed in [2, 18]. In [1], stability analysis of a delayed ring network model has been discussed.

In this paper, we study Hopf bifurcation for a kind of DDE system. In fact, we consider a general ring network with n neurons and n time delays, which is described by the following DDE system:

$$\begin{cases} \dot{x}_1(t) = -r_1 x_1(t) + g_1(x_1(t)) + f_1(x_n(t-\tau_n)), \\ \dot{x}_2(t) = -r_2 x_2(t) + g_2(x_2(t)) + f_2(x_1(t-\tau_1)), \\ \dot{x}_3(t) = -r_3 x_3(t) + g_3(x_3(t)) + f_3(x_2(t-\tau_2)), \\ \vdots \\ \dot{x}_{n-1}(t) = -r_{n-1} x_{n-1}(t) + g_{n-1}(x_{n-1}(t)) + f_{n-1}(x_{n-2}(t-\tau_{n-2})), \\ \dot{x}_n(t) = -r_n x_n(t) + g_n(x_n(t)) + f_n(x_{n-1}(t-\tau_{n-1})), \end{cases} \tag{1}$$

where $r_i \geq 0$ $(i = 1, 2, ..., n)$ denotes the stability of internal neuron processes, and $x_i(t)$ $(i = 1, 2, ..., n)$ represents the state of the ith neuron at time t. f_i and g_i $(i = 1, 2, ..., n)$ are the activation function and nonlinear feedback function, respectively. Also, $\tau_i \geq 0$ $(i = 1, 2, ...n)$ describes the synaptic transmission delay. Figure 1 shows system (1) schematically.

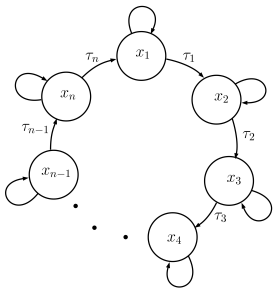

Figure 1: A general n-neuron ring with n delays

The properties of periodic solutions are also important in many applications. In fact, various local periodic solutions can arise from the different equilibrium points of ring networks by applying Hopf bifurcation technique, therefore the study of Hopf bifurcation is very important. In [16], the simplified three-neuron bidirectional ring network has been investigated. Four-neuron and five-neuron networks with multiple delays have been studied in [8, 17].

In this paper, we discuss Hopf bifurcation on system (1) generally, not for a particular n. To the best of our knowledge, it has not been done before. We would like to point out that in [1], although stability analysis of the system has been presented, but the authors didn't discuss Hopf bifurcation on the system in all cases. First, we take the sum of the delays $\tau_1 + \tau_2 + ... + \tau_n$ as a parameter τ. By classifying based on n, we study the associated characteristic equation. To calculate the critical value τ_0 for Hopf bifurcation, we generalize and modify the methods proposed in [8, 9]. Then we will show that the zero solution loses its stability and Hopf bifurcation occurs when τ passes through the critical value τ_0. We would like to point out that it is the first time to deal with Hopf bifurcation analysis of system (1). This paper is organized in five sections. In Sect.2, we give the necessary preliminaries. In Sect.3, we will study Hopf bifurcation on system (1). To illustrate the results, some numerical simulations are presented in Sect.4. Finally, in Sect.5, some main conclusions are stated.

2 Preliminaries

2.1 Delay Differential Equation

There is always a time delay in many natural phenomena, because a finite time is required to sense information and then react to it. DDEs are differential equations in which the derivatives of some unknown functions at present time are dependent on the values of the functions at previous times. A general delay differential equation for $x(t) \in \mathrm{R^n}$ takes the form

$$\dot{x}(t) = f(t, x(t), x_t), \tag{2}$$

where $x_t(\theta) = x(t + \theta)$ and $-\tau \leq \theta \leq 0$. Observe that $x_t(\theta)$ with $-\tau \leq \theta \leq 0$ represents a portion of the solution trajectory in a recent past. In this equation f is a functional operator from $\mathrm{R} \times \mathrm{R^n} \times C^1(\mathrm{R}, \mathrm{R^n})$ to $\mathrm{R^n}$. Similar to ODEs, many properties of linear DDEs can be characterized and analyzed using the characteristic equation.

The linearization of system (2) at the equilibrium point x_0 is

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_1) + \cdots + A_m x(t - \tau_m), \tag{3}$$

where $A_j = D_{j+1}f(x_0, ..., x_0), (j = 0, 1, ..., m)$ and $D_j f$ is the Jacobian of f corresponding to its jth component.

Substitude $x(t) = e^{\lambda t}v$, $v \in R^n$ into (3), we have

$$[\lambda I - A_0 - \sum_{j=1}^{m} A_j e^{-\lambda \tau_j}]e^{\lambda t}v = 0.$$

Therefore the characteristic equation associated with (2) is

$$det(\lambda I - A_0 - \sum_{j=1}^{m} A_j e^{-\lambda \tau_j}) = 0. \tag{4}$$

For further details, see [4].

2.2 Hopf bifurcation

In this section, we study bifurcations that occur in C^1 system

$$\dot{x} = f(x, \mu), \tag{5}$$

depending on a parameter $\mu \in \mathrm{R}$, at nonhyperbolic equilibrium points. In the following, we will give definition of a structurally stable vector field or dynamical system.

Definition 2.1. *Let E be an open subset of $\mathrm{R^n}$. A vector field $f \in C^1(E)$ is said to be structurally stable if there is an $\varepsilon > 0$ such that for all $g \in C^1(E)$ with $\|f - g\| < \varepsilon$, f and g are topologically equivalent on E; i.e., there is a homeomorphism $H : E \to E$ which maps trajectories of $\dot{x} = f(x)$ onto trajectories of $\dot{x} = g(x)$, and preserves their orientation by time. In this case, we also say that the dynamical system $\dot{x} = f(x)$ is structurally stable. If a vector field $f \in C^1(E)$ is not structurally stable, then f is said to be structurally unstable.*

The qualitative behavior of the solution set of system (5) depending on a parameter $\mu \in \mathrm{R}$, changes as the vector field f passes through a point in the bifurcation set or as the parameter μ varies through a bifurcation value μ_0. A value μ_0 of the parameter μ in equation (5) for which the C^1 vector field $f(x, \mu_0)$ is not structurally stable is called a bifurcation value. For more information see [13].

Theorem 2.2. *Suppose that system (5), has an equilibrium (x_0, μ_0) at which the following properties are satisfied:*

a *: $D_x f_{\mu_0}(x_0)$ has a simple pair of pure imaginary eigenvalues and other eigenvalues have negative real parts. Therefore, there exist a smooth*

curve of equilibria $(x(\mu), \mu)$ *with* $x(\mu_0) = x_0$. *The eigenvalues* $\lambda(\mu)$, $\bar{\lambda}(\mu)$ *of* $D_x f_{\mu_0}(x(\mu))$ *which are imaginary at* $\mu = \mu_0$ *vary smoothly with* μ.

Furthermore, if,

$$\mathbf{b} \ : \ \frac{d}{d\mu}(Re\lambda(\mu))|_{\mu=\mu_0} = d \neq 0,$$

then Hopf bifurcation will occur.

Proof. see [6]. □

When the parameter μ passes through the critical value μ_0, according theorem 2.2, Hopf bifurcation occurs. In fact, a family of periodic solutions appear or disappear. If the equilibrium point is asymptotically stable for $\mu < \mu_0$, when μ passes through μ_0, a family periodic solutions bifurcate from the equilibrium point. In this case, we say that supercritical Hopf bifurcation occurs. But, if a branch of periodic solutions exist for $\mu < \mu_0$ and while μ passes through μ_0, these periodic solutions disappear, we say that subcritical Hopf bifurcation happens. For more details, see [6].

3 Main results

To establish the main results for system (1), it is necessary to make the following assumption

$$f_i, g_i \in C^1, \ f_i(0) = g_i(0) = 0, \text{ for } i = 1, 2, ..., n. \tag{6}$$

It is easily seen that the origin (0,0,...,0) is an equilibrium point of (1). Under the hypothesis (6), the linearization of (1) around the origin gives

$$\begin{cases} \dot{x}_1(t) = -k_1 x_1(t) + f_1^{'}(0) x_n(t - \tau_n), \\ \dot{x}_2(t) = -k_2 x_2(t) + f_2^{'}(0) x_1(t - \tau_1), \\ \vdots \\ \dot{x}_{n-1}(t) = -k_{n-1} x_{n-1}(t) + f_{n-1}^{'}(0) x_{n-2}(t - \tau_{n-2}), \\ \dot{x}_n(t) = -k_n x_n(t) + f_n^{'}(0) x_{n-1}(t - \tau_{n-1}), \end{cases} \tag{7}$$

where $k_i = r_i - g_i^{'}(0)$, $i = 1, 2, ..., n$. The characteristic equation of (2) is

$$\lambda^n + a_1 \lambda^{n-1} + ... + a_{n-1}\lambda + a_n + be^{-\lambda\tau} = 0, \tag{8}$$

where

$$a_1=\sum_{i=1}^{n} k_i, \quad a_2=\sum_{1\leqslant i<j\leqslant n} k_i k_j, \quad \cdots,$$
$$a_{n-1}=\sum_{1\leqslant \underbrace{i<j<l<...<m}_{\text{n-1}}\leqslant n} k_i k_j k_l ... k_m,$$
$$a_n=\prod_{i=1}^{n} k_i, \quad b=-\prod_{i=1}^{n} f_i^{'}(0), \quad \tau=\sum_{i=1}^{n}\tau_i. \tag{9}$$

Denote

$$p(\lambda)=\lambda^n+a_1\lambda^{n-1}+...+a_{n-1}\lambda+a_n,$$

then equation (8) becomes

$$p(\lambda)+be^{-\lambda\tau}=0. \tag{10}$$

To study Hopf bifurcation, it is necessary to discuss the existence of pure imaginary roots of (10). Letting $\lambda=i\omega$, and substituting this into (10), we have

$$A+iB+b(cos\omega\tau-isin\omega\tau)=0, \tag{11}$$

where

$$A=Re\{p(i\omega)\}, \quad B=Im\{p(i\omega)\}. \tag{12}$$

Separating the real and imaginary parts of (11), we get

$$A+bcos\omega\tau=0, \tag{13}$$

and

$$B-bsin\omega\tau=0. \tag{14}$$

We rewrite the equations (13) and (14), as follows:

$$A=-bcos\omega\tau, \tag{15}$$

and

$$B=bsin\omega\tau. \tag{16}$$

Squaring both sides of (15) and (16), and adding them up gives

$$A^2+B^2=b^2. \tag{17}$$

Now, according to (12), it is easy to use computer to calculate the roots of (17). Then we can get the time delay τ, by substituting ω in (15).

To find the solutions for equation (17), we consider the following cases:

Case (a) : $n=4k$ $(k\in\mathbb{N})$,

Case (b) : $n=4k+1$ $(k\in\mathbb{N})$,

Case (c) : $n = 4k + 2$ $(k \in \mathbb{N} \cup \{0\})$,

Case (d) : $n = 4k + 3$ $(k \in \mathbb{N} \cup \{0\})$,

In case (a), from (12) and the definition of $p(\lambda)$, we can get

$$\begin{cases} A = \omega^n - a_2\omega^{n-2} + a_4\omega^{n-4} - ... + a_n, \\ B = -a_1\omega^{n-1} + a_3\omega^{n-3} - a_5\omega^{n-5} + ... + a_{n-1}\omega. \end{cases} \tag{18}$$

Using (17) and (18), we obtain

$$(\omega^n - a_2\omega^{n-2} + a_4\omega^{n-4} - ... + a_n)^2 + (-a_1\omega^{n-1} + a_3\omega^{n-3} - a_5\omega^{n-5} + ... + a_{n-1}\omega)^2 = b^2. \tag{19}$$

In case (b), the definition of $p(\lambda)$ and (12) lead to

$$\begin{cases} A = a_1\omega^{n-1} - a_3\omega^{n-3} + a_5\omega^{n-5} - ... + a_n, \\ B = \omega^n - a_2\omega^{n-2} + a_4\omega^{n-4} - ... + a_{n-1}\omega. \end{cases} \tag{20}$$

Substituting (20) in (17) gives

$$(a_1\omega^{n-1} - a_3\omega^{n-3} + a_5\omega^{n-5} - ... + a_n)^2 + (\omega^n - a_2\omega^{n-2} + a_4\omega^{n-4} - ... + a_{n-1}\omega)^2 = b^2. \tag{21}$$

In case (c), A and B can be calculated as follows:

$$\begin{cases} A = -\omega^n + a_2\omega^{n-2} - a_4\omega^{n-4} + ... + a_n, \\ B = a_1\omega^{n-1} - a_3\omega^{n-3} + a_5\omega^{n-5} - ... + a_{n-1}\omega. \end{cases} \tag{22}$$

From the equations (17) and (22), we have

$$(-\omega^n + a_2\omega^{n-2} - a_4\omega^{n-4} + ... + a_n)^2 + (a_1\omega^{n-1} - a_3\omega^{n-3} + a_5\omega^{n-5} - ... + a_{n-1}\omega)^2 = b^2. \tag{23}$$

In case (d), from the definition of $p(\lambda)$ and (12), we can compute

$$\begin{cases} A = -a_1\omega^{n-1} + a_3\omega^{n-3} - a_5\omega^{n-5} + ... + a_n, \\ B = -\omega^n + a_2\omega^{n-2} - a_4\omega^{n-4} + ... + a_{n-1}\omega. \end{cases} \tag{24}$$

By using the equations (17) and (24), we can get

$$(-a_1\omega^{n-1} + a_3\omega^{n-3} - a_5\omega^{n-5} + ... + a_n)^2 + (-\omega^n + a_2\omega^{n-2} - a_4\omega^{n-4} + ... + a_{n-1}\omega)^2 = b^2. \tag{25}$$

In all the above cases, after simplification, we can easily see that the equations (19), (21), (23) and (25) lead to

$$\omega^{2n} + e_1\omega^{2n-2} + e_2\omega^{2n-4} + ... + e_{n-1}\omega^2 + e_n = 0, \tag{26}$$

where the coefficients e_i $(i = 1, 2, ..., n)$, can be calculated as follows:

$$e_1 = a_1^2 - 2a_2,\ e_2 = a_2^2 - 2a_1a_3,\ \cdots,\ e_{n-1} = a_{n-1}^2 - 2a_na_{n-2},\ e_n = a_n^2 - b^2. \tag{27}$$

Letting $z = \omega^2$, then equation (26) becomes

$$z^n + e_1z^{n-1} + e_2z^{n-2} + ... + e_{n-1}z + e_n = 0. \tag{28}$$

Thus, the fact that equation (28) has positive roots is a necessary condition for the existence of the pure imaginary roots of equation (8).

Let

$$h(z) = z^n + e_1z^{n-1} + e_2z^{n-2} + ... + e_{n-1}z + e_n. \tag{29}$$

In the following, we will give lemma to establish the distribution of positive real roots of equation (28).

Lemma 3.1. *If $e_n < 0$, then equation (12) has at least one positive root.*

Proof. Since $h(0) = e_n < 0$ and $lim_{z\to\infty}h(z) = +\infty$. Hence, there exists a $z_0 > 0$ such that $h(z_0) = 0$. □

Now, from (15) we get

$$\tau_k^{(j)} = \begin{cases} \frac{1}{\omega_k}(cos^{-1}(\frac{-A}{b}) + 2j\pi) \text{ if } \frac{B}{b} \geq 0 \\ \frac{1}{\omega_k}(2\pi - cos^{-1}(\frac{-A}{b}) + 2j\pi) \text{ if } \frac{B}{b} < 0 \end{cases} (j = 0, \pm1, \pm2, ...; k = 1, 2, ..., n), \tag{30}$$

where $w_k = \sqrt{z_k^*}$, and without loss of generality, z_k^* $(k = 1, ..., n)$ are the positive roots of (28). Therefore, we can define

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,...,n\}} \{\tau_k^{(0)}\}, \quad \omega_0 = \omega_{k_0}. \tag{31}$$

So, with the help of relations (17) and (31), ω_0 and τ_0 are obtained. Now, we show that system (1) undergoes Hopf bifurcation at the origin when $\tau = \sum_{i=1}^{n}\tau_i$ passes through τ_0. In all cases, the stability and Hopf bifurcation can be analyzed analogously.

Let

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau), \tag{32}$$

be the root of equation (8) near $\tau = \tau_k^{(j)}$ satisfying $\alpha(\tau_k^{(j)}) = 0$, $\omega(\tau_k^{(j)}) = \omega_k$. Then, the following lemma holds.

Lemma 3.2. *Suppose $h'(z_k^*) \neq 0$, where $h(z)$ is defined by (4) and $z_k^* = \omega_k^2$. Then $\pm i\omega_k$ is a pair of simple purely imaginary roots of equation (3) when $\tau = \tau_k^{(j)}$. Moreover,*

$$\frac{dRe\big(\lambda(\tau)\big)}{d\tau}|_{\tau=\tau_k^{(j)}} \neq 0.$$

Proof. From (17) and (12), equation (26) can be transformed into the following form

$$p(i\omega)\overline{p(i\omega)} - b^2 = 0. \tag{33}$$

From (29), we have

$$h(\omega^2) = p(i\omega)\overline{p(i\omega)} - b^2. \tag{34}$$

Differentiating both sides of equation (34) with respect to ω, we obtain

$$2\omega h^{'}(\omega^2) = i[p^{'}(i\omega)\overline{p(i\omega)} - p(i\omega)\overline{p^{'}(i\omega)}]. \tag{35}$$

If $i\omega_k$ is not simple, then ω_k must satisfy

$$\frac{d}{d\lambda}[p(\lambda) + be^{-\lambda\tau_k^{(j)}}]\big|_{\lambda=i\omega_k} = 0,$$

that implies

$$p^{'}(i\omega_k) + b(-\tau_k^{(j)})e^{-i\tau_k^{(j)}\omega_k} = 0.$$

Together with equation (10), we have

$$\tau_k^{(j)} = -\frac{p^{'}(i\omega_k)}{p(i\omega_k)}. \tag{36}$$

Thus, by (33), (35) and (36), we obtain

$$Im(\tau_k^{(j)}) = Im\Big\{-\frac{p^{'}(i\omega_k)}{p(i\omega_k)}\Big\} = Im\Big\{-\frac{p^{'}(i\omega_k)\overline{p(i\omega_k)}}{p(i\omega_k)\overline{p(i\omega_k)}}\Big\}$$

$$= i\frac{p^{'}(i\omega_k)\overline{p(i\omega_k)} - \overline{p^{'}(i\omega_k)}p(i\omega_k)}{2p(i\omega_k)\overline{p(i\omega_k)}} = \frac{\omega_k h^{'}(\omega_k^2)}{|p(i\omega_k)|^2}.$$

Since $\tau_k^{(j)}$ is real, i.e. $Im(\tau_k^{(j)}) = 0$, we have $h^{'}(z_k^*) = h^{'}(\omega_k^2) = 0$. Therefore, we get a contradiction to the condition $h^{'}(z_k^*) \neq 0$. This proves the first conclusion in the lemma. Differentiating both sides of equation (10) with respect to τ, we obtain

$$\big[p^{'}(\lambda) - b\tau e^{-\lambda\tau}\big]\frac{d\lambda}{d\tau} - b\lambda e^{-\lambda\tau} = 0,$$

which implies

$$\frac{d\lambda(\tau)}{d\tau} = \frac{b\lambda}{p^{'}(\lambda)e^{\lambda\tau} - b\tau} = \frac{b\lambda\big[\overline{p^{'}(\lambda)}e^{-\lambda\tau} - b\tau\big]}{|p^{'}(\lambda)e^{\lambda\tau} - b\tau|^2} = \frac{\lambda\big[-\overline{p^{'}(\lambda)}p(\lambda) - b^2\tau\big]}{|p^{'}(\lambda)e^{\lambda\tau} - b\tau|^2}.$$

It follows together with (35) that

$$\begin{aligned}\frac{d(Re\lambda(\tau))}{d\tau}\Big|_{\tau=\tau_k^{(j)}} &= \frac{Re\Big\{\lambda\big[-\overline{p'(\lambda)}p(\lambda)-b^2\tau\big]\Big\}}{|p'(\lambda)e^{\lambda\tau}-b\tau|^2}\Big|_{\tau=\tau_k^{(j)}}\\ &= \frac{i\omega_k\big[+p'(i\omega_k)\overline{p(i\omega_k)}-\overline{p'(i\omega_k)}p(i\omega_k)\big]}{2|p'(i\omega_k)e^{i\omega_k\tau_k^{(j)}}-b\tau_k^{(j)}|^2}\\ &= \frac{\omega_k^2 h'(\omega_k^2)}{|p'(i\omega_k)e^{i\omega_k\tau_k^{(j)}}-b\tau_k^{(j)}|^2}\\ &= \frac{\omega_k^2 h'(z_k^*)}{|p'(i\omega_k)e^{i\omega_k\tau_k^{(j)}}-b\tau_k^{(j)}|^2}\neq 0.\end{aligned}$$

Thus, $\frac{dRe(\lambda(\tau))}{d\tau}|_{\tau=\tau_k^{(j)}}\neq 0$. This completes the proof. □

By the well-known Routh-Hurwitz criteria, we can find the following set of conditions:

$$H_1=det\,(a_1)>0, H_2=det\begin{pmatrix}a_1 & 1\\ a_3 & a_2\end{pmatrix}>0,...,H_n=det\begin{pmatrix}a_1 & 1 & 0 & 0 & \dots & 0\\ a_3 & a_2 & a_1 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & 0 & \dots & a_n\end{pmatrix}>0. \tag{37}$$

To discuss the distribution of the roots of the exponential polynomial equation (8), we need the following result from Ruan and Wei [14].

Theorem 3.3. *Consider the exponential polynomial*

$$\begin{aligned}p(\lambda,e^{-\lambda\tau_1},...,e^{-\lambda\tau_m}) &= \lambda^n+p_1^{(0)}\lambda^{n-1}+..+p_{n-1}^{(0)}\lambda+p_n^{(0)}\\ &+[p_1^{(1)}\lambda^{n-1}+...+p_{n-1}^{(1)}\lambda+p_n^{(1)}]e^{-\lambda\tau_1}\\ &+...+[p_1^{(m)}\lambda^{n-1}+...+p_{n-1}^{(m)}\lambda+p_n^{(m)}]e^{-\lambda\tau_m},\end{aligned}$$

where $\tau_i\geq 0$ $(i=1,2,...,m)$, *and* $p_j^{(i)}$ $(i=0,1,2,...,m; j=1,2,...,n)$ *are constants. As* $(\tau_1,\tau_2,...,\tau_m)$ *vary, the sum of the orders of the zeros of* $p(\lambda,e^{-\lambda\tau_1},...,e^{-\lambda\tau_m})$ *on the open right half plane can change only if a zero appears on or crosses the imaginary axis.*

Now, we can state the following main theorem:

Theorem 3.4. *Suppose that conditions* (1), (5) *and* $h'(z_k^*)\neq 0$ *hold, where* $h(z)$ *is defined by* (4) *and* $z_k^*=\omega_k^2$. *Then, system* (1) *undergoes Hopf bifurcation at the origin when* $\tau=\sum_{i=1}^n\tau_i$ *passes through* τ_0, *and it has a branch of periodic solutions bifurcating from the zero solution near* $\tau=\tau_0$, *where* τ_0 *is defined by* (31).

Proof. By the well-known Routh-Hurwitz criteria, we can conclude that when (37) holds, all the roots of equation (8) at $\tau=0$, have negative real parts.

Hence, by using theorem 3.3, we conclude that the zero solution of system (1) is asymptotically stable when $\tau < \tau_0$. By using lemma 3.2, we can see that the conditions of Hopf bifurcation are satisfied at $\tau = \tau_0$ in system (1), and so Hopf bifurcation occurs at the origin. In addition, a family of periodic solutions appear as τ passes through τ_0. □

4 Numerical simulations

In this section, numerical simulations are presented to support our theoretical results. We will study system (1) for the case $n = 7$. Consider the following system

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) + tanh(x_1(t)) + 2tanh(x_7(t-\tau_7)), \\ \dot{x}_2(t) = -2x_2(t) + tanh(x_2(t)) + tanh(x_1(t-\tau_1)), \\ \dot{x}_3(t) = -2x_3(t) + tanh(x_3(t)) + 1.2tanh(x_2(t-\tau_2)), \\ \dot{x}_4(t) = -2x_4(t) + tanh(x_4(t)) + tanh(x_3(t-\tau_3)), \\ \dot{x}_5(t) = -2x_5(t) + tanh(x_5(t)) - 0.5tanh(x_4(t-\tau_4)), \\ \dot{x}_6(t) = -2x_6(t) + tanh(x_6(t)) + 0.5tanh(x_5(t-\tau_5)), \\ \dot{x}_7(t) = -2x_7(t) + tanh(x_7(t)) + 2tanh(x_6(t-\tau_6)), \end{cases} \tag{38}$$

which has the origin as an equilibrium point. By equations (8) and (9), we obtain the associated characteristic equation for $n = 7$:

$$\lambda^7 + 7\lambda^6 + 21\lambda^5 + 35\lambda^4 + 35\lambda^3 + 21\lambda^2 + 7\lambda + 1 + 1.2e^{-\lambda\tau} = 0. \tag{39}$$

By the equations (30) and (31), we can compute $\tau_0 = 6.7067$. Choosing $\tau_1 = 0.8$, $\tau_2 = 1$, $\tau_3 = 0.6$, $\tau_4 = 1$, $\tau_5 = 1.1$, $\tau_6 = 0.7$ and $\tau_7 = 1$, Figure 2 shows that the origin is asymptotically stable, and the phase portraits for system (38) are shown in Figure 4. When $\tau = \sum_{i=1}^{7} \tau_i$ passes through the critical value $\tau_0 = 6.7067$, the origin loses its stability and Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the origin. Choosing $\tau_1 = 0.9$, $\tau_2 = 1$, $\tau_3 = 0.8$, $\tau_4 = 1$, $\tau_5 = 1.3$, $\tau_6 = 0.7$ and $\tau_7 = 1.1$, Hopf bifurcation happens at the zero solution. In Figure 3, the bifurcating periodic solutions are presented, and the phase portraits for system (38) are shown in Figure 5. Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for $\tau > \tau_0$.

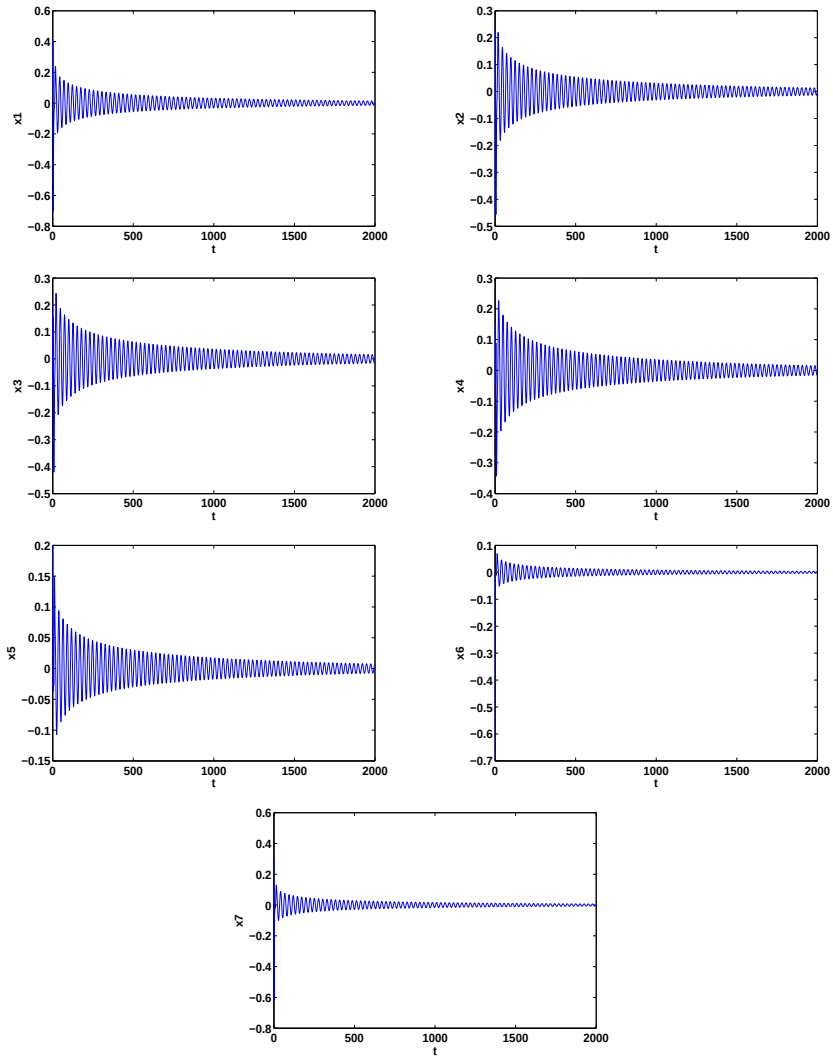

Figure 2: The origin is asymptotically stable while $\tau_1 = 0.8$, $\tau_2 = 1$, $\tau_3 = 0.6$, $\tau_4 = 1$, $\tau_5 = 1.1$, $\tau_6 = 0.7$ and $\tau_7 = 1$

5 Conclutions

In this paper, we investigated the dynamics of a general class of ring networks with n neurons and n time delays. We have chosen $\tau = \tau_1 + \tau_2 + ... + \tau_n$ as a bifurcation parameter and analyzed the corresponding characteristic equation. Then we have proved that the zero solution (the equilibrium point of the

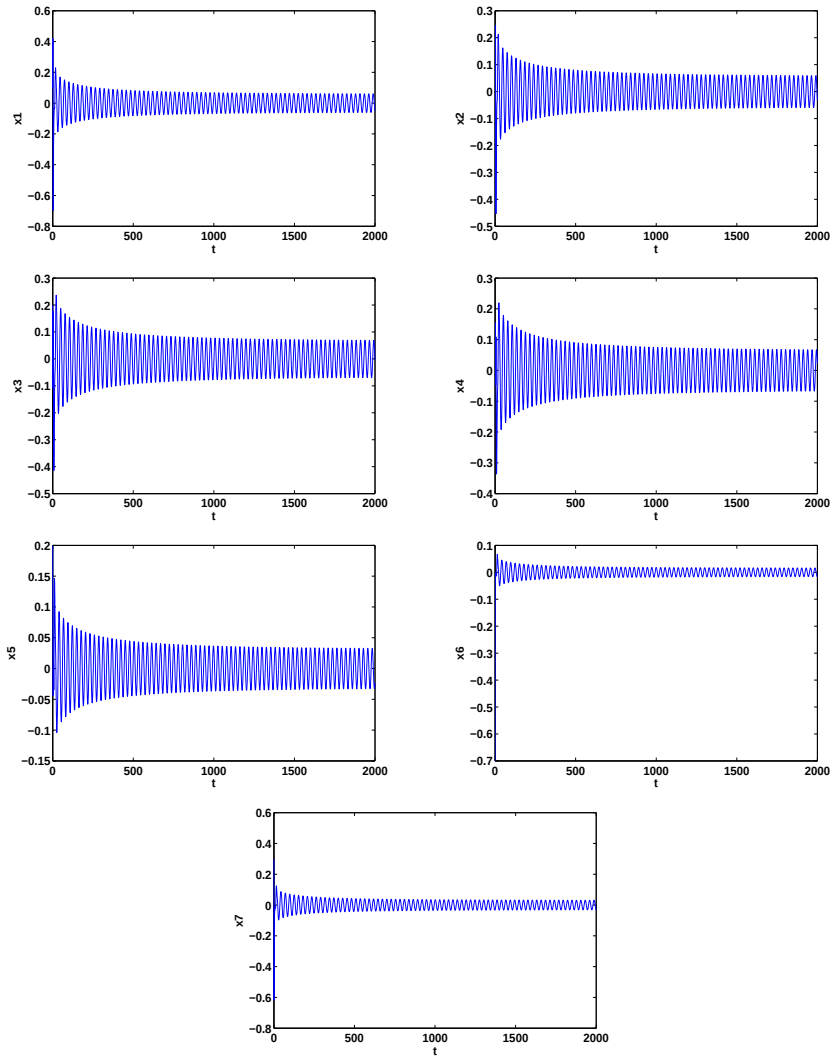

Figure 3: A family of periodic solutions bifurcates from the origin when $\tau_1 = 0.9$, $\tau_2 = 1$, $\tau_3 = 0.8$, $\tau_4 = 1$, $\tau_5 = 1.3$, $\tau_6 = 0.7$ and $\tau_7 = 1.1$

system) loses its stability and Hopf bifurcation occurs. Therefore, a family of periodic solutions bifurcate from the zero solution when τ passes through a critical value τ_0. Finally, the results have been validated by numerical simulations.

At the end, we would like to point out that it is so significant to study the networks in general case, not for a special value of n. Although, in this

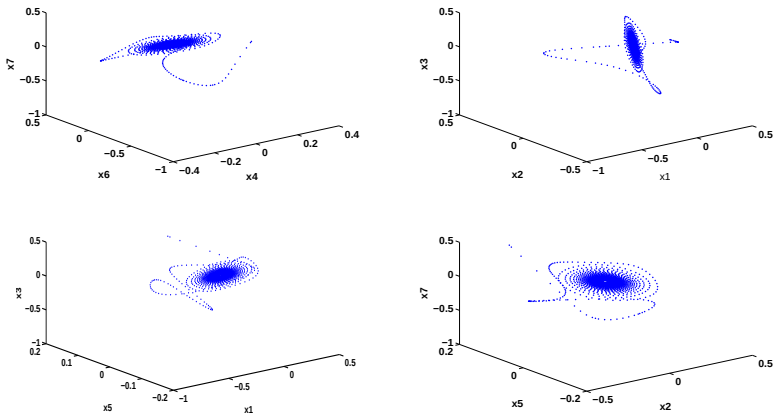

Figure 4: The phase portraits while $\tau_1 = 0.8$, $\tau_2 = 1$, $\tau_3 = 0.6$, $\tau_4 = 1$, $\tau_5 = 1.1$, $\tau_6 = 0.7$ and $\tau_7 = 1$

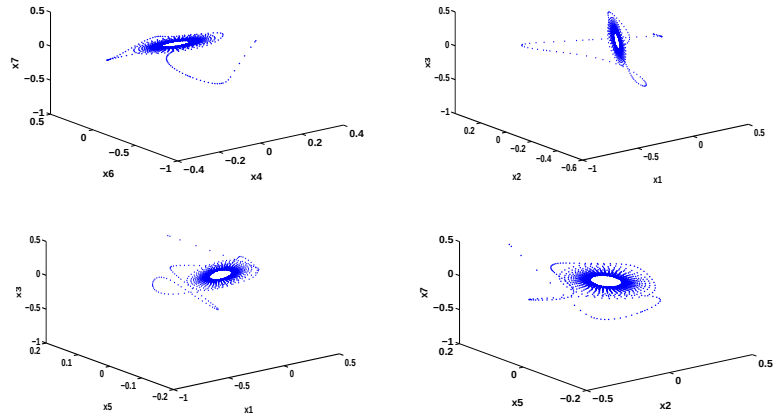

Figure 5: The phase portraits when $\tau_1 = 0.9$, $\tau_2 = 1$, $\tau_3 = 0.8$, $\tau_4 = 1$, $\tau_5 = 1.3$, $\tau_6 = 0.7$ and $\tau_7 = 1.1$

paper, we considered a general kind of ring networks, the methods, we have proposed can be generalized to be applied for other kinds of neural networks. We leave this as the future research.

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Population based algorithms for approximate optimal distributed control of wave equations

A. H. Borzabadi*, S. Mirassadi and M. Heidari

Abstract

In this paper, a novel hybrid iterative scheme to find approximate optimal distributed control governed by wave equations is considered. A partition of the time-control space is considered and the discrete form of the problem is converted to a quasi assignment problem. Then a population based algorithm, with a finite difference method, is applied to extract approximate optimal distributed control as a piecewise linear function. A convergence analysis is proposed for discretized form of the original problem. Numerical computations are given to show the proficiency of the proposed algorithm and the obtained results applying two popular evolutionary algorithms, genetic and particle swarm optimization algorithms.

Keywords: Optimal control problem; Evolutionary algorithm; Finite difference method; Wave equation

1 Introduction

In the past few decades, the science and engineering have witnessed a phenomenal growth in the field of optimal control problems (OCPs) governed by partial differential equations (PDEs), specially parabolic and hyperbolic equations. A large part of these improvements is due to the efforts of pioneer researchers such as J. L. Lions [15, 16, 17] and D. L. Russell [19, 20].

In particular, the controllability in wave equations are studied in [21, 22]. Kim and Erzberger [14] derived Riccati equation for optimum boundary con-

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trol of wave equation, with quadratic cost function. Applicability of Laplace transform for determination of time optimal control for hyperbolic class of problems has been shown in [10]. But solving optimal control problems governed by wave equations with analytical approaches, have some difficulties such as computing gradient, integrals in hyperbolic and parabolic equations and target functionals. For overcoming complexities due to the analytical approaches, the numerical approaches are created based on various techniques as regularization [9, 8, 11], measure theoretical concepts [1, 5, 6] and penalty method [12].

Recently, nature-inspired optimization methods have attracted more and more attention and these powerful tools have been applied for solving a wide range of OCPs [2, 3, 7].

In this paper, by combinating one of the population based algorithms (Evolutionary Algorithms (EAs)) and a numerical method for solving wave equations (finite difference method), an effective numerical scheme for finding approximate optimal control and state functions has been procreated for OCPs governed by wave equations with a distributed control and a non-classical boundary condition as follows:

$$\textit{minimize} \quad J(\nu(.,.)) = \int_0^T \int_0^L \Phi(t,x,\nu(x,t))\, dx\, dt \tag{1}$$

$$\textit{subject to} \quad u_{tt}(x,t) = u_{xx}(x,t) + \nu(x,t), \qquad (x,t) \in [0,L] \times [0,T] \tag{2}$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in [0,L] \tag{3}$$

$$u(0,t) = \mu(t), \qquad u_x(L,t) - u_x(0,t) = \eta(t), \qquad t \in [0,T] \tag{4}$$

where $\varphi(x), \psi(x), \mu(t), \eta(t)$ are given functions and $\nu(x,t)$ is a bounded distributed control and gets its values in the interval $\mathcal{V} \subset \mathbb{R}$. The purpose is to find the approximate optimal control $\nu(x,t)$ and state $u(x,t)$ that minimize the functional (1) and satisfy the wave equation (2) with initial conditions (3), boundary conditions (4) and terminal conditions

$$u(x,T) = \omega(x), \quad u_t(x,T) = \zeta(x). \tag{5}$$

Here $\omega(x)$ and $\zeta(x)$ are target functions.

The paper is organized as follows. In Sec. 2, we describe the discretization of optimal distributed control problem governed by wave equation. The problem is considered as a quasi assignment problem. The convergence of this modification is proved in the third section. In Sec. 4, we present the algorithm for solving OCP (1)-(5). In Sec. 5, numerical results arising from applying and comparing the given algorithm using two EAs, i.e. particle swarm optimization(PSO) and genetic algorithm(GA), are presented.

2 Description of the method

To find the optimal solution we must examine the performance index in the set of all possibilities of control-state pairs. The set of admissible pairs consisting of pairs like (u, ν) satisfying in (2)-(4) is denoted by $\mathcal{P}$. In this section we consider a control space discretization based method considering equidistant partitions of $[0, T]$, $[0, L]$ and $\mathcal{V}$ as $\triangle_t = \{0 = t_0, t_1, \cdots, t_{n-1}, t_n = T\}$ $\triangle_x = \{0 = x_0, x_1, \cdots, x_{m-1}, x_m = L\}$ and $\triangle_\nu = \{v_0, v_1, \cdots, v_{l-1}, v_l\}$, respectively. Now the main problem can be considered as a quasi assignment problem, where a performance index can be assigned corresponding to each chosen partition and choosing the best performance index can lead to determine the near optimal control of the problem. A trivial way to determine the near optimal solution is to calculate all possible partitions and compare the corresponding trade offs. This trivial method of total enumeration needs $((m+1)(n+1))^{(l+1)}$ evaluation. A typical discretization is given in Figure

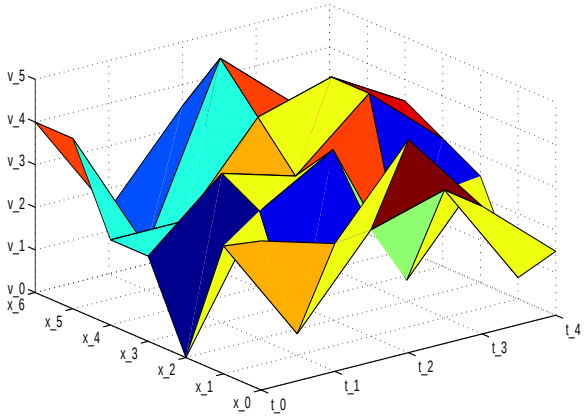

Figure 1: A typical control function in time-control space

1 with $n = 4$, $m = 6$ and $l = 5$. To avoid so many computations, we use the EAs for evaluating special partitions that guides us to the optimal one. For each partition of control we need its corresponding trajectory to evaluate the performance index. Trivially, the corresponding trajectory should be in discretized form.

For discretization of the wave equation (2)-(4), we use an approximate method like finite difference method as central difference approximation for the second partial derivative and forward difference approximation for the first partial derivative. By this method,

$$\frac{u_i^{j-1} - 2u_i^j + u_i^{j+1}}{k^2} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2} + \bar{\nu}_i^j,$$
$$u_i^0 = \varphi_i,$$
$$\frac{u_i^1 - u_i^0}{k} = \psi_i,$$
$$u_0^j = \mu^j,$$
$$\frac{u_{m+1}^j - u_m^j}{h} - \frac{u_1^j - u_0^j}{h} = \eta^j,$$

where $u_i^j = u(x_i, t_j), \varphi_i = \varphi(x_i), \psi_i = \psi(x_i), \mu^j = \mu(t_j), \eta^j = \eta(t_j)$ and $\bar{\nu}_i^j = \bar{\nu}(x_i, t_j)$. Then we have

$$u_i^0 = \varphi_i,\ u_i^1 = k\psi_i + u_i^0,\ i = 0, 1, \cdots, m, \tag{6}$$
$$u_0^j = \mu_j,\ u_{m+1}^j = h\eta^j + u_m^j + u_1^j - u_0^j,\ j = 0, 1, \cdots, n, \tag{7}$$

and

$$u_i^{j+1} = \lambda^2 u_{i+1}^j + (2 - 2\lambda^2)u_i^j + \lambda^2 u_{i-1}^j - u_i^{j-1} + k^2\bar{\nu}_i^j, \tag{8}$$

where $j = 1, 2, \cdots, n,\ i = 0, 1, \cdots, m$ and $\lambda = k/h$.

If $(u, \bar{\nu})$ be a pair of the trajectory and the control which satisfies in (10)-(11) and

$$\|u(x_i, t_n) - \omega(x_i)\| \le \epsilon_1 \quad i = 0, 1, \cdots, m \tag{9}$$
$$\|u_t(x_i, t_n) - \zeta(x_i)\| \le \epsilon_2 \quad i = 0, 1, \cdots, m \tag{10}$$

for given small numbers $\epsilon_1 > 0$ and $\epsilon_2 > 0$, then we can claim that, a good approximate pair for minimizing functional J in (1) has been found. Here $\|\cdot\|$ is the infinity norm.

Also in (1), the integral term can be estimated by a numerical method of integration, e.g. one of Newton-Cotes methods. After discretization of the OCP governed by wave equation, the problem is converted to optimization problem with two extra objective functions. We add the terms, $\sum_{i=1}^{m-1} \|u(x_i, t_n) - \omega(x_i)\|$ and $\sum_{i=1}^{m-1} \|u_t(x_i, t_n) - \zeta(x_i)\|$ to the original objective function and then, we apply EAs for this new criteria function. Therefore, by applying the above method, the OCP governed by wave equation is converted to constrained programming:

$$\text{(CP)}\min \sum_{i=1}^{m}\sum_{j=1}^{n} A_i B_j \Phi(t_j, x_i, \bar{\nu}_i^j) \tag{11}$$
$$+ M\sum_{i=1}^{m} \|u_i^n - \omega(x_i)\| + W\sum_{i=1}^{m} \|u_t(x_i, t_n) - \zeta(x_i)\|,$$

$$\textit{subject to} \quad u_i^{j+1} = \lambda^2 u_{i+1}^j + (2-2\lambda^2)u_i^j + \lambda^2 u_{i-1}^j - u_i^{j-1} + k^2 \bar{\nu}_i^j, \qquad (12)$$

$$u_i^0 = \varphi_i,\ u_i^1 = k\psi_i + u_i^0,\ i = 0, 1, \cdots, m, \qquad (13)$$

$$u_0^j = \mu_j,\ u_{m+1}^j = h\eta^j + u_m^j + u_1^j - u_0^j,\ j = 0, 1, \cdots, n, \qquad (14)$$

where, A_i and B_j are the weights of a numerical method of integration, M and W are large positive numbers (as the parameters in penalty function approach).

3 Convergence

The solution of (CP) approximates the original problem by minimizing $J(u, \nu)$ over the subset $\mathcal{P}_N$ of $\mathcal{P}$ consists of all piecewise linear functions $u(.,.)$ and $\nu(.,.)$ with nodes at $u_i^j, \bar{\nu}_i^j, j = 0, 1, \cdots, N, i = 0, 1, \cdots, N$ which satisfies (11) and the objective function (11) for this nodes called J_N, here without loss of generality, we assume that $N = m = n$. Our first aim is to show that $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3 \cdots$ in an embedding fashion.

Lemma 1. *There exists an embedding that maps $\mathcal{P}_N$ to a subset of $\mathcal{P}_{N+1}$ for all $N = 1, 2, \cdots$.*

Proof. For simplicity, we prove the case when $N = 1$. The proof for $N \geq 2$ is obtained analogously.
Let consider an arbitrary pair (u, ν) in $\mathcal{P}_1$ represented by $u_i^j, \bar{\nu}_i^j, j = 0, 1, i = 0, 1$. We have to find a corresponding pair $(\hat{u}, \hat{\nu})$ in $\mathcal{P}_2$ with $\hat{u}_i^j, \hat{\bar{\nu}}_i^j, j = 0, 1, 2, i = 0, 1, 2$, as nodes that corresponds to (u, ν). We have from (11)

$$u_i^{j+1} = \lambda^2 u_{i+1}^j + (2-2\lambda^2)u_i^j + \lambda^2 u_{i-1}^j - u_i^{j-1} + k^2 \bar{\nu}_i^j, \quad j = 0, 1, \quad i = 0, 1,$$

where $u_i^{j+1} = u(x_i, t_{j+1})$. On the other hand, a typical element $(\hat{u}, \hat{\nu})$ in $\mathcal{P}_2$ satisfies

$$\hat{u}_i^{j+1} = \lambda^2 \hat{u}_{i+1}^j + (2-2\lambda^2)\hat{u}_i^j + \lambda^2 \hat{u}_{i-1}^j - \hat{u}_i^{j-1} + k^2 \hat{\bar{\nu}}_i^j, \quad j = 0, 1, 2, \quad i = 0, 1, 2,$$

where $\hat{u}_i^{j+1} = \hat{u}(\hat{x}_i, \hat{t}_{j+1})$.
It is clear that here we have $\hat{x}_i = x_i, i = 0, 1$ and $\hat{t}_j = t_j, j = 0, 1$. Therefore we can choose $\hat{u}_i^j, \hat{\bar{\nu}}_i^j, j = 0, 1, 2, i = 0, 1, 2$ in such a way that

$$\hat{u}_i^j = u_i^j, \quad j = 0, 1, \quad i = 0, 1.$$

This shows that the constructed pair $(\hat{u}, \hat{\nu})$ corresponds to (u, ν) and belongs to $\mathcal{P}_2$. □

The above lemma has an important result in decreasing behavior of the optimal value of the objective function which leads to the following theorem.

Theorem 1. *If* $\mu_N = \inf_{\mathcal{P}_N} J_N$ *for* $N = 1, 2, \cdots$, *and* $\mu^* = \inf_{\mathcal{P}} J$ *exists, then* $\lim_{N\to\infty} \mu_N = \mu^*$.

Proof. By Lemma 1, we have $\mu_1 \geq \mu_2 \geq \cdots \geq \mu^*$. So, this decreasing and bounded sequence converges to a limit $\mu^0 \geq \mu^*$. It is enough to show that $\mu^0 = \mu^*$. If $\mu^0 > \mu^*$, then $\epsilon = \mu^0 - \mu^* > 0$ and by continuity of $J(u, \nu)$, we may find a pair $(u^j_{n_0}, \nu^j_{n_0})$, such that $|J(u^j_{n_0}, \nu^j_{n_0}) - \mu^*| < \epsilon$, then $J(u^j_{n_0}, \nu^j_{n_0}) < \mu^0$, and so $\mu_{n_0} < \mu^0$ which is incorrect and therefore $\mu^0 = \mu^*$. □

4 Algorithm of the approach

In this section, an algorithm on the basis of the previous discussions is presented. This algorithm is designed in two stages, initialization step and main steps, where the main steps contain the main structure of algorithm considering initialization step.

Initialization step:

Choose an equidistant partition for time interval $[0, T]$, with parameter discretization $k = t_{j+1} - t_j,\ j = 0, 1, \cdots, n-1$ and an equidistant partition for interval $[0, L]$, with parameter $h = x_{i+1} - x_i,\ i = 0, 1, \cdots, m-1$.

Main steps:

Step 1. Choose a population randomly.

Step 2. Compute $u^j_i,\ j = 0, 1, \cdots, n,\ i = 0, \cdots, m$, using (10)-(11).

Step 3. Fitness scores are assigned to each population using objective function of (CP).

Step 4. Apply the rules of EA for current population.

Step 5. Consider the new population as the current population.

Step 6. If the termination conditions are satisfied, stop; otherwise jump to Step 2.

5 Numerical results

In this section the proposed algorithm in the previous section is examined by one numerical example. We have applied PSO and GA as two of the most popular EAs.

Consider the following OCP:

$$\begin{aligned} \min \quad & J = \int_0^1 \int_0^1 \nu^2(x,t)\ dt\ dx \\ \textit{subject to} \quad & u_{tt}(x,t) = u_{xx}(x,t) + \nu(x,t), \qquad (x,t) \in (0,1)\times(0,1) \\ & u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad x \in (0,1) \\ & u(0,t) = 0, \qquad u_x(1,t) = u_x(0,t), \qquad t \in (0,1) \\ & u(x,T) = \sin(2\pi x), \qquad u_t(x,T) = \sin(4\pi x), \qquad t \in (0,1) \end{aligned}$$

For analytical solution of this example, see [13].

Our solutions by *PSO* and *GA* algorithms are shown in the following tables:

- where wave parameters, λ, be $0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ and the population sizes, (m), be 200 and the number of iterations,(k_{max}), be 500, results are shown in Table 1. 1.

- where the population size,(m), be $50, 100, 150, 200, 250, 300, 350$ and the number of iterations,(k_{max}), be 500 and parameter wave equation,λ, be 0.2, results are shown in Table 2. 2.

- where the number of iterations,(k_{max}), be $100, 200, 300, 400, 500, 600, 700$ and the population size,(m), be 200 and parameter wave equation,λ, be 0.2, results are shown in Table 3. 3.

Also comparison between $\omega(x)$ and $\zeta(x)$ with $u(x,T)$ and $u_t(x,T)$ are shown in Figures 2 and 3 ,respectively, when $m = 200, k_{max} = 500, \lambda = 0.2$.

Table 1: Comparison of the errors due to applying PSO and GA with increasing λ

parameter λ	$\|u_t(x,T) - \zeta(x)\|$	
	PSO	GA
0.1	$1.1662e-005$	$1.2375e-005$
0.2	$2.4418e-005$	$6.5369e-005$
0.3	$5.3563e-005$	$1.1038e-004$
0.4	$2.2052e-004$	$8.0804e-005$
0.5	$1.3987e-004$	$8.9201e-005$
0.6	$9.6621e-005$	$7.8682e-005$
0.7	$1.5991e-004$	$8.3949e-005$
0.8	$1.3152e-004$	$1.1785e-004$
0.9	$1.6093e-004$	$1.1817e-004$
1.0	$2.4100e-004$	$2.5027e-004$

Table 2: Comparison of the errors due to applying PSO and GA with increasing the number of population

number of population	$\|u_t(x,T)-\zeta(x)\|$	
	PSO	GA
50	$1.8470e-004$	$2.16684e-004$
100	$1.1513e-004$	$9.4944e-005$
150	$1.5074e-004$	$1.0690e-004$
200	$9.4195e-005$	$8.5366e-005$
250	$1.6513e-004$	$1.2451e-004$
300	$1.6264e-004$	$1.0645e-004$
350	$5.0556e-005$	$8.6299e-005$

Table 3: Comparison of the errors due to applying PSO and GA with increasing the number of iterations

number of iterations	$\|u_t(x,T)-\zeta(x)\|$	
	PSO	GA
100	$2.3324e-004$	$4.0122e-004$
200	$9.4547e-005$	$1.1504e-004$
300	$1.8058e-004$	$1.0747e-004$
400	$1.2749e-004$	$1.0134e-004$
500	$2.1751e-004$	$9.1401e-005$
600	$1.0138e-004$	$7.7138e-005$
700	$9.9944e-005$	$9.2749e-005$

6 Conclusion

In this paper, a hybrid approach for the resolution of OCPs governed by wave equations is presented. This approach is based on partitioning of the time-control space, finite difference method, penalty method and EAs. The derived results show the superiority of the approach.

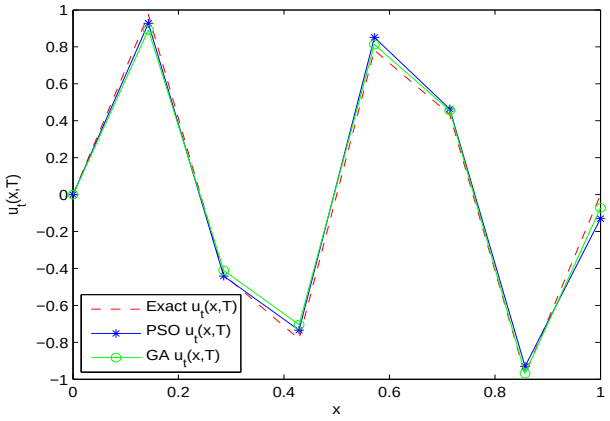

Figure 2: Diagram of $u(x,T)$ and $\omega(x)$

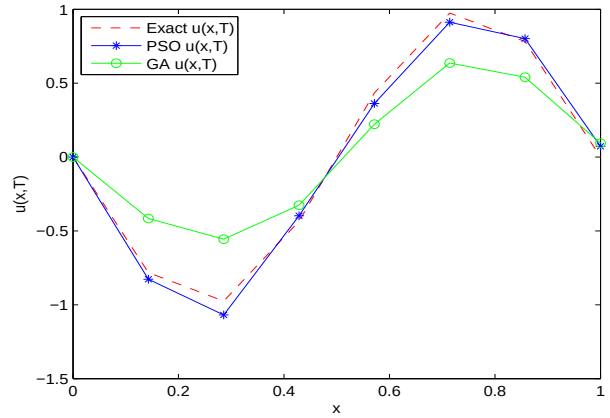

Figure 3: Diagram of $u_t(x,T)$ and $\zeta(x)$

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Operational Tau method for nonlinear multi-order FDEs

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Abstract

This paper presents an operational formulation of the Tau method based upon orthogonal polynomials by using a reduced set of matrix operations for the numerical solution of nonlinear multi-order fractional differential equations(FDEs). The main characteristic behind the approach using this technique is that it reduces such problems to those of solving a system of nonlinear algebraic equations. Some numerical examples are provided to demonstrate the validity and applicability of the method.

Keywords: Fractional differential equations(FDEs); Caputo derivative; Operational Tau method.

1 Introduction

The mathematical modelling and simulation of systems and processes based upon the description of their properties in terms of fractional derivatives, naturally leads to differential equations of fractional order and to the necessity to solve such equations. However, effective general methods for solving them can not be found even in the most useful works on fractional derivatives and integrals.

There are several approaches to the generalization of the notation of differentiation to fractional orders e.g., Riemann-Liouville, Grunwald-Letnikov and Caputo. We focus on one particular form so-called Caputo derivative.

Recently, linear FDEs based upon the fractional derivatives(such as Riemann-Liouville and Caputo schemes) with general variable coefficients have been solved by adapting various analytical and numerical methods[2, 5, 7, 20]. Nowadays, applications have included some classes of nonlinear FDEs, and this motivates us to consider their effective numerical methods for solution of these type of equations. Among the most recent works concerned with

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nonlinear initial value problems of fractional order, we can consider papers [4, 8, 12, 13, 18, 19, 21, 22, 31, 32].

Spectral methods have been studied intensively in the last two decades because of their good approximation properties. The formulation of spectral methods was first presented in the monograph of Gottlieb and Orszag [11]. The text book of Canuto, et al [3] focuses on practical and theoretical aspects of global spectral methods.

Global spectral methods use a representation of function $u(t)$ throughout the domain via a truncated series expansion with suitable basis functions. This series is then substituted into functional equation and upon the minimization of the residual function the unknown coefficients are computed.

Spectral methods can be broadly classified into three categories, Pseudospectral or Collocation, Galerkin and Tau methods. The Tau method, through which the spectral methods, as shown in [6, 23-29] has found extensive application for the numerical solution of many operator equations in the recent years.

The Tau method, firstly introduced by Lanczos[15-17], involves the projection of the residual function on the span of some appropriate set of basis functions, typically arising as the eigenfunctions of a singular Sturm-Liouville problem. The auxiliary conditions imposed as constraints on the expansion coefficients. It is well known that eigenfunctions of certain singular Sturm-Liouville problems allow the approximation of functions belong to the space $C^{\infty}[a, b]$ whose truncation error approaches zero faster than any negative power of the number of basis functions used in approximation, as that number(order of truncation N)tends to ∞. This phenomenon is usually referred to as "Spectral accuracy" (Gottlieb and Orszag [11]). A convergence analysis and error bounds for the Tau method was considered by Ortiz and Pham in the papers [24, 25]. The recursive form of the Tau method, formulated by Ortiz in [26] was extended to the case of systems of ordinary differential equations in [6]. The basic philosophy of the method was extended to the numerical solutions of the linear and nonlinear initial value, boundary value, and mixed problems for ordinary differential equations [23, 25, 27], to the eigenvalue problems [27, 28], to the "Stiff" problems [23], to the partial differential equation[29], among others.

The main objective of the present paper is to provide Ortiz and Samara[27] operational approach to the Tau method for the numerical solution of nonlinear FDEs of the general form

$$L_D(u(t)) = f(t), \tag{1}$$

on $t \in \Lambda = [0, 1]$ with initial conditions

$$u^{(i)}(0) = d_i, \qquad i = 0, 1, \ldots, \nu - 1, \tag{2}$$

where

$$L_D(u(t)) = \sum_{r=0}^{N_d} p_r(t) \prod_{k=0}^{l_r} \left(D_C^{\theta_{rk}} u(t)\right)^{\gamma_{rk}}, \quad \theta_{rk} \in \mathbb{Q}^+, \quad N_d, \gamma_{rk}, l_r \in \mathbb{N} \bigcup \{0\}, \tag{3}$$

where $\mathbb{N}, \mathbb{Q}^+$ are the collections of the all natural and positive rational numbers, respectively. d_i are constants and $\nu = \max_{0 \le r \le N_d} \{\lceil \theta_{rk} \rceil\}_{k=0}^{l_r}$. The symbol $\lceil q \rceil$ is the smallest integer greater than or equal to q. $u(t)$ is unknown function, $p_r(t)$ and $f(t)$ are algebraic polynomials or their suitable polynomial approximations. Finally, the fractional derivative is considered in the Caputo sense that is given by

$$D_C^{\theta_{rk}} u(t) = \frac{1}{\Gamma(\lceil \theta_{rk} \rceil - \theta_{rk})} \int_0^t (t-\tau)^{\lceil \theta_{rk} \rceil - \theta_{rk} - 1} u^{(\lceil \theta_{rk} \rceil)}(\tau) d\tau, \quad t \in \Lambda. \tag{4}$$

The properties of Caputo derivative can be found in [30].

In this paper we proceed as follows: In the next section, the spectral Tau method for nonlinear FDEs is described. We reduce the problem to a set of nonlinear algebraic equations using some useful operational matrices. Numerical experiments are carried out in Section 3, to illustrate the efficiency of the proposed method.

2 Numerical approach

Consider the operational Tau solution for nonlinear FDE (2- 3) as a polynomial of degree N

$$u_N(t) = \sum_{i=0}^{\infty} u_i J_i^{\alpha,\beta}(t) = \underline{u_N}\ \underline{\mathbf{J}} = \underline{u_N} \mathbf{J} \underline{X_t}, \tag{5}$$

where $\underline{u_N} = [u_0, u_1, \ldots, u_N, 0, \ldots]$. $\mathbf{J}$ is non-singular lower triangular coefficient matrix given by the shifted Jacobi polynomials in Λ, where $\{J_i^{\alpha,\beta}(t)\}_{i=0}^{\infty} = \underline{\mathbf{J}} = \mathbf{J}\underline{X_t}$ with a standard basis vector $X_t = [1, t, t^2, \ldots]^T$[3]. The effect of $u_N^{(k)}(t)$, $t^s u_N(t)$ and $(u_N(t))^p$ on the coefficients vector of polynomial (5) are

$$u_N^{(k)}(t) = \underline{u_N} \mathbf{J} \eta^k \underline{X_t},\ t^s u_N(t) = \underline{u_N} \mathbf{J} \mu^s \underline{X_t},\ (u_N(t))^p = \underline{u_N} \mathbf{J} \mathrm{E}^{p-1}(\underline{u_N}, \mathbf{J}) \underline{X_t}, \tag{6}$$

where matrices η and μ have the following simple structures ([27])

$$\eta = \begin{bmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \\ 0 & 2 & 0 & \vdots \\ 0 & 0 & 3 & \\ \vdots & & \cdots & \ddots \end{bmatrix}, \qquad \mu = \begin{bmatrix} 0 & 1 & 0 & 0\cdots \\ 0 & 0 & 1 & 0\vdots \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & & \cdots & \ddots \end{bmatrix},$$

and $\mathrm{E}(\underline{u_N}, \mathbf{J})$ is an infinite upper triangular Toeplitz matrix with the following structure

$$\mathrm{E}(\underline{u_N}, \mathbf{J}) = \begin{bmatrix} \underline{u_N}\mathbf{J_0} & \underline{u_N}\mathbf{J}_1 & \underline{u_N}\mathbf{J}_2 & \dots \\ \mathbf{0} & \underline{u_N}\mathbf{J}_0 & \underline{u_N}\mathbf{J}_1 & \dots \\ \mathbf{0} & \mathbf{0} & \underline{u_N}\mathbf{J}_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\mathbf{J_i}$ is the i-th column of the matrix $\mathbf{J}$. Details for formulation of the matrix $\mathrm{E}(\underline{u_N}, \mathbf{J})$ can be found in [10].

Now, we intend to explain details of the structure of the operational approach to the Tau method with Jacobi polynomial bases for the numerical solution of the nonlinear multi-order FDEs. Firstly, in the Lemma 2.1, we will show that the effect of Caputo fractional derivative $D_C^{\theta_{rk}}(u_N(t))$, will be represented as the product of a matrix and a vector. Secondly, in the Lemma 2.2, we will prove that the product of polynomials can be written as the product of a matrix and a vector. Finally, in the Theorem 2.3, we will give the matrix representation of $L_D(u_N(t))$ by using the Lemmas 2.1 and 2.2.

Lemma 2.1 *Let $J_j^{\alpha,\beta}(t)$ be the shifted Jacobi polynomials with respect to the weight function $\chi^{\alpha,\beta}(t) = (2-2t)^\alpha(2t)^\beta$ on Λ. Assume that the approximated solution $u_N(t)$ and the fractional derivative $D_C^{\theta_{rk}}$ are given by the relations (5) and (4) respectively, then*

$$D_C^{\theta_{rk}}(u_N(t)) = \underline{u_N}\mathbf{J}\Upsilon_{\theta_{rk}}\underline{\mathbf{J}},$$

where

$$\Upsilon_{\theta_{rk}} = \begin{pmatrix} 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \\ \Theta(\lceil\theta_{rk}\rceil)\xi_{m_{rk},0} & \cdots & \Theta(\lceil\theta_{rk}\rceil)\xi_{\lceil\theta_{rk}\rceil,N} & \cdots \\ \Theta(\lceil\theta_{rk}\rceil+1)\xi_{\lceil\theta_{rk}\rceil+1,0} & \cdots & \Theta(\lceil\theta_{rk}\rceil+1)\xi_{\lceil\theta_{rk}\rceil+1,N} & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \Theta(N)\xi_{N,0} & \cdots & \Theta(N)\xi_{N,N} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

with $\Theta(\zeta) = \frac{\zeta!}{\Gamma(\zeta-\theta_{rk}+1)}$ and

$$\xi_{k,j} = \frac{1}{\| J_j^{\alpha,\beta}(t) \|^2_{\chi^{\alpha,\beta}}}(t^{k-q}, J_j^{\alpha,\beta}(t))_{\chi^{\alpha,\beta}}, \quad \begin{matrix} k \geq \theta_{rk}, \\ j = 0, 1, \dots \end{matrix}.$$

Proof. See [9]. □

Lemma 2.2 (a) For two given polynomials $h(t) = \sum_{i=0}^{\infty} h_i v_i(t) = HV\underline{X_t}$ and $s(t) = \sum_{i=0}^{\infty} s_i v_i(t) = SV\underline{X_t}$ with $H = [h_0, h_1, h_2, \ldots], S = [s_0, s_1, s_2, \ldots]$, we have

$$s(t)h(t) = SV\mathrm{E}(H, V)\underline{X_t}.$$

(b) For given polynomials $h_i(t) = \sum_{j=0}^{\infty} a_j T_{ij}(t) = \underline{a_N} T_i \underline{X_t}, \quad i = 0, 1, ...,$ where T_i are nonsingular coefficients matrices given by $\{T_{ij}\}_{j=0}^{\infty} = T_i\underline{X_t}$, we have

$$\prod_{i=0}^{l} h_i(t) = \underline{a_N} T_0 \prod_{i=1}^{l} \mathrm{E}(\underline{a_N}, T_i)\underline{X_t}. \tag{7}$$

Proof. For proof of part**(a)** see [10]. By using part **(a)** and mathematical induction we can prove part **(b)**. □

Theorem 2.3 (Matrix representation for nonlinear part)
Assume that the approximated solution $u_N(t)$ and the nonlinear fractional operator L_D are given by the relations (5) and (2), respectively, then

$$L_D(u_N(t)) = \underline{u_N}\widehat{\Pi}\ \underline{\mathbf{J}},$$

where

$$\widehat{\Pi} = \mathbf{J}\Big(\sum_{r=0}^{N_d} \Psi_d \mathrm{E}^{\gamma_{rd}-1}(\underline{u_N}, \mathbf{J}\Psi_d) \prod_{k=d+1}^{l_r} \mathrm{E}(\underline{u_N}, F_{rk})\ p_r(\mu)\Big)\mathbf{J}^{-1},$$

$$\Psi_k = \begin{cases} \eta^{\theta_{rk}}, & \theta_{rk} \in \mathbb{N}, \\ \Upsilon_{\theta_{rk}}\mathbf{J}, & ,\theta_{rk} \in \mathbb{Q}^+ - \mathbb{N}, \end{cases} \qquad F_{rk} = \mathbf{J}\Psi_k \mathrm{E}^{\gamma_{rk}-1}(\underline{u_N}, \mathbf{J}\Psi_k),$$

and d is the smallest index that $\gamma_{rd} \neq 0$.

Proof. From Lemma 2.1 and the third relation in (6) for $k \neq \{p\ |\gamma_{rp} = 0\}$ we have

$$\prod_{k=0}^{l_r} \left(D_C^{\theta_{rk}}(u_N(t))\right)^{\gamma_{rk}} = \prod_{k=0}^{l_r} (\underline{u_N}\mathbf{J}\Psi_k\underline{X_t})^{\gamma_{rk}} = \prod_{k=0}^{l_r} (\underline{u_N}\mathbf{J}\Psi_k \mathrm{E}^{\gamma_{rk}-1}(\underline{u_N}, \mathbf{J}\Psi_k)\underline{X_t}).$$

Let d be the smallest index that $\gamma_{rd} \neq 0$, then from (6) we can write

$$\prod_{k=0}^{l_r}(\underline{u_N}\mathbf{J}\Psi_k\mathrm{E}^{\gamma_{rk}-1}(\underline{u_N},\mathbf{J}\Psi_k)\underline{X_t}) = \underline{u_N}\mathbf{J}\Psi_d\mathrm{E}^{\gamma_{rd-1}}(\underline{u_N},\mathbf{J}\Psi_d) \\ * \prod_{k=d+1}^{l_r}\mathrm{E}(\underline{\mathrm{u_N}},\mathbf{J}\Psi_\mathrm{k}\mathrm{E}^{\gamma_{\mathrm{rk}-1}}(\underline{\mathrm{u_N}},\mathbf{J}\Psi_\mathrm{k}))\underline{\mathrm{X_t}} \\ = \underline{u_N}\Pi_r\underline{\mathbf{J}},$$

where

$$\Pi_r = \mathbf{J}\Psi_d\mathrm{E}^{\gamma_{rd}-1}(\underline{u_N},\mathbf{J}\Psi_d)\prod_{k=d+1}^{l_r}\mathrm{E}(\underline{u_N},F_{rk})\mathbf{J}^{-1}\mathrm{and}F_{rk} = \mathbf{J}\Psi_k\mathrm{E}^{\gamma_{rk}-1}(\underline{u_N},\mathbf{J}\Psi_k).$$

By substituting the above relation in (3) and using the second relation in (6) we obtain

$$L_D(u_N(t)) = \underline{u_N}\Big(\sum_{r=0}^{N_d}\Pi_r p_r(\mu)\Big)\underline{\mathbf{J}},$$

that is the statement of the Theorem. □

Also for obtaining the matrix form of the initial conditions (3), we introduce vector $\underline{d} = [d_0, d_1, \ldots, d_{\nu-1}, 0, \ldots]$ where $\nu = \max\limits_{0\le r\le N_d}\{\theta_{rk}\}_{k=0}^{l_r}$. On the other hand we can write

$$u_N^{(j)}(0) = \underline{u_N}\mathbf{J}\eta^j\underline{X_t}\big|_{t=0} = \underline{u_N}\mathbf{J}\eta^j e_1 = \underline{u_N}b_j, \quad j = 0,1,2,\ldots,\nu-1,$$

where $e_1 = [1,0,0,\ldots]^T$ and

$$B = (b_j)_{j=0}^{\nu-1} = (\mathbf{J}\eta^j e_1)_{j=0}^{\nu-1}. \tag{8}$$

Now, we are ready to obtain the nonlinear algebraic system of implementing the operational Tau method to the nonlinear multi-order FDE (2-3).

Following Theorem 2.3 and the relation (7) we obtain

$$\begin{cases} \underline{u_N}\widehat{\Pi}\underline{J} = \underline{f}\underline{J}, \\ \\ \underline{u_N}B = \underline{d}, \end{cases} \tag{9}$$

where $f(t) = \underline{f}\underline{J}$ with $\underline{f} = [f_0, f_1, ...]$. Because of orthogonality of $\{J_i^{\alpha,\beta}(t)\}_{i=0}^{\infty}$, projecting (9) on the $\{J_k^{\alpha,\beta}(t)\}_{k=0}^{N}$ yields

$$\underline{u_N}\widehat{\Pi}_k = f_k, \qquad k = 0,1,2,\ldots,N.$$

By setting

$$M_N = [b_0, b_1, \ldots, b_{\nu-1}, \widehat{\Pi}_0, \widehat{\Pi}_1, \ldots, \widehat{\Pi}_N], r_N = [d_0, d_1, \ldots, d_{\nu-1}, f_0, f_1, \ldots, f_N],$$

we obtain $\underline{u_N} M_N = r_N$. We restrict this system to its first $N+1$ columns. The square system $\underline{u_N} M_{N-\nu} = r_{N-\nu}$, gives us unknown vector $\underline{u_N}$.

3 Numerical results

In this section we have considered three test problems. All of these test problems have been solved by the operational Tau method based on the Chebyshev and Legendre bases. In all cases any non-polynomial functions were replaced by a suitable polynomial approximation. All calculations were performed on a PC running Mathematica software. To report some information about the number of operations, we use function *LeafCount* in the Mathematica software, that gives us total number of indivisible subexpressions. All of achieved nonlinear algebraic systems were solved by the well known iterative Newton method.

Example 1 : *[18] Consider the nonlinear FDE with* $\alpha = 1.5$

$$D_C^{\alpha} u(t) + u^2(t) = f(t), \quad u^{(i)}(0) = 0, \quad i = 0, 1,$$

where

$$f(t) = \frac{\Gamma(6)}{\Gamma(6-\alpha)} t^{5-\alpha} - \frac{3\Gamma(5)}{\Gamma(5-\alpha)} t^{4-\alpha} + \frac{2\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} + (t^5 - 3t^4 + 2t^3)^2.$$

the exact solution is $u(t) = t^5 - 3t^4 + 2t^3$.

We apply the proposed operational Tau method to obtain the approximated solution of the problem. The maximal error, with the Chebyshev and Legendre bases have been given in Table 1. A comparison of the Tau method with fractional high order method proposed by R. Lin and F. Liu in [18] shows that our method produces powerful superiority with respect to the proposed method in [18].

Table 1: Numerical results of Example 1, using operational Tau method with different bases

N	Maximal Error	
	Chebyshev Tau	Legendre Tau
5	4.58×10^{-16}	7.28×10^{-17}
7	1.50×10^{-16}	7.67×10^{-17}
10	8.75×10^{-17}	5.52×10^{-16}
Lin and Liu scheme	Max. error is 1.544×10^{-5} in $t = 1$	

Example 2 : *Consider the nonlinear FDE*

$$D_C^{\frac{1}{4}}u(t)D_C^{\frac{1}{2}}u(t)+u(t)=e^t+e^{2t}\mathrm{erf}(\sqrt{t})\big(1-\frac{\Gamma(\frac{3}{4},t)}{\Gamma(\frac{3}{4})}\big),\quad u(0)=1,$$

where $\mathrm{erf}(z)$ *gives the error function and* $\Gamma(a,z)$ *is the incomplete Gamma function.*

the exact solution of this problem is $u(t)=e^t$. We apply the proposed operational Tau method to obtain the approximate solution of this example. We have reported the obtained numerical results in Table 2 and Fig. 1 with the Chebyshev and Legendre bases. In Fig. 1, obtained numerical errors are plotted for several values of approximation degree N in the L_∞ norm. From Table 2 and Fig. 1, we can conclude that desired spectral accuracy is obtained for this nonlinear problem and the approximate solutions are in high agreement with the exact solution. In addition, by using the function $\backslash$*LeafCount* in the Mathematica software, for $N=4,8,12$ and $N=16$, we need $213, 812, 1261$ and 2153 operations, respectively, to obtain the operational Tau solution with the reported errors in the Table 2 and Fig. 1 based on the Chebyshev polynomial bases.

Table 2: Numerical results of Example 2, using operational Tau method with different bases

	Maximal Error	
N	Chebyshev Tau	Legendre Tau
4	6.79×10^{-5}	5.13×10^{-5}
8	7.19×10^{-11}	5.85×10^{-11}
12	7.31×10^{-16}	7.06×10^{-16}
16	2.85×10^{-16}	2.81×10^{-16}

Example 3 : *[14] Consider the following equation of fractional order* $\theta=0.5$*:*

$$D_C^{\theta}u(t)=\lambda t^{\beta}(u(t))^2,\quad (0<\theta<1), \tag{10}$$

with $\lambda,\beta\in\mathbb{R}(\lambda\neq0)$*. If* $\theta+\beta<1$*, this equation has the exact solution*

$$u(t)=\frac{\Gamma(1-\theta-\beta)}{\lambda\Gamma(1-2\theta-\beta)}t^{-(\theta+\beta)}. \tag{11}$$

If $\beta\leq-2\theta$, then the equation (10), has unique solution $u(t)\in C[a,b]$ given by (11) and if $\beta=-(k+\theta), k\in\mathbb{N}$, then the equation (10) , has unique solution $u(t)\in C^{\infty}[a,b]$. (See [14, Chapter 3]).

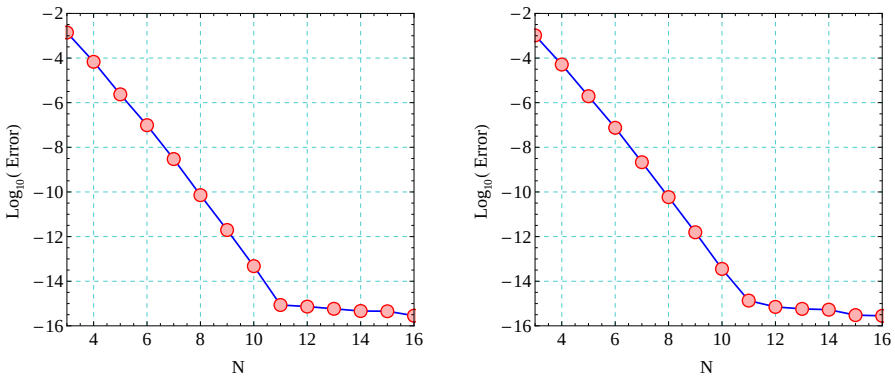

Figure 1: An illustration of the rate of convergence for the Tau method with various N. We observe the errors of Example 2 using Chebyshev bases (left) and Legendre bases (right)

The numerical results for example 3 with the Chebyshev and Legendre bases are presented in Fig. 2 and Table 3. Fig. 2, shows the rate of convergence for various β. Each part of the figure contains numerical errors for several values of N, which are plotted for a special value of β in L_∞ norm. As we can see from Table 3., and Fig. 2, the performance of the spectral Tau method with the Chebyshev and Legendre bases for $\beta \in [-1.5, -2.5]$ almost same, but when β tends to the $\beta = -2.5$(smooth solution) the rate of convergence increases and we have accurate numerical solutions. For $\beta = -1.5, -2.5$, numerical results have not been presented, since the exact solution is obtained. In addition, by using the function $\backslash LeafCount$ in the Mathematica software, for $N = 15$ we need 917 operations to obtain the operational Tau solution based on the Legendre polynomial bases.

Table 3: The numerical results of Example 3 with different β, $\lambda = 1$ and $N = 15$

	Maximal Error for Chebyshev Tau			Maximal Error for Legendre Tau		
x	$\beta = -2$	$\beta = -2.2$	$\beta = -2.4$	$\beta = -2$	$\beta = -2.2$	$\beta = -2.4$
0.2	9.92×10^{-6}	3.38×10^{-6}	5.74×10^{-7}	1.29×10^{-5}	4.27×10^{-6}	7.01×10^{-7}
0.4	7.45×10^{-6}	2.57×10^{-6}	4.41×10^{-7}	1.02×10^{-5}	3.46×10^{-6}	5.80×10^{-7}
0.6	4.39×10^{6}	1.44×10^{-6}	$2.38e \times 10^{-7}$	7.77×10^{-6}	2.61×10^{-6}	4.31×10^{-7}
0.8	3.47×10^{-6}	1.12×10^{-6}	1.82×10^{-7}	5.41×10^{-6}	1.70×10^{-6}	2.63×10^{-7}
1	2.00×10^{-6}	5.10×10^{-7}	9.23×10^{-8}	5.20×10^{-6}	1.67×10^{-6}	2.64×10^{-7}

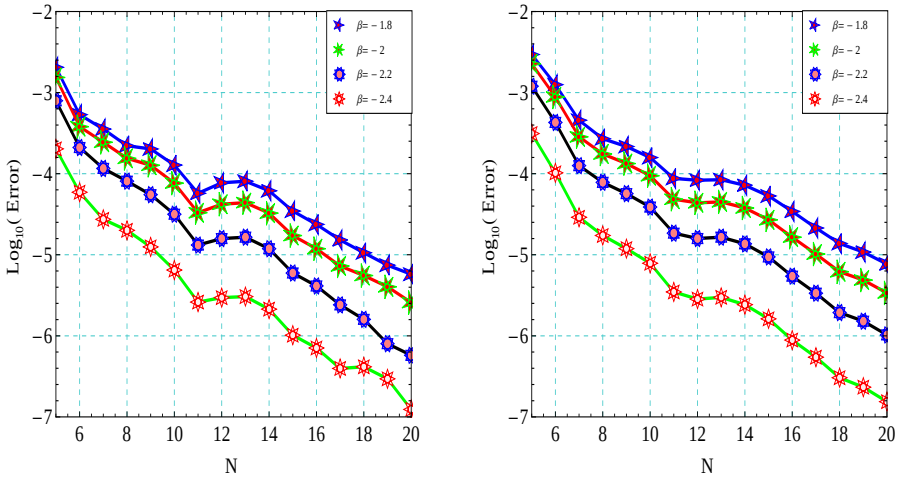

Figure 2: An illustration of the rate of convergence for the Tau method with various β. We observe the errors of Example 3 using Chebyshev bases (left) and Legendre bases (right)

4 Conclusion

In this paper, we presented a numerical scheme for solving nonlinear multi-order fractional differential equations. The operational Tau method was employed. Also, several test problems were used to show the applicability and efficiency of the method. The obtained results indicate that the new approach can solve the problem effectively.

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A new approach for solving nonlinear system of equations using Newton method and HAM

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Abstract

A new approach utilizing Newton Method and Homotopy Analysis Method (HAM) is proposed for solving nonlinear system of equations. Accelerating the rate of convergence of HAM, and obtaining a global quadratic rate of convergence are the main purposes of this approach. The numerical results demonstrate the efficiency and the performance of proposed approach. The comparison with conventional homotopy method, Newton Method and HAM shows the great freedom of selecting the initial guess, in this approach.

Keywords: Homotopy Analysis Method; Zero order deformation equations; Control convergence parameter; Newton's method; Iterative method; Multi-step iterative method; Order of convergence.

1 Introduction

Solving algebraic and transcendental equations is an interesting mathematical problem that has been occupied an important place in mathematical history. This problem arises in different applications of mathematics in sciences and engineering. Analytical solution of this problem is reserved to a small category of equations. For this reason and the exigencies of those increasing applications, from the beginning of era of electronic computing numerical methods of these problems have been progressed.

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Actually there is a vast group of conventional methods to solve algebraic and transcendental equations, but yet there exist enormous difficulty due to local convergence of these methods that make the new research inevitable. Particular numerical solution of system of nonlinear equations is realized by different methods. A traditional method is Newton method that can have quadratic order of convergence, but the convergence is local [16]. There is a variety of modified Newton methods which make a global convergence possible [16]. Many new one-step and multi-step methods are used to solve these system of equations (for more details one can refer to [4, 7, 8]). There is also acceleration methods and multi-step methods but these methods are also very dependent to initial guess and have local convergence in the most of the cases [16]. Recently the homotopy method using the notion of homotopy and functional series are applied to solve the system of nonlinear equations [1, 3, 6, 15, 11, 17]. Some methods are very suitable, but in practice they need to solve a system of differential equations with initial conditions [14]. One of the most important of Homotopy methods which is principally used for solving the nonlinear differential equations is Homotopy Analysis Method (HAM), that can be applied for solving nonlinear equations, but it is normally slow with local convergence [14]. In this paper a combination of Newton Method and HAM is considered to solve the algebraic and transcendental system of equations with the aim of improving the both mentioned methods, in view of local convergence and the rate of convergence. The results of proposed method will be compared with other methods.

The organization of the paper is as follows. In Section 2 a concise description of the Newton Method, the Homotopy Method are presented. In Section 3 the fundamental of HAM and proposed approach is discussed. In Section 4 the numerical results for 3 methods are given and compared. Finally Section 5 ends the paper with conclusion and discussion.

2 Description of problem and the methods

Consider the following nonlinear algebraic or transcendental system of equations

$$F(x) = 0 \ , \qquad F = (f_1, f_2, \cdots, f_n) \ , \tag{1}$$

where $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$, that D is an open region in $\mathbb{R}^n$ and $F \in C^1(D)$ such that $F(\widehat{\mathbf{x}}) = 0$. The vector $\widehat{\mathbf{x}}$ is called the zero of F or the solution of the equation (1). Recalling that the Newton Method for solving (1) is formulated as follows

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - [DF(\mathbf{x}^{(k)})]^{-1} F(\mathbf{x}^{(k)}), \qquad k = 0, 1, 2, \cdots \ , \tag{2}$$

where DF is the Jacobian matrix of F and $\mathbf{x}^{(0)}$ is an initial guess of $\widehat{\mathbf{x}}$. For more details see [16]. The Newton method is a suitable technique for

differentiable functions. In general, the rate of convergence is quadratic in a neighborhood of the solution $\widehat{\mathbf{x}}$, with local convergence property. As a second choice for solving (1), the homotopy method for the system of nonlinear equation is recalled [6]. The Homotopy function

$$\mathcal{H} : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n \ ,$$

is defined by

$$\mathcal{H}(q,\mathbf{x}) = qF(\mathbf{x}) + (1-q)(F(\mathbf{x}) - F(\mathbf{x}^{(0)})) \tag{3}$$

$$= F(\mathbf{x}) + (q-1)F(\mathbf{x}^{(0)}) \ , \tag{4}$$

here $\mathbf{x}^{(0)}$ is an initial guess of $\widehat{\mathbf{x}}$ and q is called Homotopy parameter or embedding parameter. Obviously, at $q = 0$ and $q = 1$,

$$\mathcal{H}(0,\mathbf{x}) = F(\mathbf{x}) - F(\mathbf{x}^{(0)}) \ , \qquad \mathcal{H}(1,\mathbf{x}) = F(\mathbf{x}).$$

If q increases from 0 to 1 then the function $\mathcal{H}(q,\mathbf{x})$ varies continuously from $F(\mathbf{x}) - F(\mathbf{x}^{(0)})$ to $F(\mathbf{x})$. In topology, such a kind of continuous variation is called deformation. The function $\mathcal{H}$ respect to parameter q, provides us a family of functions that can lead from the known value $\mathbf{x}^{(0)}$, to solution $\widehat{\mathbf{x}}$. The function $\mathcal{H}$ is a Homotopy between $\mathcal{H}(0,\mathbf{x}) = F(\mathbf{x}) - F(\mathbf{x}^{(0)})$ and $\mathcal{H}(1,\mathbf{x}) = F(\mathbf{x})$. Accepting that $\phi : [0,1] \to \mathbb{R}^n$, $\mathbf{x} = \phi(q)$ is a unique solution of the equation

$$\mathcal{H}(q,\mathbf{x}) = 0 \ , \qquad q \in [0,1] \ , \tag{5}$$

or

$$\mathcal{H}(q,\phi(q)) = 0 \ , \qquad q \in [0,1] \ . \tag{6}$$

The set $\{\phi(q) | 0 \leq q \leq 1\}$ can be viewed as a family of parameterized curves respect to q in $\mathbb{R}^n$ from $\phi(0)$ to $\phi(1) = \widehat{\mathbf{x}}$. The solution $\widehat{\mathbf{x}}$ of $F(\mathbf{x}) = 0$ can be obtained by solving the following system of equations

$$\phi'(q) = -[J(\phi(q))]^{-1} F(\phi(0)), \qquad 0 \leq q \leq 1,$$

with the initial condition $\phi(0) = \mathbf{x}^{(0)}$, where $J(\phi(q))$ is jacobian matrix of $\mathcal{H}$ respect to $\mathbf{x}$ [6]. This method will be reffered as HM.

3 HAM combined with Newton method

The Homotopy Analysis Method (HAM) is proposed by Liao [2]. In this method one introduces a homotopy function for solving (1). To be more precise, the following homotopy function is considered:

$$\mathcal{H}[q,\phi(q)] = (1-q)\mathcal{L}[\phi(q) - \mathbf{x}^{(0)}] + q\mathcal{N}[\phi(q)] \ , \tag{7}$$

where $q \in [0,1]$ is an embedding parameter and $\phi(q)$ is a function of q, and $\mathbf{x}^{(0)} \in \mathbb{R}^n$ is an initial estimation of $\widehat{\mathbf{x}}$, the solution of (1). Also, $\mathcal{N}$ is a nonlinear operator and $\mathcal{L}$ is a linear operator and

$$\mathcal{N}(\mathbf{x}) \equiv F(\mathbf{x}) \ . \tag{8}$$

If $q = 0$ and $q = 1$, then considering $\phi(0) = \mathbf{x}^{(0)}$, yields

$$\mathcal{H}[q, \phi(q)]\big|_{q=0} = \mathcal{L}[\phi(0) - \mathbf{x}^{(0)}] = 0 \ , \tag{9}$$

and

$$\mathcal{H}[q, \phi(q)]\big|_{q=1} = \mathcal{N}[\phi(1)] \ , \qquad \phi(1) = \widehat{\mathbf{x}} \ .$$

By using (9), the vector

$$\phi(1) = \widehat{\mathbf{x}} \ ,$$

is obviously the solution of the equation

$$\mathcal{H}[q, \phi(q)]\big|_{q=1} = 0 \ .$$

As the embedding parameter q increases from 0 to 1, the solution $\phi(q)$ of equation

$$\mathcal{H}[q, \phi(q)] = 0 \ ,$$

depends upon the embedding parameter q and varies from initial approximation $\mathbf{x}^{(0)}$ to the solution $\widehat{\mathbf{x}}$ of equation (9). Now by using homotopy function (7) we construct a family of equations

$$(1-q)\mathcal{L}[\phi(q) - \mathbf{x}^{(0)}] = q\mathcal{N}(\phi(q)) \ , \qquad q \in [0,1] \ , \tag{10}$$

subject to the initial condition

$$\phi(0) = \mathbf{x}^{(0)} \ . \tag{11}$$

Consider equation (1) and let A be a non-singular matrix which will be determined later. We construct following deformation equation that is called zeroth-order deformation equation:

$$(1-q)A(\phi(q) - \mathbf{x}^{(0)}) = qF(\phi(q)) \ . \tag{12}$$

Suppose $\widehat{\mathbf{x}}$ is solution of $F(\mathbf{x}) = 0$ and the sequence $\left\{\mathbf{x}^{(i)}\right\}_{i\in\mathbb{N}}$ exist with the following property

$$\widehat{\mathbf{x}} = \sum_{m=0}^{\infty} \mathbf{x}^{(m)} \ ,$$

and

$$\mathbf{x}^{(i)} = (x_1^{(i)}, \cdots, x_n^{(i)}) \in \mathbb{R}^n \ , \ i = 0, 1, 2, \ldots \ .$$

Next a function $\phi : [0,1] \to \mathbb{R}^n$ is defined as follows

$$\mathbf{x} = \phi(q) = \sum_{m=0}^{\infty} \mathbf{x}^{(m)} q^n \ , \qquad q \in [0,1],$$

Subject to

$$\phi(0) = \mathbf{x}^{(0)} \ , \tag{13}$$
$$\phi(1) = \widehat{\mathbf{x}} \ . \tag{14}$$

By differentiating (12) with respect to q, the following equation is obtained:

$$-A(\phi(q) - \mathbf{x}^{(0)}) + (1-q)(A\frac{d}{dq}\phi(q)) = F(\phi(q)) + q\frac{d}{dq}F(\phi(q)) \ . \tag{15}$$

Putting $q = 0$ in (15) yields

$$A\frac{d}{dq}\phi(q)\Big|_{q=0} = F(\phi(0)) \ . \tag{16}$$

Matrix A being non-singular, it deduces

$$\frac{d}{dq}\phi(q)\Big|_{q=0} = A^{-1}F(\phi(0)) \ . \tag{17}$$

On the other hand,

$$\frac{d}{dq}\phi(q) = \sum_{m=1}^{\infty} m\mathbf{x}^{(m)} q^{m-1}.$$

Then

$$\frac{d}{dq}\phi(q)\Big|_{q=0} = \mathbf{x}^{(1)} = A^{-1}F(\mathbf{x}^{(0)}) \ . \tag{18}$$

The equation of (15) is called first-order deformation equation. By differentiating equation (15) with respect to q, the following equation is obtained

$$-2A\frac{d}{dq}\phi(q) + (1-q)A\frac{d^2}{dq^2}\phi(q) = 2\frac{d}{dq}F(\phi(q)) + q\frac{d^2}{dq^2}F(\phi(q)) \ . \tag{19}$$

Putting $q = 0$, the second-order deformation equation is obtained as follows

$$-2A\mathbf{x}^{(1)} + 2A\mathbf{x}^{(2)} = 2D_x F(\mathbf{x}^{(0)})\mathbf{x}^{(1)} \ , \tag{20}$$

or

$$\mathbf{x}^{(2)} = (A^{-1}D_x F(\mathbf{x}^{(0)}) + I)\mathbf{x}^{(1)} \ .$$

By repeating the same procedure the m-th order deformation equation can be obtained. Indeed, the following proposition can be proved.

Proposition 3.1. *If $F : \mathbb{R}^n \to \mathbb{R}^n$, and $F \in C^m(\mathbb{R}^n)$, $A \in \mathbb{R}^{n\times n}$ a given matrix, and*

$$x = \phi(q) = \sum_{m=0}^{\infty} \boldsymbol{x}^{(m)} q^m, \qquad \phi : [0,1] \to \mathbb{R}^n \ ,$$

$$(1-q)A(\phi(q) - \boldsymbol{x}^{(0)}) = qF(\phi(q)).$$

where ϕ is an analytic function, then

$$A(\boldsymbol{x}^{(m)} - \chi_m \boldsymbol{x}^{(m-1)}) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dq^{m-1}}F(\phi(q))\Big|_{q=0} \ , \tag{21}$$

where

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & o.w. \end{cases} \ .$$

If $m \ge 2$ and A be a nonsingular matrix then

$$\boldsymbol{x}^{(m)} = \boldsymbol{x}^{(m-1)} + \frac{1}{(m-1)!}A^{-1}\frac{d^{m-1}}{dq^{m-1}}F(\phi(q,\boldsymbol{x}))\Big|_{q=0} \ . \tag{22}$$

The equation (22) is called m-th order deformation equation.

For solving system of algebraic equations in general one can use the above equations to determine the vectorial terms $\mathbf{x}^{(i)}$ of $\widehat{\mathbf{x}} = \sum_{i=0}^{\infty} \mathbf{x}^{(i)}$, i.e. the following equations.

$$\mathbf{x}^{(m)} = \begin{cases} \mathbf{x}^{(0)} & m = 0 \\ A^{-1}F(\mathbf{x}^{(0)}) & m = 1 \\ \mathbf{x}^{(m-1)} + \frac{1}{(m-1)!}A^{-1}\frac{d^{m-1}}{dq^{m-1}}F(\phi(q))\Big|_{q=0} & m \ge 2 \end{cases} \ . \tag{23}$$

In practice, one can obtain a finite number of $\mathbf{x}^{(i)}$. Then by considering partial sum of above series one can determine $\phi(1)$ approximately by a K^{th} order partial sum as follows:

$$\widehat{\mathbf{x}} = \phi(1) \approx \mathbf{x}^{(0)} + \mathbf{x}^{(1)} + \cdots + \mathbf{x}^{(K)} \ ,$$

Unfortunately, the homotopy series

$$\phi(q) = \sum_{m=0}^{\infty} \mathbf{x}^{(m)} q^m \ ,$$

may be divergent at $q = 1$. To overcome this restriction, Liao [14] introduced an auxiliary parameter $h \neq 0$ to construct a kind of deformation equations based on

$$(1-q)A(\phi(q,h) - \mathbf{x}^{(0)}) = qhF(\phi(q,h)) \ ,$$

where

$$\phi(q,h) = \sum_{m=0}^{\infty} \mathbf{x}^{(m)}(h) q^m,$$

the vectors $\mathbf{x}^{(m)}$ are dependent on h. In particular if series is convergent for at least one $\widehat{h}$, it is deduced [9],

$$\widehat{\mathbf{x}} = \sum_{m=0}^{\infty} \mathbf{x}^{(m)}(\widehat{h}), \phi(0,h) = \mathbf{x}^{(0)}, \phi(1,h) = \widehat{\mathbf{x}} \ .$$

Therefore, the equation (23) is transformed to

$$\mathbf{x}^{(m)} = \begin{cases} \mathbf{x}^{(0)} & m = 0 \\ hA^{-1}F(\mathbf{x}^{(0)}) & m = 1 \\ \mathbf{x}^{(m-1)} + \frac{h}{(m-1)!} A^{-1} \frac{d^{m-1}}{dq^{m-1}} F(\phi(q,\mathbf{x}))\Big|_{q=0} & m \geq 2 \end{cases} . \quad (24)$$

The parameter h is called convergence control parameter. The convergence rate and region of series solution depend on the convergent control parameter. This parameter provides a convenient way to adjust and control convergence region and rate of convergence of series solution given by the HAM. For finding a suitable h, some approaches are proposed in [2, 5]. The traditional approach gives the possibility of estimation a suitable value of h, by plotting the h-curves (for more details see [14]). Following [9], we use a more systematic approach in this work. Consider

$$\phi(q,h) \approx \widetilde{\phi}(q,h) = \sum_{m=0}^{K} \mathbf{x}^{(m)} q^m = \mathbf{x}^{(0)} + \mathbf{x}^{(1)} q + \mathbf{x}^{(2)} q^2 + \cdots + \mathbf{x}^{(K)} q^K \ .$$

The value $\widetilde{\phi}(1,h)$ is only a function of h, which is denoted by

$$\psi_k(h) = \widetilde{\phi}(1,h) \ .$$

As proved by Liao in general [14], if the series solution converges, then there exists at least an h_0 such that

$$\lim_{k\to\infty} \parallel F(\psi_k(h_0)) \parallel = 0 \ ,$$

where denote $||.||$ is Euclidian norm in $\mathbb{R}^n$. Accordingly, we let

$$||F(\psi_k(h_0))|| = \min_{h\in R_h} ||F(\psi_k(h))|| \ , \tag{25}$$

where R_h is a valid region that lie on a horizontal segment of the h-curves. The $\psi_k(h_0)$ is a vector in $\mathbb{R}^n$ that can be regarded as an approximation of $\widehat{\mathbf{x}}$. So, we can apply $\psi_k(h_0)$ as initial point for Newton method, if Newton method converges, the desired approximate solution is found, otherwise, after some iterations, the result of Newton method is considered as an initial point for a new HAM procedure and so on.

The proof of convergence is an open problem [14] . The numerical examples show that proposed method is more efficient than Newton method.

The proposed HAM is convergent for many examples but this method spends a lot of time during each iteration. For accelerating the convergence this method, we suggest the combination of HAM and Newton method. At the beginning, a new initial point can be obtained by utilizing the proposed method, then the process continues by Newton method with this new initial point. If Newton method does not converge to solution after some iterations, the HAM method can be applied again by using this new initial point. If $DF(\mathbf{x}^{(0)})$ is non-singular, this matrix is practically profitable as a good selection of A, so

$$A = DF(\mathbf{x}^{(0)}).$$

Using the above choice it is observed when $h = -1$ the first step of the homotopy consists of the first iteration of Newton method, in fact, one has

$$\widehat{\mathbf{x}} = \phi(1) \approx \overset{\sim}{\phi}(1) = \left[\sum_{m=0}^{1} \mathbf{x}^{(m)} q^m\right]_{q=1} = \mathbf{x}^{(0)} + \mathbf{x}^{(1)} \ ,$$

where by using (24)

$$\mathbf{x}^{(1)} = -DF(\mathbf{x}^{(0)})^{-1} F(\mathbf{x}^{(0)}) \ .$$

This result demonstrate the validity of choosing $A = DF(\mathbf{x}^{(0)})$. Application and implementation of this hybrid method allow us improving local convergence of newton method , and choosing $\mathbf{x}^{(0)}$ arbitrary.

4 Numerical experiments

In this section, several examples are considered and the numerical results for mentioned methods: Homotopy Method(HM), HAM, Newton method

Table 1: Numerical results for Example 4.1 with $x^{(0)} = 1$

Method	NI	$\|\|F(\mathbf{x}^{(m)})\|\|$	CPU time	result
NHAM	4	$4.335133e-008$	$3.333667e+000$	Convergent
Newton	4	$1.691234e-008$	$7.639678e-002$	Convergent
HAM	3	$5.775537e-008$	$2.129251e-001$	Convergent
HM	–	$7.457211e-005$	$4.041735e-001$	Convergent

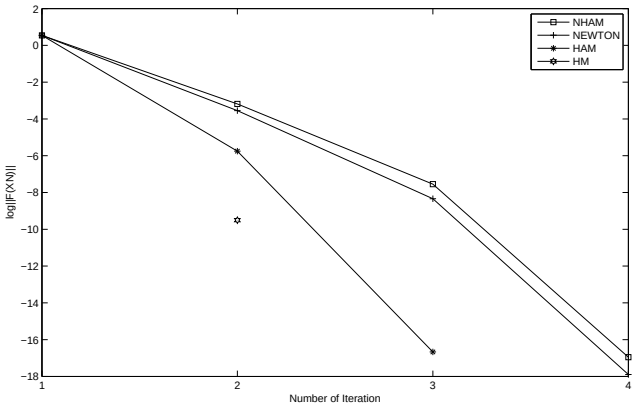

Figure 1: The graph of $\ln(||F(X)||)$ for Example 4.1 with $x^{(0)} = 1$

and Newton-HAM (NHAM) are reported. We utilize MATLAB 8. In Tables and Figures, the number of iterations (NI), the Euclidean norm of residual of government equation and CPU time, are presented.

Example 4.1. Consider the following equation:

$$f(x) = xe^x - 1 = 0, \tag{26}$$

The function f has at least one zero between 0 and 1. For $x^{(0)} = 1$, the numerical results are shown in Table 1. For this initial point all methods are convergent, but the Newton method is apparently faster than other methods. For $x^{(0)} = 10$, the numerical results are shown in Table 2.

In this case, HM method is divergent, Newton method is faster than NHAM and HAM and results are more accurate than others. The number of iterations for NHAM is less than the others. For $x^{(0)} = -400$, the numerical results are shown in the Table 3. In this example NHAM method is convergent and other methods are divergent.

Table 2: Numerical results for Example 4.1 with $x^{(0)} = 10$

Method	NI	$\|\|F(\mathbf{x}^{(m)})\|\|$	CPU time	result
NHAM	6	$2.160101e-010$	$3.148814e+000$	Convergent
Newton	16	$5.107026e-014$	$2.622956e-001$	Convergent
HAM	31	$3.594237e-008$	$2.876829e+000$	Convergent
HM	–	$5.790573e+003$	$4.081270e-001$	Divergent

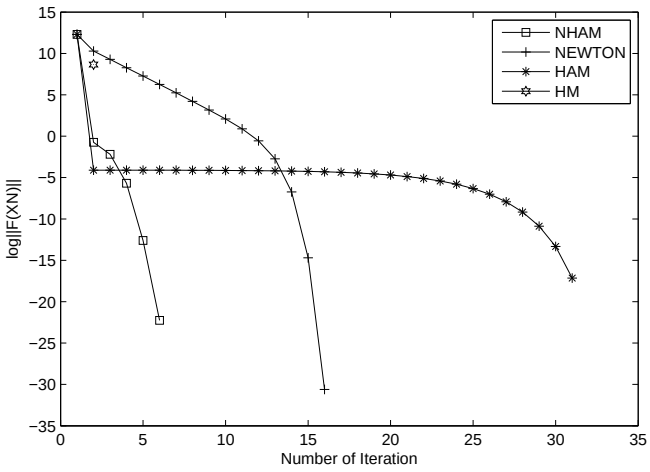

Figure 2: The graph of $ln(||F(X)||)$ for Example 4.1 with $x^{(0)} = 10$

Table 3: Numerical results for Example 4.1 with $x^{(0)} = -400$

Method	NI	$\|\|F(\mathbf{x}^{(m)})\|\|$	CPU time	result
NHAM	212	$2.403109e-007$	$6.452634e+000$	Convergent
Newton	3	Nan	–	Divergent
HAM	100	$Infinity$	–	Divergent
HM	–	Nan	–	Divergent

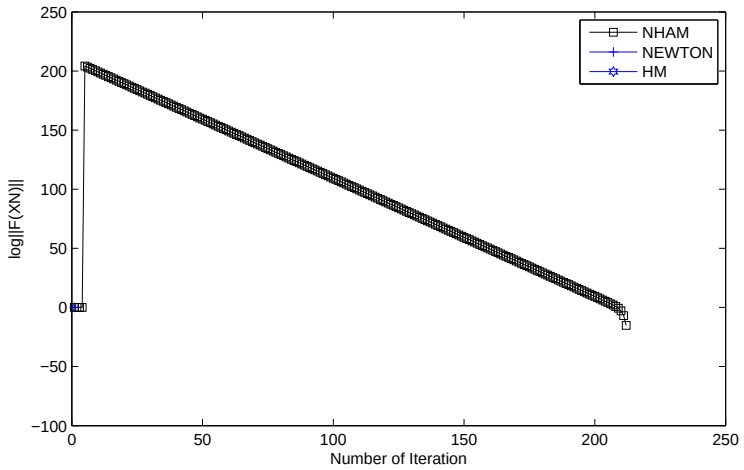

Figure 3: The graph of $ln(||F(X)||)$ for Example 4.1 with $x^{(0)} = -400$

Table 4: Numerical results for Example 4.2

Method	NI	$\|\|F(\mathbf{x}^{(m)})\|\|$	CPU time	result
NHAM	6	$2.085579e-008$	$2.116125e+000$	Convergent
Newton	1	NaN	–	Divergent
HAM	8	$2.247981e-010$	$3.105904e+000$	Convergent
HM	–	$1.967763e+009$	$7.584255e-001$	Divergent

Example 4.2. Consider following equations:

$$\begin{cases} f_1(x,y,z,d) = xyz + d - 31 = 0, \\ f_2(x,y,z,d) = x + y + z + d - 11 = 0, \\ f_3(x,y,z,d) = 2x + 3y + 4z + d - 35 = 0, \\ f_4(x,y,z,d) = x + z - y + d - 1 = 0, \end{cases} \tag{27}$$

where $F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}^T$.

We know $\hat{X_1} = (2,3,5,1)$ and $\hat{X_2} = (\frac{29}{5}, \frac{11}{10}, 5, \frac{-9}{10})$ are two solutions of $F(X) = 0$. For $X^{(0)} = (1,1,1,1)$, numerical results are shown in Table 4.

Newton Method is divergent because $det(DF(X^{(0)})) = 0$. But HAM and NHAM methods converge, and NHAM is faster than HAM.

Example 4.3. Consider the following equations:

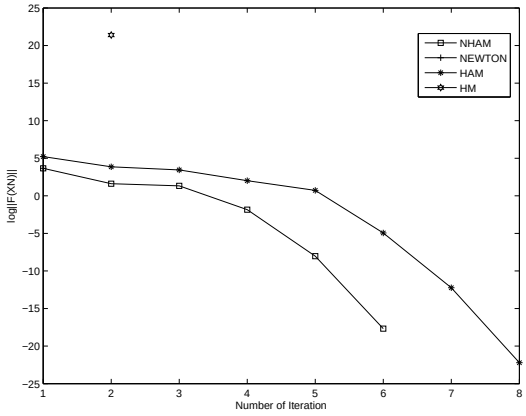

Figure 4: The graph of $ln(||F(X)||)$ for Example 4.2

Table 5: Numerical results for Example 4.3

Method	NI	$\|\|F(\mathbf{x}^{(m)})\|\|$	CPU time	result
NHAM	8	$6.567317e-010$	$8.196840e+001$	Convergent
Newton	101	$5.030214e+003$	$6.042236e+000$	Divergent
HAM	18	$5.830347e-008$	$3.106638e+003$	Convergent
HM	–	$5.242329e+002$	$2.483993e+000$	Divergent

$$\begin{cases} f_1(x_1, x_2, \cdots, x_n) = (3 - \frac{1}{2}x_1)x_1 - 2x_2 + 1 = 0 \ , \\ f_i(x_1, x_2, \cdots, x_n) = (3 - \frac{1}{2}x_i)x_i - x_{i-1} - 2x_{i+1} + 1 = 0 \ , \quad 1 < i < n \ , \\ f_n(x_1, x_2, \cdots, x_n) = (3 - \frac{1}{2}x_n)x_n - 2x_{n-1} + 1 = 0, \end{cases} \tag{28}$$

that $F = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}^T$.

For $n = 50$ and $X^{(0)} = (100, 100, \cdots, 100)$, numerical results are shown in Table 5.

In this example HAM and NHAM are convergent to the exact solution $\hat{X} = (1, ..., 1)$, but Newton method is divergent. Also NHAM is faster than HAM. Results are shown in Figure 5.

Example 4.4. Consider the following equations:

$$\begin{cases} f_k(x_1, x_2, \cdots, x_n) = 10000x_k x_{k+1} - 1 = 0, \quad mod(k, 2) = 1, \\ f_k(x_1, x_2, \cdots, x_n) = exp(-x_{k-1}) + exp(-x_k) - 1.0001 = 0, \ mod(k, 2) = 0, \end{cases} \tag{29}$$

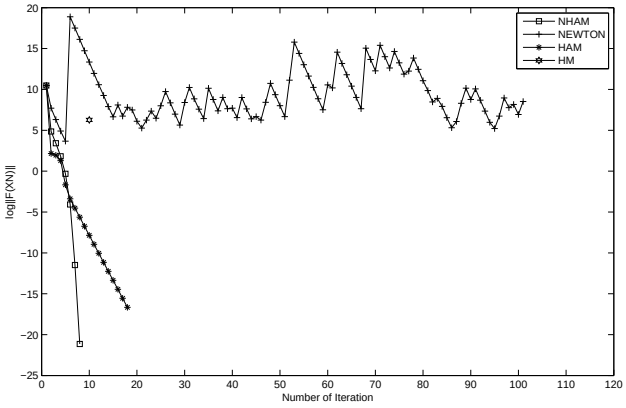

Figure 5: The graph of $ln(||F(X)||)$ for Example 4.3

Table 6: Numerical results for Example 4.4

Method	NI	$\|\|F(\mathbf{x}^{(m)})\|\|$	CPU time	result
NHAM	13	$1.059758e-009$	$5.152483e+001$	Convergent
Newton	12	$1.112530e-010$	$1.972995e+001$	Convergent
HAM	23	$2.728830e+056$	$1.128194e+002$	Divergent
HM	–	$4.211734e-001$	$1.092275e+002$	Convergent

that $F = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}^T$.

For $n = 100$ and $X^{(0)} = (1, 0, 1, 0, 1, 0, 1, \cdots, 0)$, numerical results are shown in Table 6. In this example Newton method, NHAM and HM are convergent, but HAM is divergent. Results are shown in Figure 6.

Example 4.5. Consider the following equations:

$$\begin{cases} f_1(x_1, x_2) = exp(x_1) + x_1 x_2 - 1 = 0, \\ \\ f_2(x_1, x_2) = sin(x_1 x_2) + x_1 + x_2 - 1 = 0, \end{cases} \tag{30}$$

that $F = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix}^T$.

For $n = 100$ and $X^{(0)} = (1, 0, 1, 0, 1, 0, 1, \cdots, 0)$, numerical results are shown in Table 7. In this example all the methods are convergent. Results are shown in Figure 7.

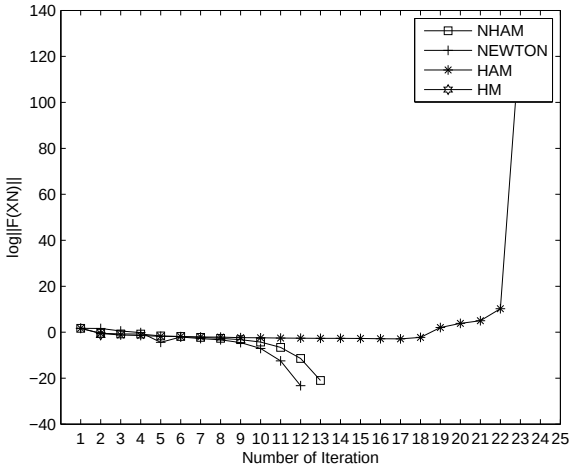

Figure 6: The graph of $ln(||F(X)||)$ for Example 4.4

Table 7: Numerical results for Example 4.5

Method	NI	$\|\|F(\mathbf{x}^{(m)})\|\|$	CPU time	result
NHAM	4	$1.993082e-010$	$1.848117e+000$	Convergent
Newton	4	$1.405720e-012$	$2.472726e-001$	Convergent
HAM	4	$1.187309e-008$	$1.719704e+000$	Convergent
HM	–	$5.368713e-006$	$1.380355e+000$	Convergent

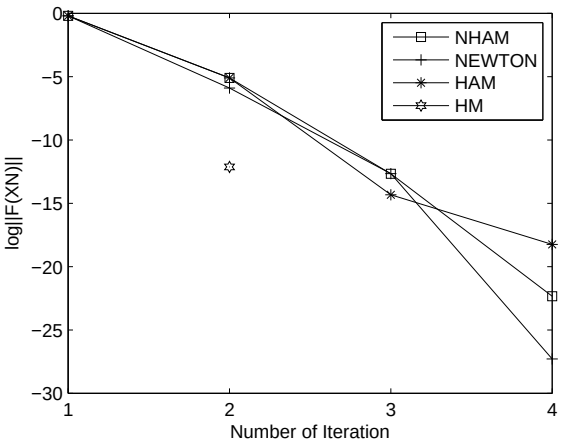

Figure 7: The graph of $ln(||F(X)||)$ for Example 4.5

5 Conclusion

In this paper, Newton-HAM (NHAM) applying control parameter h are proposed for solving systems of nonlinear equations. The results for all examples are convergent and also NHAM is faster than Homotopy method. The results demonstrate that by choosing a suitable h, HAM and NHAM methods are convergent. The numerical results show in general that the proposed method is effective and efficient and provides highly accurate results in a less number of iterations as compared by other methods. The main advantage of NHAM is the relative freedom of choosing initial guess. The appropriate proof convergence of NHAM can be continuation of the present work.

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Solving nonlinear Volterra integro-differential equation by using Legendre polynomial approximations

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Abstract

In this paper, we construct a new iterative method for solving nonlinear Volterra Integral Equation of the second kind, by approximating the Legendre polynomial basis. Error analysis is worked using property of interpolation. Finally, some examples are given to compare the results with some of the existing methods.

Keywords: Nonlinear Volterra integro-differential equation; Legendre polynomial; Error analysis.

1 Introduction

The area of orthogonal polynomials is an active research area in mathematics as well as with applications in mathematical physics, engineering, and computer science [6, 16]. Several numerical methods were used to solve integro-differential equations such as successive approximation method, Adomian decomposition method, Chebyshev and Taylor collocation methods, Haar Wavelet method, Wavelet Galerkin method, monotone iterative technique, Tau method, Walsh series method and Bezier curves method [2, 3, 4, 6, 13, 22]. One of the most common set of orthogonal polynomials is the set of the Legendre polynomials $L_0(x), L_1(x), ..., L_M(x)$, which are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = 1$. The

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Legendre polynomials $L_n(x)$, for $-1 \leq x \leq 1$ and $n \geq 0$, are given by the forms [6, 7, 11, 12, 21]

$$L_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}, n = 0, 1, \ldots, \qquad (1)$$

where $[n/2] = n/2$ if n is even, otherwise $\frac{n-1}{2}$. To use the Legendre polynomials for our purposes, it is preferable to map this to $[0, 1]$. Then we can also define them by the following recursive formula [11, 12]: $L_0(x) = 1$; and $L_1(x) = 2x - 1$ and for $n = 1, 2, ...$

$$(n+1)L_{n+1}(x) = (2n+1)(2x-1)L_n(x) - nL_{n-1}(x). \qquad (2)$$

On the other hand, the methods based on Legendre polynomials may be more appropriate for solving linear and nonlinear differential and Fredholm-Volterra integral and integro-differential-difference equations [5, 6, 7, 15, 18, 21]. Legendre polynomials are examples of eigen functions of singular Strum Liouville problems and have been used extensively in the solution of the boundary value problems and in computational fluid dynamics [5, 20] . Several ways for solving nonlinear integro differential equations are exist, for example Ghasemi et al. [8] with homotopy perturbation method and in [9] with wavelet Galerkin method and in [10] with sine−cosine wavelet method, Zhao and Corless in [23] adopted finite difference method, Lepik and Tamme in [19] with Haar wavelet method. In this paper, by means of the matrix relations between the Legendre polynomials and their derivatives, the mentioned methods above are modified and developed for solving the following nonlinear Volterra Integro-differential equation with variable coefficients

$$f_1(x)u(x) + f_2(x)u'(x) = g(x) + \int_0^x K(x, t, u(t))dt, \qquad (3)$$

where $u \in X := C([0, 1], \mathbb{R})$, $f : [0, 1] \to \mathbb{R}$, $K : [0, 1]^2 \times \mathbb{R} \to \mathbb{R}$, also is assumed K is a continuous function, and $u : [0, 1] \to \mathbb{R}$ is an unknown function. We have obtained a solution expressed in the form

$$u(x) \approx \sum_{n=0}^{M} a_n L_n(x). \qquad (4)$$

Next sections of this paper are organized as follows. In Section 2, expansion of Legendre basis properties and matrix relations, and its discretization of a integro-differential equation are given. In Section 3, the convergence of the method is described. In Section 4, the efficiency of the method by solving some examples and comparison of the numerical solutions with some other existing methods, is shown. A short conclusion is given in Section 5.

2 Expansion of Legendre basis and method of solution

If we define $\mathbf{L}(\mathbf{x}) = [L_0(x)\ L_1(x)\ ...\ L_M(x)]$ and $\mathbf{A} = [a_0\ a_1\ ...\ a_M]^T$ then

$$u(x) \approx \sum_{n=0}^{M} a_n L_n(x) = \mathbf{L}(\mathbf{x})\mathbf{A}. \tag{5}$$

Simillary if we define $\mathbf{L}'(\mathbf{x}) = [L_0'(x)\ L_1'(x)\ ...\ L_M'(x)]$ we have

$$u'(x) \approx \sum_{n=0}^{M} a_n L_n'(x) = \mathbf{L}'(\mathbf{x})\mathbf{A}, \tag{6}$$

where $'$ denotes the derivative with respect to x. By using Legendre recursive formula (1) for $n = 0, 1, 2, ..., M$, we can also obtain the matrix form of the equation as follows

$$\mathbf{L}'(\mathbf{x}) = \mathbf{L}(\mathbf{x})\mathbf{\Omega}^{\mathbf{T}}, \tag{7}$$

where $\mathbf{\Omega}$ has two forms different for odd and even values of M, that is, for odd values of M we have

$$\mathbf{\Omega} = \begin{pmatrix} 0 & 0 & 0 & 0 \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 3 & 0 & 7 \cdots & 2M-3 & 0 & 0 \\ 1 & 0 & 5 & 0 \cdots & 0 & 2M-1 & 0 \end{pmatrix},$$

and for even values of M

$$\mathbf{\Omega} = \begin{pmatrix} 0 & 0 & 0 & 0 \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0 \cdots & 2M-3 & 0 & 0 \\ 0 & 3 & 0 & 7 \cdots & 0 & 2M-1 & 0 \end{pmatrix}.$$

From (6) and (7) we get

$$u'(x) \approx \mathbf{L}(\mathbf{x})\mathbf{\Omega}^{\mathbf{T}}\mathbf{A}.$$

We use this method to approximate left hand side of Volterra integro-differential equation (3) as follows

$$f_1(x)u(x) + f_2(x)u'(x) \approx f_1(x)\mathbf{L}(\mathbf{x})\mathbf{A} + f_2(x)\mathbf{L}(\mathbf{x})\mathbf{\Omega^T}\mathbf{A} \\ = (f_1(x)\mathbf{L}(\mathbf{x}) + f_2(x)\mathbf{L}(\mathbf{x})\mathbf{\Omega^T})\mathbf{A}. \tag{8}$$

Thus

$$f_1(x)u(x) + f_2(x)u'(x) \approx \mathbf{S}(\mathbf{x})\mathbf{A}, \tag{9}$$

where $\mathbf{S}(\mathbf{x}) = [s_0(x)\ s_1(x)\ ...\ s_M(x)]$, and for $i = 0, 1, ..., M$, we define

$$s_i(x) = f_1(x)L_i(x) + f_2(x)\mathbf{L}(\mathbf{x})(\mathbf{\Omega^T})^i.$$

To obtain a solution of the problem (3), for each $x, t \in [0, 1]$ we define

$$K(x, t, u(t)) \approx \mathbf{L}(\mathbf{x})\mathbf{K}^*\mathbf{L^T}(\mathbf{t})$$

where $\mathbf{K}^* = [k_{nm}]$, and

$$k_{nm} = \frac{< L_n(x), < K(x, t, u(t)), L_m(t) >>}{\|L_n\|^2\|L_m\|^2}.$$

We use this method to approximate the right hand side of Volterra integro-differential equation (3) as follows

$$g(x) + \int_0^x \mathbf{L}(\mathbf{x})\mathbf{K}^*\mathbf{L^T}(\mathbf{t})dt = g(x) + \mathbf{L}(\mathbf{x})\mathbf{K}^* \int_0^x \mathbf{L^T}(\mathbf{t})dt.$$

By using Legendre formulas

$$\int_0^x \mathbf{L}(\mathbf{t})dt = (\mathbf{L}(\mathbf{x}) - \mathbf{L}(\mathbf{0}))(\mathbf{\Omega^T})^{-1},$$

we have

$$g(x) + \int_0^x K(x, t, u(t))dt = g(x) + \mathbf{L}(\mathbf{x})\mathbf{K}^*(\mathbf{\Omega})^{-1}(\mathbf{L}(\mathbf{x})^\mathbf{T} - \mathbf{L}(\mathbf{0})^\mathbf{T}). \tag{10}$$

Let

$$h(x) = g(x) + \mathbf{L}(\mathbf{x})\mathbf{K}^*(\mathbf{\Omega})^{-1}(\mathbf{L}(\mathbf{x})^\mathbf{T} - \mathbf{L}(\mathbf{0})^\mathbf{T}), \tag{11}$$

then from (9) and (11) we have

$$\mathbf{S}(\mathbf{x})\mathbf{A} = h(x). \tag{12}$$

We can use a matrix method based on Legendre collocation points defined by

$$x_i = \frac{i}{M} \quad i = 0, 1, \ldots, M. \tag{13}$$

Now, by substituting the collocation points into Eq. (12) we have the following system

$$\mathbf{S}(\mathbf{x_i})\mathbf{A} = h(x_i) \quad i = 0, 1, \ldots, M. \tag{14}$$

Thus, we use this numerical method to approximate the solutions of nonlinear Volterra integro-differential equation, which correspond to a system of $(M+1)$ algebraic equations for $(M+1)$ unknown Legendre coefficients $a_0, a_1, ..., a_M$. Briefly, Eq. (14) in the matrix form is as follows

$$\mathbf{SA} = \mathbf{H}, \tag{15}$$

where for $i = 0, 1, ..., M$

$$\mathbf{S} = [\mathbf{S}(x_0)\ \mathbf{S}(x_1)\ ...\ \mathbf{S}(x_M)]^T,$$

and

$$\mathbf{H} = [h(x_1)\ h(x_2)\ ...\ h(x_M)].$$

3 Error analysis

We assume that $u(x)$ is a sufficiently smooth function and $P_M(x)$ is the polynomial that interpolates u at points x_i, $i = 0, 1, ..., M$ that are the roots of $M+1$ degree shifted Chebyshev polynomial in $[0, 1]$. Then we have

$$u(x) - P_M(x) = \frac{d^{M+1}u}{dx^{M+1}}(\xi)\frac{\prod_{i=0}^{M}(x - x_i)}{(M+1)!}, \tag{16}$$

where $\xi \in [0, 1]$, therefore

$$|u(x) - P_M(x)| \leq max|\frac{d^{M+1}u(x)}{dx^{M+1}}|\frac{\prod_{i=0}^{M}(x - x_i)}{(M+1)!}. \tag{17}$$

If we assume that c is an upper bound for $max\frac{d^{M+1}u(x)}{dx^{M+1}}$, then

$$|u(x) - P_M(x)| \leq c\frac{1}{(M+1)!2^{2M+1}}. \tag{18}$$

Theorem 3.1. *Let* $u_M(x) = U_M^T\mathbf{L}(\mathbf{x})$ *where* $U_M = [u_0\ u_1\ ...\ u_M]^T$ *and*

$$u_m = (2m+1)\int_0^1 u(x)L_m(x)dx,$$

then, there exists a real number c' *such that*

$$\|u(x) - u_M(x)\|_2 \leq c'\frac{1}{(M+1)!2^{2M+1}}. \tag{19}$$

Proof. Suppose $f : [0, 1] \to \mathbf{R}$ be an arbitrary continuous function. We define

$$||f||_2 = \int_0^1 |f(x)|^2 dx. \tag{20}$$

Let X_M be the space of all polynomials that their degrees are equal or less than M and $f(x)$ be an arbitary function. Since X_M is a finite dimensional vector space, f has an uniqe best approximation u_M, such that

$$\|u(x) - u_M\|_2 \leq \|u - g\|_2 \qquad \forall g \in X_M. \tag{21}$$

In particular, we have

$$\|u(x) - u_M(x)\|_2^2 = \int_0^1 |u(x) - u_M(x)|^2 dx \leq \int_0^1 |u(x) - P_M(x)|^2 dx, \tag{22}$$

where P_M interpolates f. Thus

$$\|u(x) - u_M(x)\|_2^2 = \int_0^1 (c\frac{1}{(M+1)!2^{2M+1}})^2 dx, \tag{23}$$

so

$$\|u(x) - u_M(x)\|_2 \leq c\frac{1}{(M+1)!2^{2M+1}}. \tag{24}$$

4 Numerical examples

In this section, several numerical examples are given to show the efficiency of our proposed method for approximating the solution of Volterra integro-differential equation by comparing with other methods. In all examples N denotes the number of itereations

Example 4.1. *Consider the following nonlinear Volterra integro-differential equation of the second kind with the exact solution* $u(x) = x^3$

$$(x-1)u'(x)+xu(x) = 3(x-1)x^2-\frac{1}{3}x+\frac{1}{3}xcos(x^3)+\int_0^x xt^2 sin(u(t))dt. \tag{25}$$

Comparison of the absolute errors between Block-Pulse functions method [1] and the proposed method for $N = 7$ is shown in Table 2 . Also, Figure 2 shows the comparison between exact and approximate solutions for $N = 2$ and $N = 7$.

Example 4.2. *Consider the following nonlinear Volterra integro-differential equation of the second kind with the exact solution* $u(x) = x - x^2$

$$3(x-1)u(x) + x^2u'(x) = f(x) + \int_0^x (x-t)u(t)dt, \tag{26}$$

Table 1: Absolute errors for Example 4.1

t	BPFs method [1] for N=7	proposed method for N=7
0.0100	4.654×10^{-4}	3.585×10^{-6}
0.3537	8.098×10^{-5}	1.726×10^{-7}
0.6101	6.675×10^{-5}	6.052×10^{-7}
0.9500	3.581×10^{-5}	2.138×10^{-7}

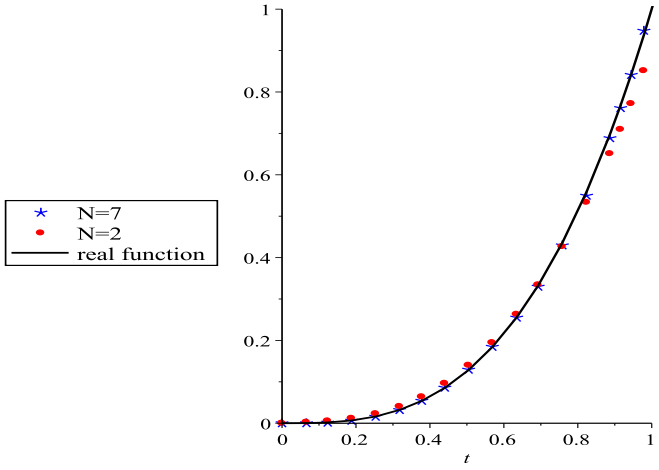

Figure 1: Comparison between exact and approximate solutions for Example 4.1

where
$f(x) = 3(x-1)(x-x^2) + x^2(1-2x) - \frac{1}{4}x^4 + \frac{1}{3}(x+1))x^3 - \frac{1}{2}(1/2)x^3$.

Comparison of absolute errors between CAS wavelet method [3] and the proposed method for $N = 7$ is shown in Table 2 . Also, Figure 2 shows the comparison between exact and approximate solutions for $N = 2$ and $N = 7$.

Example 4.3. *Consider the following nonlinear Volterra integral equation of the second kind with the exact solution* $u(x) = ln(x+1)$

$$u'(x) = f(x) + \int_0^x xt^2(u(t))^2 dt, \tag{27}$$

where

$$f(x) = \frac{1}{x+1} + (\frac{11}{9} + \frac{2}{3}x - \frac{1}{3}x^2 + \frac{2}{9}x^3)x\ln(x+1) - \frac{1}{3}(1+x^3)x(\ln(x+1))^2 - \frac{1}{9}x^2(11 - \frac{5}{2}x + \frac{2}{3}x^2). \tag{28}$$

Table 2: Absolute errors for Example 4.2

t	CAS wavelet method [3] for N=7	proposed method for N=7
0.0100	3.37×10^{-3}	4.8×10^{-4}
0.3446	4.72×10^{-3}	5.7×10^{-5}
0.7075	5.87×10^{-3}	3.4×10^{-4}
0.9178	3.42×10^{-2}	2.13×10^{-6}
1.0000	6.20×10^{-2}	5.8×10^{-5}

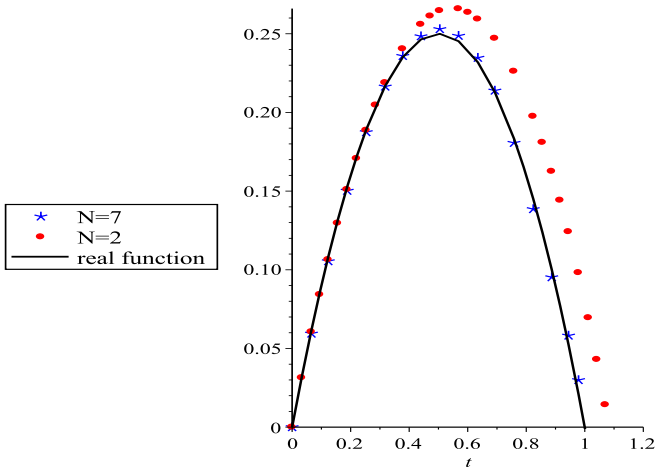

Figure 2: Comparison between exact and approximate solutions for Example 4.2

Comparison of absolute errors between DT wavelet method [4] and proposed method for $N = 7$ is shown in Table 3 . Also, Figure 3 shows the comparison between exact and approximate solutions for $N = 2$ and $N = 7$.

5 Conclusion

In this paper, we have solved nonlinear Volterra integro-differential equation of the second kind by using Legendre polynomial. A considerable advantage of this method is to find the approximation of analytical solution that is a polynomial of degree up to N. An other advantage of the method is that Legendre coefficients of the solution can be found very easily by using computer programs. The convergence of this method has been presented by Theorem 3.1.

Table 3: Absolute errors for Example 4.3

t	DT wavelet method [4] for N=7	proposed method for N=7
0.054	3.29×10^{-2}	3.90×10^{-6}
0.600	1.49×10^{-2}	1.83×10^{-6}
0.851	1.82×10^{-1}	2.88×10^{-5}
1.000	4.71×10^{-1}	7.52×10^{-6}

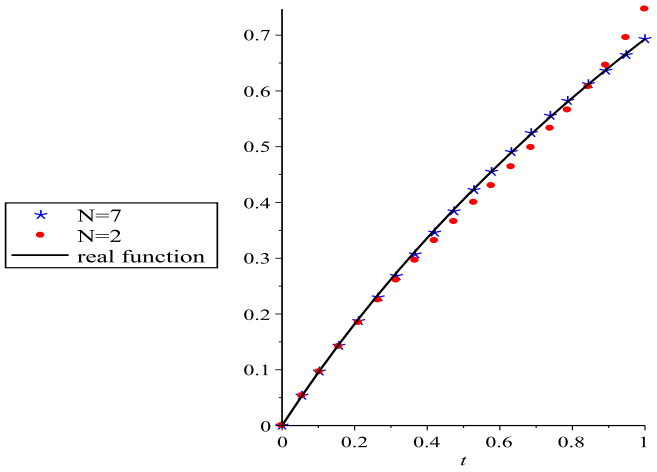

Figure 3: Comparison between exact and approximate solutions for Example 4.3

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Persian Translation of Abstracts

تحلیل فلاتر پانل در جریان فرا صوت بوسیله انشعاب هاف

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چکیده : این مقاله به مطالعه یک معادله دیفرانسیل مشتقات جزئی حاکم بر حرکت پانل در جریان فراصوت اختصاص داده شده است. این معادله دیفرانسیل مشتقات جزئی می تواند بوسیله یک روش گالرکین به یک معادله دیفرانسیل معمولی تبدیل شود. در اینجا با استفاده از معیاری که وابسته به معیار روت–هورویتز می باشد، معادله دیفرانسیل معمولی ذکر شده را از نقطه نظر انشعاب هاف مورد بررسی قرار می دهیم. در حقیقت ما ناحیه ای برای وجود انشعاب هاف ساده برای آن را می یابیم. به کمک نرم افزار مطلب و ابزار انشعاب هاف، فلاتر ونوسانات چرخه حدی پانل بررسی شده اند. همچنین نظریه انشعاب هاف برای تحلیل سرعت فلاتر سیستم مورد استفاده قرار گرفته است.

کلمات کلیدی : فلاتر پانل؛ چرخه حدی؛ انشعاب هاف؛ معیار روت–هورویتز؛ نوسانات.

بررسی انشعاب هاف در شبکه حلقه ای n نرونی با n تاخیر زمانی

الهام جاویدمنش و محسن خورشیدی

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چکیده : در این مقاله، یک شبکه حلقه ای n نرونی با n تاخیر زمانی را در نظر می گیریم و به مطالعه ی انشعاب هاف سیستم حاصل از این شبکه می پردازیم. با طبقه بندی بر حسب باقیمانده های تقسیم n بر ۴، به مطالعه ی معادله مشخصه حاصل از این دسته بندی می پردازیم. مجموع تاخیرها را به عنوان پارامتر سیستم در نظر می گیریم. نشان می دهیم وقتی پارامتر تاخیر از یک مقدار بحرانی می گذرد انشعاب هاف رخ می دهد. در واقع، زمانی که نقطه تعادل سیستم (جواب صفر) پایداری مجانبی اش را از دست می دهد، یک خانواده از جواب های دوره ای از مبدا منشعب می شوند.

کلمات کلیدی : شبکه حلقه ای؛ پایداری؛ جواب های دوره ای؛ انشعاب هاف؛ تاخیر زمانی.

الگوریتمی بر پایه جمعیت برای تقریب کنترل بهینه توزیعی تحت معادلات موج

اکبر هاشمی برزآبادی، سکینه بیگم میراسدی، و محمد حیدری

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چکیده : در این مقاله، یک روش تلفیقی تکراری نوین برای یافتن کنترل بهینه تقریبی توزیعی تحت معادله موج مورد بررسی قرار گرفته است. افرازی از فضای زمان–کنترل در نظر گرفته شده که مساله را گسسته می کند و سپس این شکل گسسته شده به یک مساله شبه تخصیص تبدیل شده است. آنگاه یک الگوریتم بر پایه جمعیت همراه با یک روش تفاضل متناهی برای استخراج کنترل بهینه تقریبی توزیعی به صورت یک تابع قطعه ای خطی به کار گرفته شده است. یک تحلیل همگرایی برای شکل گسسته مساله ابتدایی ارائه شده است. همچنین برای نشان دادن توانایی الگوریتم داده شده، نتایج عددی ارائه شده با نتایج حاصل از بکارگیری دو الگوریتم بر پایه جمعیت، الگوریتم ژنتیک و الگوریتم ازدحام ذرات مقایسه شده اند.

کلمات کلیدی : مسال کنترل بهینه؛ الگوریتم ارزیابی؛ روش تفاضلات متناهی؛ معادله موج.

روش تاو عملیاتی برای معادلات دیفرانسیل کسری چند مرتبه ای

پیام مختاری

دانشگاه صنعتی سهند، گروه ریاضی

چکیده : این مقاله روش تاو عملیاتی مبتنی بر چندجمله ایهای متعامد را برای حل عددی معادلات دیفرانسیل با مشتقات کسری چند مرتبه ای غیر خطی ارائه می دهد. مشخصه اصلی این روش این است که جواب عددی معادله مورد نظر را با استفاده از حل دستگاه غیر خطی جبری بدست می آورد. برخی از مثال های عددی به منظور نمایش کارایی و کاربردی بودن روش ارائه شده است.

کلمات کلیدی : معادلات دیفرانسیل با مشتقات کسری؛ مشتق کاپاتو؛ روش تاو عملیاتی.

یک روش جدید برای حل دستگاه معادلات غیر خطی با استفاده از روش نیوتن و HAM

جلال ایزدیان[1] ، *غلامرضا ابریشمی*[1] ، *و مریم جلیلی*[2]

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چکیده :
یک روش جدید برای حل دستگاه معادلات غیر خطی با استفاده از روش نیوتن و روش آنالیز هموتوپی (HAM) ارائه می شود. افزایش سرعت نرخ همگرایی HAM و بدست آوردن سرعت درجه دوم همگرایی کلی اهداف اصلی این روش است. نتایج عددی کارایی و عملکرد روش پیشنهادی را در مقایسه با روش هموتوپی معمولی، روش نیوتن و HAM نشان می دهد، که در این روش آزادی بسیاری در انتخاب حدس اولیه داریم.

کلمات کلیدی : روش آنالیز هموتوپی؛ معادلات دگردیسی مرتبه صفر؛ پارامتر کنترل همگرایی؛ روش نیوتن؛ روش تکراری؛ روش تکراری چند گام؛ مرتبه همگرایی.

حل تقریبی معادلات دیفرانسیل انتگرال غیر خطی با استفاده از چندجمله ای های لژاندر

مرتضی گچ پزان، مجید عرفانیان، و حسین بیگلو

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چکیده : در این مقاله، یک روش تکراری برای بدست آوردن جوابهای معادلات دیفرانسیل انتگرالی ولترای نوع دوم بر اساس چندجمله ای های لژاندر ساخته شده است. تحلیل خطا با استفاده از درونیابی انجام گرفته است. سرانجام چند مثال با استفاده از روش پیشنهادی حل و با برخی روش های موجود مقایسه شده است.

کلمات کلیدی : معادله دیفرانسیل انتگرال ولترای غیر خطی؛ چندجمله ای های لژاندر؛ تحلیل خطا.

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