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Quasi-permutation representations of Borel and parabolic subgroups of Steinberg's triality groups*

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Abstract

If G is a finite linear group of degree n, that is, a finite group of automorphisms of an n-dimensional complex vector space, or equivalently, a finite group of non-singular matrices of order n with complex coefficients, we shall say that G is a quasi-permutation group if the trace of every element of G is a non-negative rational integer. By a quasi-permutation matrix we mean a square matrix over the complex field C with non-negative integral trace. Thus every permutation matrix over C is a quasi-permutation matrix. For a given finite group G, let $c(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the complex numbers and let $r(G)$ denote the minimal degree of a faithful rational valued complex character of G. The purpose of this paper is to calculate $c(G)$ and $r(G)$ for the Borel and parabolic subgroups of Steinberg's triality groups.

Keywords and phrases: Borel subgroup, character table, parabolic subgroup, quasi-permutation, Steinberg's triality group.
AMS Subject Classification 2000: Primary 20E10; Secondary 20C15.

1 Introduction

If F is a subfield of the complex numbers C, then a square matrix over F with non-negative integral trace is called a quasi-permutation matrix over F. Thus

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every permutation matrix over C is a quasi-permutation matrix. For a given finite group G, let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices.

By a rational valued character we mean a character χ corresponding to a complex representation of G such that $\chi(g) \in Q$, for all $g \in G$. As the values of the character of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of G is then simply a complex representation of G whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module. We will call a homomorphism from G into $GL(n, Q)$ a rational representation of G and its corresponding character will be called a rational character of G. Let $r(G)$ denote the minimal degree of a faithful rational valued character of G.

Finding the above quantities have been carried out in some papers, for example in [3], [4], [5] and [6] we found these for the groups $GL(2, q)$, $SU(3, q^2)$, $PSU(3, q^2)$, $SP(4, q)$ and $G_2(2^n)$, respectively. In [2] we found the rational character table and the values of $r(G)$ and $c(G)$ for the group $PGL(2, q)$.

In this paper we will apply the algorithms in [1] to the Borel and parabolic subgroups of Steinberg's triality groups.

2 Notation and preliminary results

Let ${}^3D_4(q)$ be the Steinberg's simple triality group defined over a finite field with $q = p^n$ elements, where p is a prime number and n a positive integer.

Let $\mathbf{B}$ be the F-stable Borel subgroup TU of G, where U is the product of all root subgroups of G to positive roots, and let $B = \mathbf{B}^{\mathbf{F}}$ be the corresponding Borel subgroup of G^F.

The group B is the semidirect product of $T = \mathbf{T}^{\mathbf{F}}$ by the unipotent normal subgroup

$$U = \mathbf{U}^{\mathbf{F}} = X_{\alpha} X_{\beta} X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}.$$

The elements of T form a set of representatives for the semisimple conjugacy classes of B and we parametrize these classes according to Table $A.3$ in Appendix A of [9].

The character table of the Borel subgroup B is given by Tables $A.5$ and $A.6$ in Appendix A of [8] and [9].

Let $\mathbf{P} = \langle \mathbf{B}, n_{r_1}, n_{r_3}, n_{r_4} \rangle$ be the F-stable maximal parabolic subgroup of G corresponding to the subset $\{r_1, r_3, r_4\} \subseteq \Delta$ and $P := \mathbf{P^F}$ be the corresponding maximal parabolic subgroup of $G^F =^3 D_4(q)$. Then P is generated by B and n_α and $|P| = q^{12}(q^6-1)(q-1)$.

P is the semidirect product of the Levi complement $L_P = \langle \mathbf{T^F}, X_\alpha, X_{-\alpha} \rangle$ and the unipotent radical $U_P := X_\beta X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$. The character table of the parabolic subgroup P is given by Tables $A.9$ and $A.10$ in Appendix A of [8] and [9].

Let $Q = \langle \mathbf{B}, \mathbf{n_{r_2}} \rangle$ be the F-stable maximal parabolic subgroup of G corresponding to the subset $\{r_2\} \subseteq \Delta$ and $Q := \mathbf{Q^F}$ be the corresponding maximal parabolic subgroup of $G^F =^3 D_4(q)$. Then Q is generated by B, n_β, and $|Q| = q^{12}(q^3-1)(q^2-1)$.

Q is the semidirect product of the Levi complement $L_Q = \langle \mathbf{T^F}, X_\beta, X_{-\beta} \rangle$ by the unipotent radical $U_Q := X_\alpha X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$.

The character table of the parabolic subgroup Q is given by Tables $A.13$ and $A.14$ in Appendix A of [8] and [9], respectively.

Assume E is a splitting field for G and that F is a subfield of E. If $\chi, \psi \in Irr_E(G)$ we say that χ and ψ are Galois conjugate over F if $F(\chi) = F(\psi)$ and there exists $\sigma \in Gal(F(\chi)/F)$ such that $\chi^\sigma = \psi$, where $F(\chi)$ denotes the field obtained by adding the values $\chi(g)$, for all $g \in G$, to F. It is clear that this defines an equivalence relation on $Irr_E(G)$.

Let η_i for $0 \le i \le r$ be the Galois conjugacy classes of irreducible complex characters of G. For $0 \le i \le r$, let φ_i be a representative of the class η_i, with $\varphi_o = 1_G$. Write $\Psi_i = \sum_{\chi_i \in \eta_i} \chi_i$ and $K_i = ker\varphi_i$. We know that $K_i = ker\Psi_i$. For $I \subseteq \{0, 1, 2, \cdots, r\}$, put $K_I = \bigcap_{i \in I} K_i$. By definitions of $r(G)$, $c(G)$ and using the above notations we have

$r(G) = \min\{\xi(1) : \xi = \sum_{i=1}^{r} n_i\Psi_i, n_i \geq 0, K_I = 1,\ for\ I = \{i, i \neq 0, n_i > 0\}\}$,

$c(G) = \min\{\xi(1) : \xi = \sum_{i=0}^{r} n_i\Psi_i, n_i \geq 0, K_I = 1,\ for\ I = \{i, i \neq 0, n_i > 0\}\}$,

where $n_0 = -\min\{\xi(g)|g \in G\}$.

$d(\chi), m(\chi)$ and $c(\chi)$ have been defined in [1] [see Definition 3.4]. Here we may redefine them as follows.

Definition 2.1. Let χ be a complex character of G, such that $\ker\chi = 1$ and $\chi = \chi_1 + \cdots + \chi_n$, for some $\chi_i \in Irr(G)$. Then

(1) $d(\chi) = \sum_{i=1}^{n} |\Gamma_i(\chi_i)|\chi_i(1)$

(2) $m(\chi) = \begin{cases} 0, & \text{if } \chi = 1_G, \\ |\min\{\sum_{i=1}^{n}\sum_{\alpha\in\Gamma_i(\chi_i)}\chi_i^{\alpha}(g) : g \in G\}|, & \text{otherwise,} \end{cases}$

(3) $c(\chi) = \sum_{i=1}^{n}\sum_{\alpha\in\Gamma_i(\chi_i)} \chi_i^{\alpha} + m(\chi)1_G$.

So

$$r(G) = \min\{d(\chi) : \ker\chi = 1\},$$

and

$$c(G) = \min\{c(\chi)(1) : \ker\chi = 1\}.$$

The proofs of the following statements may be found in [1].

proposition 2.2. Let $\chi \in Irr(G)$, then $\sum_{\alpha\in\Gamma(\chi)} \chi^{\alpha}$ is a rational valued character of G. Moreover, $c(\chi)$ is a non-negative rational valued character of G and $c(\chi) = d(\chi) + m(\chi)$.

Lemma 2.3. Let $\chi \in Irr(G), \chi \neq 1_G$. Then $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$.

Lemma 2.4. Let $\chi \in Irr(G)$. Then

(**1**) $c(\chi)(1) \geq d(\chi) \geq \chi(1)$;

(**2**) $c(\chi)(1) \leq 2d(\chi)$. Equality holds if and only if $Z(\chi)/ker\chi$ is of even order.

3 Quasi-permutation representations

In this section, we calculate $r(G)$ and $c(G)$ for Borel and parabolic subgroups of Steinbergs triality groups ${}^3D_4(q)$. First we shall determine these quantities for odd q.

Theorem 3.1. Let q be a power of an odd prime number. Then

A) If G is a Borel subgroup B of ${}^3D_4(q)$, then

1) $r(G) = \begin{cases} mq^4(q-1) & \text{if } m \leq \frac{q^3-1}{2}, \\ \frac{1}{2}q^4(q^3-1)(q-1) & \text{otherwise,} \end{cases}$

2) $c(G) = \begin{cases} mq^5 & \text{if } m \leq \frac{q^3-1}{2}, \\ \frac{1}{2}q^5(q^3-1) & \text{otherwise,} \end{cases}$

where $m = |\Gamma(B\chi_{17}(k))|$.

B) If G is the maximal parabolic subgroup P of ${}^3D_4(q)$, then

1) $r(G) = q^4(q-1)$;

2) $c(G) = q^5$.

C) If G is the maximal parabolic subgroup Q of ${}^3D_4(q)$, then

1) $r(G) = \begin{cases} mq^3(q^2-1) & if \ \frac{m}{n} \leq q-1 \\ nq^3(q-1)^2(q+1), & otherwise, \end{cases}$

2) $c(G) = \begin{cases} mq^5 & if \ \frac{m}{n} \leq q-1, \\ nq^5(q-1), & otherwise; \end{cases}$

where $m = |\Gamma(Q\chi_{16}(k))|$ and $n = |\Gamma(Q\chi_{17}(k))|$.

Proof. In order to calculate $r(G)$ and $c(G)$, we need to determine $d(\chi), m(\chi)$, and $c(\chi)(1)$, for all characters which are faithful or $\bigcap_{\chi} Ker\chi = 1$. Since the degrees of faithful characters are minimal, we only need to consider the faithful characters and by Lemmas 2.3 and 2.4 we have

A) Using the character table $A.6$ of [9] for the Borel subgroup B, we have $d(B\chi_{17}(k)) = |\Gamma(B\chi_{17}(k))|B\chi_{17}(k)(1) \geq q^4(q-1)$ and so $c(B\chi_{17})(k)(1) \geq q^5$, $d(B\chi_{18}) = d(B\chi_{19}) = d(B\chi_{20}) = d(B\chi_{21}) = |\Gamma(B\chi_{18})|B\chi_{18}(1) = \frac{1}{2}q^4(q^3-1)(q-1)$ and so $c(B\chi_{18})(1) = c(B\chi_{19})(1) = c(B\chi_{20})(1) = c(B\chi_{21})(1) = \frac{1}{2}q^5(q^3-1)$. The values are set out in Table (I):

Table (I)

χ	$d(\chi)$	$c(\chi)(1)$
$B\chi_{17}(k)$	$\geq q^4(q-1)$	$\geq q^5$
$B\chi_{18}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$\frac{1}{2}q^5(q^3-1)$
$B\chi_{19}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$\frac{1}{2}q^5(q^3-1)$
$B\chi_{20}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$\frac{1}{2}q^5(q^3-1)$
$B\chi_{21}$	$\frac{1}{2}q^4(q^3-1)(q-1)$	$\frac{1}{2}q^5(q^3-1)$

For the character $B\chi_{17}(k)$, as $|\Gamma(B\chi_{17}(k))| \leq q^3-1$, where $\Gamma(B\chi_{17}(k)) = \Gamma(Q(B\chi_{17}(k))Q)$, we have

$$q^4(q-1) \leq d(B\chi_{17}(k)) \leq q^4(q-1)(q^3-1).$$

Now by Table (I) and the above inequality we have
min $\{d(\chi): \ Ker\chi = 1\} = d(B\chi_{17}(k)) = mq^4(q-1)$ if $m \leq \frac{q^3-1}{2}$, otherwise, $\min\{d(\chi): \ Ker\chi = 1\} = \frac{1}{2}q^4(q^3-1)(q-1)$. Also
min $\{c(\chi)(1): \ Ker\chi = 1\} = c(B\chi_{17}(k))(1) = mq^5$, if $m \leq \frac{q^3-1}{2}$, otherwise, $\min\{c(\chi): \ Ker\chi = 1\} = \frac{1}{2}q^5(q^3-1)$, where $m = |\Gamma(B\chi_{17}(k))|$.

B) By the character table $A.10$ of [9], we have
$d(P\chi_{15}) = |\Gamma(P\chi_{15})|P\chi_{15}(1) = q^4(q-1)$ and so $c(P\chi_{15})(1) = q^5$,
$d(P\chi_{16}) = |\Gamma(P\chi_{16})|P\chi_{16}(1) = q^7(q-1)$ and so $c(P\chi_{16})(1) = q^8$,
$d(P\chi_{17}) = |\Gamma(P\chi_{17})|P\chi_{17}(1) = \frac{1}{2}q^4(q-1)(q^3+1)$ and so $c(P\chi_{17})(1) = \frac{1}{2}q^5(q^3+1)$,
$d(P\chi_{18}) = |\Gamma(P\chi_{18})|P\chi_{18}(1) = \frac{1}{2}q^4(q-1)(q^3+1)$ and so $c(P\chi_{18})(1) = \frac{1}{2}q^5(q^3+1)$,
$d(P\chi_{19}) = |\Gamma(P\chi_{19})|P\chi_{19}(1) = \frac{1}{2}q^4(q-1)(q^3-1)$ and so $c(P\chi_{19})(1) = \frac{1}{2}q^5(q^3-1)$,
$d(P\chi_{20}) = |\Gamma(P\chi_{20})|P\chi_{20}(1) = \frac{1}{2}q^4(q-1)(q^3-1)$ and so $c(P\chi_{20})(1) = \frac{1}{2}q^5(q^3-1)$,
$d(P\chi_{21}(k)) = |\Gamma(P\chi_{21}(k))|P\chi_{21}(k)(1) \geq q^4(q-1)(q^3+1)$ and so $c(P\chi_{21})(k)(1) \geq q^5(q^3+1)$,
$d(P\chi_{22}(k)) = |\Gamma(P\chi_{22}(k))|P\chi_{22}(k)(1) \geq q^4(q-1)(q^3-1)$ and so $c(P\chi_{22})(k)(1) \geq q^5(q^3-1)$.
The values are set out in Table (II):

Table (II)

χ	$d(\chi)$	$c(\chi)(1)$
$P\chi_{15}$	$q^4(q-1)$	q^5
$P\chi_{16}$	$q^7(q-1)$	q^8
$P\chi_{17}$	$\frac{1}{2}q^4(q-1)(q^3+1)$	$\frac{1}{2}q^5(q^3+1)$
$P\chi_{18}$	$\frac{1}{2}q^4(q-1)(q^3+1)$	$\frac{1}{2}q^5(q^3+1)$
$P\chi_{19}$	$\frac{1}{2}(q^7-q^4)$	$\frac{1}{2}q^5(q^3-1)$
$P\chi_{20}$	$\frac{1}{2}q^4(q-1)(q^3-1)$	$\frac{1}{2}q^5(q^3-1)$
$P\chi_{21}(k)$	$\geq q^4(q-1)(q^3+1)$	$\geq q^5(q^3+1)$
$P\chi_{22}(k)$	$\geq q^4(q-1)(q^3-1)$	$\geq q^5(q^3-1)$

Now by Table (II) and the above inequality we have
$\min\{d(\chi):\ Ker\chi=1\}=d(P\chi_{15})=q^4(q-1))$ and
$\min\{c(\chi)(1):\ Ker\chi=1\}=c(P\chi_{15})(1)=q^5$.

C) By the character table $A.14$ of [9], we have
$d(Q\chi_{16}(k))=|\Gamma(Q\chi_{16}(k))|Q\chi_{16}(k)(1)\geq q^3(q^2-1)$ and so $c(Q\chi_{16}(k))(1)\geq q^5$,
$d(Q\chi_{17}(k))=|\Gamma(Q\chi_{17}(k))|Q\chi_{17}(k)(1)\geq q^3(q^2-1)(q-1)$ and so $c(Q\chi_{17}(k))(1)\geq q^5(q-1)$,
$d(\sum_{k=0}^{1}Q\chi_{18}(k))=|\Gamma(\sum_{k=0}^{1}Q\chi_{18}(k))|(\sum_{k=0}^{1}Q\chi_{18}(k))(1)=2q^3(q^2-1)(q^3-1)$ and so $c(\sum_{k=0}^{1}Q\chi_{18}(k))(1)=2q^5(q^3-1)$,
$d(\sum_{k=0}^{1}Q\chi_{19}(k))=|\Gamma(\sum_{k=0}^{1}Q\chi_{19}(k))|(\sum_{k=0}^{1}Q\chi_{19}(k))(1)=2q^3(q^2-1)(q^3-1)$ and so $c(\sum_{k=0}^{1}Q\chi_{19}(k))(1)=2q^5(q^3-1)$,
$d(\sum_{k=1}^{q-1}Q\chi_{20}(k))=|\Gamma(\sum_{k=1}^{q-1}Q\chi_{20}(k))|(\sum_{k=1}^{q-1}Q\chi_{20}(k))(1)\geq q^3(q-1)(q^2-1)(q^3-1)$ and so $c(\sum_{k=1}^{q-1}Q\chi_{20}(k))(1)\geq q^5(q-1)(q^3-1)$.

The values are set out in Table (III):

Table (III)

χ	$d(\chi)$	$c(\chi)(1)$
$Q\chi_{16}(k)$	$\geq q^3(q^2-1)$	$\geq q^5$
$Q\chi_{17}(k)$	$\geq q^3(q^2-1)(q-1)$	$\geq q^5(q-1)$
$\sum_{k=0}^{1} Q\chi_{18}(k)$	$2q^3(q^2-1)(q^3-1)$	$2q^5(q^3-1)$
$\sum_{k=0}^{1} Q\chi_{19}(k)$	$2q^3(q^2-1)(q^3-1)$	$2q^5(q^3-1)$
$\sum_{k=1}^{q-1} Q\chi_{20}(k)$	$\geq q^3(q-1)(q^2-1)(q^3-1)$	$\geq q^5(q-1)(q^3-1)$

For the character $Q\chi_{16}(k)$, as $|\Gamma(Q\chi_{16}(k))| \leq q^3-1$, where $\Gamma(Q\chi_{16}(k)) = \Gamma(Q(Q\chi_{16}(k))Q)$, we have

$$q^3(q^2-1) \leq d(Q\chi_{16}(k)) \leq q^3(q^2-1)(q^3-1).$$

Thus, for the character $Q\chi_{17}(k)$, as $|\Gamma(Q\chi_{17}(k))| \leq q^2+q+1$, where $\Gamma(Q\chi_{17}(k)) = \Gamma(Q(Q\chi_{17}(k)) : Q)$, we have

$$q^3(q-1)(q^2-1) \leq d(Q\chi_{17}(k)) \leq q^3(q^2-1)(q^3-1).$$

Now by Table (III) and the above inequality, we have
$\min\{d(\chi):\ Ker\chi = 1\} = d(Q\chi_{16}(k)) = mq^3(q^2-1)$ if $\frac{m}{n} \leq q-1$, otherwise,
$\min\{d(\chi):\ Ker\chi = 1\} = d(Q\chi_{17}(k)) = nq^3(q^2-1)(q-1)$, and
$\min\{c(\chi)(1):\ Ker\chi = 1\} = c(Q\chi_{16}(k))(1) = mq^5$, if $\frac{m}{n} \leq q-1$, otherwise ,
$\min\{c(\chi):\ Ker\chi = 1\} = c(Q\chi_{17}(k))(1) = nq^5(q-1)$, where $m = |\Gamma(Q\chi_{16}(k))|$ and $n = |\Gamma(Q\chi_{17}(k))|$.

In the following theorem, we have constructed the values of $r(G)$ and $c(G)$ for the case when q is even.

Theorem 3.2. **A)** Let G be the Borel subgroup B of ${}^3D_4(2^n)$, then

1) $r(G) = |\Gamma(B\chi_{15}(k))|q^4(q-1)$

2) $c(G) = |\Gamma(B\chi_{15}(k))|q^5$.

B) Let G be the maximal parabolic subgroup P of ${}^3D_4(2^n)$, then

1) $r(G) = q^4(q-1)$

2) $c(G) = q^5$.

C) Let G be the maximal parabolic subgroup Q of ${}^3D_4(2^n)$, then

1) $r(G) = \begin{cases} mq^3(q^2-1), & \text{if } \frac{m}{n} \leq q-1, \\ nq^3(q-1)^2(q+1), & \text{otherwise,} \end{cases}$

2) $c(G) = \begin{cases} mq^5, & \text{if } \frac{m}{n} \leq q-1, \\ nq^5(q-1) & \text{otherwise,} \end{cases}$

where $m = |\Gamma(Q\chi_{14}(k))|$ and $n = |\Gamma(Q\chi_{15}(k))|$.

Proof. The quasi-permutation representations of Borel subgroup B and maximal parabolic subgroups P and Q of ${}^3D_4(2^n)$ are constructed by the same method as in Theorem 3.1. So in order to calculate $r(G)$ and $c(G)$, we need to determine $d(\chi), m(\chi)$, and $c(\chi)(1)$, for all characters which are faithful or $\bigcap_\chi Ker\chi = 1$. Since the degrees of faithful characters are minimal, we only need to consider the faithful characters. By Lemmas 2.3, 2.4, and the character table $A.6$ of [8], we have

A) $d(B\chi_{15}(k)) = |\Gamma(B\chi_{15}(k))|B\chi_{15}(k)(1) \geq q^4(q-1)$ and so $c(B\chi_{15})(k)(1) \geq q^5$, $d(B\chi_{16}) = |\Gamma(B\chi_{16})|B\chi_{16}(1) = q^4(q^3-1)(q-1)$ and so $c(B\chi_{16})(1) = q^5(q^3-1)$. For the character $B\chi_{15}(k)$, as $|\Gamma(B\chi_{15}(k))| \leq q^3 - 1$, where $\Gamma(B\chi_{15}(k)) = \Gamma(Q(B\chi_{15}(k)) : Q)$, we have

$$q^4(q-1) \leq d(B\chi_{15}(k)) \leq q^4(q-1)(q^3-1).$$

Now, we have

$\min\{d(\chi) : Ker\chi = 1\} = d(B\chi_{15}(k)) = |\Gamma(B\chi_{15}(k))|q^4(q-1)$ and $\min\{c(\chi)(1) : Ker\chi = 1\} = c(B\chi_{15}(k))(1) = |\Gamma(B\chi_{15}(k))|q^5$.

B) By the character table $A.10$ of [8] we obtain

$d(P\chi_{15}) = |\Gamma(P\chi_{15})|P\chi_{15}(1) = q^4(q-1)$ and so $c(P\chi_{15})(1) = q^5$,

$d(P\chi_{16}) = |\Gamma(P\chi_{16})|P\chi_{16}(1) = q^7(q-1)$ and so $c(P\chi_{16})(1) = q^8$,

$d(P\chi_{17}) = |\Gamma(P\chi_{17})|P\chi_{17}(1) \geq q^4(q-1)(q^3+1)$ and so $c(P\chi_{17})(1) \geq q^5(q^3+1)$,

$d(P\chi_{18}) = |\Gamma(P\chi_{18})|P\chi_{18}(1) \geq q^4(q-1)(q^3-1)$ and so $c(P\chi_{18})(1) \geq q^5(q^3-1)$

The values are set out in Table (IV):

Table (IV)

χ	$d(\chi)$	$c(\chi)(1)$
$P\chi_{15}$	$q^4(q-1)$	q^5
$P\chi_{16}$	$q^7(q-1)$	q^8
$P\chi_{17}$	$\geq q^4(q-1)(q^3+1)$	$\geq q^5(q^3+1)$
$P\chi_{18}$	$\geq q^4(q-1)(q^3-1)$	$\geq q^5(q^3-1)$

Now by Table (IV) we have
$\min\{d(\chi):\ Ker\chi=1\}=d(P\chi_{15})=q^4(q-1))$ and
$\min\{c(\chi)(1):\ Ker\chi=1\}=c(P\chi_{15})(1)=q^5$.
C) By the character table $A.14$ of [8] we may calculate the following
$d(Q\chi_{14}(k))=|\Gamma(Q\chi_{14}(k))|Q\chi_{14}(k)(1)\geq q^3(q^2-1)$ and so $c(Q\chi_{14}(k))(1)\geq q^5$,
$d(Q\chi_{15}(k))=|\Gamma(Q\chi_{15}(k))|Q\chi_{15}(k)(1)\geq q^3(q-1)^2(q+1)$ and so $c(Q\chi_{15}(k))(1)\geq q^5(q-1)$ and
$d(\sum_{k=1}^{q}Q\chi_{16}(k))=|\Gamma(\sum_{k=1}^{q}Q\chi_{16}(k))|(\sum_{k=1}^{q}Q\chi_{16}(k))(1)=q^4(q-1)(q+1)(q^3-1)$ so $c(\sum_{k=1}^{q}Q\chi_{16}(k))(1)=q^6(q-1)(q^2+q+1)$.
For the character $Q\chi_{14}(k)$, as $|\Gamma(Q\chi_{14}(k))|\leq q^3-1$, where $\Gamma(Q\chi_{14}(k))=\Gamma(Q(Q\chi_{14}(k)):Q)$, we have

$$q^3(q^2-1)\leq d(Q\chi_{14}(k))\leq q^3(q^2-1)(q^3-1).$$

So for the character $Q\chi_{15}(k)$, as $|\Gamma(Q\chi_{15}(k))|\leq q^2+q+1$, where $\Gamma(Q\chi_{15}(k))=\Gamma(Q(Q\chi_{15}(k)):Q)$, we have

$$q^3(q-1)(q^2-1)\leq d(Q\chi_{15}(k))\leq q^3(q^2-1)(q^3-1).$$

Now by the above inequality we have
$\min\{d(\chi):\ Ker\chi=1\}=d(Q\chi_{14}(k))=mq^3(q^2-1)$ if $\frac{m}{n}\leq q-1$, otherwise,
$\min\{d(\chi):\ Ker\chi=1\}=d(Q\chi_{15}(k))=nq^3(q^2-1)(q-1)$. and
$\min\{c(\chi)(1):\ Ker\chi=1\}=c(Q\chi_{14}(k))(1)=mq^5$, if $\frac{m}{n}\leq q-1$, otherwise ,
$\min\{c(\chi):\ Ker\chi=1\}=c(Q\chi_{15}(k))(1)=nq^5(q-1)$, where $m=|\Gamma(Q\chi_{14}(k))|$ and $n=|\Gamma(Q\chi_{15}(k))|$.

References

[1] Behravesh, H., Quasi-Permutation representations of p-groups of class 2, *Journal of London Math. Soc.* **55**(2)(1997), 251–260.

[2] Behravesh, H., Daneshkhah, A., Darafsheh, M.R. and Ghorbany, M., The rational character table and quasi-permutation representations of the group $PGL(2, q)$, *Italian Journal of Pure and Applied Mathematics* **11**(2001), 9–18.

[3] Darafsheh, M.R., Ghorbany, M., Daneshkhah, A. and Behravesh, H., Quasi-permutation representation of the group $GL(2, q)$, *Journal of Algebra* **243**(2001), 142–167.

[4] Darafsheh, M.R. and Ghorbany, M., Quasi-permutation representations of the groups $SU(3, q^2)$ and $PSU(3, q^2)$, *Southeast Asian Bulletin of Mathematics* **26**(2002), 395–406.

[5] Darafsheh, M.R. and Ghorbany, M., Special representations of the group $SP(4, q)$ with minimal degrees, *Acta Math. Hungar.* **102**(4)(2004), 287–296.

[6] Ghorbany, M., Special representations of the group $G_2(2^n)$ with minimal degrees, *Southeast Asian Bulletin of Mathematics* **30**(2006), 663–670.

[7] Gow, R., Schur indices of some groups of Lie type, *Journal of Algebra* **42**(1976), 102–120.

[8] Himstedt,F., Character tables of parabolic subgroups of Steinberg's triality groups ${}^3D_4(2^n)$, *Journal of Algebra* **281**(1)(2007), 254-283.

[9] Himstedt, F., Character tables of parabolic subgroups of Steinberg's triality groups, *Journal of Algebra* **281**(2004), 774–822.

[10] Isaacs, I.M., Character Theory of Finite Groups, Academic Press, New York, 1976.

[11] Wong, W.J., Linear groups analogous to permutation groups, *Journal of Austral. Math. Soc (Sec. A)* **3**(1963), 180–184.

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Properties of groups with points[*,†]

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Abstract

In this paper, we consider groups with points which were introduced by V.P. Shunkov in 1990. In Novikov-Adian's group, Adian's periodic products of finite groups without involutions and Olshansky's periodic monsters every non-unit element is a point. There exist groups without points. In this article we shall prove some properties of the groups with points.

Keywords and phrases: Adian's periodic products, Chernikov group, locally finite, locally soluble groups, Novikov-Adian's group, points of group.
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1 Introduction

Finiteness conditions in groups which are connected with finiteness of systems of subgroups were traditionally studied in Krasnoyarsk group theory School. An element in a group is a point if the sets of finite subgroups in special system of subgroups connected with this element are finite. More precisely, an element of finite order of a group G of the following types is called a *point* of G

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a) The identity element is a point if and only if the set of elements of finite orders of G is finite;

b) Non-identity element a of G is a point if for every non-identity finite subgroup K of G normalized by the element a, then the set of finite subgroups of $N_G(K)$ containing a is finite.

The definition of a point was introduced by V.P. Shunkov in 1990(see, for example [13]).

The concept of the points in groups give us the possibility of studying infinite groups. In particular, by using this concept, the sign of non-simplicity of an infinite group came to exist in [12]. In this article, we establish some properties of groups with points. We start by proving some properties of the common character (Lemmas and Theorems 1–5). Theorem 6 gives us the sign of placement of a point in an infinite group outside of infinite locally finite subgroups. Simultaneously, it will be proved that there are no points in an infinite locally finite group.

Theorems 7–10 have more special character. In Theorem 7, we construct an infinite subset of the set of finite subgroups with intersection that contains some points, such that every infinite subset of it has the same intersection. Theorem 8 describes a construction of an infinite subset of a set of finite subgroups with intersection contains a point of second order. Theorem 9 describes the centralizer of a point of second order if one more finiteness condition is valid for this point. Theorem 10 is about Sylow 2-subgroups of groups with point of second order.

Now, we recall some definitions, which we use frequently in this article.

A point a is called a *trivial point*, if the set of finite subgroups of G containing a is finite.

A group G is said to be *locally finite*, if any finite subset of G generates a finite subgroup.

A group G is called *Chernikov group*, if it is a finite group and a finite extension of direct product of a finite number of quasi-cyclic groups.

Let π' be the complement of the set of prime numbers π. The periodic group G is called a *π'–group*, if all the prime divisors of orders of non-unit elements of the group G belong to the set of π'.

An element of order two is called an *involution*.

A group G of the form $G = F\lambda H$ is called a *Frobenius group* with the kernel F and the complement H, if $H \cap H^g = 1$, for any $g \in G \setminus H$ and $F \setminus 1 = G \setminus \cup_{g \in G} H^g$, where H is a proper subgroup of G.

A group G is called *locally solvable*, if every finite set of its elements generates a solvable subgroup. A maximal normal p'-subgroup of G is denoted by $O_{p'}(G)$.

For the elements $a, b \in G$, the group G is carried out (a,b)*-finiteness condition*, if the subgroup $L_g = \langle a, b^g \rangle$ is finite, for almost all $g \in G$ (i.e., except may be finite number). An (a,b)-finiteness condition called *strong*, if L_g is a finite group for all $g \in G$.

An element a of G is called strictly real with respect to the involution i, if $iai = a^{-1}$.

A subgroup H of the group G is called *strongly embedded*, if H contains an involution and for any element $g \in G \setminus H$ there are no involutions in the subgroup $H \cap H^g$.

2 Examples of groups with points

Here, we give some examples of groups with different sets of points.

All finite groups are examples of groups, in which every element is considered as a point.

Novikov-Adian's group, Adian's periodic products of finite groups without involution [1] and Olshansky's periodic monster [11] are examples of groups, in which every non-unit element is a point.

Unit group and torsion-free group are groups with unique point.

Groups with a finite periodic part is a group, in which every element of finite order is a point.

Free product of a non-trivial finite group by any other non-trivial group is a group with infinite set of points.

Let $T_1, T_2, ..., T_n, ...$ be infinite sequence of finite Frobenius groups with the same complement H, where $T_n = F_n \lambda H,\ n = 1, 2, 3, ...$ Then, the free product G of groups in this sequence by the joined subgroup H is a group with a non-trivial point.

3 Some properties of groups with points

In this section, we study some properties of groups with points.

All the necessary known results are listed in Section 4 at the end of the article. We refer to these results with the appropriate numbers.

Lemma 3.1 *If a is a point of the group G, then a is a point of any subgroup of G, containing a.*

Proof. Let a be a point of the group G, H be an arbitrary subgroup of G containing a and L be a non-trivial finite subgroup of H. By the definition of a point, the set of finite subgroups of normalizer $N_G(L)$ containing a point a is finite as the set of finite subgroups of normalizer $N_H(L)$ containing a is also finite. Hence, the element a is a point of the subgroup H.

Proposition 3.2 *No group G can contain simultaneously an infinite set of finite subgroups with non-trivial intersection containing a point a and a non-trivial finite normal subgroup.*

Proof. Let the group G contain a non-trivial finite normal subgroup K and an infinite set of finite subgroups with non-trivial intersection L containing a point a. Then, the group G contains infinite number of elements of finite orders and $a \neq e$, by the definition of a point. As K is a normal subgroup of the group G, then $N_G(K) = G$ and the set of finite subgroups in $N_G(K)$ containing a is infinite. Thus a is not a point of the group G. This proves the proposition.

Proposition 3.3 *If a group G contains a point a, then for every element b of finite order of the normalizer $N_G(a)$, the intersection $N_G((a)) \cap C_G(b)$ has finite index in $N_G(a)$.*

Proof. By the way of contradiction, we assume that there is an element b of the normalizer $N_G(a)$, such that the index of the intersection $N_G((a)) \cap C_G(b)$ in the normalizer $N_G(a)$ is infinite. We then consider two cases: $a = e$ and $a \neq e$ in G.

If $a = e$, then $N_G((a)) = G$ and by the assumption $|G : C_G(b)| = \infty$. It means that the number of elements, conjugate with b in the group G is infinite. This is a contradiction to the definition of points.

Now we consider the second case and assume a is a non-identity element of G. By the assumption, the intersection $N_G((a)) \cap C_G(b)$ has infinite index in the normalizer $N_G(a)$. Then, the number of elements conjugated with b in the normalizer $N_G(a)$ is infinite. Hence, the normalizer $N_G(a)$ contains infinite number of finite subgroups of the form $< a, b^c >, c \in N_G(a)$, which contradicts Proposition 3.2. So, the result holds.

Proposition 3.4 *No group may have simultaneously an infinite set of finite subgroups containing a point a and a finite non-trivial invariant set of elements of finite orders.*

Proof. Let the group G have a finite non-trivial invariant set of elements of finite orders. By Ditsman's Lemma (see Theorem 1), this set generates a finite normal subgroup in G. However, the group G can not have infinite set of finite subgroups containing the point a, by Proposition 3.2.

Theorem 3.5 *Infinite Chernikov's group has no points.*

Proof. By the properties of Chernikov's groups, in infinite Chernikov's group, every element is contained in an infinite set of finite subgroups. As every infinite Chernikov's group has a finite normal subgroup, then the statement follows from Proposition 3.2.

The following lemma is already proved in [8].

Lemma 3.6 *Every group has no infinite locally finite subgroup containing a point a.*

Proposition 3.7*Let a be a point of a group G, $\mathfrak{M}$ be an infinite set of finite subgroups of G and $a \in \cap_{H \in \mathfrak{M}} H$. Then, $\mathfrak{M}$ contains an infinite subset $\mathfrak{B}$ such that for any infinite subset $\mathfrak{U}$ of $\mathfrak{B}$, $\cap_{H \in \mathfrak{U}} H = \cap_{H \in \mathfrak{B}} H$.*

Proof. Let $T = \cap_{H \in \mathfrak{M}} H$ and assume that the claim is not true. Then, $\mathfrak{M}$ has an infinite subset $\mathfrak{M}_1$ with intersection $T_1 = \cap_{H\in\mathfrak{M}_1} H \neq T$, $\mathfrak{M}_1$ has a subset $\mathfrak{M}_2$ with intersection $T_2 = \cap_{H\in\mathfrak{M}_2} H \neq T_1$ and etc. As a result of such choices of subsets $\mathfrak{M}_n (n = 1, 2, \dots)$ from $\mathfrak{M}$, we obtain a strictly ascending chain of finite subgroups $T < T_1 < T_2 < ... < T_n < ....$

Clearly the union V of this chain is an infinite locally finite subgroup containing the point a, which contradicts Lemma 2.6. Hence, the chain breaks off after finite number of steps. This proves the result.

Theorem 3.8 *Any infinite set $\mathfrak{M}$ of finite subgroups of a group G with intersection $T = \cap_{H\in\mathfrak{M}} H$, where i is a point of the second order, almost all (for exception, may be, of finite number) consists of subgroups isomorphic to Frobenius groups with complements containing T or subgroups isomorphic to groups $Sz(Q), SL_2(Q)$, where Q is a field of characteristic two, $T = P\lambda(c)$ and P is some Sylow 2-subgroup of such subgroups.*

Proof. In view of Proposition 3.7 and without loss of generality, the statement is valid for $\mathfrak{M}$.

1) If $\mathfrak{B}$ is an infinite subset of $\mathfrak{M}$ such that $T = \cap_{H\in\mathfrak{B}} H$.

Assume that for some infinite subset $\mathfrak{N}$ of $\mathfrak{M}$ and for some (i)-invariant subgroup $K \neq 1$ of T we have $N_H(K) \not\leq T (H \in \mathfrak{N})$.

The set $\{N_H(K)|H \in \mathfrak{N}\}$ can not be infinite, as in this case we come to the contradiction of conditions $K \neq 1$, $i \in N_H(K)$ and the involution i is a point of G. Hence, $\{N_H(K)|H \in \mathfrak{N}\}$ is finite and by statement 1) $\mathfrak{N}$ has such infinite subset $\mathfrak{U}$, that $N_H(K) \leq T (H \in \mathfrak{U})$ contrary to the definition of the set $\mathfrak{N}$. The contradiction means, that the condition $N_H(K) \not\leq T$ can be only valid for finite number of subgroups $H \in \mathfrak{M}$. Therefore without loss of generality, we may suppose that

2) $N_H(K) \leq T \neq H$, for any non-trivial (i)-invariant subgroup K of T and any subgroup H of $\mathfrak{M}$.

Let M be some subgroup of $\mathfrak{M}$ and $O_{2'}(M) \neq 1$. Then, we are able to prove that

3) M is a Frobenius group with complement $C_M(i)$, containing T.

Let R be a nilpotent radical of $O_{2'}(M)$, then by Theorem 3.7, $R \neq 1$. If $T \cap R \neq 1$, then using the normalizer condition for nilpotent groups ([9], Theorem 17.1.4) and statement 2), we show that $R \leq T$ and $M \leq T$ contrary to the condition $T \neq M$ from statement 2). Hence, $T \cap R = 1$ and, in particular, $C_M(i) \cap R = 1$. If $C_M(R)$ has an involution k. Clearly, R can be chosen so that $k \in C_G(i)$. Now by statement 2), $R < C_M(K) \leq T$ and we obtain a contradiction to the above, $R \cap T = 1$. From here we have, that $C_M(R)$ does not contain any involutions, and as $C_M(R) \triangleleft M$, that $C_M(R) \leq O_{2'}(M)$. Furthermore, in view of Theorems 11 and 12, $C_M(R) = R$ and $M = RC_M(i)$. Using this and statement 2), it obviously follows that $C_M(i)$ is a complement of Frobenius group M. Hence the statement 3) is proved.

Now we show that:

4) If H is a subgroup of $\mathfrak{M}$, then all involutions of T are conjugate with i in H.

Let j be an involution from T. If $V = \langle \{j\}^H \rangle \leq T$, then by statement 2) $H \leq N_H(V) \leq T$ and $T = H$, but this is impossible, as in view of statement 2) $T \neq H$. Hence, $k = j^g \notin T$ for some element $g \in H$. If the element ik has even order, then by Theorem 13 and statement 2), it follows that $k \in T$ contrary to the above that $k \notin T$. This contradiction means that the element ik has odd order and so by Theorem 13, i and $k = j^g$ are conjugate in H. Hence i and j are conjugate in H and thus the statement 4) is proved.

Finally, we shall prove that:

5) If $H \in \mathfrak{M}$, then T is strongly embedded subgroup in H. By statement 4), every involution of T is a point and therefore statement 2) is valid for every involution of T. Using this remark, it is easy to show that if, for some $g \in H$, the intersection $T \cap T^g$ contains an involution, then it contains also some Sylow 2-subgroup S of T. Then in view of Sylow Theorem [9] $tg \in N_H(S)$, where t is some element of T. By the above remark and statement 2), $tg \in N_G(S) \leq T$ and $g \in T$. So, the statement 3) is stablished.

Now, having applied the statements 2) – 5) and Theorems 14 and 15 to every subgroup of the set $\mathfrak{M}$, we obtain the following theorem.

Theorem 3.9 *Let G be a group with infinite set of elements of finite orders and i its point of the second order satisfying (i,i)-finiteness condition. Then, $H = C_G(i)$ is a strongly embedding subgroup in G and H has a finite periodic part that is not contained in any larger subgroup with such a property.*

Proof. By Proposition 3.4, $C_G(i)$ has finite periodic part and $|G : C_G(i)|$ is infinite. The set $\mathfrak{M}$ of all subgroups with periodic part containing $C_G(i)$ is partially ordered and obviously, the union of any chain of $\mathfrak{M}$ belongs to it. By Zorn's Lemma, $\mathfrak{M}$ has a maximal element, i.e., there exists a subgroup H of $\mathfrak{M}$ which is not contained in any larger subgroup of $\mathfrak{M}$. Let V be a periodic part of H. As $i \in V$, Proposition 3.7 implies that V is a finite subgroup. It is obvious that V is normal in H and V is automorphic permissible in H. In view of maximality of H in $\mathfrak{M}$, it follows that $N_G(V) = N_G(H) = H$.

Take an involution $k \in V$. If $\langle \{k^g | g \in G\} \rangle \leq V$, then we would obviously arrive to a contradiction with the definition of point i and (i,i)-finiteness condition. Hence, for some $c \in G$, the involution $t = k^c \notin H$. Now, we consider the dihedral subgroup $L = \langle i, t \rangle$ and assume that L is not a finite Frobenius group with complement (i) and kernel (d), where $d = it$. In this case $|d| = \infty$, or $|d|$ is even. The case $|d| = \infty$ is impossible in view of (i,i)-finiteness condition and Theorem 13. If $|d|$ is even, then by Theorem 13, (d) contains an involution j where $j \in C_G(i) \cap C_G(t)$. Obviously, $|H : C_G(j) \cap H|$ is finite and as i is a point and (i,i)-finiteness condition is valid in $C_G(j)$, then $C_G(j)$ has a finite periodic part R (using Theorem 16 and Proposition 3.4). The intersection $H \cap C_G(j)$ contains such subgroup X, so that $|H : X| < \infty$, $X \triangleleft H$ and $V, R < C_G(X) \leq N_G(X)$. But $t \in R$, and therefore $t \in N_G(X)$. On the other hand, $t \notin H$ and $H < N_G(X)$.

Hence, $H \neq N_G(X)$ and in view of the definition of H a subgroup $M = N_G(X)$ has no periodic part. Furthermore, $X \leq C_G(i) \leq H$, $|H : X| < \infty$ and X has a finite periodic part. But then $i \in X$ would mean that $|M : C_G(i)| < \infty$ and hence M would have a finite periodic part which contradicts the above. Hence, $i \notin X$ and obviously in $\bar{M} = M/X$ the centralizer $C_{\bar{M}}(iX)$ is finite and (iX, iX)-

finiteness condition is valid. By Theorem 18, $\bar{M}$ is a locally finite group. Now, as $H/X \leq \bar{M}$ and H/X is a finite subgroup of $\bar{M}$, H/X is contained in a larger finite subgroup K/X of $\bar{M}$, where K is a subgroup of M and $X < H < K$. Obviously, $|K : C_G(i)| < \infty$, means that K has a finite periodic part. But $K \neq H$ and $H < K$. Hence, we obtain a contradiction to the definition of the subgroup H. This contradiction means that d is an element of odd order and the involutions i and k are conjugate in G (Theorem 13), so k and i are also conjugate in G.

Now we prove that H is a strongly embedding subgroup in G and we assume that it is not so. Then $H \neq H^g$, for some $g \in G$ and $H \cap H^g$ has an involution k. As it is proved above, k is a point of G and, besides, $|H : C_G(k) \cap H|$, $|H^g : C_G(k) \cap H^g|$ are finite. Again as proved above, $C_G(k) \leq H \cap H^g$ and $H = H^g$, i.e. $g \in N_G(H) = H$. Hence, H is a strongly embedding subgroup in G. If H has more than one involution, then by Theorem 17 and in view of (i, i)-finiteness condition in H there would be a non-unit element c of finite order, strictly real concerning to some involution $j \in G \setminus H$. By Theorem 17, i and j are conjugate in G and therefore j is a point. Now consider a subgroup $M = C_G(c)\lambda(j)$. As j is a point of M and M is satisfied to (j, j)-finiteness condition, then by Proposition 3.4, M has a finite periodic part. It is obvious that $|H : M \cap H| < \infty$ and as proved above, we obtain a contradiction to $j \notin M$. Hence, H has a unique involution and so the theorem is proved.

Theorem 3.10 *Let G be a group with infinite set of elements of finite orders and i be its point of the second order satisfying (i, i)-finiteness condition. Then, all Sylow 2-subgroups of $G = C_G(i)$ are cyclic or generalized quaternion groups.*

Proof. By Theorem 3.9 and Theorem 10, it follows that all Sylow 2-subgroups of H are cyclic or generalized quaternion groups. By Theorem 17, they are also Sylow subgroups in G, so they are conjugate in G. This completes the proof.

4 Known results

In this final section, we have collected some known results, which were used in proving our results and we referred to them as theorems with their appropriate numbers.

1. **Ditsman's Lemma.** Let M be a finite invariant set of elements of finite orders in a group, then the subgroup generated by this set is finite [10].

2. **Remak's Theorem**. Let G be a group, $H_i, i \in I$, be its normal subgroups and H be their intersection. Then the factor-group G/H is isomorphic to some sub cartesian product of the factor-groups G/H_i [9].

3. **Feit-Thompson Theorem**. Any finite group of odd order is solvable [5].

4. Let H be a periodic locally solvable group and k an element of prime order p of H such that $C_G(k)$ is finite. Then all Sylow p-subgroups of H are Chernikov groups [16].

5. Let H be a periodic locally solvable group with Chernikov Sylow p-subgroups for some $p \in \pi(H)$. Then $H/O_{p'}(H)$ is a Chernikov group [4].

6. **Blackburn Theorem**. If G is a locally finite p-group and the centralizer of some finite subgroup of G is a Chernikov group, then G is also a Chernikov group [3].

7. **Higman-Thompson Theorem.** Any finite group with regular automorphism of the prime order p is a nilpotent group. The length of its upper central series is also terminated after a finite number of steps, which only depends on p [7, 17].

8. Subgroups of a Chernikov group are Chernikov [9].

9. Extension of Chernikov group by a Chernikov group is also a Chernikov group [15].

10. A 2-group with only one involution is either a locally cyclic group (cyclic or quasi-cyclic), or a generalized quaternion group (finite or infinite) [16].

11. Let G be a finite group and H be its subgroup with $H \cap H^g = 1$ (for all $g \in G \setminus H$). Then

a) $G = F\lambda H$, where $F \setminus 1 = G \setminus \cup_{g\in G} H^g$ (Frobenius Theorem);

b) $(|F|, |H|) = 1$;

c) Sylow p-subgroups of H are cyclic or generalized quaternion groups;

d) If H has involution i, then $H = C_G(i)$, F is an abelian subgroup and $i = f^{-1} (f \in F)$;

e) If H has odd order, then all elements of prime orders of H generate a cyclic subgroup;

f) F is a nilpotent subgroup (Thompson Theorem);

g) If $p \in \pi(H)$, then the nilpotent length of subgroups F is only limited to a number depends on p (Higman Theorem);

h) If $h \in H$ and $f \in F$, then the elements h, fh are conjugate by some element of F [13].

12. let G be a finite solvable group and L its nilpotent radical. Then $C_G(L) < L$ [2].

13. Let $G = \langle i, k\rangle$ and i, k be involutions of G. Then a) $G = (c)\lambda(i) = (c)\lambda(k)$, where $c = ik$; b) $i^{-1}ci = ici = c^{-1}$, $k^{-1}ck = kck = c^{-1}$; c) i, ic^{2m} (or k, kc^{2m}) are conjugate in G, where m is an integer; d) if c is an element of odd order, then i and k are conjugate in G; e) if c is an element of even order and t is an involution of $\langle c\rangle$, then G is an elementary Abelian group of 4-th order or $Z(G) = \langle t\rangle$ [15].

14. **Bender Theorem**. Let G be a finite group and H be its strongly embedded subgroup. Then $G/O_{2'}(G) = T$ has a unique involution or normal subgroup of an odd index in T, which is isomorphic to one of the groups of type $SL(2, Q)$, $S_Z(Q)$ or $PSU(3, Q)$, where Q is a finite field of characteristic two [2].

15. Let $G \simeq PSU(3, Q)$, where Q is a finite field of characteristic two, S be a Sylow 2-subgroup of G and $H = N_G(S)$. Then H is a strongly embedded subgroup in G and H has a non-trivial element b such that $C_G(b) \not\leq H$ and $C_G(b) \cap S \neq 1$ [2].

16. If some involution $i \in G$ satisfies the (i, i)-finiteness condition, then every involution $k \in G$ is carried out strong (k, i)-finiteness condition [13].

17. Let G be a group, H be its strongly embedded subgroup, and i be

an involution of H satisfying the condition that for almost all elements $g^{-1}ig$ ($g \in G \setminus H$), the subgroups $\langle i^g \rangle$ are finite, then

a) if k is an involution of $G \setminus H$, then $|ki|$ is finite and odd number;

b) all involutions of H are conjugate in H;

c) all involutions of G are conjugate in G;

d) any element g of $G \setminus H$ has the form $g = h_g j_g$, where $h_g \in H$ and j_g is an involution of $G \setminus H$;

e) for every involution j of $G \setminus H$, the set of elements of H strictly to j, have the same power as the set of involutions in H [13].

18. Let G be a group and i an involution of it with finite centralizer $C_G(i)$. If G satisfies the (i,i)-finiteness condition, then G is a locally finite and almost solvable group [13].

19. If G has a locally finite group containing an element with finite centralizer, then G has locally soluble normal subgroup of finite index [6].

References

[1] Adjan, S.I., The Burnside problem and identities in groups, Moscow: Nauka, (1975) (in Russian).

[2] Bender, H., Transitive gruppen gerader ordnung, in denen jede involution genau einen punkt festlasst, *Journal of Algebra* **17**(4)(1971), 527–554.

[3] Blackburn, N., Some remarks on Chernikov p-groups III, *Journal of Math.* **6**(1962), 421–433.

[4] Chernikov, S.N., Conditions of finiteness in the general theory of groups, *Uspehy Math. Nauk.* **14**(5)(1959), 45–96.

[5] Feit, W. and Thompson, J.G., Solvability of groups of odd order, *Pacific Journal of Math.* **13**(3)(1963), 775–1029.

[6] Hartley, B., A general Brauer-Fowler Theorem and centralizers in locally finite groups, *Pacific Journal of Math.* **152**(1992), 101–117.

[7] Higman, G., Groups and rings having automorphisms without nontrivial fixed points, *Journal of London Math. Soc.* **32**(1957), 321–334.

[8] Iakovleva, E.N., On infinite groups with points, *Discrete Mathematic* **14**(4)(2001), 153–157.

[9] Kargapolov, M.I. and Merzlyakov, Yu.I., The Foundations of Group Theory, Moscow Nauka, 1982 (in Russian).

[10] Kurosh, A.G., Group Theory, Moscow, Nauka, 1967 (in Russian).

[11] Ol'shanskii, A.Yu., Geometry of definite relations in a group, Moscow Nauka, 1989 (in Russian).

[12] Senashov, V.I. and Shunkov, V.P., On one characterization of groups with finite periodic part, *Algebra i logika* **22**(1)(1983), 484–496.

[13] Shunkov, V.P., M_p-groups, Moscow Nauka, 1990 (in Russian).

[14] Shunkov, V.P., On locally finite groups with extreme Sylow p-subgroups on some prime number p, *Sib. Mat. Zh.* **8**(1)(1967), 213–229.

[15] Shunkov, V.P., On embedding of prime order elements in a group, Moscow Nauka, 1992 (in Russian).

[16] Shunkov, V.P., On one class of p-groups, *Algebra i logika* **9**(4)(1970), 484–496.

[17] Thompson, J.G., Finite groups with fixed point free automorphisms of prime order, *Proc. Nat. Amer. Sci.* **45**(1959), 578–581.

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A sufficient condition for null controllability of nonlinear control systems*

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Abstract

Classical control methods such as Pontryagin Maximum Principle and Bang-Bang Principle and other methods are not usually useful for solving *optimal control systems* (OCS) specially *optimal control of nonlinear systems* (OCNS). In this paper, we introduce a new approach for solving OCNS by using some combination of atomic measures. We define a criterion for controllability of lumped nonlinear control systems and when the system is nearly null controllable, we determine controls and states. Finally we use this criterion to solve some numerical examples.

Keywords and phrases: Approximation theory, controllability, fuzzy theory, measure theory, optimal control.
AMS Subject Classification 2000: Primary 49J37; Secondary 49M99.

1 Introduction

We consider a nonlinear time-variant system as follows:

$$\dot{x} = g(t, x(t), u(t)), \qquad \forall t \in J, \tag{1}$$

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$$x(t_0) = x_0, x(t_f) = x_f, \tag{2}$$

where $\Omega_1 = J \times A \times U \times D$, here J is a known closed interval $[t_0, t_f]$, A and D are compact and peicewise connected sets in R^n such that $x(t) \in A$ and $\dot{x}(t) \in D, \forall t \in J$, and U is a compact set in R^m such that $u(t) \in U, \forall t \in J$, and g is continuous on J. If there are $u(.)$ and $x(.)$ that satisfy equation (1)-(2) we call the system is controllable.

In the following, by means of a process of embedding and using measure theory, this problem is replaced by another one in the space of Borel measures, that we seek to minimize to a linear form over a compact subset of the measure space. The theory allows us to convert the new problem to an infinite-dimensional linear programming problem. Later on the infinite-dimensional linear programming problem is approximated by a finite dimensional one. Then by the solution of the linear programming problem one can find approximate functions for states $x(\cdot)$ and control $u(\cdot)$.

If the system has an objective function we can use this process for solving the systems defined by multi-objective control systems.

There are some literature on nonlinear optimal control for lumped and distributed parameter systems, see for example, [2]–[12].

2 Defining the problem

Let us define in (1), for all t in $J = [t_0, t_f]$

$$y(t) \triangleq \dot{x}(t), \tag{3}$$

Then the equations can be rewritten as

$$y(t) = g(t, x(t), u(t)), \tag{4}$$

$$x(t_0) = x_0, x(t_f) = x_f. \tag{5}$$

Now we define the function $h : \Omega_1 \to R$ as

$$h(t, x(t), u(t), y(t)) \triangleq \|y(t) - g(t, x(t), u(t))\|, \tag{6}$$

and let the functional $I(., x(.), u(.), y(.))$ be as follows:

$$I(., x(.), u(.), y(.)) \triangleq \int_J h(t, x(t), u(t), y(t))dt.$$

Now, we investigate a necessary and sufficient condition for controllability of control system (1)-(2).

Theorem 1. *A necessary and sufficient condition for controllability of control system (1)-(2) is*

$$\text{Min} \quad I(\cdot, \cdot, \cdot, \cdot) = 0,$$

that is equation (1) and boundary conditions (2) are valid on Ω_1.

Proof. Since $h \geq 0$ and it is continuous, h is Rimman integrable. If

$$\text{Min} \quad I(\cdot, \cdot, \cdot, \cdot) = 0,$$

$u^*(.)$ and $x^*(.)$ are the corresponding control and trajectory and $x(t_0) = x_0, x(t_f) = x_f$, then

$$\int_J h(t, x^*(t), u^*(t), y^*(t))dt = 0$$

and we will have $h = 0$. So

$$y^*(t) = g(t, x^*(t), u^*(t)),$$

or

$$\dot{x}^*(t) = g(t, x^*(t), u^*(t)).$$

In other words, in this case $u^*(.)$ and $x^*(.)$ satisfy equations (1)-(2) and the system will be controllable.

Conversely, if the system is controllable; that is, if (1)-(2) are satisfied, then $h = 0$, for all t in J. So

$$\int_J h(t, x(t), u(t), y(t))dt = 0$$

and then $I(., x, u, y) = 0$, hence $\text{Min} \quad I(., x, u, y) = 0$.

Note In practice we usually obtain suboptimal solution for $I(.,.,.,.)$ in Theorem 1, that is we have many errors for controllability of the system, for example computational errors. So usually I is not exactly equal to zero, in this case let the total permissible errors be at most $\epsilon > 0$, where ϵ is a known positive number. If

$$e(t) \triangleq \|y^*(t) - g(t, x^*(t), u^*(t))\|_{L_2} = (\int_J \|y^*(t) - g(t, x^*(t), u^*(t))\|^2 dt)^{1/2} \quad (7)$$

and $e < \epsilon$, then the system is almost controllable, so we define fuzzy controllability.

Fuzzy controllability Let $\widetilde{C}$ be the fuzzy set of permissible controls and trajectories as follows:

$$\widetilde{C} \cong \{(x, u, y) : C(x, u, y) \text{ is as follows}\}$$

$$C(x, u, y) = \begin{cases} \frac{(\epsilon - e)}{\epsilon} & , e < \epsilon, \\ 0 & , \text{otherwise.} \end{cases}$$

Then we say the system is controllable of grade C.

Controllability of multi-objective systems

Let our Multi-Objectives System be the minimization of

$$I_i(t, x(t), u(t), \dot{x}(t)) = \int_J f_i(t, x(t), u(t), \dot{x}(t))dt, \quad i = 1, 2, \cdots, k,$$

subject to the conditions (1)-(2), also we would like to be sure that our system is controllable or fuzzy controllable.

If we consider $y(t)$ as before and

$$w(t) \triangleq (w_1(t), w_2(t), \cdots, w_{k+1}(t))$$

such that

$$w(t) \in E, \text{ where } E \subset R^{k+1} \text{ and } E = [0, 1] \times [0, 1] \times \cdots \times [0, 1],$$

and also we define an objective function which is a convex combination of the above objectives, that is, we assume the weights for objectives, as follows:

$$I(\cdot,\cdot,\cdot,\cdot,\cdot) \cong \sum_{i=1}^{k} w_i(t) \int_J f_i(t,x(t),u(t),y(t))dt + w_{(k+1)}(t) \int_J h(t,x(t),u(t),y(t))dt,$$

$$\sum_{i=1}^{k+1} w_i(t) = 1, \quad 0 \leq w_i(t) \leq 1, \quad i = 1,2,...,k+1,$$

and we consider f in the following way

$$f(t,x(t),u(t),y(t),w(t)) \triangleq \sum_{i=1}^{k} w_i(t) f_i(.,x,u,y) + w_{(k+1)}(t) h(.,x,u,y),$$

then $I(.,.,.,.,.)$ will be

$$I(.,x,u,y,w) \triangleq \int_J f(t,x(t),u(t),y(t),w(t))dt,$$

then the minimization of $I(.,x,u,y,w)$ will be a criterion for controllability and also multi-objective performances functional. In the special case, when $w_i(t) = 0, \quad i = 1,2,...,k$ and $w_{(k+1)}(t) = 1$, it is just a criterion for controllability.

3 Metamorphism

We define $[x(\cdot),u(\cdot),y(\cdot)]$ to be an *admissible triple,* provided for all t in J,

- the function $x(\cdot)$ is continuous, and $x(t) \in A$;
- the function $x(\cdot)$ is continuous, and $y(t) \in D$;
- the function $u(\cdot)$ is Lebesgue measurable, and $u(t) \in U$;
- the triple satisfies the system of differential equations (4)-(5) and a.e. on $J^\circ = (t_0, t_f)$ in the sense of Caratheodory.

We denote the set of admissible triples by V. The problem has no solution unless $V \neq \emptyset$.

Using the above assumption, the problem is now as follows:
Find an optimal admissible triple $v \in V$ which minimizes the functional

$$I(\cdot,\cdot,\cdot,\cdot,\cdot) = \int_J f(t,x(t),u(t),y(t),w(t))dt. \tag{8}$$

Assume that B is an open ball in R^{n+1} containing $J \times A$, denote the space of all differentiable functions on B by $C'(B)$, and for $\phi \in C'(B)$ define

$$\phi^g(t,x,u,y) \cong \phi_x(t,x).g + \phi_t(t,x), \tag{9}$$

where $\phi(\cdot)$ and $g(\cdot)$ are n-vectors and the first term in the right-hand side of (9) is an inner product and ϕ^g is in the space $C(\Omega)$ of real-valued continuous functions defined on the compact set Ω, where $\Omega = \Omega_1 \times E$. Then by the definitions of g and ϕ and using the chain rule we have

$$\int_J \phi^g(t,x(t),u(t),y(t))dt = \int_J \dot{\phi}(t,x(t))dt$$
$$= \phi(t_f,x(t_f)) - \phi(t_0,x(t_0)) = \delta\phi.$$

Therefore

$$\int_J \phi^g(t,x(t),u(t),y(t))dt = \delta\phi, \forall \phi \in C'(B). \tag{10}$$

Since A may have an empty interior in R^n, we need to introduce the set B and space $C'(B)$. Suppose $D(J^\circ)$, is the space of infinitely differentiable real valued functions with compact support in J° and each x and g have n components such as x_j and $g_j, j = 1,2,\ldots$,n. For each $\psi \in D(J^\circ)$, define

$$\psi_j(t,x,u,y) \cong x_j\psi'(t) + g_j\psi(t), j = 1,2,\ldots,n. \tag{11}$$

If w is an admissible pair, then for any $\psi \in D(J^\circ)$ we have

$$\int_J \psi_j(t,x(t),u(t),y(t))dt = \int_J x_j\psi'(t)dt + \int_J g_j\psi(t)dt$$
$$= x_j(t)\psi(t)\mid_{t_0}^{t_f} - \int_J \{x_j' - g_j(t,x(t),u(t))\}\psi(t)dt,$$

Since ψ has compact support on J°, it follows that

$$\psi(t_0) = \psi(t_f) = 0,$$

and hence

$$\int_J \psi_j(t, x(t), u(t), y(t))dt = 0. \tag{12}$$

Now, we choose those functions in $C'(B)$ which depend on the time variable only and denote this subspace by $C_1(\Omega)$. Set

$$\beta(t, x, u, y, w) = \beta(t), (t, x, u, y) \in \Omega.$$

Thus

$$\int_J \beta^g(t, x(t), u(t), y(t))dt = a_\beta, \beta \in C_1(\Omega),$$

where a_β is the Lebegue integral of $\beta(t, x, u, y)$ on J.

In a given classical problem, the set of admissible triples is fixed. If we add some elements to it, we have changed the problem and considered a new one, inspired classically, but a different formulation nevertheless.

Consider the mapping

$$\Lambda_w : F \in C(\Omega) \to \int_J F(t, x(t), u(t), y(t), w(t))dt,$$

which is a linear and positive functional. Let us rewrite (8) subject to the conditions (4)-(5) in the new representation as follows:

$$\text{Minimize} \quad \Lambda_v(f) \tag{13}$$

subject to

$$\Lambda_v(\phi^g) = \delta\phi, \phi \in C'(B)$$

$$\Lambda_v(\psi_j) = 0, j = 1, 2, ..., n; \psi \in D(J^\circ) \tag{14}$$

$$\Lambda_v(\beta) = a_\beta, \beta \in C_1(\Omega).$$

We mention that Λ_w is a positive Radon measure on the set $C(\Omega)$. We denote the space of all positive Radon measures on Ω by $M^+(\Omega)$. A Radon measure on Ω can be identified with a regular Borel measure on this set (see [13], Riesz

Representation Theorem). Thus, for a given positive functional on $C(\Omega)$, there is a positive Borel measure on Ω such that

$$\Lambda_v(F) = \int_\Omega F d\mu = \mu(F).$$

Now, the problem (13)-(14) can be replaced by a new problem as follows. We seek a measure in $M^+(\Omega)$ which minimizes the functional

$$\mu \in M^+(\Omega) \to \mu(f) \in R \tag{15}$$

and satisfies the following constraints:

$$\begin{aligned} &\mu(\phi^g) = \delta\phi, \phi \in C'(B) \\ &\mu(\psi_j) = 0, j = 1, 2, ..., n; \psi \in D(J^\circ) \\ &\mu(\beta) = a_\beta, \beta \in C_1(\Omega). \end{aligned} \tag{16}$$

Thus, we consider the extension of our problem: Minimization of (15) over the set Q of all positive Radon measures on Ω satisfying (16). Considering such measure theoretic form of the problem has teo main adventages, namely

- The existence of an optimal measure in the set Q, which satisfies (16) can be studied in a straightforward manner without having to impose conditions such as convexity, which may be artificial.
- The functional in (15) is linear, although $f(\cdot,\cdot,\cdot,\cdot,\cdot)$ may be nonlinear.

By the Proposition II.1, Theorem II.1 and Proposition II.3 of [14], we are able to prove the existence of the optimal measure.

4 First approximation

The problem (15)-(16) is an infinite dimensional linear programming(LP) problem, because all the functionals in (15)-(16) are linear in the variable μ even if the original problem is nonlinear and furthermore, the measure μ is required to be positive. Of course, (15)-(16) is an infinite dimensional LP problem, because $M^{+}(\Omega)$ is an infinite dimensional space. It is possible to approximate the solution of this problem by the solution of a finite-dimensional LP of sufficiently large dimension. Also, from the solution of this new finite dimensional LP we induce an approximated admissible triple in a suitable manner. We shall first develop an intermediate problem, still infinite-dimensional, by considering the minimization (15), not over the set Q but over a subset of $M^{+}(\Omega)$ defined by requiring that only a finite number of the constraints in (16) are satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in the set Q, and then selecting a finite number of them. Consider the first set of equalities in (16). Let the set $\{\phi_i, i = 1, 2, ...\}$ be such that the linear combinations of the functions $\phi_i \in C'(B)$ are uniformly dense. For instance, these functions can be taken to be monomials in the components of the n-vectors x and variable t.

Now, we consider the functions in $D(J^{\circ})$ defined as below

$$\sin[\frac{2\pi r(t-t_0)}{\delta t}], \quad 1-\cos[\frac{2\pi r(t-t_0)}{\delta t}], r = 1, 2, ... \tag{17}$$

where $\delta t = t_f - t_0$, if ψ's are chosen as (17), and the sequence $\{\chi_{\ l}\}, l = 1, 2, ...$ is of type ψ_j in (11). Then the first approximation will be completed by using the above subjects and Proposition III.1 of [14].

5 Second approximation

By Proposition III.2 of [14] the optimal measure has the form

$$\mu^* = \sum_{k=1}^{N} \alpha_k^* \delta(z_k^*), \tag{18}$$

where $z_k^* \in \Omega$ and $\alpha_k^* \geq 0, k = 1, 2, ..., N$, where $\delta(\cdot)$ is unitary atomic measure with the support being the singleton set $\{z_k^*\}$, characterized by

$$\delta(z)(F) = F(z), z \in \Omega.$$

This structural result points the way towards a nonlinear problem in which the unknowns are the coefficients α_k^* and supports $\{z_k^*\}, k = 1, 2, ..., N$.

To change this problem to a LP problem, we use another approximation. If ω^N is a countable dense subset of Ω, we can approximate μ^* by a measure $\nu \in M^+(\Omega)$ such that

$$\nu = \sum_{k=1}^{N} \alpha_k^* \delta(z_k),$$

where $z_k \in \omega^N = \{z_1, z_2, ..., z_N\}$(Proposition III.3 of [14]).

This result suggests the following LP problem

Given $\epsilon > 0$ and $z_j \in \omega^N,\ j = 1, 2, ..., N$,

$$\text{Minimize} \qquad \sum_{j=1}^{N} \alpha_j f(z_j) \tag{19}$$

subject to

$$\begin{gathered} \mid \sum_{j=1}^{N} \alpha_j \phi_i^g(z_j) - \delta\phi_i \mid \leq \epsilon, i = 1, 2, \cdots, M_1, \\ \mid \sum_{j=1}^{N} \alpha_j \chi_l(z_j) \mid \leq \epsilon, l = 1, 2, ..., M_2, \\ \mid \sum_{j=1}^{N} \alpha_j \beta_s(z_j) - a_{\beta_s} \mid \leq \epsilon, s = 1, 2, \cdots, L, \\ \alpha_j \geq 0, j = 1, 2, ..., N. \end{gathered} \tag{20}$$

Assume $P(M_1, M_2, L)^\epsilon$ in R^N is defined by $\alpha_j \geq 0, j = 1, 2, ..., N$ satisfies (20), then by Theorem III.1 of [14], for every $\epsilon \geq 0$ the problem of minimizing the functional (19) on the set $P(M_1, M_2, L)^\epsilon$ has a solution for $N = N(\epsilon)$ sufficiently large, and the solution satisfies

$$\eta(M_1, M_2, L) + \rho(\epsilon) \leq \sum_{j=1}^{N} \alpha_j f(z_j) \leq \eta(M_1, M_2, L) + \epsilon,$$

where $\rho(\epsilon) \to 0$ as $\epsilon \to 0$.

Let $\theta_r \in C_1(\Omega)$,

$$\theta_r(t,x,u,y,w) = t^r, r = 0,1,..., \tag{21}$$

then the set of θ_r's is dense in $C_1(\Omega)$. Assume that there are a number L of them in the set $\{\phi_i^g\}_{i=1}^{M_1}$. It is necessary to choose L number of functions of the time only, to replace the functions $\theta_r, r = 0,1,...$ which were not found suitable, so we have chosen some suitable functions, to be denoted by f_s, $s = 1,2,...L$, as follows:

$$f_s(t) = \begin{cases} 1 & \text{if } t \in J_s \\ 0 & \text{otherwise,} \end{cases}$$

where $J_s = (t_0 + (s-1)d, t_0 + sd)$, $d = \frac{\delta t}{L}$ Since every continuous function can be written as a linear combination of monomials of type $1, x, x^2, ...$ We assume

$$\phi_1 = x_1, \phi_2 = x_2, ..., \phi_n = x_n,$$

$$\phi_{n+1} = {x_1}^2, \phi_{n+2} = {x_2}^2, ..., \phi_{2n} = {x_n}^2,$$

until M_1 functions are chosen, also assume

$$\psi^r(t) = \sin[\frac{2\pi r(t-t_0)}{\delta t}], r = 1,2,...,M_{21}$$

or

$$\psi^r(t) = 1 - \cos[\frac{2\pi r(t-t_0)}{\delta t}], r = M_{21}+1, M_{21}+2, ..., 2M_{21},$$

where χ_l are chosen as ψ_j^r in (11), then we have $M_2 = 2nM_{21}$ number of type χ_l.

Now, if in the problem (19)-(20), $\epsilon \to 0$ and $z_j \in \omega^N, j = 1,2,...,N$, then we have

$$\text{Minimize} \quad \sum_{j=1}^{N} \alpha_j f(z_j) \tag{22}$$

subject to

$$\sum_{j=1}^{N} \alpha_j \phi_i^g(z_j) = \delta\phi_i, i = 1,2,\cdots,M_1,$$

$$\sum_{j=1}^{N} \alpha_j \chi_l(z_j) = 0, l = 1,2,...,M_2, \tag{23}$$

$$\sum_{j=1}^{N} \alpha_j f_s(z_j) = a_s, s = 1, 2, ..., L,$$
$$\alpha_j \geq 0, j = 1, 2, ..., N,$$

where a_s is the integral of f_s on J. By solving this finite dimensional LP problem we obtain the nearly optimal α^*'s.

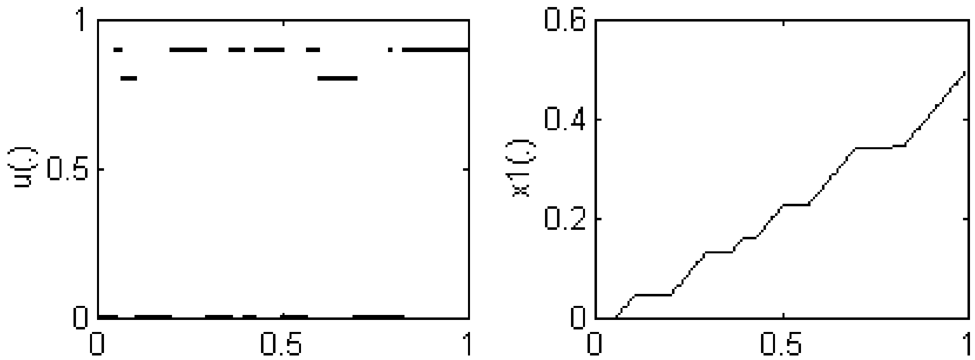
6 Numerical examples

Example 1. Consider the nonlinear time-variant problem

$$\dot{x} = x^2 sin(x) + u$$

$$x(0) = 0, x(1) = 0.5.$$

We let $\epsilon = 0.1$ and partition respectively the sets $J = [0, 1], A = [0, 0.5], D = [0, 1]$, and $U = [0, 1]$ into $p_t = p_A = p_D = p_u = 10$ and $M_1 = 6, M_2 = 4$, and $L = 10$.



Optimal Control State

We used Revised Simplex method to solve such problem and found $f^* = 0.0065$, $x^*(0) = 0$ and $x^*(1) = 0.4995$, and degree of controllability of this example is $C = 0.9349$. Below, the figures of $x(\cdot)$ and $u(\cdot)$ are given.

Example 2. Consider the nonlinear time-variant optimal control problem

$$\text{Minimize} \quad \int_0^1 u^2(t)dt,$$

subject to the conditions

$$\dot{x} = x^2 sin(x) + u$$

$$x(0) = 0, x(1) = 0.5.$$

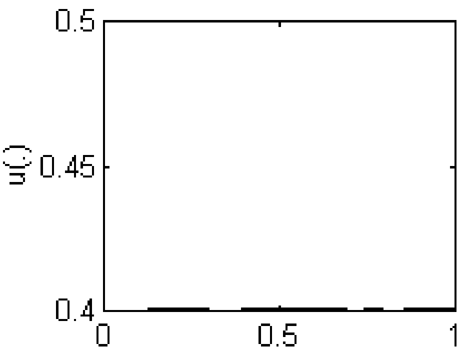
Then

$$h(t, x(t), u(t), y(t)) = \|y(t) - (x^2 sin(x) + u)\|, \quad \forall t \in J$$

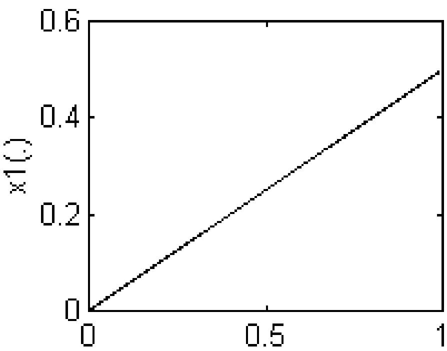
and

$$I(\cdot, \cdot, \cdot, \cdot, \cdot) = w_1(t) \int_0^1 u^2(t)dt + w_2(t) \int_0^1 h(t, x(t), u(t), y(t))dt.$$

We let $w_1(t) = w_2(t) = \frac{1}{2}$ and $\epsilon = 0.1$ and divide respectively the sets $J = [0, 1], A = [0, 0.5], D = [0, 1]$, and $U = [0, 1]$ into $p_t = p_A = p_D = p_u = 10$ and $M_1 = 6, M_2 = 4$, and $L = 10$.
We used Revised Simplex method to solve this problem and found $f^* = 0.1133$, $x^*(0) = 0$ and $x^*(1) = 0.4981$. Below, the figures of $x(\cdot)$ and $u(\cdot)$ are given.



Optimal Control



State

Example 3 (A system of coupled hydraulic tanks [1]) A state-space model can be set up with the inlet flow-rate u as input, the depths of liquid (x_1, x_2) in the respective tanks as state variables and the output taken as x_2, since the objective is to control the level in tank2. With tanks of the same dimensions, and orifices of equal size, the state-space equations expressed in suitably normalized variables become

$$\dot{x_1} = u - \sqrt{x_1 - x_2},$$

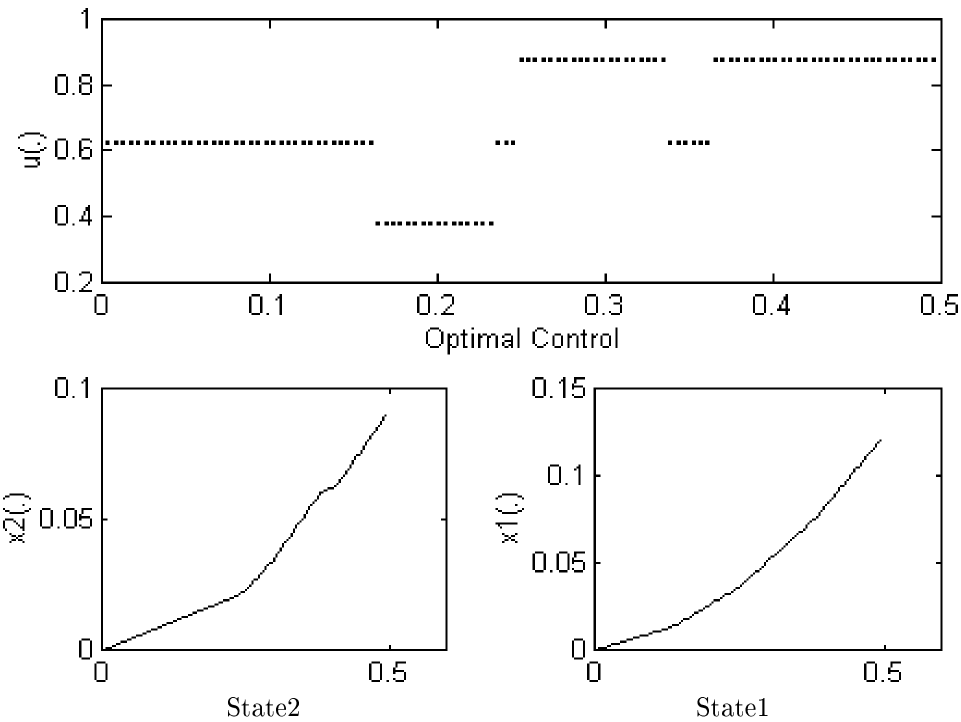
$$\dot{x_2} = \sqrt{x_1 - x_2} - \sqrt{x_2},$$

where it is understood that the system operates only in the region

$$x_1 > x_2 > 0.$$

We assume $\epsilon = 0.1$ and divide respectively the sets $J = [0, 0.5], A1 = [0.05, 0.45], D1 = [0.05, 0.45], A2 = [0.05, 0.35], D2 = [0.05, 0.35]$, and $U = [0, 1]$ into $p_t = p_{A_1} = p_{D_1} = p_{A_2} = p_{D_2} = p_u = 4$ and $M_1 = 2, M_2 = 8$, and $L = 4$.

We solve this problem and found $f^* = 0.0465$, $x_1^*(0) = 0.02, x_1^*(0.5) = 0.1214, x_2^*(0) = 0.02$ and $x_2^*(0.5) = 0.0902$, and degree of controllability of this example is $C = 0.5355$. Below, the figures of $u(\cdot), x_1(\cdot)$, and $x_2(\cdot)$ are given.



References

[1] Cook, P.A., Nonlinear Dynamical Systems, Printice Hall International Series in Systems and Control Engineering, 1994.

[2] Farahi, M.H., Rubio, J.E. and Wilson, D.A., The optimal control of the linear wave equation, *Int. J. control* **63**(5)(1996), 833–848.

[3] Farahi, M.H., Rubio, J.E. and Wilson, D.A., The global control of a nonlinear wave equation, *Int. J. control* **65**(1)(1996), 1–15.

[4] Heydari, A., Kamyad, A.V. and Farahi, M.H., On the existence and numerical estimation of the minimum integral functionals, *Journal of Institute of Mathematics and Computer Sciences* **12**(1999), 263–272.

[5] Heydari, A., Farahi, M.H. and Heydari, A.A., Chemotherapy in an HIV model by a pair of optimal control, Proceeding of the 7th WSEAS International Conference (SMO)(2007), 58–63.

[6] Heydari, A., Farahi, M.H. and Heydari, A.A., Optimal control of treatment of tuberculosis, *International Journal of Applied Mathematics (IJAM)* **19**(4)(2006), 389–402.

[7] Heydari, A., Kamyad, A.V. and Farahi, M.H., A new approach for solving robust control problems, *J. of Inst. of Math. and Comp. Sci* **15**(1)(2002), 1–7.

[8] Heydari, A., Kamyad, A.V. and Farahi, M.H., A new method for solving time optimal control problems, *Iranian Int. J. Sci.* **4**(2)(2003), 241–254.

[9] Heydari, A., Kamyad, A.V. and Farahi, M.H., Using measure theory for the numerical solution of optimal control problems with bounded state variables, *Engineering Simulation* **19**(3)(2001), 47–59.

[10] Kamyad, A.V., Rubio, J.E. and Wilson, D.A., The optimal control of the multidimensional diffusion equation, *Journal of Optimization Theory and Applications* **70**(1991), 191–209.

[11] Kamyad, A.V., Rubio, J.E. and Wilson, D.A., An optimal control problem for the multidimensional diffusion equation with a general control variable, *Journal of Optimization Theory and Aplications* **75**(1992), 101–132.

[12] Kamyad, A.V., Strong controllability of the diffusion equation in n-dimensions, *Bulletin of the Iranian Mathematical Society* **18**(1)(1992), 39–49.

[13] Royden, H.L., Real Analysis, The Macmillan Company, London, 1970.

[14] Rubio, J.E., Control and optimisation, The Linear Treatment of Nonlinear Problems, Manchester University Press, England, 1986.

[15] Young, L.C., Calculus of Variation and Optimal Control Theory, Philadelphia: W.B. Saunders, 1969.

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The variational iteration method for solving linear and nonlinear Schrodinger equations*

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Abstract

In this paper, the variational iteration method which proposed by Ji-Huan He is applied to solve both linear and nonlinear Schrodinger equations. The main property of the method is in its flexibility and ability to solve linear and nonlinear equations accurately and conveniently. In this method, general Lagrange multipliers are introduced to construct correction functionals to the problems. The multipliers in the functionals can be identified optimally via the variational theory. Numerical results show that this method can readily be implemented with excellent accuracy to linear and nonlinear Schrodinger equations. This technique can be extended to higher dimensions linear and nonlinear Schrodinger equations without a serious difficulties.

Keywords and phrases: General Lagrange multipliers, linear and nonlinear Schrodinger equations, variational iteration method.
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1 Introduction

The variational iteration method (VIM) was first proposed by Ji-Huan He in 1998 [6,7] and systematically illustrated in 1999 [11]. Since then, it has been success-

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fully applied to various engineering problems [15,16]. This method is employed in [18] to solve the Klein-Gordon equation which is the relativistic version of the Schrodinger equation, which is used to describe spinless particles. Application of He's variational iteration technique to an inverse parabolic problem is described in [4]. In [2] the VIM is employed to solve the time dependent reaction-diffusion equation which has special importance in engineering and sciences and constitutes a good model for many systems in various fields. This technique is also employed in [5] to solve the Fokker-Planck equation and in [3] to solve a biological population model. For more application of the method, the interested reader is referred to [1,17,19,21]. The VIM [11,12] is a powerful tool to search for approximate solutions of linear and nonlinear equations without requirement of linearization or perturbation. Another important advantage is that the VIM is capable of greatly reducing the size of calculation while still maintaining high accuracy of the numerical solution. Moreover, the power of the method gives a wider applicability in handling a huge number of analytical and numerical applications. The convergence of He'variational iterative method is investigated in [20]. Here, we apply VIM to one and two dimensional linear and nonlinear Schrodinger equations. This paper is organized as follows: In Section 2, we introduce the model of the problems. In Section 3, first we describe VIM method and then we apply VIM in a direct manner to establish exact solutions for linear and nonlinear Schrodinger equations. In Section 4, we describe the numerical solution of linear and nonlinear Schrodinger equations to show the power of the method in a unified manner without requiring any additional restriction.

2 The model of the problem

In this paper, the linear Schrodinger equation is considered as follows:

$$\frac{\partial \psi}{\partial t}(x,t) + i\frac{\partial^2 \psi}{\partial x^2}(x,t) = 0, \quad \psi(x,0) = f(x), \;\; x \in \mathbb{R}, \;\; t \geq 0, \;\; i^2 = -1, \qquad (1)$$

and we consider the nonlinear Schrodinger equation of the form

$$i\frac{\partial\psi}{\partial t}(X,t)=-\frac{1}{2}(\frac{\partial^2\psi}{\partial x^2}(X,t)+\frac{\partial^2\psi}{\partial y^2}(X,t))+\beta|\psi|^2\psi,\ \ X\in\mathbb{R}^2,\ \ t\geq 0, \qquad (2)$$

where $X=(x,y)$, $|\psi|^2=\psi\overline{\psi}$, and β is a real constant.

3 Basic ideas of He's variational iteration method

In this section, the application of the VIM is discussed for linear and nonlinear Schrodinger equations. Considering the following general differential equation:

$$L\psi(x,t)+R\psi(x,t)+N\psi(x,t)=g(x,t), \qquad (3)$$

where L is a first order partial differential operator, R is a linear operator, N is a nonlinear operator and $g(x,t)$ is a known analytical function. According to the VIM[8–10], we can construct the following correction functional:

$$\psi_{n+1}(x,t)=\psi_n(x,t)+\int_0^t\lambda(L\psi_n(\xi)+R\tilde{\psi_n}(\xi)+N\tilde{\psi_n}(\xi)-g(\xi))d\xi \quad ,n\geq 0, \quad (4)$$

where λ is a general Lagrange multiplier [14], which should be identified optimally via the variational theory [14], the subscript n denotes the nth approximation, and $\tilde{\psi_n}$ is considered as a restricted variation [6,7,11] and [13] i.e $\delta\tilde{\psi_n}=0$. We first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $\psi_{n+1}(x,t)$ $,n\geq 0$ of the solution $\psi(x,t)$ will be readily obtained using the derived Lagrange multiplier and by using any selective function ψ_0. The initial values $\psi(x,0)$ and $\psi_t(x,0)$ are usually used for selecting the zeroth approximation ψ_0 . With λ determined, several approximation $\psi_j(x,0)$, $j\geq 0$ follow immediately. Consequently, the exact solution may be obtained by using (see [20])

$$\psi=\lim_{n\to\infty}\psi_n. \qquad (5)$$

According to the VIM, we consider linear Schrodinger equation (1) in the following form[8–10]:

$$\psi_{n+1}(x,t)=\psi_n(x,t)+\int_0^t\lambda(\xi)(\frac{\partial\psi_n}{\partial\xi}(x,\xi)+i\frac{\partial^2\widetilde{\psi}_n}{\partial x^2}(x,\xi))d\xi. \qquad (6)$$

To find the optimal value of λ, we have

$$\delta\psi_{n+1}(x,t)=\delta\psi_n(x,t)+\delta\int_0^t\lambda(\xi)(\frac{\partial\psi_n}{\partial\xi}(x,\xi)+i\frac{\partial^2\widetilde{\psi}_n}{\partial x^2}(x,\xi))d\xi=0, \quad (7)$$

or

$$\delta\psi_{n+1}(x,t)=\delta\psi_n(x,t)+\delta\int_0^t\lambda(\xi)(\frac{\partial\psi_n}{\partial\xi}(x,\xi))d\xi=0. \quad (8)$$

which follows

$$\delta\psi_{n+1}(x,t)=\delta\psi_n(x,t)(1+\lambda(t))-\delta\int_0^t\lambda'(\xi)\psi_n(x,\xi)d\xi=0, \quad (9)$$

The following stationary conditions

$$1+\lambda(t)=0, \quad (10)$$

$$\lambda'(\xi)=0, \quad (11)$$

follow immediately. This in turn gives

$$\lambda(\xi)=-1. \quad (12)$$

Substituting this value of the Lagrange multiplier $\lambda=-1$ into the functional (6) gives in the following iteration formula

$$\psi_{n+1}(x,t)=\psi_n(x,t)-\int_0^t(\frac{\partial\psi_n}{\partial\xi}(x,\xi)+i\frac{\partial^2\widetilde{\psi}_n}{\partial x^2}(x,\xi))d\xi. \quad (13)$$

Similarly, we obtain the correction functional for (2). Hence we have

$$\begin{aligned}\psi_{n+1}(X,t) &= \psi_n(X,t)+\int_0^t\lambda(\xi)(i\frac{\partial\psi_n}{\partial\xi}(X,\xi)+\frac{1}{2}(\frac{\partial^2\widetilde{\psi}_n}{\partial x^2}(X,\xi)+\frac{\partial^2\widetilde{\psi}_n}{\partial y^2}(X,\xi))\\ &- \beta\psi^2\overline{\psi})d\xi. \quad (14)\end{aligned}$$

The stationary conditions are of the following form

$$1+i\lambda(t)=0, \quad (15)$$

$$\lambda'(\xi)=0, \quad (16)$$

and so we have

$$\lambda(\xi)=i. \quad (17)$$

Substituting this value of the Lagrange multiplier $\lambda = i$ into the functional (14) gives the following iteration formula

$$\begin{aligned}\psi_{n+1}(X,t) &= \psi_n(X,t) + i\int_0^t \Big(i\frac{\partial \psi_n}{\partial \xi}(X,\xi) + \frac{1}{2}\Big(\frac{\partial^2 \widetilde{\psi}_n}{\partial x^2}(X,\xi) + \frac{\partial^2 \widetilde{\psi}_n}{\partial y^2}(X,\xi)\Big) \\ &- \beta\psi^2\overline{\psi}\Big)d\xi. \qquad (18)\end{aligned}$$

Here, we will use this method to solve linear and nonlinear Schrodinger equations to establish exact solutions for these equations.

4 Examples

To illustrate the solution procedure and show the ability of the method, some examples are provided.

Example 4.1 Consider the following linear Schrodinger equation :

$$\frac{\partial \psi}{\partial t}(x,t) + i\frac{\partial^2 \psi}{\partial x^2}(x,t) = 0, \qquad (19)$$

$$\psi_0(x) = \sinh 2x. \qquad (20)$$

Using (13), we obtain the following successive approximations:

$$\begin{aligned}\psi_1(x,t) &= (1-4it)\sinh 2x, \\ \psi_2(x,t) &= (1-4it+\frac{(-4it)^2}{2!})\sinh 2x, \\ &\vdots \\ \psi_n(x,t) &= (1-4it+\frac{(-4it)^2}{2!}+\frac{(-4it)^3}{3!}+...+\frac{(-4it)^n}{n!})\sinh 2x.\end{aligned}$$

Consequently, the exact solution is

$$\psi_{exact}(x,t) = e^{-4it}\sinh 2x. \qquad (21)$$

Example 4.2 Consider the nonlinear Schrodinger equation

$$i\frac{\partial \psi}{\partial t}(x,t) + \frac{\partial^2 \psi}{\partial x^2}(x,t) + 2|\psi|^2\psi = 0, \qquad (22)$$

$$\psi_0(x) = e^{-ix}. \qquad (23)$$

Using (18), we obtain the following successive approximations:

$$\begin{aligned}
\psi_1(x,t) &= (1+it)e^{-ix}, \\
\psi_2(x,t) &= (1+(it)+\frac{(it)^2}{2!})e^{-ix}, \\
\psi_3(x,t) &= (1+(it)+\frac{(it)^2}{2!}+\frac{(it)^3}{3!})e^{-ix}, \\
&\vdots \\
\psi_n(x,t) &= (1+(it)+\frac{(it)^2}{2!}+\frac{(it)^3}{3!}+\frac{(it)^4}{4!}+...+\frac{(it)^n}{n!})e^{-ix}.
\end{aligned}$$

Consequently, the exact solution is

$$\psi_{exact}(x,t) = e^{i(t-x)}. \tag{24}$$

Example 4.3 Consider the nonlinear Schrodinger equation

$$i\frac{\partial\psi}{\partial t}(X,t) + \frac{1}{2}(\frac{\partial^2\psi}{\partial x^2}(X,t) + \frac{\partial^2\psi}{\partial y^2}(X,t)) + 2|\psi|^2\psi = 0 \ , \tag{25}$$

$$\psi_0(x,y) = e^{i(x+y)}. \tag{26}$$

Using (9), we obtain the following successive approximations:

$$\begin{aligned}
\psi_1(X,t) &= (1+it)e^{i(x+y)}, \\
\psi_2(X,t) &= (1+(it)+\frac{(it)^2}{2!})e^{i(x+y)}, \\
\psi_3(X,t) &= (1+(it)+\frac{(it)^2}{2!}+\frac{(it)^3}{3!})e^{i(x+y)}, \\
&\vdots \\
\psi_n(X,t) &= (1+(it)+\frac{(it)^2}{2!}+\frac{(it)^3}{3!}+\frac{(it)^4}{4!}+...+\frac{(it)^n}{n!})e^{i(x+y)}.
\end{aligned}$$

Consequently, the exact solution is

$$\psi_{exact}(X,t) = e^{i(t+x+y)}.$$

Conclusions

In this paper, He's variational iteration method has been successfully applied to find the solution of the linear and nonlinear Schrodinger equations. The main advantage of the method is the fact that it provides an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed term. Analytical solutions enable researchers to study the effect of different variables or parameters on the function under study easily. A clear conclusion can be drawn from the numerical results which VIM provides with highly accurate numerical solution without spatial discretizations for linear and nonlinear Schrodinger equations.

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References

[1] Biazar, J. and Ghazvini, H., He's variational iteration method for solving hyperbolic differential equations, *International Journal of Nonlinear Sciences and Numerical Simulation* **8**(2007), 311–314.

[2] Dehghan, M. and Shakeri, F., Application of He's variational iterative method for solving the Cauchy reaction-diffusion problem, *Journal of Comput. Appl. Math.* **214**(2008), 435–446.

[3] Dehghan, M. and Shakeri, F., Numerical solution of a biological population model using He's variational iterative method, *Comput. Math. Appl.* **54**(2007), 1197–1209.

[4] Dehghan, M. and Tatari, M., Identifying an unknown function in a parabolic equation with overspecified data via He's variational iteration method, *Chaos Solitons Fractals* **36**(2008), 157–166.

[5] Dehghan, M. and Tatari, M., The use of He's variational iterative method for solving the Fokkre-Planck equation, *Physica Scripta* **74**(2006), 310–316.

[6] He, J.H., Approximate analytical solution for seepage flow with fractional derivatives in porous media, *Comput. Methods. Appl. Mech. Eng.* **167**(1998), 57–68.

[7] He, J.H., Approximate solution of nonlinear differential equations with convolution product non-linearities, *Comput. Methods Appl. Mech. Eng* **167**(1998), 69–73.

[8] He, J.H., A generalized variational principle in micromorphic thermoelasticity, *Mech. Res. Commun* **3291**(2005), 93–98.

[9] He, J.H. and Wu, X.H., Variational iteration method: New development and applications, *Computers and Mathematics with Applications* **54**(2007), 881–894.

[10] He, J.H., Some asymptotic methods for strongly nonlinear equations, *Int J Mod Phys B* **20**(2006), 1141–1199.

[11] He, J.H., Variational iteration method a kind of non-linear analytical technique: some examples, *Int J Nonlinear Mech* **34**(1999), 699–708.

[12] He, J.H., Variational iteration method for autonomous ordinary differential systems, *Applied Mathematics and Computation* **114**(2000), 115–123.

[13] He, J.H., Variational iteration method-Some recent results and new interpretations, *Journal of Computational and Applied Mathematics* **207**(2007), 3–17.

[14] Inokuti, M., Sekine, H. and Mura, T., General use of the Lagrange multiplier in non-linear mathematical physics. In: S. Nemat-Nasser, Editor, Variational Method in the Mechanics of Solids, Pergamon Press, Oxford (1978), 156–162.

[15] Momani, S. and Abuasad, S., Application of He's variational iteration method to Helmholtz equation, *Chaos Solitons Fractals* **27**(2006), 1119–1123.

[16] Odibat, Z.M. and Momani, S., Application of variational iteration method to nonlinear differential equations of fractional order, *Int. Journal of Nonlinear Sci. Numer. Simul.* **7**(2006), 27–36.

[17] Ozer, H., Application of the variational iteration method to the boundary value problems with jump discontinuities arising in solid mechanics, *International Journal of Nonlinear Sciences and Numerical Simulation* **8**(2007), 513–518.

[18] Shakeri, F. and Dehghan, M., Numerical solution of the Klein-Gordon equation via He's variational iteration method, *Nonlinear Dynam.* **51**(2008), 89–97.

[19] Tari, H., Ganji, D.D. and Rostamian, M., Approximate solutions of K (2,2), KdV and modified KdV equations by variational iteration method, homotopy perturbation method and homotopy analysis method, *International Journal of Nonlinear Sciences and Numerical Simulation* **8**(2007), 203–210.

[20] Tatari, M. and Dehghan, M., On the convergence of He's variational iteration method, *Journal of Comput. Appl. Math.* **207**(2007), 121–128.

[21] Yusufoglu, E., Variational iteration method for construction of some compact and noncompact structures of Klein-Gordon equations, *International Journal of Nonlinear Sciences and Numerical Simulation* **8**(2007), 153–158.

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Asymptotic normality of the truncation probability estimator for truncated dependent data*

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Abstract

In some long term studies, a series of dependent and possibly truncated lifetimes may be observed. Suppose that the lifetimes have a common marginal distribution function. In left-truncation model, one observes data (X_i, T_i) only, when $T_i \leq X_i$. Under some regularity conditions, we provide a strong representation of the $\hat{\beta}_n$ estimator of $\beta = P(T_i \leq X_i)$, in the form of an average of random variables plus a remainder term. This representation enables us to obtain the asymptotic normality for $\hat{\beta}_n$.

Keywords and phrases: α-mixing, left-truncation, product-limit estimator, strong representation, truncation probability.
AMS Subject Classification 2000: Primary 12J15, 26A03; Secondary 26E30.

1 Introduction

In medical follow-up or in engineering life testing studies, one may not be able to observe the variable of interest, referred to hereafter as the lifetime. Among the different forms in which incomplete data appears, right censoring and left-truncation are two common ones. Left truncation may occur if the time origin of

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the lifetime precedes the time origin of the study. Only the subjects which are failed after the start of the study are followed, otherwise they are left truncated. Woodroofe [9] reviews examples from astronomy and economy where such data may occur.

Let $X_1, X_2, \ldots$ be a sequence of the lifetime variables which may not be mutually independent, but have a common continuous marginal distribution function F. Let $T_1, T_2, \ldots$ be a sequence of independent and identically distributed random variables with continuous distribution function G. They are also assumed to be independent of the random variables X_i's. In the left-truncation model, (X_i, T_i) is observed when $T_i \le X_i$. Let $(X_1, T_1), \ldots, (X_n, T_n)$ be only the sample one observes (i.e., $T_i \le X_i$), and $\beta > 0$, where

$$\beta = P(T_1 \le X_1) = \int_{-\infty}^{\infty} G(s)dF(s), \tag{1}$$

is the truncation probability (TP).

Assume, without loss of generality, that X_i and $T_i, i = 1, \ldots, n$, are non-negative random variables. For any distribution function H, we denote the left and right endpoints of its support by $a_H = \inf\{z : H(z) > 0\}$ and $b_H = \sup\{z : H(z) < 1\}$, respectively. Then under the current model, as discussed by Woodroofe [9], we assume that $a_G \le a_F$ and $b_G \le b_F$. Equation (1) suggests estimating β by

$$\beta_n = \int_{-\infty}^{\infty} G_n(s)dF_n(s), \tag{2}$$

provided good estimates F_n and G_n for F and G can be obtained.

For the case in which the lifetime observations are mutually independent, Woodroofe [9] proved that if F_n and G_n are product-limit estimates (given by (4) below), β_n converges in probability to β as $n \to \infty$. Under similar conditions as in Woodroofe [9], the asymptotic normality of $\sqrt{n}(\beta_n - \beta)$ has been investigated by several authors using different methods. Chao [7] used influence curves and Keiding and Gill [6] used finite Markov processes and the well-known delta method. Since F_n and G_n have complicated product-limit forms, the properties of β_n is generally not easy to study. Let $I(A)$ denotes the indicator function of the event A. He and Yang [4] proposed, instead, another estimate of β as

$$\widehat{\beta}_n = \frac{G_n(x)(1 - F_n(x))}{C_n(x)},$$

for all x for which $C_n(x) > 0$, where

$$C_n(x) = n^{-1} \sum_{i=1}^{n} I(T_i \leq x \leq X_i),$$

is the empirical distribution of

$$C(x) = P(T_1 \leq x \leq X_1 | T_1 \leq X_1) = \beta^{-1}(1 - F(x))G(x).$$

Using $\widehat{\beta}_n$, He and Yang [4] proved the almost sure convergence of the estimate of β and obtained a manageable i.i.d. representation for $\widehat{\beta}_n$, hence the asymptotic normality of the estimate.

Our basic aim in this article is to express the TP estimator $\widehat{\beta}_n$ as an average of a sequence of bounded random variables plus a remainder of order $O(n^{-1/2}(\log n)^{-\delta})$ for some $\delta > 0$, for the case in which the underlying lifetimes are assumed to be α-mixing whose definition is given below. As a result, the asymptotic normality of TP estimator is obtained.

Let $\mathcal{F}_i^k$ denote the σ-field of events generated by $\{Y_j; i \leq j \leq k\}$. For easy reference, let us recall the following definition.

Definition. Let $\{Y_i, i \geq 1\}$ denote a sequence of random variables. Given a positive integer n, set

$$\alpha(n) = \sup_{k \geq 1}\{|P(A \cap B) - P(A)P(B)| : A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+n}^{\infty}\}. \tag{3}$$

The sequence is said to be α-mixing (strongly mixing) if the mixing coefficient $\alpha(n) \to 0$ as $n \to \infty$. Among various mixing conditions used in the literature, α-mixing is reasonably weak and has many practical applications (see, e.g. [1] for more details). In particular, the stationary autoregressive-moving average (ARMA) processes, which are widely applied in time series analysis, are α-mixing with exponential mixing coefficient, i.e., $\alpha(n) = e^{-\nu n}$, for some $\nu > 0$.

The rest of the present paper is organized as follows. In Section 2, we provide the strong representation results for the TP estimator. The proofs are given in Section 3.

2 Strong representation for the TP estimator

We first introduce some notation before stating the strong representation result. The random truncation model is defined by the joint distribution

$$H(x,t)=P(X_1\le x, T_1\le t|T_1\le X_1)$$

with marginal distributions,

$$F^*(x)=P(X_1\le x|T_1\le X_1)=\beta^{-1}\int_0^x G(s)dF(s),$$

and

$$G^*(x)=P(T_1\le x|T_1\le X_1)=\beta^{-1}\int_0^x (1-F(s))dG(s).$$

Let F_n^* and G_n^* be the empirical distributions of F^* and G^* defined by $F_n^*(x)=n^{-1}\sum_{i=1}^n I(X_i\le x)$ and $G_n^*(x)=n^{-1}\sum_{i=1}^n I(T_i\le x)$.
The well-known product-limit (PL) estimates of F_n and G_n are defined by

$$\widehat{F}_n(x)=1-\prod_{i:X_i\le x}(1-\frac{1}{nC_n(X_i)}),\qquad \widehat{G}_n(x)=\prod_{i:T_i>x}(1-\frac{1}{nC_n(T_i)}). \quad (4)$$

For construction of these estimates, see [9] or [7]. Suppose

$$\int\frac{dF^*(s)}{C^2(s)}<\infty \qquad \text{and} \qquad \int\frac{dG^*(s)}{C^2(s)}<\infty. \quad (5)$$

Let

$$\psi_1(x,t,y)=\frac{I(x\le y)}{C(x)}-\int_0^y\frac{I(t\le s\le x)}{C^2(s)}dF^*(s),$$

and

$$\psi_2(x,t,y)=\frac{I(t>y)}{C(t)}-\int_y^\infty\frac{I(t\le s\le x)}{C^2(s)}dG^*(s).$$

Then, $E\psi_1(X_i,T_i,y)=E\psi_2(X_i,T_i,y)=0$, and

$$Cov(\psi_1(X_i,T_i,y_1),\psi_1(X_i,T_i,y_2))=\int_{a_{F^*}}^{y_1\wedge y_2}\frac{dF^*(s)}{C^2(s)},$$

and

$$Cov(\psi_2(X_i, T_i, y_1), \psi_2(X_i, T_i, y_2)) = \int_{y_1 \wedge y_2}^{b_{G^*}} \frac{dG^*(s)}{C^2(s)}.$$

The following theorem provides the strong representation for $\widehat{\beta}_n$.

Theorem 2.1. Suppose that $\{X_n; n \geq 1\}$ is a sequence of stationary α-mixing random variables with $\alpha(n) = O(n^{-\upsilon})$, for some $\upsilon > 3$. If $a_G < a_F$, then

$$\widehat{\beta}_n - \beta = -\beta \frac{1}{n} \sum_{i=1}^{n} \psi(X_i, T_i) + R_n(y), \tag{6}$$

is uniformly in $0 \leq y \leq b < b_F$, where

$$\sup_{0 \leq y \leq b} |R_n(y)| = O(n^{-1/2} (\log n)^{-\delta}) \qquad a.s.$$

for some $\delta > 0$ depending only on υ. We next present the asymptotic normality of the TP estimator based on our strong representation result.

Theorem 2.2. Under the assumptions of Theorem 2.1, if $a_G < a_F$, then for $0 \leq y \leq b < b_F$,

$$\sqrt{n}(\widehat{\beta}_n - \beta) \xrightarrow{\mathcal{L}} N(0, \sigma^2), \tag{7}$$

where

$$\sigma^2 = \beta^2 \{Var(\psi(X_1, T_1, y)) + 2 \sum_{i=2}^{\infty} Cov(\psi(X_1, T_1, y), \psi(X_i, T_i, y))\}.$$

3 Proofs

In order to prove Theorem 2.1, we need the following lemma which is Theorem 2.1 in Sun and Zhou [8]. Note that the proof of (9) is similar to that of (8) and is therefore omitted.

Lemma 3.1. Suppose that $\{X_n; n \geq 1\}$ is a sequence of α-mixing random variables with $\alpha(n) = O(n^{-\upsilon})$, for some $\upsilon > 3$. If $a_G < a_F$, then

$$\widehat{F}_n(y) = F(y) + (1 - F(y)) \frac{1}{n} \sum_{i=1}^{n} \psi_1(X_i, T_i, y) + R_{n1}(y) \quad a.s. \tag{8}$$

and

$$\widehat{G}_n(y) = G(y) - G(y)\frac{1}{n}\sum_{i=1}^{n}\psi_2(X_i, T_i, y) + R_{n2}(y) \quad a.s., \tag{9}$$

uniformly in $0 \leq y \leq b < b_F$, where

$$\sup_{0\leq y\leq b}|R_{n1}(y)| = O(n^{-1/2}(\log n)^{-\delta}) \qquad a.s.$$

and

$$\sup_{0\leq y\leq b}|R_{n2}(y)| = O(n^{-1/2}(\log n)^{-\delta}) \qquad a.s.,$$

for some $\delta > 0$ depending only on υ.

Proof of Theorem 2.1. Using Lemma 3.1, for $0 \leq y \leq b < b_F$, with probability 1 for large n, we have

$$\begin{aligned}
\widehat{\beta}_n - \beta &= \frac{G_n(y)(1-F_n(y))}{C_n(y)} - \frac{G(y)(1-F(y))}{C(y)} \\
&= \frac{(1-F(y))C(y)G(y)}{C_n(y)C(y)}\{-\frac{1}{n}\sum_{i=1}^{n}\psi_1(X_i,T_i,y) - \frac{1}{n}\sum_{i=1}^{n}\psi_2(X_i,T_i,y) \\
&- \frac{1}{nC(y)}\sum_{i=1}^{n}[I(T_i \leq y \leq X_i) - C(y)]\} + O(n^{-1/2}(\log n)^{-\delta}) \\
&= -\beta\frac{1}{n}\sum_{i=1}^{n}\psi(X_i,T_i,y) + O(n^{-1/2}(\log n)^{-\delta}) \quad a.s.,
\end{aligned}$$

where

$$\begin{aligned}
\psi(X_i,T_i,y) &= \psi_1(X_i,T_i,y) + \psi_2(X_i,T_i,y) + \frac{1}{C(y)}[I(T_i \leq y \leq X_i) - C(y)] \\
&= \frac{1}{C(X_i)} - \int_0^{b_{F^*}}\frac{I(T_i \leq s \leq X_i)}{C^2(s)}dF^*(s) - 1 \qquad a.s.
\end{aligned}$$

It is easy to see from Lemma 1 of Cai [1] that $\{\psi(X_i,T_i,y); T_i \leq X_i,\ i = 1,2,\ldots\}$ is a sequence of stationary α-mixing bounded random variables. The random variable $\psi(X_i,T_i,y)$ does not depend on y, therefore, the proof of Theorem 2.1 is complete.

Proof of Theorem 2.2. An application of Theorem 18.5.4 of Ibragimov and Linnik Yu [5] and Theorem 2.1 gives (2.4). It can be easily checked that

$$Var(\psi(X_1,T_1,y) = \int_{a_{F^*}}^{x}\frac{dF^*(s)}{C^2(s)} + \int_{x}^{b_{G^*}}\frac{dG^*(s)}{C^2(s)} - \frac{1}{C(s)} + 2\alpha - 1,$$

which is finite under (5). On the other hand, $\alpha(n) = O(n^{-\nu}), \nu > 3$ implies $\sum \alpha(n) < \infty$ and therefore $\sum_{i=2}^{\infty} Cov(\psi(X_1, T_1, y), \psi(X_i, T_i, y)) < \infty$. So, σ^2 is a positive finite number and the proof of Theorem 2.2 is complete.

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References

[1] Cai, Z., Kernel density and hazard rate estimation for censored dependent data, *J. Multivariate. Anal.* **67**(1998a), 23–34.

[2] Cai, Z., Asymptotic properties of Kaplan–Meier estimator for censored dependent data, *Statist. Probab. Lett.* **37**(1998b), 381–389.

[3] Chao, M.T., Influence curves for randomly truncated data, *Biometrika.* **74**(1987), 426–429.

[4] He, S. and Yang, G., Estimation of the truncation probability in the random truncation model, *Ann. Statist.* **26**(1998), 1011–1027.

[5] Ibragimov, I.A. and Linnik Yu, V., Independent and Stationary Sequences of Random Variables, Walters-Noordhoff, Groningen, The Netherlands, 1971.

[6] Keiding, N. and Gill, R.D., Random truncation models and Markov processes *Ann. Statist.* **18**(1990), 582–602.

[7] Wang, M.C., Jewell, N.P. and Tsai, W.Y., Asymptotic properties of the product limit estimate under random truncation, *Ann. Statist.* **14**(1986), 1597–1605.

[8] Sun, L. and Zhou, X., Survival function and density estimation for truncated dependent data, *Statist. Probab. Lett.* **52**(2001), 47–57.

[9] Woodroofe, M., Estimating a distribution function with truncated data, *Ann. Statist.* **13**(1985), 163–177.

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Asymptotic fisher information in order statistics of geometric distribution*

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Abstract

In this paper, the geometric distribution is considered. The means, variances, and covariances of its order statistics are derived. The Fisher information in any set of order statistics in any distribution can be represented as a sum of Fisher information in at most two order statistics. It is shown that, for the geometric distribution, it can be further simplified to a sum of Fisher information in a single order statistic. Then, we derived the asymptotic Fisher information in any set of order statistics.

Keywords and phrases: Fisher information, geometric distribution, order statistics, percentile, Quantile.
AMS Subject Classification 2000: Primary 94A15; Secondary 41A60.

1 Introduction

The geometric distribution with parameter θ is given by the probability mass function (pmf)

$$f(x;\theta) = (1-\theta)\theta^x,\ x = 0, 1, 2, \ldots\ ,\ 0 < \theta < 1. \tag{1}$$

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In general, the distribution theory for order statistics is complex when the parent distribution is discrete. However, order statistics from a geometric distribution exhibit some interesting properties. The geometric distribution possesses several properties (like lack of memory) of exponential distribution. Due to the relationship between the geometric and the exponential distributions, there also exists a close relationship between the dependence structure of order statistics from a geometric distribution and those from the exponential distribution. To this end, we may first note that when Y has an exponential distribution, i.e., it's probability density function is given by

$$f(y;\lambda) = \frac{1}{\lambda}e^{-\frac{1}{\lambda}y}, \ y > 0, \ \lambda > 0, \tag{2}$$

then $X = [Y]$, when $[.]$ stands for integer part, is distributed as geometric with parameter $\theta = 1 - e^{-1/\lambda}$. Also, the geometric distribution is the only discrete distribution for which the first order statistic and the sample range are independent [2].

The Fisher information plays an important role in statistical inference in connection with estimation, sufficiency and properties of variance of estimators. It is well known that Fisher information serves as a valuable tool for derivation of variance in the asymptotic distribution of maximum likelihood estimators (MLE). For a random variable X, discrete or continuous, which pmf or pdf is $f(x;\theta)$, where $\theta \in \Theta$ is a real value and Θ is the space parameter, the exact Fisher information contained in X is defined as

$$I_X(\theta) = E\Big(\frac{\partial \log f(x;\theta)}{\partial\theta}\Big)^2 = -E\Big(\frac{\partial^2 \log f(x;\theta)}{\partial^2\theta}\Big)\ , \tag{3}$$

under certain regularity condition (see [7]). Let $X_i, \ i = 1, ..., n$ be a sample from F_θ, the exact Fisher information about θ in any k order statistics, $X_{r_1:n} \leq X_{r_2:n} \leq ...X_{r_k:n}, \ 1 \leq r_1 < r_2 < ... < r_k \leq n$, is defined as

$$I_{r_1 r_2 \ldots r_k:n}(\theta) = E\Big\{\frac{\partial}{\partial\theta}\log f_{r_1 r_2 \ldots r_k:n}(\theta)\Big\}^2, \tag{4}$$

where $f_{r_1 r_2 \ldots r_k:n}(\theta)$ is joint pmf or pdf of $(X_{r_1:n}, X_{r_2:n}, ...X_{r_k:n})$. The problem of obtaining the Fisher information in order statistics was described in [2] with the words: "while the recipe for $I_Y(\theta)$is simple, the details are messy in most cases" where Y is an arbitary collection of order statistics. Several results have been published in this direction in recent years. For example, Mehtoria et al. [8] presented the Fisher information in the first r order statistics. Park [9] used an indirect approach to obtain the Fisher information in r order statistics, and presented very information plots to demonstrate which order statistics have more Fisher information. Zheng and Gastwirth [14] calculated the Fisher information contained in any collection of order statistics. Abo-Eleneen and Nagaraja [1] studied the Fisher information in collections of order statistics and their concomitants from bivariate samples. Park and Zheng [12] derived a necessary and sufficient condition under which two distribution have equal Fisher information in any set of order statistics. Hofman et al. [6] used the Fisher information in minima and upper record values for characterization of hazard function. Park [10] considered the optimal spacing based on the Fisher information. Park and Kim [11] considered the Fisher information in exponential distribution and simplified the Fisher information in any set of order statistics to a sum of single integrals. In other application, such as life testing surveys (see [3]) and optimal spacing (see [4] and [10]), the asymptotic Fisher information is used.

The rest of the paper is organized as follows. In Section 2, the means, variances, and covariances of geometric order statistics are derived. We derived the asymptotic Fisher information about θ contained in the rth sample quantile ($X_{r:n}$) of geometric distribution in Section 3. In Section 4, we provide the simple method for obtaining the Fisher information and asymptotic Fisher information in any set of order statistics of geometric distribution.

2 Calculating means, variances, and covariances

Since the Fisher information is related to the variance-covariance matrix of the estimate of ϑ, being its inverse under certain conditions, we derive variances, and covariances of order statistics come from a geometric population.

Let $X_1, ..., X_n$ be a sample from (1) and denote the corresponding order statistics by $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$.

Lemma 2.1 *Let* $\mu_{r:n}$, *and* $\sigma^2_{r:n}$, *and* $\sigma_{r,s:n}$ *be mean of* $X_{r:n}$, *and variance of* $X_{r:n}$, *and covariance of* $X_{r:n}$ *and* $X_{s:n}$, *respectively. Then, we have, for* $1 \leq r < s \leq n$

$$\mu_{r:n} = \sum_{j=0}^{r} \frac{\theta^{n-j+1}}{1-\theta^{n-j+1}}, \tag{5}$$

$$\sigma^2_{r:n} = \sum_{j=0}^{r} \frac{\theta^{n-j+1}}{(1-\theta^{n-j+1})^2}, \tag{6}$$

and

$$\sigma_{r,s:n} = \sigma^2_{r:n} \ . \tag{7}$$

Proof. Under the transformation $Z_i = (n-i+1)(X_{i:n} - X_{i-1:n})$ for $i = 1, 2, ..., n$, one can see that the variables $Z_1, Z_2, ..., Z_n$ are independent random variables and pmf each of Z_i is given by (see [2])

$$f_{z_i}(z; \theta) = (1-\theta^{n-i+1})\theta^z, \ z = 0, \ n-i+1, \ 2(n-i+1), ... \ . \tag{8}$$

The equivalent transformation can be written as

$$X_{r:n} = \sum_{j=0}^{r} \frac{Z_j}{n-j+1}. \tag{9}$$

From (8) and (9) we immediately, conclude that

$$\mu_{r:n} = \sum_{j=0}^{r} \frac{\theta^{n-j+1}}{1-\theta^{n-j+1}},$$

$$\sigma^2_{r:n} = \sum_{j=0}^{r} \frac{\theta^{n-j+1}}{(1-\theta^{n-j+1})^2},$$

and

$$\sigma_{r,s:n} = \sum_{j=0}^{r} \frac{\theta^{n-j+1}}{(1-\theta^{n-j+1})^2}.$$

We may similarly derive the higher-order moments of $X_{r:n}$, if needed.

3 Asymptotic Fisher information in the rth order statistic

Definition ([13]) Assume $\frac{r_i}{n} \to p_i$ (for i=1, 2, ..., k) as $n \to \infty$, where $0 \leq p_1 < p_2 < ... < p_k \leq 1$. The asymptotic Fisher information about θ contained in k sample quantiles $(X_{r_1:n}, X_{r_2:n}, ...X_{r_k:n})$, denoted by $I_{p_1 p_2 ... p_k}(\theta)$, is defined as

$$I_{p_1 p_2 ... p_k}(\theta) = \lim_{n\to\infty} \frac{1}{n} I_{r_1 r_2 ... r_k : n}(\theta), \tag{10}$$

which can be written as ([13])

$$I_{p_1 p_2 ... p_k}(\vartheta) = \sum_{i=0}^{k} \frac{1}{p_{i+1} - p_i} \left\{ \int_{\xi_{p_i}}^{\xi_{p_{i+1}}} \frac{\partial}{\partial \vartheta} f(x;\vartheta) dx \right\}^2 , \tag{11}$$

where $p_0 = 0, p_{k+1} = 1$, and $\xi_p = F^{-1}(p;\theta)$.

The asymptotic Fisher information in a single order statistic can be obtain rapidly by substituting $k = 1$ in (11). Thus, we get

$$I_p(\vartheta) = \frac{1}{p(1-p)} \left\{ \int_{-\infty}^{\xi_p} \frac{\partial}{\partial \vartheta} f(x;\vartheta) \right\}^2 . \tag{12}$$

In what follows, we find the asymptotic Fisher information in a single quantile of geometric distribution. From (1), we have $F(x;\theta) = (1 - \theta^{x+1})$ so $F^{-1}(p;\theta) = (\frac{\log(1-p)}{\log\theta} - 1)$. By (12), $I_p(\theta)$ for geometric distribution can be calculated as follows

$$\begin{aligned} I_p(\theta) &= \frac{1}{p(1-p)} \left\{ \sum_{x=0}^{\xi_p} \frac{\partial}{\partial \theta} f(x;\theta) \right\}^2 \\ &= \frac{1}{p(1-p)} \left\{ \sum_{x=0}^{\xi_p} x\theta^{x-1} - (x+1)\theta^x \right\}^2 \\ &= \frac{1}{p(1-p)} \left\{ \left([\frac{\log(1-p)}{\log\theta} - 1] + 1 \right)^2 \theta^{2[\frac{\log(1-p)}{\log\theta} - 1]} \right\} , \end{aligned} \tag{13}$$

where $[\cdot]$, denotes the integer part.

Remark 3.1 *By using (13) for geometric distribution, we can approximate the Fisher information contained in $X_{r:n}$ about θ by nothing that $I_{r:n}(\theta) \simeq nI_p(\theta)$ for large values of sample size and $r \leq n$ as follows*

$$I_{r:n}(\theta) \simeq \frac{n^3}{r(n-r)} \left\{ \left([\frac{\log(n-r)-\log(r)}{\log\theta} - 1] + 1 \right)^2 \theta^{2[\frac{\log(n-r)-\log(r)}{\log\theta} - 1]} \right\}. \quad (14)$$

4 Asymptotic Fisher information in k order statistics

Park [10] has shown that the Fisher information in any set of order statistics can be written as

$$I_{r_1 r_2 \ldots r_k:n}(\theta) = \sum_{i=2}^{k} I_{r_{i-1} r_i:n}(\theta) - \sum_{i=2}^{k-1} I_{r_i:n}(\theta), \quad (15)$$

where $0 \leq r_1 < r_2 < \ldots < r_k \leq n$. We will show that it can be further simplified to a sum of Fisher information in a single order statistics while the parent distribution is geometric.

Theorem 4.1 *If the random sample comes from a geometric population, then*

$$I_{r_1 r_2 \ldots r_k:n}(\theta) = \sum_{i=1}^{k} I_{r_i - r_{i-1}:n-r_{i-1}}(\theta), \quad (16)$$

where $r_0 = 0$.

Proof. The proof follows by using the lack of memory property of the geometric distribution. As it has been shown in [2], $X_{r_i:n} - X_{r_{i-1}:n}$ is distributed as $X_{r_i - r_{i-1}:n-r_{i-1}}$ in geometric distribution, $(X_{r_1:n}, X_{r_2:n}, ..., X_{r_k:n})$ and $(X_{r_1:n}, X_{r_2:n} - X_{r_1:n}, ..., X_{r_k:n} - X_{r_{k-1}:n})$ are equivalent statistics and $(X_{r_1:n}, X_{r_2:n} - X_{r_1:n}, ..., X_{r_k:n} - X_{r_{k-1}:n})$ are independently and geometrically distributed, therefore the proof is completed.

Theorem 4.2 *If the random sample has geometric distribution, then*

$$I_{p_1 p_2 \ldots p_k}(\theta) = \sum_{i=0}^{k} \frac{1-p_{i-1}}{(1-p_i)(p_i - p_{i-1})} \left\{ \left(\left[\frac{\log(\frac{1-p_i}{1-p_{i-1}})}{\log\theta} - 1 \right] + 1 \right)^2 \theta^{2[\frac{\log(\frac{1-p_i}{1-p_{i-1}})}{\log\theta} - 1]} \right\}. \quad (17)$$

Proof. By using (13), the asymptotic Fisher information of $I_{r_i - r_{i-1}:n-r_{i-1}}(\theta)$ can be written as $\frac{1}{1-p_{i-1}} I_{\frac{p_i - p_{i-1}}{1-p_{i-1}}}(\theta)$. Thus, the the proof is completed by considering Theorem 4.1.

Remark 4.1 *By using (17) for geometric distribution, we can approximate the Fisher information contained in* $(X_{r_1:n}, X_{r_2:n}, ... X_{r_k:n})$ *about* θ *by nothing that* $I_{r_1 r_2 ... r_k:n}(\theta) \simeq n I_{p_1 p_2 ... p_k}(\theta)$ *for large values of sample size and* r_i *(for* $i = 1, ..., k$*) as follows*

$$I_{r_1 r_2 ... r_k:n}(\theta) \simeq \sum_{i=0}^{k} \frac{n^2(n - r_{i-1})}{(n - r_i)(n - r_{i-1})} \left\{ \left(\left[\frac{\log(\frac{n-r_i}{n-r_{i-1}})}{\log\theta} - 1 \right] + 1 \right)^2 \theta^{2\left[\frac{\log(\frac{n-r_i}{n-r_{i-1}})}{\log\theta} - 1 \right]} \right\}, \tag{18}$$

where $0 \leq r_1 < r_2 < ... < r_k \leq n$*.*

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References

[1] Abo-Eleneen, Z.A. and Nagaraja, H.N., Fisher information in order statistics and its concomitant, *Ann. Inst. Statist. Math.* **54**(2002), 667–680.

[2] Arnold, B.C., Balakrishnan, N. and Nagaraga, H.N., *A First Course in Order Statistics*, John Wiley, New York, 1992.

[3] Battacharyya, G.K., The asymptotic of maximum likelihood and related estimators based on Type II censored data, *J. Amer. Statist. Assoc.* **80**(1985), 398–404.

[4] Cheng, S.W., A unified approach to choosing optimum quantiles for ABLE's. *J. Amer. Statist. Assoc.* **70**(1975), 155–159.

[5] David, H.A. and Nagaraja, H.N., *Order Statistics*, 3^{rd} Ed. John Wiley, New York, 2003.

[6] Hofmann, G., Balakrishnan, N. and Ahmadi, J., Characterization of hazard function factorization by Fisher information in minima and upper record values, *Statist. Probab. Lett.* **72**(2005), 51–57.

[7] Lehmann, E.L., Theory of Point Estimation, Second Edition, John Wiley, New York, 1998.

[8] Mehtoria, K.G., Johnson, R.A. and Battacharyya, G.K., Exact Fisher information for censored samples and the extended hazard rate function, *Comm. Statist. - Theory Methods* **15**(1979), 1493–1510.

[9] Park, S., Fisher information in order statistics, *J. Amer. Statist. Assoc.* **91**(1996), 385–390.

[10] Park, S., On simple calculation of the Fisher information in order statistics, *Statist. Papers* **46**(2005), 293–301.

[11] Park, S. and Kim, C.E., A note on the Fisher information in exponential distribution, *Comm. Statist. -Theory Methods* **35**(2006), 13–19.

[12] Park. S. and Zheng, G., Equal Fisher information in order statistics, *Sankhya Statist.* **66**(2004), 20–34.

[13] Zheng, G., On the rate of convergence of Fisher information in multiple type II censored data, *J. Japan Statist. Soc.* **30**(2000), 197–204.

[14] Zheng, G. and Gastwirth, J.L., Where is the Fisher information in the ordered sample?, *Statist. Sinica* **10**(2000), 1267–1280.

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Estimation of $P[Y < X]$ for generalized exponential distribution in presence of outlier*

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Abstract

This paper deals with the estimation of $P(Y < X)$, where Y has generalized exponential distribution with parameters α and λ and X has mixture generalized exponential distribution (or marginal distribution of $X_1, X_2, ..., X_n$, in presence of one outlier with parameters β_1 and β_2) such that X and Y are independent. when the scale parameter (λ) is known the maximum likelihood estimator of $R = P(Y < X)$ is derived. Analysis of a simulated data set has also been presented for illustrative purposes.

Keywords and phrases: Maximum likelihood estimator, outlier, stress-strength model.

AMS Subject Classification 2000: Primary 62G05; Secondary 62E10.

1 Introduction

Recently the two-parameter generalized exponential (GE) distribution has been proposed by many authors. It has been studied extensively by Gupta and Kundu ([11]–[17]), Raqab [26], Raqab and Ahsanullah [27], Zheng [34] and Kundu *et*

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al. [21]. Note that the generalized exponential distribution is a submodel of the exponentiated weibull distribution introduced by Mudholkar and Srivastava [22] and later studied by Mudholkar *et al.* [24] and Mudholkar and Huston [23].

The two-parameter GE distribution has the following density function

$$f(x,\alpha,\lambda) = \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}, \qquad x>0. \tag{1}$$

and the distribution function

$$F(x,\alpha,\lambda) = (1-e^{-\lambda x})^{\alpha}, \qquad x>0 \tag{2}$$

Here $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively. For different values of the shape parameter, the density function can take different shape. For detail description of the distribution, one is referred to the original paper of Gupta and Kundu [11]. From now one GE distribution with the shape parameter α and scale parameter λ will be denoted by $GE(\alpha,\lambda)$.

Let the random variables $X_1, X_2, ..., X_{n-1}$ are independent, each having the probability density function f(x),

$$f_1(x,\beta_2) = \beta_2 e^{-x}(1-e^{-x})^{\beta_2-1}, \qquad x>0. \tag{3}$$

and the one random variable (As outlier) is also independent, has the probability density function $g(x)$.

$$f_2(x,\beta_1) = \beta_1 e^{-x}(1-e^{-x})^{\beta_1-1}, \qquad x>0 \tag{4}$$

The joint density of $X_1, X_2, ..., X_n$ is given as

$$\begin{aligned} f(x_1,x_2,...,x_n) &= \frac{1}{n}\prod_{i=1}^{n} f(x_i).\sum_{A_1}^{n}\frac{f_2(x_{A_1})}{f_1(x_{A_1})} \\ &= \frac{1}{n}{\beta_2}^n e^{-\sum_{i=1}^n x_i}\prod_{i=1}^{n}(1-e^{-x_i})^{\beta_2-1}\sum_{A=1}^{n}\frac{\beta_1 e^{-x_{A_1}}(1-e^{-x_{A_1}})^{\beta_1-1}}{\beta_2 e^{-x_{A_1}}(1-e^{-x_{A_1}})^{\beta_2-1}} \\ &= \frac{1}{n}\beta_1{\beta_2}^{n-1}e^{-\sum_{i=1}^n x_i}\prod_{i=1}^{n}(1-e^{-x_i})^{\beta_2-1}\sum_{A_1=1}^{n}(1-e^{-x_{A_1}})^{\beta_1-\beta_2} \\ ,x>0 \end{aligned} \tag{5}$$

(see [7] and [8])

From equation (5), the marginal distribution of X is,

$$h(x, \beta_1, \beta_2) = \frac{1}{n}\beta_1 e^{-x}(1-e^{-x})^{\beta_1-1} + \frac{n-1}{n}\beta_2 e^{-x}(1-e^{-x})^{\beta_2-1}, \qquad x > 0 \quad (6)$$

The main purpose of this paper is to focus on the inference of $R = P(Y < X)$, where $Y \sim GE(\alpha, \lambda)$ and X has mixture GE or marginal distribution of $X_1, X_2, ..., X_n$ with presence of one outlier. For simplify we consider $\lambda = 1$. The estimation of R is very common in the statistical literature. For example, if X is the strength of a component which is subject to a stress Y, when R is a measure of system performance and arises in the context of mechanical reliability of a system. We obtain the maximum likelihood estimator (MLE) of R. It may be mentioned here that related problems have been widely used in the statistical literature. The MLE of $P(Y < X)$, when X and Y have bivariate exponential distribution, has been considered by Awad *et al.* [2]. Church and Harris [4], Downtown [6], Govidarajulu [9], Woodward and Kelley [33] and Owen, Craswell and Hanson [25] considered the estimation of $P(Y < X)$, when X and Y are normally distributed. Similar problem for the multivariate normal distribution has been considered by Gupta and Gupta [10]. Kelley, Kelley and Schucany [18], Sathe and Shah [29], Tong [31], [32] considered the estimation of $P(Y < X)$ when X and Y are independent exponential random variables. Constantine and Karson [5] considered the estimation of $P(Y < X)$, when X and Y are independent gamma random variables. Sathe and Dixit [8] have been estimate of $P(Y < X)$ in the negative binomial distribution. Ahmad *et al.* [1] and Surles and Padgett [30] considered the estimation of $P(Y < X)$, where X and Y are Burr Type random variables. Baklizi and Dayyeh [3] have done shrinkage estimation of $P(Y < X)$ in exponential case.

The rest of the paper is organized as follows. In section 2, we derive the MLE of R. Analysis of a real life data set has been presented in section 3 and finally we draw conclusion in section 4.

2 Maximum likelihood estimator of R

Let $Y_1, Y_2, ..., Y_m$ be a random sample for Y with pdf

$$g(y,\alpha) = \alpha e^{-y}(1-e^{-y})^{\alpha-1}, \qquad y > 0 \tag{7}$$

and $X_1, X_2, ..., X_n$ be random sample for X with pdf

$$f(x,\beta_1,\beta_2) = \frac{1}{n}\beta_1 e^{-x}(1-e^{-x})^{\beta_1-1} + \frac{n-1}{n}\beta_2 e^{-x}(1-e^{-x})^{\beta_2-1}, \qquad x > 0 \tag{8}$$

Then

$$\begin{aligned}
R &= P(Y < X)\\
&= \int_0^\infty \int_0^x g(y,\alpha) f(x,\beta_1,\beta_2,)dydx\\
&= \int_0^\infty [\int_0^x \alpha e^{-y}(1-e^{-y})^{\alpha-1}dy] \times\\
&\quad [\frac{1}{n}\beta_1(1-e^{-x})^{\beta_1-1}\frac{n-1}{n}\beta_2 e^{-x}(1-e^{-x})^{\beta_2-1}]dx\\
&= \int_0^\infty \int_0^x [\frac{1}{n}\beta_1 e^{-x}(1-e^{-x})^{\beta_1-1}][\alpha e^{-y}(1-e^{-y})^{\alpha-1}]dydx\\
&+ \int_0^\infty \int_0^x [\frac{n-1}{n}\beta_2 e^{-x}(1-e^{-x})^{\beta_2-1}][\alpha e^{-y}(1-e^{-y})^{\alpha-1}]dydx\\
&= \int_0^\infty \frac{1}{n}\beta_1 e^{-x}(1-e^{-x})^{\alpha+\beta_1-1}dx\\
&+ \int_0^\infty \frac{n-1}{n}\beta_2 e^{-x}(1-e^{-x})^{\alpha+\beta_2-1}dx\\
&= \frac{1}{n}.\frac{\beta_1}{\alpha+\beta_1} + \frac{n-1}{n}.\frac{\beta_2}{\alpha+\beta_2}
\end{aligned} \tag{9}$$

Therefore, the MLE of R becomes

$$\hat{R} = \frac{1}{n}\frac{\hat{\beta}_1}{\hat{\alpha}+\hat{\beta}_1} + \frac{n-1}{n}\frac{\hat{\beta}_2}{\hat{\alpha}+\hat{\beta}_2}$$

Now, to compute the MLE of R, we first consider the joint distribution of $X_1, X_2, ..., X_n$with presence of one outlier in (5), so

$$\begin{aligned}
L(\alpha,\beta_1,\beta_2) &= g(y_1,y_2,...,y_m).f(x_1,x_2,...,x_n)\\
&= \alpha^m e^{-\sum_{i=1}^m y_i}\pi_{i=1}^m(1-e^{-y_i})^{\alpha-1}\\
&\times \frac{1}{n}\beta_1\beta_2^{n-1}e^{-\sum x_i}\Pi_{i=1}^n(1-e^{-x_i})^{\beta_2-1}\sum_{A_1=1}^n(1-e^{-x_{A_1}})^{\beta_1-\beta_2}.
\end{aligned}$$

The Log-likelihood function of the observed sample is

$$\begin{aligned}\ln L(\alpha,\beta_1,\beta_2) &= m\ln(\alpha) - \sum_{i=1}^{m} y_i + (\alpha-1)\sum_{i=1}^{m}\ln(1-e^{-y_i})\\ &+ \ln[\frac{\beta_1\beta_2^{n-1}}{n} e^{-\sum_{i=1}^{n} x_i}\Pi_{i=1}^{n}(1-e^{-x_i})^{\beta_2-1}\sum_{A_1=1}^{n}(1-e^{-x_i})^{\beta_1-\beta_2}]\end{aligned} \tag{10}$$

The MLE's of α, β_1 and β_2 say $\hat{\alpha}, \hat{\beta}_1$ and $\hat{\beta}_2$, respectively, which is obtained as the solutions of

$$\frac{\partial \ln L}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m}\ln(1-e^{-y_i}) = 0$$

or

$$\frac{m}{\alpha} = -\sum_{i=1}^{m}\ln(1-e^{-y_i})$$

Hence

$$\hat{\alpha} = \frac{-m}{\sum_{i=1}^{m}\ln(1-e^{-y_i})} \tag{11}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \beta_1} &= \frac{1}{\beta_1} + \frac{\frac{\partial}{\partial\beta_1}\sum_{A_1=1}^{n}(1-e^{-x_{A_1}})^{\beta_1-\beta_2}}{\sum_{A_1=1}^{n}(1-e^{-x_{A_1}})^{\beta_1-\beta_2}}\\ &= \frac{1}{\beta_1} + \frac{\sum_{A_1=1}^{n}(1-e^{-x_{A_1}})^{\beta_1-\beta_2} ln(1-e^{-x_{A_1}})^{\beta_1-\beta_2}}{\sum_{A_1=1}^{n}(1-e^{-x_{A_1}})^{\beta_1-\beta_2}} = 0\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \beta_2} &= \frac{n-1}{\beta_2} + \sum_{i=1}^{n}\ln(1-e^{-x_i}) + \frac{\frac{\partial}{\partial\beta_2}\sum_{A_1=1}^{n}(1-e^{-x_{A_1}})^{\beta_1-\beta_2}}{\sum_{A_1=1}^{n}(1-e^{-x_{A_1}})^{\beta_1-\beta_2}}\\ &= \frac{n-1}{\beta_2} + \sum_{i=1}^{n}\ln(1-e^{-x_i}) - \frac{\sum_{A_1=1}^{n}(1-e^{-x_{A_1}})^{\beta_1-\beta_2}\ln(1-e^{-x_{A_1}})^{\beta_1-\beta_2}}{\sum_{A_1=1}^{n}(1-e^{-x_{A_1}})^{\beta_1-\beta_2}}\\ &=\end{aligned} \tag{13}$$

Form (13), (14), $\hat{\beta}_1$ and $\hat{\beta}_2$ can be obtained as the solution of non-linear equations.

For $\beta_1 = \beta_2 = \beta$ in homogenous case, $\hat{\alpha}$ and $\hat{\beta}$ can be obtained as

$$\hat{\alpha} = \frac{-m}{\sum_{i=1}^{m}\ln(1-e^{-y_i})} \qquad \hat{\beta} = \frac{-n}{\sum_{i=1}^{m}\ln(1-e^{-y_i})}. \tag{14}$$

These equations proposed by Kundu and Gupta (2006).

3 Numerical experiments and discussions

In order to have some idea about Bias and Mean Square Error (MSE) of MLE, we perform sampling experiments using Maple.
To have inference about R, we consider the following small sample sizes; (m,n)=

$$(15, 15), (20, 20), (25, 25), (15, 20), (20, 15), (15, 25), (25, 15), (20, 25), (25, 20).$$

Here, we take $\alpha = 1.50$ and $\beta_1 = 2.5$ and $\beta_2 = 2.75$, respectively. As we know, the generated sample size n from $f(x, \beta_1, \beta_2)$, (n-1) sample generated from the equation (3) and one sample generated from the equation (4). All the results are based on 1000 replications. Here we present a complete analysis of a simulated data. The data has been generated using $m = n = 20$, $\alpha = 1.5, \beta_1 = 2.5$ and $\beta_2 = 2.75$.

The data has been truncated after two decimal places and it has been presented below. The Y values are,

0.74	1.41	0.86	0.20	0.72	3.11	0.73	0.44
1.31	0.86	0.27	2.27	0.88	1.32	4.41	1.17
0.86	2.19	0.53	0.08				

and the corresponding X values are,

1.12	5.30	0.65	1.46	1.27	0.74	1.51	0.81
1.79	2.11	1.33	1.50	1.57	1.26	0.49	2.93
0.85	0.85	1.73	1.83				

Now, we obtain the MLE of $\hat{\alpha} = 1.671$, $\hat{\beta}_1 = 0.315$ and $\hat{\beta}_2 = 2.18$. Therefore, $\hat{R} = 0.5457$.

4 Conclusions

In this paper, we have addressed the problem of estimating $P(Y < X)$ for the Generalized Exponential distribution with presence of one outlier, when the scale parameter is know. The results are given in table 1, 2 and 3. It is observed that the maximum likelihood estimator R work quit well. We report the average estimates and the MSEs based on 1000 replications. The results are reported in the following Tables. In this case, as expected when m=n and m, n increase then the average biases and the MSEs decrease . For fixed m as n increases the MSEs decrease and also for fixed n as m increases the MESs decreases.

Table 1

$\alpha = 1.5, \beta_1 = 2.5, \beta_2 = 2.75$

(n,m)	R	$\hat{R}$	Bias $\hat{R}$	MSE $\hat{R}$
(15,15)	0.6455882354	0.3358738948	-0.3097143406	0.1837793490
(20,20)	0.6459558824	0.3335933151	-0.3123625673	0.1882900337
(25,25)	0.6461764706	0.3448594453	-0.3013170253	0.1811570295
(15,20)	0.6455882354	0.3489987431	-0.2965894923	0.1767666919
(20,15)	0.6459558824	0.3433466835	-0.3026091989	0.1804719689
(15,25)	0.6455882354	0.3547599579	-0.2908282775	0.1719679412
(25,15)	0.6461764706	0.3489031863	-0.2972732843	0.1791734199
(20,25)	0.6459558824	0.3373103044	-0.3086455780	0.1854977684
(25,20)	0.6461764706	0.3412694950	-0.3049069756	0.1847561247

Table 2

$\alpha = 1.5, \beta_1 = 2.5, \beta_2 = 2.75$

(n,m)	$\hat{\alpha}$	Bias $\hat{\alpha}$	MSE $\hat{\alpha}$
(15,15)	1.642093877	0.142093877	0.2526986879
(20,20)	1.580778465	0.080778465	0.1554931754
(25,25)	1.566022760	0.066022760	0.1135263151
(15,20)	1.575140968	0.075140968	0.1467095745
(20,15)	1.618209236	0.118209236	0.2182551913
(15,25)	1.563346657	0.063346657	0.1069492224
(25,15)	1.606350563	0.106350563	0.2132163252
(20,25)	1.562139231	0.62139231	0.1176980818
(25,20)	1.581781725	0.081781725	0.1455059822

Table 3

$\alpha = 1.5, \beta_1 = 2.5, \beta_2 = 2.75$

(n,m)	$\hat{\beta}_1$	Bias $\hat{\beta}_1$	MSE $\hat{\beta}_1$
(15,15)	2.257869912	-0.242130088	383.2417437
(20,20)	1.239779251	-1.260220749	14.25371765
(25,25)	3.353135569	0.853135569	1509.645815
(15,20)	6.881500401	4.381500401	19357.50094
(20,15)	1.228829792	-1.271170208	20.41377913
(15,25)	1.634796843	-0.865203157	33.74240776
(25,15)	2.291221924	-0.208778076	628.5131946
(20,25)	14.49625222	11.99625222	86704.72425
(25,20)	4.531374567	2.031374567	5162.227553

Table 4

$\alpha = 1.5, \beta_1 = 2.5, \beta_2 = 2.75$

(n,m)	$\hat{\beta_2}$	Bias $\hat{\beta_2}$	MSE $\hat{\beta_2}$
(15,15)	3.002631681	0.252631681	0.9558988122
(20,20)	2.932442797	0.182442797	0.6090520559
(25,25)	2.891735451	0.141735451	0.4601051183
(15,20)	3.002309839	0.252309839	0.9969076983
(20,15)	2.900321238	0.150321238	0.5981276207
(15,25)	2.969478335	0.219478335	0.8879485964
(25,15)	2.863795246	0.113795246	0.4227174191
(20,25)	2.935928610	0.185928610	0.6328966532
(25,20)	2.871473618	0.121473618	0.4015675626

References

[1] Ahmad, K.E., Fakhry, M.E. and Jaheen, Z.F., Empirical Bayes estimation of $P(Y < X)$ and charaterizations of Burr-type X model, *Journal of Statistical Planning and Inference* **64** (1997), 297-308.

[2] Awad, A.M., Azzam, M.M. and Hamadan, M.A., Some inference results in $P(Y < X)$ bivariate expoential model, *Communications in Statistics-Theroy and Methods* **10**(1981), 2515–2524.

[3] Baklizi, A. and Dayyh, W.A., Shrinkage estimation of $P(Y < X)$ in the exponential case, *Comm. in Statistics, Theory and Methods* **32**(1)(2003), 31–42.

[4] Church, J.D. and Harris, B., The estimation of reliablity from stress strength relationships, *Technometrics* **12**(1970), 49–54.

[5] Constantine, K. and Karson, M., The estimation of $P(Y < X)$ in gamma case, *Comm. in Statistics-Computations and Simulations* **15**(1986), 365–388.

[6] Downtown, F., The estimation of $P(Y < X)$ in normal case, *Technometrics* **15**(1973), 551–558.

[7] Dixit, U.J., Estimation of parameters of the Gamma distribution in the presence of outliers, *Comm. in Statistics, Theory and Methods* **18**(1989), 3071–3085.

[8] Dixit, U.J. and Nasiri, P., Estimation of parameters of the exponential distribution in the presence of outliers generated from uniform distribution, *Metron* **49**(3-4)(2001), 187–198.

[9] Govidarajulu, Z., Two sided confidence limits for $P(X > Y)$ based on normal samples of X and Y, *Sankhya B* **29** (1967), 35-40.

[10] Gupta, R.D. and Gupta, R.C., Estimation of $P(aX > bY)$ in the multivarite normal case, *Statistics* **1**(1990), 91–97.

[11] Gupta, R.D. and Kundu, D., Generalied exponential distributions, *Australian and New zealand Journal of Stastistics* **41**(1999), 173–188.

[12] Gupta, R.D. and Kundu, D., Generalied exponential distributions; Different Method of Esimations, *Journal of Stastistics Computation and Simulation* **69**(2001a), 315–338.

[13] Gupta, R.D. and Kundu, D., Generalied exponential distributions; An alternative to gamma or Weibull distribution, *Biometrical Journal* **43**(2001b), 117–130.

[14] Gupta, R.D. and Kundu, D., Generalied exponential distributions; Statistical Inferences, *Journal of Statistical Theory and Applications* **1**(2002), 101–118.

[15] Gupta, R.D. and Kundu, D., Closeness between the gamma and generalized exponentioal distributions, *Communications in Statistcs-Theory and Methods* **32**(2003a), 705–722.

[16] Gupta, R.D. and Kundu, D., Discriminating between the Weibull and Generalied exponential distributions, *Computational Statistics and Data Analysis* **43**(2003b), 179–196.

[17] Gupta, R.D. and Kundu, D., Discriminating between gamma and the Generalied exponential distributions, *Journal of Statistical Computation and Simulation* **74**(2004), 107–122.

[18] Kelley, G.D., Kelley, J.A. and Schucany, W.R., Efficient estimation of $P(Y < X)$ in expoential case, *Technometrics* **15**(1976), 359–360.

[19] Kundu, D. and Gupta R.D., Estimation of $P[Y < X]$ for generalied exponentioal distribution, *Metrika* **61**(3)(2005), 291–308.

[20] Kundu, D. and Gupta R.D., Estimation of $P[Y < X]$ for Weibull distribution, *IEEE Transactions on Reliability* **55**(2)(2006), 270-280.

[21] Kundu, D., Gupta, R.D. and Manglick, A., Discriminating between lognormal and generalized exponential distribution,*Journal of Statistical Planning and Inference* **127** (2005), 213–227.

[22] Mudholkar, G.S. and Srivastava, D.K., Exponentiated Weibull family for analyzing bathtub failure-rate data, *IEEE Transactions on Reliability* **42**(1993), 299–302.

[23] Mudholkar, G.S. and Hutson, A.D., The exponentiated Weibull family: some properties and a flood data applications, *Communications in Statistics-Theory and Methods* **25**(1996), 3059–3083.

[24] Mudholkar, G.S. and Srivastava, D.K. and Freimer, M., The exponentiated Weibull family: a reanalysis of the bus-motor failure data, *Technometrics* **37**(1995), 436–445.

[25] Owen, D.B., Craswell, K.J. and Hanson, D.L., Non-parametric upper confidence bounds for $P[Y < X]$ and confidence limits for $P(Y < X)$ when X

and Y are normal, *Journal of the American Statistical Association* **59**(1977), 906–924.

[26] Ragab, M.Z., Inference for generalized exponential distribution based on record statistics, *Journal of Statistical Planning and Inference* **104**(2002), 339–350.

[27] Ragab, M.Z. and Ahsanullah, M., Estimation of the location and parameters of the generalized exponential distribution based on order statistics, *Journal of Statistical Computation and Simulation* **69**(2001), 109–124.

[28] Sathe, Y.S. and Dixit, U.J., Estimation of $(X \leq Y)$ in negative binomial distribution, *Journal of Statistical Planning and Inference* **03**(2001), 83–92.

[29] Sathe, Y.S. and Shah, S.P., On estimation $P(Y < X)$ for the exponential distribution, *Communications in Statistics-Theory and Methods* **10**(1981), 39–47.

[30] Surles, J.G. and Padgett, W.J., Inference for reliablitiy and stress-strength for a scaled Burr-type X distribution, *Lifetime Data Analysis* **7**(2001), 187–200.

[31] Tong, H.A., A note on the estimation of $P(Y < X)$ in the exponential case, *Technometrics* **17**, 625. (Errara 1975, Vol. **17**(1975), 395).

[32] Tong, H.A., On the of $P(Y < X)$ for exponential families, *IEEE Trunsactions on Reliablility* **26**(1977), 54–56.

[33] Woodward, W.A. and Kelley, G.D., Minimum variance unbiased estimation of $P[Y < X]$ in the normal case, *Technometrics* **19**(1977), 95–98.

[34] Zheng, G., On the Fisher information matrix in type-II censored data from the exponentiaed exponetial family, *Biometrical Journal* **44**(2002), 353–357.

Abstracts

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مجله پژوهش‌های علوم ریاضی مشهد، جلد ۲، شماره ۱، (۱۳۸۸)

نمایش های شبه جایگشتی از زیر گروه‌های بورل و سهمی از گروه‌های سه‌گان اشتاینبرگ

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چکیده

اگر G یک گروه متناهی از درجه n باشد، یعنی گروهی متناهی از خودریختی‌های فضای برداری مختلط n—بعدی است، یا به طور معادل گروهی متناهی از ماتریس‌های نامنفرد از رتبه n با ضرایب مختلط باشد. گوییم G گروهی شبه جایگشتی است هر گاه اثر هر عنصر G یک عدد صحیح گویای نامنفی باشد. یک ماتریس شبه جایگشتی، ماتریس مربعی روی میدان اعداد مختلطC با اثر صحیح نامنفی است. بنابراین، هر ماتریس جایگشتی روی C یک ماتریس شبه جایگشت است. به ازای گروه متناهی مفروض G، فرض کنید $c(G)$ درجه کمین یک نمایش برای G به وسیله ماتریس‌های شبه جایگشتی روی اعداد مختلط و $r(G)$ نمایانگر درجه کمین یک شاخص مختلط با وفا با مقدار گویااز G باشد. هدف از این مقاله محاسبه $c(G)$ و $r(G)$ برای زیر گروه‌های بورل و سهمی از گروه‌های سه گان اشتاینبرگ است.

واژه های کلیدی: جدول شاخص، شبه جایگشت، زیر گروه بورل، زیر گروه سهمی، گروه‌های سه‌گان اشتاینبرگ.

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مجله پژوهش‌های علوم ریاضی مشهد، جلد ۲، شماره ۱، (۱۳۸۸)

خواص گروه‌های با نقاط

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چکیده

در این مقاله گروه‌های حاوی نقاطی را در نظر می گیریم که توسط وی.پی.شانکف در سال ۱۹۹۰ معرفی شده‌اند. در گروه نوویکف—آدیان، حاصل‌ضرب های دوره ای آدیان از گروه‌های متناهی بدون برگردان و گروه‌های غول دوره ای اولشانسکی، هر عنصر نایکه آن ها یک نقطه می باشد. یادآور می شود که گروه های بدون نقطه نیز وجود دارد. در این مقاله، برخی از خواص گروه‌های با نقطه را مورد بحث و بررسی قرار می دهیم.

واژه های کلیدی: حاصلضرب های دوره‌ای آدیان، گروه‌های موضعاً متناهی، گروه های موضعاً حل پذیر، گروه‌های چرنیکوف، گروه نوویکوف—آدیان.
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مجله پژوهش‌های علوم ریاضی مشهد، جلد ۲، شماره ۱، (۱۳۸۸)

شرط کافی برای کنترل‌پذیری صفر از سیستم‌های کنترل غیرخطی

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چکیده

روش‌های کنترل کلاسیک نظیر اصل بیشینه پونتریاگین و اصل کنترل بنگ–بنگ و دیگر روش ها معمولاً برای حل مسائل کنترل بهینه‌ی غیر خطی مفید نیستند. در این مقاله با استفاده از ترکیب اندازه‌های اتمی، رهیافت جدیدی را برای حل سیستم های کنترل بهینه‌ی غیر خطی معرفی می کنیم. ما محکی برای کنترل پذیری سیستم‌های کنترل غیر خطی فشرده تعریف می کنیم. هنگامی که سیستم تقریباً کنترل پذیر به‌مبدأ است، کنترل‌ها و حالت‌ها را معین می کنیم. در پایان این محک را برای چند مثال عددی به کار می‌بریم.

واژه های کلیدی: کنترل بهینه نظریه اندازه، کنترل پذیری، نظریه تقریب، نظریه فازی.
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مجله پژوهش‌های علوم ریاضی مشهد، جلد ۲، شماره ۱، (۱۳۸۸)

روش تکراری تغییرات برای حل معادلات خطی و غیر خطی شرودینگر

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چکیده

در این مقاله، روش تکراری تغییرات که به وسیله ژی هووان هی معرفی شده برای حل معادلات خطی و غیر خطی شرودینگر به کار گرفته شده است. مزیت اصلی این روش انعطاف پذیری و توانمندی آن در حل دقیق و راحت معادلات خطی و غیر خطی است. در این روش، ضرایب لاگرانژ عمومی برای ساخت تابعی‌های اصلاحی مسائل معرفی می شوند. ضرایب در تابعی ها را می توان از طریق نظریه تغییرات به صورت بهینه شناسایی کرد. نتایج عددی نشان می دهند که این روش می تواند به سرعت برای حل دقیق معادلات خطی و غیر خطی شرودینگر به کار گرفته شود. این روش را می توان بدون مشکلات جدی برای حل معادلات خطی و غیر خطی شرودینگر با ابعاد بالاتر توسعه داد.

واژه های کلیدی: روش تکرار تغییرات، معادلات خطی و غیر خطی شرودینگر، ضرایب لاگرانژ عمومی.

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مجله پژوهش‌های علوم ریاضی مشهد، جلد ۲، شماره ۱، (۱۳۸۸)

خاصیت مجانبی نرمال بودن برآوردگر احتمال برش برای داده‌های بریده وابسته

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چکیده

گاهی در برخی از مطالعات طولانی مدت، دنباله ای از طول عمرهای وابسته و بریده از چپ مشاهده می شوند. فرض کنید طول عمرها دارای توزیع مشترک باشند. در مدل برش چپ، زوج مرتب (X_i, T_i) وقتی $T_i \leq X_i$ مشاهده می شوند. تحت برخی شرایط نظم، یک نمایش قوی از برآوردگر $\hat{\beta_n}$ پارامتر $\beta = P(T_i \leq X_i)$ برحسب مجموعی از متغیرهای تصادفی به علاوه یک جمله باقی مانده ارائه خواهیم کرد. با این نمایش خاصیت مجانبی نرمال بودن برای $\hat{\beta_n}$ ثابت خواهد شد.

واژه های کلیدی: احتمال برش، $\alpha-$ آمیزنده، برش چپ، برآوردگر حد حاصلضربی، نمایش قوی.

رده بندی موضوعی ریاضیات AMS (۲۰۰۰): اولیه ۱۲J۱۵، ۲۶A۰۳؛ ثانویه ۲۶E۳۰.

مجله پژوهش‌های علوم ریاضی مشهد، جلد ۲، شماره ۱، (۱۳۸۸)

اطلاع فیشر مجانبی در آماره های مرتب توزیع هندسی

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چکیده

در این مقاله توزیع هندسی مدنظر است. میانگین‌ها، واریانس‌ها و کواریانس‌های آماره های مرتب این توزیع محاسبه می شود. اطلاع فیشر در هر مجموعه از آماره‌های مرتب از هر توزیع را می توان به عنوان مجموعی از اطلاع فیشر در حداکثر دو آماره مرتب تبیین کرد. در موارد هندسی نشان داده شده است که این موضوع به خوبی به مجموعی از اطلاع فیشر در یک آماره مرتب تسری می یابد. بنابراین در این مقاله ما اطلاع فیشر مجانبی در هر مجموعه از آماره‌های مرتب را ارائه می کنیم.

واژه های کلیدی: آماره‌های مرتب، اطلاع فیشر، توزیع هندسی، چارک، صدک.
رده بندی موضوعی ریاضیات AMS (۲۰۰۰): اولیه ۹۴A۱۵؛ ثانویه ۴۱A۶۰.

مجله پژوهش‌های علوم ریاضی مشهد، جلد ۲، شماره ۱، (۱۳۸۸)

برآورد $P(Y < X)$ برای توزیع نمایی تعمیم یافته در مواجهه با داده‌های پرت

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چکیده

در این مقاله، برآورد $P(Y < X)$ را مورد بررسی قرار می دهیم که در آن Y دارای توزیع نمایی تعمیم یافته با پارامتر های α و λ است و X دارای توزیع آمیخته نمایی تعمیم یافته (یا توزیع حاشیه ای $X_1, X_2, \ldots, X_n$ در معرض یک مشاهده پرت با پارامترهای β_1 و β_2) است طوری که X و Y مستقل هستند. وقتی پارامتر مقیاس (λ) معلوم است، برآوردگر درستنمایی ماکسیمم $R = P(Y < X)$ مشخص می شود. تحلیل مجموعه‌ای از داده‌های شبیه‌سازی شده نیز برای تبیین اهداف مورد نظر ارائه می شود.

واژه های کلیدی: برآوردگر درستنمایی ماکسیمم، داده پرت، مدل عصبی—قدرتی.
رده بندی موضوعی ریاضیات AMS (۲۰۰۰): اولیه ۶۲G۰۵؛ ثانویه ۶۲E۱۰.

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[1] Ahlfors, L.V., Finitely generated Kleinian group, Amer. J. Math., **86** (1964), 413-424.

[2] Lehmann, E.L. and Casella, G., Theory of point estimation, Second edition, Springer-Verlag, New York, 1998.

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