ISSN: 1735-7144

Mashhad Research Journal of Mathematical Sciences

Faculty of Mathematical Sciences
Ferdowsi University of Mashhad

Vol. 1 No. 1 Winter 2008

http://jm.um.ac.ir/index.php/math

In the Name of God

Mashhad Research Journal of Mathematical Sciences

This Journal is authorized under the registration No. 174/853 dated 1386/2/26, by the Ministry of Cultural and Islamic Guides.

Volume 1, Number 1, Winter 2008.

ISSN: 1735-7144

Publisher: Faculty of Mathematical Sciences, Ferdowsi University of Mashhad

Published by: Ferdowsi University of Mashhad Press

Circulation: 500

Subscription Information:

The author(s) receive 20 offprints free of charge. Additional offprints are provided in lots of 50, when the order form is received on time.

Personal: 20,000 Rials and $60 for international

Institutional: 30,000 Rials and $100 for international

Student: 10,000 Rials and $30 for international

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E-mail: mjms@um.ac.ir

Website: http://jm.um.ac.ir/index.php/math

This journal is indexed by:

- Mathematical Reviews
- Zenteralblatt
- Shiraz Reginal Library of Science and Technology

Mashhad Research Journal of Mathematical Sciences

Volume 1, Number 1, Winter 2008

Ferdowsi University of Mashhad - Iran

Mashhad Research Journal of Mathematical Sciences

This journal is published under the auspices of Ferdowsi University of Mashhad

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Letter from the Editor in Chief

I welcome you to the international Mashhad Research Journal of Mathematical Sciences (MRJMS). This journal is published biannually and supported by Faculty of Mathematical Sciences at Ferdowsi University of Mashhad. Faculty of Mathematical Sciences with three centers of excellence and three research centers is well-known in mathematical communities in Iran.

The main aim of the journal is to facilitate discussions and collaborations between specialists in mathematics and statistics, in the region and worldwide.

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Mohammad Reza R. Moghaddam

Mashhad R. J. Math. Sci., Vol.1(1)(2008) 1-22.

On algebraic characterizations for finiteness of the dimension of $\underline{E}G$

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Abstract

Certain algebraic invariants of the integral group ring $\mathbb{Z}G$ of a group G were introduced and investigated in relation to the problem of extending the Farrell-Tate cohomology, which is defined for the class of groups of finite virtual cohomological dimension. It turns out that the finiteness of these invariants of a group G implies the existence of a generalized Farrell-Tate cohomology for G which is computed via complete resolutions.

In this article we present these algebraic invariants and their basic properties and discuss their relationship to the generalized Farrell-Tate cohomology. In addition we present the status of conjecture which claims that the finiteness of these invariants of a group G is equivalent to the existence of a finite dimensional model for $\underline{E}G$, the classifying space for proper actions.

Keywords and phrases: Farrell-Tate cohomology, virtual cohomological dimension, complete resolution, finitistic dimension of the integral group ring, classifying space for proper action.

AMS Subject Classification 2000: Primary 20G10, 20J06; Secondary 18G10.

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†This project is cofunded by the European Social Fund and National Resources, EPEAEK II–Pythagoras, grant # 70/3/7298. It was also supported by grant ELKE # 70/4/6411 (Univ. of Athens).

1 Introduction

In their efforts to generalize the Farrell-Tate cohomology, which was defined for the class of groups of finite virtual cohomological dimension, Ikenaga in [12] and Gedrich and Gruenberg in [10] considered certain algebraic invariants of a group and showed that if these were finite then generalized Tate cohomology is defined for the group.

In particular, Ikenaga defined the generalized cohomological dimension of a group G, $\underline{\mathrm{cd}}\, G$, to be

$$\underline{\mathrm{cd}}\, G = \sup\{k : \mathrm{Ext}^k_{\mathbb{Z}G}(M, F) \neq 0,\ M\ \mathbb{Z}\text{-free},\ F\ \mathbb{Z}G\text{-free}\}$$

and showed that if G admits a complete resolution and $\underline{\mathrm{cd}}\, G < \infty$ then generalized Tate cohomology is defined for G.

A complete resolution of G is an acyclic complex $\{P_k\}_{k\in\mathbb{Z}}$ of projective $\mathbb{Z}G$-modules which agree with an ordinary projective resolution of G in sufficiently high (positive) dimensions.

Gedrich and Gruenberg considered the supremum of the projective lengths of injective $\mathbb{Z}G$-modules, $\mathrm{spli}\, \mathbb{Z}G$, and the supremum of the injective lengths of projective $\mathbb{Z}G$-modules, $\mathrm{silp}\, \mathbb{Z}G$. Then showed that if $\mathrm{spli}\, \mathbb{Z}G < \infty$ then G admits a complete resolution and moreover $\mathrm{silp}\, \mathbb{Z}G < \infty$ which implies that any two complete resolutions are homotopy equivalent, so generalized Tate cohomology is defined for G.

Note that $\mathrm{silp}\, \mathbb{Z}G$ and $\underline{\mathrm{cd}}\, G$ are closely related, namely $\underline{\mathrm{cd}}\, G \leq \mathrm{silp}\, \mathbb{Z}G \leq 1 + \underline{\mathrm{cd}}\, G$.

Mislin in [19] generalized these ideas and defined generalized Tate cohomology, $\hat{H}^n(G, -)$, for any group G and any integer n as follows: $\hat{H}^n(G, -) = \varinjlim_{j\geq 0} S^{-j} H^{n+j}(G, -)$ where $S^{-j} H^{n+j}(G, -)$ denotes the jth left satellite of the functor $H^{n+j}(G, -)$. Alternative but equivalent definitions were also given by Benson and Carlson [1] and Vogel (see [11]).

Note that the generalized Tate cohomology can not always be calculated via complete resolutions as they do not always exist. It turns out that the generalized Tate cohomology can be calculated via complete resolutions if and only if $\operatorname{spli}\mathbb{Z}G<\infty$ [24].

This article is a survey on the algebraic invariants of G that appeared in the search for the definition of generalized Tate cohomology for G.

We first discuss their basic properties and interrelations.

We then discuss the state of a conjecture (Conj. A in [26]) which claims that the finiteness of the above algebraic invariants, which imply that the generalized Tate cohomology can be calculated via complete resolutions, is the algebraic characterization of those groups G which admit a finite dimensional model for $\underline{E}G$, the classifying space for proper actions of G.

2 $\operatorname{spli}\mathbb{Z}G$

First we will establish some notation.

Let G be a group, $H\leq G$ and $i:\mathbb{Z}H\to\mathbb{Z}G$ the ring homomorphism induced from $H\hookrightarrow G$. Then the ring homomorphism i gives rise to the following functors:

1. $r:{}_{\mathbb{Z}G}\mathrm{Mod}\to{}_{\mathbb{Z}H}\mathrm{Mod}$, where any (left) $\mathbb{Z}G$-module can be regarded as a $\mathbb{Z}H$-module via i. If $M\in{}_{\mathbb{Z}G}\mathrm{Mod}$, then we denote $r(M)$ by $M\mid_H$.

2. $e:{}_{\mathbb{Z}H}\mathrm{Mod}\to{}_{\mathbb{Z}G}\mathrm{Mod}$

$N\to\mathbb{Z}G\underset{\mathbb{Z}H}{\otimes}N$, where the left $\mathbb{Z}G$-action on $\mathbb{Z}G\underset{\mathbb{Z}H}{\otimes}N$ is inherited from the $(\mathbb{Z}G,\mathbb{Z}H)$-bimodule structure of $\mathbb{Z}G$.

The module $e(N)=\mathbb{Z}G\underset{\mathbb{Z}H}{\otimes}N$ is called induced and we denote it by $\overset{\searrow}{\mathbb{Z}}G\underset{\mathbb{Z}H}{\otimes}N$.

3. $c:{}_{\mathbb{Z}H}\mathrm{Mod}\to{}_{\mathbb{Z}G}\mathrm{Mod}$

$N\to\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G,N)$, where the left $\mathbb{Z}G$-action on $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G,N)$ is inherited from the $(\mathbb{Z}H,\mathbb{Z}G)$-bimodule structure of $\mathbb{Z}G$.

The (left) $\mathbb{Z}G$-module $c(N)=\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G,N)$ is called co-induced and we denote it by $\mathrm{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}G},N)$.

Let now G be a group and $A,B\in{}_{\mathbb{Z}G}\mathrm{Mod}$.

We denote by $\mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{A},\overset{\searrow}{B})$ (resp. $\overset{\searrow}{A}\underset{\mathbb{Z}}{\otimes}\overset{\searrow}{B}$) the (left) $\mathbb{Z}G$-module $\mathrm{Hom}_{\mathbb{Z}}(A,B)$

(resp. $A \underset{\mathbb{Z}}{\otimes} B$) with the diagonal action $(gf)(\alpha) = gf(g^{-1}\alpha)$, $g \in G$, $f \in \mathrm{Hom}_{\mathbb{Z}}(A, B)$, $\alpha \in A$ (resp. $g(\alpha \otimes \beta) = g\alpha \otimes g\beta$, $g \in G$, $\alpha \in A$, $\beta \in B$).

The following Proposition states the well-known relation between the diagonal action and the induced and co-induced actions. The Corollary after it, states some of the Proposition's well-known consequences.

We state both without proofs.

Proposition 2.1. *Let G be a group, $H \leq G$ and $M \in {}_{\mathbb{Z}G}\mathrm{Mod}$. If $\mathbb{Z}(G/H)$ is the permutation module, where G/H is the set of cosets gH and G acts on G/H by left translations then*

(i) $\mathbb{Z}(G \overset{\searrow}{/} H) \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{M} \cong \overset{\searrow}{\mathbb{Z}}G \underset{\mathbb{Z}H}{\otimes} M/H$

(ii) $\mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}\overset{\searrow}{(}G/H), \overset{\searrow}{M}\right) \cong \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}\overset{\swarrow}{G}, M|_H)$.

Corollary 2.2. *Let $A \in {}_{\mathbb{Z}G}\mathrm{Mod}$ with* $\mathrm{proj.dim}_{\mathbb{Z}G} A \leq m$. *Then*

(i) If $B \in {}_{\mathbb{Z}G}\mathrm{Mod}$ with B $\mathbb{Z}$-free then $\mathrm{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{A} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{B} \leq m$;

(ii) If $B \in {}_{\mathbb{Z}G}\mathrm{Mod}$ with B $\mathbb{Z}$-injective then $\mathrm{inj.dim}\,\mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{A}, \overset{\searrow}{B}) \leq m$.

The following proposition and theorem state some basic properties of $\mathrm{spli}\,\mathbb{Z}G$ [10].

$\mathrm{Spli}\,\mathbb{Z}G$ is the supremum of the projective lengths of the injective $\mathbb{Z}G$-modules. It is not difficult to see that $\mathrm{spli}\,\mathbb{Z}G < \infty$ iff every injective $\mathbb{Z}G$-module has finite projective dimension.

Proposition 2.3.

(i) If G is a finite group then $\mathrm{spli}\,\mathbb{Z}G = 1$

(ii) If G is a group with $\mathrm{cd}_Z G = n$ *then* $\mathrm{spli}\,\mathbb{Z}G \leq n + 1$

(iii) Let G be a group and $H \leq G$. If I is an injective $\mathbb{Z}G$-module then $I|_H$ is an injective $\mathbb{Z}H$-module. Moreover $\mathrm{spli}\,\mathbb{Z}H \leq \mathrm{spli}\,\mathbb{Z}G$

(iv) If $H \leq G$ *and* $\mid G : H \mid < \infty$*, then* $\operatorname{spli} \mathbb{Z}G = \operatorname{spli} \mathbb{Z}H$.

Proof.

(i) If I is an injective $\mathbb{Z}G$-module, with G finite, then I is cohomologically trivial [e.g. [2]] and hence $\operatorname{proj.dim} I \leq 1$ and since I is not $\mathbb{Z}$-free it follows that $\operatorname{proj.dim} I = 1$.

(ii) Since $\operatorname{cd}_{\mathbb{Z}} G = n$ we have that $\operatorname{proj.dim}_{\mathbb{Z}G} \mathbb{Z} = n$ hence by Corollary 2.2 (i), for any $\mathbb{Z}G$-module A with A $\mathbb{Z}$-free we have that $\operatorname{proj.dim}_{\mathbb{Z}G} A \leq n$.

Now if M is any $\mathbb{Z}G$-module and one takes a projective presentation of M

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

then K, being a submodule of P, is $\mathbb{Z}$-free. Hence $\operatorname{proj.dim}_{\mathbb{Z}G} K \leq n$ and since P is projective, it follows that $\operatorname{proj.dim}_{\mathbb{Z}G} M \leq n+1$. In particular if I is an injective $\mathbb{Z}G$-module then $\operatorname{proj.dim}_{\mathbb{Z}G} I \leq n+1$.

(iii) If I is an injective $\mathbb{Z}G$-module, then $I|_H$ is an injective $\mathbb{Z}H$-module since

$$\operatorname{Hom}_{\mathbb{Z}G}\left(\overset{\searrow}{\mathbb{Z}}G \underset{\mathbb{Z}H}{\otimes} -, I \right) \cong \operatorname{Hom}_{\mathbb{Z}H}(-, I|_H)$$

and $\overset{\searrow}{\mathbb{Z}}G \underset{\mathbb{Z}H}{\otimes} -$ is an exact functor: ${}_{\mathbb{Z}H}\mathrm{Mod} \to {}_{\mathbb{Z}G}\mathrm{Mod}$.

Now if K is an injective $\mathbb{Z}H$-module, then K is a $\mathbb{Z}H$-direct summand of the injective $\mathbb{Z}G$-module $\operatorname{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}G}, K)$. Hence

$$\operatorname{proj.dim}_{\mathbb{Z}H} K \leq \operatorname{proj.dim}_{\mathbb{Z}H} \operatorname{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}G}, K)|_H \leq \operatorname{proj.dim}_{\mathbb{Z}G} \operatorname{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}G}, K)$$

, which implies that $\operatorname{spli} \mathbb{Z}H \leq \operatorname{spli} \mathbb{Z}G$.

(iv) Let $\mid G : H \mid < \infty$ and let $\operatorname{spli}\mathbb{Z}H = m$. By (iii), to show that $\operatorname{spli} \mathbb{Z}G = m$, it is enough to prove that every injective $\mathbb{Z}G$-module has projective dimension $\leq m$.

Let I be an injective $\mathbb{Z}G$-module, then by (iii) $I|_H$ is an injective $\mathbb{Z}H$-module and since $\operatorname{spli}\mathbb{Z}H = m$, there is a $\mathbb{Z}H$-projective resolution

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_0 \longrightarrow I|_H \longrightarrow 0,$$

which implies that $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{\mathbb{Z}G} \underset{\mathbb{Z}H}{\otimes} I|_H \leq m$.

Since $\mid G : H \mid < \infty$, it follows that

$$\overset{\searrow}{\mathbb{Z}G} \underset{\mathbb{Z}H}{\otimes} I|_H \cong \operatorname{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}G}, I|_H).$$

But I is a $\mathbb{Z}G$-direct summand of $\operatorname{Hom}_{\mathbb{Z}H}(\overset{\swarrow}{\mathbb{Z}G}, I|_H)$, hence $\operatorname{proj.dim}_{\mathbb{Z}G} I \leq m$.

□

We will show that $\operatorname{spli}\mathbb{Z}G < \infty$ is an extension closed property. For this we need the following lemma.

Lemma 2.4. *Let G be a group and $J = \operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Z})$. Then*

(i) $\operatorname{inj.dim}_{\mathbb{Z}G} J \leq 1$*;*

(ii) if $\operatorname{spli}\mathbb{Z}G = m$ *then* $\operatorname{proj.dim}_{\mathbb{Z}G} J \leq m$*;*

(iii) $\operatorname{spli}\mathbb{Z}G < \infty$ *iff* $\operatorname{proj.dim}_{\mathbb{Z}G} J < \infty$.

Proof. The exact sequence of abelian groups $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ gives rise to the following exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Q}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Q}/\mathbb{Z})$$

from which follows (i) and (ii), since $\operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Q})$ and $\operatorname{Hom}_{\mathbb{Z}}(\overset{\swarrow}{\mathbb{Z}G}, \mathbb{Q}/\mathbb{Z})$ are injective $\mathbb{Z}G$-modules.

Now let $\operatorname{proj.dim}_{\mathbb{Z}G} J < \infty$. We will show that every injective $\mathbb{Z}G$-module I has finite projective dimension.

From the $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to IG \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, where ε is the augmentation map, we obtain the $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to J \to \mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{I}G, \overset{\searrow}{\mathbb{Z}}) \to 0$, which gives rise to the $\mathbb{Z}G$-exact sequence $0 \to I \to \overset{\searrow}{I} \otimes \overset{\searrow}{J} \to \overset{\searrow}{I} \otimes C \to 0$, where $C = \mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{I}G, \overset{\searrow}{\mathbb{Z}})$. Note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\overset{\swarrow}{G}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\overset{\searrow}{\mathbb{Z}}G, \overset{\searrow}{\mathbb{Z}})$. Since I is a $\mathbb{Z}G$-direct summand of $\overset{\searrow}{I} \otimes \overset{\searrow}{J}$ it is enough to show that $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{I} \otimes \overset{\searrow}{J} < \infty$.

Let $0 \to K \to P \to I \to 0$ be a $\mathbb{Z}G$-projective presentation of I. Since J is $\mathbb{Z}$-torsion-free we obtain the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \overset{\searrow}{K} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} \longrightarrow \overset{\searrow}{P} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} \longrightarrow \overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} \longrightarrow 0.$$

Since $\operatorname{proj.dim}_{\mathbb{Z}G} J < \infty$ and P, K are $\mathbb{Z}$-free it follows from Corollary 2.2 (i) that $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{K} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} < \infty$ and $\operatorname{proj.dim} \overset{\searrow}{P} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} < \infty$, hence $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} < \infty$. □

It is clear from the proof of (iii) of the above lemma that we have

Corollary 2.5. $\operatorname{spli} \mathbb{Z}G < \infty$ *iff there is a $\mathbb{Z}$-split, $\mathbb{Z}G$-monomorphism* $0 \to \mathbb{Z} \to M$ *with* $\operatorname{proj.dim} M < \infty$ *and M $\mathbb{Z}$-torsion free.*

Theorem 2.6. *[10] Let* $1 \to N \to G \xrightarrow{\pi} K \to 1$ *be an extension of groups. Then* $\operatorname{spli} \mathbb{Z}G \leq \operatorname{spli} \mathbb{Z}N + \operatorname{spli} \mathbb{Z}K$.

Proof. Let $\operatorname{spli} \mathbb{Z}N = n$ and $\operatorname{spli} \mathbb{Z}K = m$ and let I be an injective $\mathbb{Z}G$-module. We will show that $\operatorname{proj.dim}_{\mathbb{Z}G} I \leq n + m$.

We consider the $\mathbb{Z}$-split $\mathbb{Z}K$-exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\overset{\swarrow}{K}, \mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(IK, \mathbb{Z}) \longrightarrow 0$$

as a $\mathbb{Z}G$-exact sequence via $\pi : G \to K$ and tensoring it with I, we obtain the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow I \longrightarrow I \otimes \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\overset{\swarrow}{K}, \mathbb{Z}).$$

Since I is a $\mathbb{Z}G$-direct summand of $\overset{\searrow}{I} \otimes \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\overset{\swarrow}{K}, \mathbb{Z})$ it is enough to show that $\mathrm{proj.\,dim}_{\mathbb{Z}G}\, I \underset{\mathbb{Z}}{\otimes} J \leq n+m$, where J is the $\mathbb{Z}G$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\overset{\swarrow}{K}, \mathbb{Z})$.

Now by Lemma 2.4 (ii), $\mathrm{proj.\,dim}_{\mathbb{Z}K}\, J \leq m$ and since $\mathrm{spli}\,\mathbb{Z}N = n$ it follows that $\mathrm{proj.\,dim}_{\mathbb{Z}N}\, I|_N \leq n$.

Hence there exists $Q : 0 \to Q_m \to \cdots \to Q_0 \to J \to 0$ a $\mathbb{Z}K$-projective resolution of J of length m and $P : 0 \to P_n \to \cdots \to P_0 \to I \to 0$ a $\mathbb{Z}G$-exact sequence with P_i $\mathbb{Z}G$-projective modules for all $0 \leq i \leq n-1$ and $P_n|_N$ a projective $\mathbb{Z}N$-module.

Consider the following $\mathbb{Z}G$-complexes $Q' : 0 \to Q_m \to \cdots \to Q_0 \to 0$, a $\mathbb{Z}G$-complex via $\pi : G \to K$ and
$P' : 0 \to P_n \to \cdots \to P_0 \to 0$ and let $Q' \underset{\mathbb{Z}}{\otimes} P'$ be their tensor product.

Since J is $\mathbb{Z}$-torsion free it follows from the Künneth formula that we obtain a $\mathbb{Z}G$-exact sequence $0 \to B_{m+n} \to \cdots \to B_0 \to \overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J} \to 0$, where

$$B_\lambda = \left(Q' \underset{\mathbb{Z}}{\otimes} P'\right)_\lambda = \bigoplus_{r+s=\lambda} \overset{\searrow}{Q}_r \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{P}_s.$$

□

By Proposition 2.1 (i), B_λ is a projective $\mathbb{Z}G$-module for $0 \leq \lambda \leq m+1$. Since $P_s|_N$ is a projective $\mathbb{Z}N$-module for all s, we obtain a $\mathbb{Z}G$-projective resolution of $\overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{J}$ of length $m+n$.

3 $\mathrm{spli}\,\mathbb{Z}G$, $\mathrm{silp}\,\mathbb{Z}G$, $\mathrm{fin.\,dim}\,\mathbb{Z}G$, $K(\mathbb{Z}G)$

Silp $\mathbb{Z}G = \sup\{\mathrm{inj.\,dim}_{\mathbb{Z}G}\, P | P \mathrm{proj.}\mathbb{Z}G\text{-module}\}$ and it is not difficult to see that $\mathrm{silp}\,\mathbb{Z}G < \infty$ iff every projective $\mathbb{Z}G$-module has finite injective dimension.
Note that $\mathrm{silp}\,\mathbb{Z}G \leq m$ is equivalent to the following extension condition [12]:

For every exact sequence

$$0 \longrightarrow \ker \partial_m \longrightarrow P_m \xrightarrow{\partial_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with P_i projective $\mathbb{Z}G$-modules for $0 \leq i \leq m$, any map $\ker \partial_m \to P$, P a projective $\mathbb{Z}G$-module, extends to a map $P_m \to P$.

It is not difficult to see that if $\operatorname{silp}\mathbb{Z}G$ and $\operatorname{spli}\mathbb{Z}G$ are both finite then they are equal.

The following Proposition, which we state without proof, gives some basic properties of $\operatorname{silp}\mathbb{Z}G$.

Proposition 3.1.

(i) If G is a finite group, then $\operatorname{silp}\mathbb{Z}G = 1$.

(ii) If G is a group with $\operatorname{cd}_{\mathbb{Z}}G = n$ *then* $\operatorname{silp}\mathbb{Z}G \leq n+1$.

(iii) If G is a group and $H \leq G$ then $\operatorname{silp}\mathbb{Z}H \leq \operatorname{silp}\mathbb{Z}G$.

Moreover, if $|G:H| < \infty$ then $\operatorname{silp}\mathbb{Z}G = \operatorname{silp}\mathbb{Z}H$.

Theorem 3.2. *[10] For any group G,* $\operatorname{silp}\mathbb{Z}G \leq \operatorname{spli}\mathbb{Z}G$.

Proof. It is enough to show that if $\operatorname{spli}\mathbb{Z}G < \infty$ then $\operatorname{silp}\mathbb{Z}G < \infty$. By Lemma 2.4 (iii), it is enough to show that if $\operatorname{proj.dim}_{\mathbb{Z}G} J < \infty$ then $\operatorname{silp}\mathbb{Z}G < \infty$, where $J = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\overset{\swarrow}{G}, \mathbb{Z})$.

Let $\operatorname{proj.dim}_{\mathbb{Z}G} J < \infty$ and consider a projective $\mathbb{Z}G$-module P. The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ gives rise to the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow P \longrightarrow P \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \longrightarrow P \underset{\mathbb{Z}}{\otimes} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Hence to show that $\operatorname{inj.dim}_{\mathbb{Z}G} P$ is finite, it is enough to show that $\operatorname{inj.dim} P \underset{\mathbb{Z}}{\otimes} D$ is finite, where D is a $\mathbb{Z}$-injective abelian group.

Let $\widetilde{P} = P \underset{\mathbb{Z}}{\otimes} D$, where D is a divisible abelian group, then $\widetilde{P}$ is a direct summand of an induced module hence it is relative projective i.e. if

$$0 \longrightarrow A \longrightarrow B \longrightarrow \widetilde{P} \longrightarrow 0 \qquad (*)$$

is an exact sequence of $\mathbb{Z}G$-modules which is $\mathbb{Z}$-split, then $(*)$ is $\mathbb{Z}G$-split.

Consider the $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to J \to C \to 0$ where $C = \operatorname{Hom}_{\mathbb{Z}}(\overset{\searrow}{J}G, \overset{\searrow}{\mathbb{Z}})$. This gives rise to the following $\mathbb{Z}$-split, $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\overset{\searrow}{C}, \overset{\searrow}{\widetilde{P}}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\overset{\searrow}{J}, \overset{\searrow}{\widetilde{P}}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\overset{\searrow}{\mathbb{Z}}, \overset{\searrow}{\widetilde{P}}) \longrightarrow 0.$$

But $\operatorname{Hom}_{\mathbb{Z}}(\overset{\searrow}{\mathbb{Z}}, \overset{\searrow}{\widetilde{P}}) \cong \widetilde{P}$, hence $\widetilde{P}$ is a $\mathbb{Z}G$-direct summand of $\operatorname{Hom}_{\mathbb{Z}}(\overset{\searrow}{J}, \overset{\searrow}{\widetilde{P}})$.

Since $\operatorname{proj.dim}_{\mathbb{Z}G} J < \infty$ it follows from Corollary 2.2 (ii) that
$\operatorname{inj.dim}_{\mathbb{Z}G} \operatorname{Hom}_{\mathbb{Z}}(\overset{\searrow}{J}, \overset{\searrow}{\widetilde{P}}) < \infty$. □

Open questions 3.3.

a) It is not known if $\operatorname{silp} \mathbb{Z}G < \infty$ is an extension closed property.

b) It is not known if there is a group G such that $\operatorname{silp} \mathbb{Z}G < \infty$ and $\operatorname{spli} \mathbb{Z}G$ infinite.

c) It is conjectured in [6] that for any group G, $\operatorname{silp} \mathbb{Z}G = \underline{\operatorname{cd}}\, G + 1 = \operatorname{spli} \mathbb{Z}G$. This is proved in [6] for certain classes of groups.

Two more algebraic invariant of G, the finiteness dimensions of $\mathbb{Z}G$, and $k(G)$ are related to $\operatorname{spli} \mathbb{Z}G$, and $\operatorname{spli} \mathbb{Z}G$. The finiteness dimension of $\mathbb{Z}G$, $\operatorname{fin.dim} \mathbb{Z}G$, which is the supremum of the projective dimensions of the $\mathbb{Z}G$-modules of finite projective dimension and

$k(G) = \sup\{\operatorname{proj.dim}_{\mathbb{Z}G} M | \operatorname{proj.dim}_{\mathbb{Z}H} M < \infty$ for every finite subgroup $H \leq G\}$.

Proposition 3.4. *[26] Let G be any group, then*

$$\operatorname{fin.dim} \mathbb{Z}G \leq \operatorname{silp} \mathbb{Z}G \leq \operatorname{spli} \mathbb{Z}G \leq k(\mathbb{Z}G).$$

Moreover, if any of the above invariants is finite then it is equal to the ones less than equal to it.

Proof. It is easy to see that $\operatorname{fin.dim} \mathbb{Z}G \leq \operatorname{silp} \mathbb{Z}G\mathbb{Z}G$ and by Theorem 3.2, $\operatorname{silp} \mathbb{Z}G \leq \operatorname{spli} \mathbb{Z}G$. Now by Proposition 2.3 (i) and (iii) it follows that $\operatorname{spli} \mathbb{Z}G \leq k(G)$.

Now if $k(G) < \infty$ then clearly $k(G) \leq \operatorname{fin.dim} \mathbb{Z}G$, hence

$$\operatorname{fin.dim} \mathbb{Z}G = \operatorname{silp} \mathbb{Z}G = \operatorname{spli} \mathbb{Z}G = k(\mathbb{Z}G).$$

In [4], it was shown that if G is an $H\mathcal{F}$-group then $\operatorname{silp}\mathbb{Z}G = \operatorname{spli}\mathbb{Z}G = \operatorname{fin.dim}\mathbb{Z}G = k(\mathbb{Z}G)$.

The class $H\mathcal{F}$ of groups was defined by Kropholler in [14] as follows. Let $H_0\mathcal{F}$ be the class of finite groups. Now define $H_\alpha\mathcal{F}$ for each ordinal α by transfinite recursion: if α is a successor ordinal then $H_\alpha\mathcal{F}$ is the class of groups G which admits a finite dimensional contractible G-CW-complex with cell stabilizers in $H_{\alpha-1}\mathcal{F}$, and if α is a limit ordinal then $H_\alpha\mathcal{F} = \bigcup\limits_{\beta<\alpha} H_\beta\mathcal{F}$. A group belongs to $H\mathcal{F}$ if it belongs to $H_\alpha\mathcal{F}$, for some ordinal α. □

Note that a G-CW-complex is a CW-complex on which G acts by self-homeomorphisms in such a way that the set-wise stabilizer of each cell coincides with its point-wise stabilizer.

The class $H\mathcal{F}$ contains among others all groups of finite virtual cohomological dimension and all countable linear groups of arbitrary characteristic. Moreover, it is extension closed, subgroup closed, closed under directed unions and closed under amalgamated free products and HNN-extensions.

4 Another characterization of $\operatorname{spli}\mathbb{Z}G < \infty$

Definition. A complete resolution for a group , $(\mathcal{F}, \mathcal{P}, n)$, consists of an acyclic complex $\mathcal{F} = \{(F_i, \partial_i) | i \in \mathbb{Z}\}$ of projective modules and a projective resolution $\mathcal{P} = \{(P_i, d_i) | i \leq 0\}$ of G such that $\mathcal{F}$ and $\mathcal{P}$ coincide in sufficiently high dimensions

$$\begin{array}{ccccccccccccccc}
\cdots \longrightarrow & F_{n+1} & \longrightarrow & F_n & \xrightarrow{\partial_n} & F_{n-1} & \longrightarrow \cdots \longrightarrow & F_0 & \longrightarrow & F_{-1} & \longrightarrow & F_{-2} & \longrightarrow \cdots \\
 & \| & & \| & & & & & & & & & \\
\cdots \longrightarrow & P_{n+1} & \longrightarrow & P_n & \xrightarrow{d_n} & P_{n-1} & \longrightarrow \cdots \longrightarrow & P_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 &
\end{array}$$

The number n is called the coincidence index of the complete resolution.

Inekaga in [12] defined the notion of generalized cohomological dimension of a group G, $\underline{\operatorname{cd}}\, G = \sup\{k : \operatorname{Ext}^k_{\mathbb{Z}G}(M, F) \neq 0,\ M\ \mathbb{Z}\text{-free},\ F\ \mathbb{Z}G\text{-free}\}$.

Note that $\underline{\mathrm{cd}}\, G \le \mathrm{silp}\, \mathbb{Z}G \le \underline{\mathrm{cd}}\, G + 1$.

He showed in [12] that if G admits a complete resolution then G admits a complete resolution of coincidence index $\underline{\mathrm{cd}}\, G$. In particular a group G with $v\underline{\mathrm{cd}}\, G < \infty$ admits a complete resolution of coincidence index $v\underline{\mathrm{cd}}\, G$.

Moreover, it was shown in [12] that if a group G admits a complete resolution of coincidence index n, then

(i) $H^i(G, P) \neq 0$ for some $\mathbb{Z}G$-projective module P and some $i \le n$

(ii) $\mathrm{fin.\,dim}\, \mathbb{Z}G \le n + 1$.

Since admitting a complete resolution is a subgroup closed property, and since if $\mathcal{A}$ is a free abelian group of infinite rank, $H^i(\mathcal{A}, P) = 0$ for any projective $\mathbb{Z}\mathcal{A}$-module and any i, it follows from (i) that if a group G contains a free abelian subgroup of infinite rank then G does not admit a complete resolution.

Proposition 4.1. *[24] If* $\mathrm{spli}\, \mathbb{Z}G < \infty$ *then there is a* $\mathbb{Z}$*-split* $\mathbb{Z}G$*-exact sequence* $0 \to \mathbb{Z} \to A$ *with* A $\mathbb{Z}$*-free and* $\mathrm{proj.\,dim}_{\mathbb{Z}G} A < \infty$.

Proof. It was shown in [24] that if $\mathrm{spli}\, \mathbb{Z}G < \infty$ then G admits a complete resolution.

Now consider a complete resolution for G

$$\begin{array}{ccccccccccc}
\longrightarrow F_{n+1} & \xrightarrow{\partial_{n+1}} & F_n & \xrightarrow{\partial_n} & F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow F_{-1} \longrightarrow \cdots \\
\Vert & & \Vert & & \\
\longrightarrow P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0
\end{array}$$

Let $R_n = \mathrm{im}\partial_n$, $n \in \mathbb{Z}$. If $\lambda : R_n \to P$ is a $\mathbb{Z}G$-homomorphism with P a projective $\mathbb{Z}G$-module, then by Theorem 3.2 $\mathrm{silp}\, \mathbb{Z}G < \infty$ hence there is a positive integer m_0 and an integer m so that λ represents the zero element in $\mathrm{Ext}^{m_0}_{\mathbb{Z}G}(R_m, P)$. Hence we obtain the following commutative diagram

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & F_{n+1} \arrow[r] \arrow[d, equal] & F_n \arrow[r] \arrow[d, equal] & F_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & F_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] & \cdots \arrow[r] & F_1 \arrow[r] \arrow[d, "f_1"] & F_0 \arrow[rr] \arrow[dr] \arrow[d, "f_0"] & & F_{-1} \arrow[r] \arrow[d, "f"] & {} \\
 & {} & {} & {} & {} & & {} & {} & R_0 \arrow[ur] & {} &
\end{tikzcd}$$

$$\cdots \longrightarrow F_{n+1} \longrightarrow F_n \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z}$$

where $R_0 = \operatorname{im} \partial_0$.

Clearly $[f] \in \operatorname{Ext}_{\mathbb{Z}G}(R_{-1}, \mathbb{Z})$ and Yoneda product with $[f]$ induces an isomorphism: $\operatorname{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, -) \to \operatorname{Ext}^{i+1}_{\mathbb{Z}G}(R_{-1}, -)$. This implies (c.f. [27]) that $[f]$ is represented by an extension $0 \to \mathbb{Z} \to A \to R_{-1} \to 0$ with $\operatorname{proj.dim}_{\mathbb{Z}G} A < \infty$. The result now follows since R_{-1} is $\mathbb{Z}$-free as a $\mathbb{Z}G$-submodule of a projective $\mathbb{Z}G$-module. □

Theorem 4.2. *[24] The following statements are equivalent for any group G.*

(i) $\operatorname{spli} \mathbb{Z}G < \infty$*;*

(ii) There is a $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to A$ with A $\mathbb{Z}$-free and $\operatorname{proj.dim}_{\mathbb{Z}G} A < \infty$*.*

Proof. (i)⇒(ii) is Proposition 4.1.

For (ii)⇒(i). Let I be an injective $\mathbb{Z}G$-module and consider a $\mathbb{Z}G$-projective presentation of I

$$0 \longrightarrow K \longrightarrow P \longrightarrow I \longrightarrow 0.$$

Since A is $\mathbb{Z}$-free we obtain the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \overset{\searrow}{K} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} \longrightarrow \overset{\searrow}{P} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} \longrightarrow \overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} \longrightarrow 0.$$

By Corollary 2.2 (i), $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{K} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} < \infty$ and $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{P} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} < \infty$ hence $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} < \infty$, but tensoring $0 \to \mathbb{Z} \to A$ with I we obtain that I is a $\mathbb{Z}G$-direct summand of $\overset{\searrow}{I} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A}$, and the result follows. □

The following proposition states some of the properties of such a module A.

Proposition 4.3. *[26] Let G be a group and let $0 \to \mathbb{Z} \to A$ be a $\mathbb{Z}$-split, $\mathbb{Z}G$-exact sequence with A $\mathbb{Z}$-free and* $\operatorname{proj.dim} A = n$*. Then*

(i) If $\operatorname{proj.dim}_{\mathbb{Z}G} M < \infty$ *and M is $\mathbb{Z}$-free then* $\operatorname{proj.dim}_{\mathbb{Z}G} M \leq n$

(ii) $\operatorname{spli}\mathbb{Z}G \le n+1$

(iii) For any finite subgroup H of G, $A|_H$ is a projective $\mathbb{Z}H$-module.

Proof. (i) Consider the $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \to A \to \bar{A} \to 0$. Clearly $\bar{A}$ is $\mathbb{Z}$-free.

Now let $\operatorname{proj.dim}_{\mathbb{Z}G} M = m$. By Corollary 2.2 (i) we have that $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{M} \otimes \overset{\searrow}{A} \le n, m$ and $\operatorname{proj.dim}_{\mathbb{Z}G} \overset{\searrow}{M} \otimes \overset{\searrow}{\bar{A}} \le m$. It now follows from the long exact Ext-sequence associated to

$$0 \longrightarrow M \longrightarrow \overset{\searrow}{M} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{A} \longrightarrow \overset{\searrow}{M} \underset{\mathbb{Z}}{\otimes} \overset{\searrow}{\bar{A}} \longrightarrow 0$$

that if $\operatorname{proj.dim}_{\mathbb{Z}G} M > n$ then $\operatorname{proj.dim}_{\mathbb{Z}G} M \otimes \bar{A} \ge m+1$, which is a contradiction and hence $\operatorname{proj.dim}_{\mathbb{Z}G} M \le n$.

(ii) Let I be an injective $\mathbb{Z}G$-module and $0 \to K \to P \to I \to 0$ a $\mathbb{Z}G$-projective presentation of I. By Theorem 4.2 $\operatorname{spli}\mathbb{Z}G < \infty$ hence $\operatorname{proj.dim}_{\mathbb{Z}G} K < \infty$ and by (i) $\operatorname{proj.dim}_{\mathbb{Z}G} K \le n$ which implies that $\operatorname{proj.dim}_{\mathbb{Z}G} I \le n+1$.

(iii) Since $A|_H$ is $\mathbb{Z}$-free and has $\operatorname{proj.dim}_{\mathbb{Z}H} A < \infty$ it follows that A is a projective $\mathbb{Z}H$-module (c.f. [2], Ch. VI). $\square$

Theorem 4.4. $\operatorname{spli}\mathbb{Z}G < \infty$ *is a Weyl-group closed property i.e. if* $\operatorname{spli}\mathbb{Z}G < \infty$ *and H is a finite subgroup of G then* $\operatorname{spli}\mathbb{Z}(N_G(H)/H) < \infty$.

Proof. Assume that $\operatorname{spli}\mathbb{Z}G < \infty$ and let H be a finite subgroup of G. Let $N = N_G(A)$, then by Proposition 2.3 (iii) $\operatorname{spli}\mathbb{Z}N < \infty$ hence by Theorem 4.2 there is a $\mathbb{Z}$-split $\mathbb{Z}N$-exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow A \tag{$*$}$$

with A $\mathbb{Z}$-free and $\operatorname{proj.dim}_{\mathbb{Z}N} A = n$.

Consider a $\mathbb{Z}N$-projective resolution of A of length n

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0. \tag{$*'$}$$

Since $A|_H$ is a projective $\mathbb{Z}H$-module, $(*')$ gives rise to the following $\mathbb{Z}(N/H)$-exact sequence

$$0 \longrightarrow P_n^H \longrightarrow P_{n-1}^H \longrightarrow \cdots \longrightarrow P_0^H \longrightarrow A^H \longrightarrow 0.$$

It is not difficult to see that P_i^H are projective $\mathbb{Z}(N/H)$-modules since $\mathbb{Z}N^H \cong \mathbb{Z}(N/H)$ as $\mathbb{Z}(N/H)$-modules, hence $\operatorname{proj.dim}_{\mathbb{Z}(N/H)} A^H \leq n$.

Moreover, $(*)$ gives rise to the $\mathbb{Z}$-split and $\mathbb{Z}(N/H)$-exact sequence $0 \to \mathbb{Z} \to A^H$. Hence by Theorem 4.2 $\operatorname{spli} \mathbb{Z}(N/H) < \infty$. □

5 The classes of groups $H_1\mathcal{F}$ and $\underline{E}G$

A group G belongs to $H_1\mathcal{F}$ if there is a finite dimensional contractible G-CW-complex with finite cell stabilizers.

By a theorem of Serre (see also Exercise in [2], p. 191) it follows that $H_1\mathcal{F}$ contains all groups of finite virtual cohomological dimension.

It also contains infinite torsion groups, for example a countable locally finite group G is in $H_1\mathcal{F}$, since G acts on a tree with finite vertex stabilizers. It was proved in [5] that if G is a locally finite group of cardinality less than N_w then G is in $H_1\mathcal{F}$.

For sufficiently large e it is known [13] that the free Burnside groups of exponent e admit actions on contractible 2-dimensional complexes with cyclic stabilizers, hence these groups are in $H_1\mathcal{F}$.

If G is in $H_1\mathcal{F}$ and X is a finite dimensional contractible G-CW-complex with finite cell stabilizers, then the argumented cellular chain complex of X gives rise to the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \underset{i_n \in I_n}{\otimes} \mathbb{Z}(G/G_{i_n}) \longrightarrow \cdots \longrightarrow \underset{i_0 \in I_0}{\otimes} \mathbb{Z}(G/G_{i_0}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

with G_{i_j} finite for all i_j.

So if G is in $H_1\mathcal{F}$ and G is torsion free then $\mathrm{cd}_{\mathbb{Z}} G < \infty$.

In particular a free abelian group of infinite rank is not in $H_1\mathcal{F}$.

Proposition 5.1. *If G is in $H_1\mathcal{F}$ then*

$$\operatorname{fin.dim} \mathbb{Z}G = \operatorname{silp} \mathbb{Z}G = \operatorname{spli} \mathbb{Z}G = k(\mathbb{Z}G) < \infty.$$

Proof. Since G is in $H_1\mathcal{F}$, there is a $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \bigotimes_{i_n \in I_n} \mathbb{Z}(G/G_{i_n}) \longrightarrow \cdots \longrightarrow \bigotimes_{i_0 \in I_0} \mathbb{Z}(G/G_{i_0}) \longrightarrow \mathbb{Z} \longrightarrow 0$$

with G_{i_j} finite subgroups of G for all i_j.

If M is a $\mathbb{Z}G$-module such that $\operatorname{proj.dim}_{\mathbb{Z}H} M|_H < \infty$ for every finite subgroup H of G, and $0 \to K \to P \to M \to 0$ is a $\mathbb{Z}G$-projective presentation of M then $K|_H$ is a projective $\mathbb{Z}H$-module for every finite subgroup H of G.

Hence if we tensor $(*)$ with K we obtain the following $\mathbb{Z}G$-exact sequence

$$0 \longrightarrow \bigotimes_{i_n \in I_n} \mathbb{Z}(G/G_{i_n}) \underset{\mathbb{Z}}{\otimes} K \longrightarrow \cdots \longrightarrow \bigotimes_{i_0 \in I_0} \mathbb{Z}(G/G_{i_0}) \underset{\mathbb{Z}}{\otimes} K \longrightarrow K \longrightarrow 0$$

which by Proposition 2.1 (i), is a $\mathbb{Z}G$-projective resolution of K, since $K|_H$ is a projective $\mathbb{Z}H$-module for every finite subgroup H of G.

Hence $\operatorname{proj.dim}_{\mathbb{Z}G} K \leq n$ which implies that $k(\mathbb{Z}G) \leq n$. The result now follows from Proposition 3.4. $\square$

In [15] Kropholler and Mislin proved

Theorem A. Every $H\mathcal{F}$-group of type FP_∞ is in $H_1\mathcal{F}$.

A group G is said to be of type FP_∞ if there is a $\mathbb{Z}G$-projective resolution of G

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

with P_i finitely generated $\mathbb{Z}G$-modules for all $i \geq 0$.

Notation. If $\mathcal{X}$ is a class of groups, we denote by $\mathcal{X}_b$ the subclass of $\mathcal{X}$ consisting of those groups in $\mathcal{X}$, for which there is a bound on the orders of the finite subgroups.

To prove Theorem A, they first considered the following two properties of $H\mathcal{F}$-groups of type FP_∞, which were both shown using complete cohomology.

• If G is an $H\mathcal{F}$-group of type FP_∞ then G is in $H\mathcal{F}_b$.

In particular, if $|\Lambda(G)|$ is the G-simplicial complex determined by the poset of the non-trivial finite subgroups of G, then $\dim|\Lambda(G)| < \infty$.

• If G is an $H\mathcal{F}$-group of type FP_∞, then $\operatorname{proj.dim}_{\mathbb{Z}G} B(G,\mathbb{Z}) < \infty$, where $B(G,\mathbb{Z})$ is the $\mathbb{Z}G$-module of bounded functions from G to $\mathbb{Z}$.

They then proved, by induction on $\dim|\Lambda(G)|$

Theorem B. If G is an $H\mathcal{F}$-group such that $\dim|\Lambda(G)| < \infty$ and $\operatorname{proj.dim}_{\mathbb{Z}G} B(G,\mathbb{Z}) < \infty$ then G is in $H_1\mathcal{F}$.

Clearly Theorem A follows from Theorem B.

Generalizations of this Theorem were obtained in [17], [20], [26].

Note that $B(G,\mathbb{Z})$, the $\mathbb{Z}G$-module of bounded functions from G to $\mathbb{Z}$, is a $\mathbb{Z}$-free $\mathbb{Z}G$-module and there is a $\mathbb{Z}$-split $\mathbb{Z}G$-exact sequence $0 \to \mathbb{Z} \xrightarrow{i} B(G,\mathbb{Z})$ where $i(n) : G \to \mathbb{Z}$ is the constant function c_n [16].

By Theorem 4.2 $\operatorname{proj.dim}_{\mathbb{Z}G} B(G,\mathbb{Z}) < \infty$ implies that $\operatorname{spli}\mathbb{Z}G < \infty$. Now if G is in $H\mathcal{F}$ then it is known [4] that $\operatorname{spli}\mathbb{Z}G = k(\mathbb{Z}G)$.

So if G is in $H\mathcal{F}$ and $\operatorname{proj.dim}_{\mathbb{Z}G} B(G,\mathbb{Z}) < \infty$ then $k(\mathbb{Z}G) < \infty$. It is easy to see that $k(\mathbb{Z}G) < \infty$ is a subgroup closed property and by Theorem 4.4 a Weyl-group closed property [26].

These properties, which are implications of the finiteness of the proj. dim of $B(G,\mathbb{Z})$, for G in $H\mathcal{F}$, are crucial for the proof of Theorem B.

The following Conjecture, (Conj. A in [26]), claims that the finiteness of the algebraic invariants we've studied here, give an algebraic characterization for the class $H_1\mathcal{F}$.

Conjecture A. The following statements are equivalent for a group G:

(1) G is in $H_1\mathcal{F}$;

(2) G is of type Φ;

(3) $\operatorname{spli}\mathbb{Z}G < \infty$;

(4) $\operatorname{silp}\mathbb{Z}G < \infty$;

(5) $\operatorname{fin.dim} \mathbb{Z}G < \infty$,

where, a group G is said to be of type Φ if it has the property that for every $\mathbb{Z}G$-module M, $\operatorname{proj.dim}_{\mathbb{Z}G} M < \infty$ if and only if $\operatorname{proj.dim}_{\mathbb{Z}H} M|_H < \infty$ for every finite subgroup H of G.

Note that G is of type Φ it if has the property that for every $\mathbb{Z}G$-module M, $\operatorname{proj.dim}_{\mathbb{Z}G} M < \infty$ if and only if $M|_H$ is a cohomologically trivial $\mathbb{Z}H$-module, for every finite subgroup H of G.

Proposition 5.1 shows that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) in Conjecture A.

Kropholler and Mislin's Theorem show essentially that (5) $\Rightarrow$ (1) if G is in $(H\mathcal{F})_b$.

In [26] it was shown that (5) $\Rightarrow$ (1) if G is a torsion-free locally soluble group.

In support of Conj. A is also a result obtained in [5] which says that a group G is finite if and only if $\operatorname{spli} \mathbb{Z}G = 1$. It is worth mentioning that its proof uses the theory of groups acting on trees.

If G is in $H_1\mathcal{F}$ then there is a $\mathbb{Z}G$-resolution of G by direct sums of permutation modules of finite subgroups of G, i.e.

$$0 \longrightarrow \underset{i_n \in I_n}{\otimes} \mathbb{Z}(G/G_{i_n}) \longrightarrow \cdots \longrightarrow \underset{i_0 \in I_0}{\otimes} \mathbb{Z}(G/G_{i_0}) \longrightarrow \mathbb{Z} \longrightarrow 0 \qquad (*)$$

with G_{i_j} finite subgroups of G for all i_j.

It follows from $(*)$ that if G is in $H_1\mathcal{F}$ then $\operatorname{cd}_{\mathbb{Q}} G < \infty$.

It is likely that the existence of $(*)$ is another algebraic characterization for the $H_1\mathcal{F}$-class of groups.

As we mentioned before if G is a group of finite virtual cohomological dimension, $v\underline{\operatorname{cd}}\, G < \infty$ then G is in $H_1\mathcal{F}$, actually G is in $H_1\mathcal{F}_b$.

We consider the class $H_1\mathcal{F}$ or rather the class $H_1\mathcal{F}_b$ as a more "natural class" than the class of groups of finite $v\underline{\operatorname{cd}}$.

The class $H_1\mathcal{F}_b$ is closed under extensions and taking fundamental groups of finite graphs of groups [21] unlike the class of groups of finite $v\underline{\operatorname{cd}}$.

The following example of a group G, which was constructed by Dyer in [8] as a counter example to a conjecture related to residual finiteness, has the following properties

• G is a free product with amalgamation of groups of finite $v\underline{\mathrm{cd}}$,

• G is an extension of a finite group by a group of finite cohomological dimension and yet G is not of finite $v\underline{\mathrm{cd}}$.

$G = A \underset{H,\varphi}{*} B$ where

$$A = <a_1, a_2, a_3, a, d \mid [a_i, a_j] = [a_i, d] = [a, d] = d^p = 1,$$
$$a_1^a = a_2, a_2^a = a_3, a_3^a = a_1 a_2^{-3} a_3^2 >$$

$$B = <b_1, b_2, b_3, b, e \mid [b_i, b_j] = [b_i, e] = [b, e] = e^p = 1,$$
$$b_1^b = b_2, b_2^b = b_3, b_3^b = b_1 b_2^{-3} b_3^2 >$$

and

$$H = < a_1, a_2^p, a_3, d > \qquad \varphi(H) = < b_1^p, b_2, b_3^p, e >$$

and

$$\varphi(a_1) = b_1^p e \quad \varphi(a_2^p) = b_2 \quad \varphi(a_2) = b_3^p \quad \varphi(d) = e.$$

Note that $A \cong B \cong (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times C_p) \lhd \mathbb{Z}$, hence $v\mathrm{cd}A = v\mathrm{cd}B = 4$. It follows that $< d > \subseteq \cap\{N | G : N| < \infty\}$ hence G does not have a torsion-free subgroup of finite index.

Moreover we have the group extension

$$1 \longrightarrow < d > \longrightarrow G \longrightarrow K \longrightarrow 1 \qquad (**)$$

where K is a group with $\mathrm{cd}_{\mathbb{Z}} K < \infty$ which implies that the class of groups of finite $v\mathrm{cd}$ is not extension closed.

Now since K is in $H_1\mathcal{F}$ and $< d >$ is finite it follows from $(**)$ that G is in $H_1\mathcal{F}$.

It is worth mentioning that, it is not known whether $H_1\mathcal{F}$ is extension closed.

The class $H_1\mathcal{F}$ is closely related to the class of groups which admit a finite dimensional model for $\underline{E}G$, the classifying space for proper actions.

For every group G, there exists up to G-homotopy a unique G-CW-complex $\underline{E}G$ such that the fixed point space $\underline{E}G^H$ is contractible for every finite subgroup H of G and empty for infinite H. A G-CW-complex is called proper if all point stabilizers are finite (equivalently, if all its G-cells are of the form $G/H \times \sigma$ with H a finite subgroup of G). The space $\underline{E}G$ is an example of a proper G-CW-complex and it is referred to as the classifying space for proper actions, because it has the universal property, "for any proper G-CW-complex X there is a unique G-homotopy class of G-maps $X \to \underline{E}G$".

For a survey on classifying spaces see [18]. It is clear that the class of groups that admit a finite dimensional model for $\underline{E}G$ is a subclass of $H_1\mathcal{F}$.

Kropholler and Mislin in [15] actually proved that if G is an $H\mathcal{F}$-group such that $\dim|\Lambda(G)| < \infty$ and $\operatorname{proj.dim}_{\mathbb{Z}G} B(G,\mathbb{Z}) < \infty$ then G admits a finite dimensional model for $\underline{E}G$.

Moreover, it was shown in [26] that the condition (5) of Conjecture A implies that G admits a finite dimensional $\underline{E}G$, if G is a torsion-free locally soluble group. However it is an open question whether the class of groups which admit a finite dimensional $\underline{E}G$ is indeed a proper subclass of $H_1\mathcal{F}$.

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Mashhad R. J. Math. Sci., Vol.1(1)(2008) 23-40.

New classes of infinite groups

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Abstract

In this article, we consider some new classes of groups, namely, M_p-groups, T_0-groups, ϕ-groups, ϕ_0-groups, groups with finitely embedded involution, which were appeared at the end of twenties century. These classes of infinite groups with finiteness conditions were introduced by V.P. Shunkov. We give some review of new results on these classes of groups.

Keywords and phrases: ϕ-groups, involution, Frobenius group, Chernikov groups, quasi cyclic groups

AMS Subject Classification 2000: Primary 20E34; Secondary 20C99.

1 Introduction

At the beginning of twentieth century the classes of Frobenius groups and Chernikov groups were introduced.

A group of the form $G = F \rightthreetimes H$ is called a *Frobenius group*, if the following conditions are satisfied:

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†The work was supported by the Russian Fund of Fundamental Researches (grant 05-01-00576).

1) $H^g \cap H = 1$, $g \in G \setminus H$;

2) $G \setminus F = \bigcup_{g \in G} H^g \setminus \{1\}$.

Any finite extension of a direct product of finite number of quasi cyclic groups is called a *Chernikov group*.

In this article we consider some new classes of groups: M_p-groups, T_0-groups, Φ-groups, Φ_0-groups, groups with finitely embedded involution, which were introduced at the end of twentieth century. These classes of groups are closely connected with Frobenius and Chernikov groups. One of these classes, namely, T_0-groups firstly was discussed in Iran, while V.P. Shunkov took part as an invited speaker at the 22nd Annual Iranian Mathematics Conference held in Mashhad in 1991 and gave couple of talks on the concept of "T_0-groups" [20, 21].

Let G be a group, then an element $a \in G$ is called an *involution* or *almost regular* if it is of order two or its centralizer $C_G(a)$ is finite, respectively. The element $b \in G$ is called strictly real with respect to involution a, say, if it is transferred to its inverse by conjugating with a.

2 M_p-groups

Definition 2.1 *A group G is called an M_p-group, if for its infinite normal complete Abelian p-subgroup B with minimality condition and for any element a of order p the following conditions are satisfied:*

a) locally finite p-subgroups of $C_G(a)B/B$ are finite;

b) if some complete Abelian p-subgroup C of the group G contained in the set $\bigcup_{g \in G} \langle a, a^g \rangle$, then $C \leq B$.

The subgroups B and $\langle a \rangle$ of the group G are called the kernel and the hand of M_p-group, respectively.

We remind that the concept of M_p-groups was introduced by V.P. Shunkov at the end of 1983.

If a is the element of order p in an M_p-group G then the following properties hold:

1) all locally finite p-subgroups of $C_G(a)$ are finite;

2) all locally finite p'-subgroups of $C_G(a)$ are finite;

3) all locally finite subgroups of $C_G(a)$ are finite.

In the corresponding properties of the M_p-group G, the hand $\langle a \rangle$ is called p-finite hand, p'-finite hand and finite hand, respectively.

The following are the examples of M_p-groups (with finite hands):

Any Chernikov group with infinite p-subgroup and with almost regular element of order p is an M_p-group [15].

Any golomorfic extension of an infinite Chernikov p-group using the group of external automorphisms is also an M_p-group [15].

The kernel of an M_p-group may be different from the maximal complete Abelian p-subgroup of the group. Indeed, let $G = H \times T$, where $H = P \leftthreetimes (c)$, let P be an infinite complete Abelian p-group, $|c| = p$, $C_H(c)$ be finite, T is a free product of quasicyclic p-group S and cyclic group (b) of order p. Clearly G is an M_p-group with kernel P and with finite hand (bc), besides P is included in the maximal complete Abelian p-subgroup $P \times S \neq P$ [15].

Next theorem gives us the sign of non simplicity of an infinite group.

Theorem 2.2 *(Shunkov V.P. [15]). Let G be a group without involutions and B be its infinite complete Abelian p-subgroup, which satisfies the following conditions:*

1) $H = N_G(B)$ is an M_p-group with kernel B and p-finite hand $\langle a \rangle$;

2) for an arbitrary element $g \in G \setminus H^{\#}$, the subgroup $\langle a, a^g \rangle$ is finite;

3) $|C_G(a) : H \cap C_G(a)| < \infty$ and $H \cap C_G(a)$ contains all p'-elements of finite order from $C_G(a)$;

4) if Q is a finite $\langle a \rangle$-invariant q-subgroup from H with $Q \cap C_G(a) \neq 1$ and $q \neq p$, then $N_G(Q) \leq H$. Then $B \triangleleft G$.

Theorem 2.3 *(Shunkov V.P. [15]). Let G be a group without involutions, a an element of prime order p of G with centralizer $C_G(a)$, which is a finite p-subgroup, satisfies the condition that all subgroups $\langle a, a^g \rangle$, $g \in G$ are finite. Then G has* *complete Abelian normal p-subgroup B such that in G/B Sylow p-subgroups, containing element aB are finite and conjugate, and the number of such subgroups is finite.*

Condition of finiteness of subgroups $\langle a, a^g \rangle$ in the above theorem is necessary. Infact, it is enough to consider example of free product of infinite Chernikov p-group with almost regular element of order p ($p \neq 2$) and some non-trivial periodic group without involutions. Periodic product of such groups give us an example of periodic group without involutions, in which all the conditions of the theorem are valid, except the mentioned condition, but the statement of the theorem for such a group is incorrect.

The hand $\langle a \rangle$ is called *reduced* in M_p-group G with kernel B, if $B \cap C_G(a)$ is a reduced Abelian group. Remind that a *reduced Abelian group* is an Abelian group which dose not have any complete Abelian subgroups.

Theorem 2.4 *(Shunkov V.P. [16]). Let G be a group without involutions, B be its complete Abelian p-subgroup, a be an element of order p of G, satisfies the following conditions:*

1) $H = N_G(B)$ is an M_p-group with p-kernel B and reduced hand $\langle a \rangle$;

2) for every element $g \in G \setminus H^{\#}$, the subgroup $\langle a, a^g \rangle$ is finite;

3) $|C_G(a) : H \cap C_G(a)| < \infty$ and H contains all p'-elements of finite orders from $C_G(a)$;

4) if Q is a finite $\langle a \rangle$-invariant q-subgroup of H with condition $Q \cap C_G(a) \neq \langle 1 \rangle$ and $q \neq p$, then $N_G(Q) \leq H$. Then B is normal in G and G is an M_p-group with kernel B and reduced hand $\langle a \rangle$.

In M_p-group G the hand $\langle a \rangle$ is called *regular* if $C_G(a)$ is finite and every locally finite $\langle a \rangle$-invariant primary subgroup of G is finite.

In all well-known examples of M_p-groups with regular hand, which are not periodic almost nilpotent groups, satisfy the condition that any regular hand generates an infinite subgroup with some conjugates with it.

In this connection the next question will be occurred: Is M_p-group with regular hand exist which does not satisfy this condition and is not a periodic almost nilpotent group?

The answer to this question is negative in the class of groups without involutions (see the following theorem), that is in the class of groups without involutions periodic almost nilpotent M_p-groups with regular hand and only such groups do not satisfy this condition. But for M_p-groups with regular hand with involutions ($p \neq 2$), this question is still open even in the class of locally finite groups. The solution of this question is connected with the characterization of well-known simple groups in the class of periodic groups.

Now, we give some examples of M_p-groups with regular hands, see [17]:

— Infinite dihedral group is an M_2-group with regular hand;

— Novikov-Adian group [1] is an M_p-group with regular hand for any prime number p from the set of prime divisors of orders of elements of the group;

— Free product of non-trivial finite groups is an M_p-group with regular hand;

— Periodic product of groups without involutions [2] is an M_p-group with regular hand.

Theorem 2.5 *(Shunkov V.P. [17]). A group G is an M_p-group without involutions with regular hand $\langle a \rangle$ if and only if it is a periodic almost nilpotent group, when it satisfies the condition that: the subgroups $\langle a, a^g \rangle$, $g \in G$, are finite.*

Theorem 2.6 *(Shunkov V.P. [17]). A group G is an M_2-group with regular hand $\langle a \rangle$ if and only if is periodic almost Abelian group with finite Sylow subgroups, whenever all the subgroups $\langle a, a^g \rangle$, $g \in G$ are finite.*

For more information of the properties of the class of M_p-groups one may refer to V.P. Shunkov's monograph [18].

In [9], there was the sign of non-simplicity of an infinite group. As a corollary from that result the characterization of M_p-groups with involutions was studied.

M_p-groups with hands of orders not equal to two, were studied in the class of groups without involutions in [15] by V.P. Shunkov. M_p-groups with hands of orders 2 were studied by V.O. Gomer [6]. In the next theorem, the characterization of non-simplicity of M_p-group with hand of order not equal to three is discussed.

If G is an M_p-group with finite hand $\langle a\rangle$, then $C_G(a)$ can have infinite p-subgroups. For example, it is enough to take direct product of the above mentioned groups and Novikov-Adian free periodic group [1].

Theorem 2.7 *(Kozulin S.N., Senashov V.I., Shunkov V.P. [9]) Let G be a group, B be its infinite complete Abelian p-subgroup ($p \neq 3$), satisfying the following conditions:*

1) $H = N_G(B)$ is an M_p-group with a kernel B and p-finite hand $\langle a\rangle$;

2) for every $g \in G \setminus H^{\#}$, subgroups of the form $\langle a, a^g\rangle$ are finite and solvable;

3) $|C_G(a) : H \cap C_G(a)| < \infty$ and $H \cap C_G(a)$ contains all p'-elements of finite order from $C_G(a)$;

4) if Q is a finite $\langle a\rangle$-invariant q-subgroup from H with the condition that $Q \cap C_G(a) \neq 1$ and $q \neq p$, then $N_G(Q) \leq H$;

5) in G all finite $\langle a\rangle$-invariant p'-subgroups are solvable subgroups.

Then $B \lhd G$.

We shall remind, that the hand of M_p-group is called p-finite, if in $C_G(a)$, locally finite p-subgroups are finite.

This theorem generalizes V.O. Gomer's Theorem [6], for $p = 2$.

3 T_0-groups

At the beginning of last century the concept of a T_0-group appeared in the articles of V.P. Shunkov. This class of groups is defined by finiteness conditions. We recall the definition of the class of T_0-groups.

Definition 3.1 *Let G be a group with involutions, i be some of its involution. We call G to be a T_0-group, if it satisfies the following conditions:*

1) all subgroups of the form $< i, i^g >$, $g \in G$, are finite;

2) Sylow 2-subgroups of G are cyclic or generalized quaternions groups ;

3) the centralizer $C_G(i)$ is infinite and has a finite periodic part;

4) the normalizer of any non-trivial $< i >$-invariant finite subgroup of G is either contained in $C_G(i)$, or has a periodic part being a Frobenius group with Abelian kernel and with finite complement of even order;

5) $C_G(i) \neq G$ and for any element c of $G \setminus C_G(i)$, strictly real relating to i (i. e. such that $c^i = c^{-1}$), there exists an element s_c in $C_G(i)$, such that the subgroup $< c, c^{s_c} >$ is infinite.

Now we give the construction of Shunkov's example of T_0-group from [22] based on well-known example of S.P. Novikov and S.I. Adjan [1].

Example of T_0-group (a). Let $A = A(m,n)$ be a torsion-free group $A(m,n)$, which is a central extension of cyclic group with the group $B(m,n)$, for $m > 1$, $n > 664$ an odd number [1]. The group $A(m,n)$ has non-trivial center $Z(A) =< d >$ and $A/ < d >$ is isomorphic with $B(m,n)$ [1]. Let's consider a group $B = A \wr < x >$, where x is an involution.

Now take an element $u = d \cdot d^{-x}$ from $A \times A^x$. It is obvious, that $u \in Z(A \times A^x)$ and $u^x = u^{-1}$. As it is shown in [22], the group $G = B/ < u >$ and its involution $i = x \cdot < u >$ satisfy conditions (1)–(5) from the definition of T_0-group and $G = V \leftthreetimes < i >$, $C_G(i)$ is an infinite group with periodic part $< i >$, the all subgroups $< i, i^g >$ in G are finite and every maximal finite subgroup G with involution i is a dihedral group of order $2n$ and hence G is a T_0-group.

(b). Let $V = O(p)$ (see the definition of groups of the type $O(p)$, $C(\infty)$ in [11]). The group V has non-trivial center $Z(V) = (t)$ and $V/Z(V) = V/(t) \simeq C(\infty)$ [8].

Consider the group $T = V \wr (k) = (V \times V) \leftthreetimes (k)$, where k is an involution. Take the element $b = (t, t^{-1})$ of the group $V \times V$. Obviously, $b \in Z(V \times V)$ and $b^k = b^{-1}$. Assume $M = T/(b)$ be the factor group of T and take an involution $j = k(b)$ in M. From abstract properties of the groups $V = O(p)$, $C(\infty)$ [11], it is easy to show that the group M and its involution j satisfy the conditions 1) —

5) of the definition of T_0-group. Hence, $M = T/(b)$ is a T_0-group (with respect to the involution $j = k(b)$). Also note that in M any maximal periodic subgroup containing the involution j is the dihedral group of order $2p$.

Now, we deduce some results on T_0-groups. The detail of such kind of results can be found in [24].

Theorem 3.2 *(Shunkov V.P. [23, 26]). Let G be a group and a be an element of prime order p, satisfying the following conditions:*

(1) subgroups of the form $< a, a^g >$, $g \in G$, are finite and almost all are solvable;

(2) in the centralizer $C_G(a)$ the set of elements of finite order is finite;

(3) the normalizer of any non-trivial $< a >$-invariant finite subgroup of G has periodic part;

(4) for $p \neq 2$ and $q \in \pi(G)$, $q \neq p$, any $< a >$-invariant elementary Abelian q-subgroup of G is finite.

Then either G has almost nilpotent periodic part, or G is a T_0-group and $p = 2$.

Corollary 3.3 *Let G be a group and a an element of prime order $p \neq 2$, satisfying conditions 1) – 4) of previous Theorem. Then G has almost nilpotent periodic part.*

The following statement is equivalent to previous theorem and gives an abstract characterization of T_0-groups in the class of all groups.

Corollary 3.4 *Let G be a group and a an element of prime order p. The group G is a T_0-group and $p = 2$ if and only if for the pair (G, a) the conditions (1) – (4) of previous theorem are satisfied and the subgroup $< a^g | g \in G >$ is not periodic almost nilpotent.*

The particular case when $p = 2$ requires special consideration, since in this case condition (4) of previous theorem is superfluous, i.e. the following statements are true.

Corollary 3.5 *Let G be a group with involutions and i be one of its involutions, satisfying the following conditions:*

(1) subgroups of the form $< i, i^g >$, $g \in G$, are finite;

(2) the set of elements of finite order of $C_G(i)$ is finite;

(3) the normalizers of non-trivial $< i >$-invariant finite subgroups of G have periodic parts;

Then either G has almost nilpotent periodic part, or G is a T_0-group.

The conditions (1) – (3) of the above corollary are independent, i.e. each of them does not follow from the other two.

Theorem 3.6 *(Shunkov V.P. [23]). Let G be a group and a be an element of prime order p, satisfying the following conditions:*

1) subgroups of the form $< a, a^g >$, $g \in G$, are finite and almost all are solvable;

2) the centralizer $C_G(a)$ is finite;

3) $p \neq 2$ and for $q \in \pi(G)$, $q \neq p$, any $< a >$-invariant elementary Abelian q-subgroup of G is finite.

Then G is a periodic almost nilpotent group.

Theorem 3.7 *(Shunkov V.P. [23]). A non-trivial finitely generated group G is finite if and only if there exists an element $a \in G$ of prime order p satisfying the following conditions:*

(1) the subgroups of the form $< a, a^g >$, $g \in G$ are finite and almost all are solvable;

(2) the centralizer $C_G(a)$ is finite;

(3) when $p \neq 2$ and $q \in \pi(G)$, $q \neq p$, any $\langle a \rangle$-invariant elementary Abelian q-subgroup is finite.

Theory of T_0-groups is created by V.P.Shunkov [24].

4 Φ-groups

We devote this section for investigating the properties of a class of Φ-groups.

This class group is rather broad: among them are groups of Burnside type [1], Ol'shanskii monsters [11]. It is very closely connected with the groups of Burnside type of odd period $n \geq 665$.

Definition 4.1 *Let G be a group, and i be an involution of G, satisfying the following conditions:*

(1) all subgroups of the form $< i, i^g >, g \in G$, are finite;

(2) $C_G(i)$ is infinite and has a layer-finite periodic part;

(3) $C_G(i) \neq G$ and $C_G(i)$ is not contained in any other subgroups of G with a periodic part;

(4) if K is a finite subgroup of G, which is not contained in $C_G(i)$ and $V = K \cap C_G(i) \neq 1$, then K is a Frobenius group with complement V.

A group G with some involution i satisfying these conditions (1–(4) is called a Φ-group.

This class of groups has been introduced by V.P.Shunkov.

The example from previous section is an example of Φ-group

Theorem 4.2 *(Senashov V.I. [7]). An Φ-group G satisfies the properties:*

(1) all involutions are conjugate;

(2) Sylow 2-subgroups are locally cyclic or finite generalized quaternions groups;

(3) there are infinitely many elements of finite orders in G, which are strictly real with respect to the involution i and for every such element c of this set there exists an element s_c from the centralizer of i such that $< c, c^{s_c} >$ is an infinite group.

V.P. Shunkov posed the problem of studying groups with some additional limitations provided that for a given finite subgroup B, the following condition is valid: normalizer of any non-trivial B-invariant finite subgroup has a layer-finite periodic part.

This problem is partly solved for the class of locally soluble groups and for the case $\mid B \mid = 2$ under more general limitations, it is solved with Φ-groups accuracy.

Theorem 4.3 *(Ivko M.N., Senashov V.I. [7]). A periodic locally soluble group is layer-finite if and only if for some of its finite subgroup B, the normalizer of any non-trivial B-invariant finite subgroup is layer-finite.*

Theorem 4.4 *(Senashov V.I. [7]). Let G be a group and a be an involution of G, satisfying the following conditions:*

1. All subgroups of the form $< a, a^g >, g \in G$, are finite;

2. Normalizer of every non-trivial $< a >$-invariant finite subgroup has a layer-finite periodic part.

Then either the set of all elements of finite orders forms a layer-finite group or G is a Φ-group.

In the last two theorems, layer-finite groups are characterized for the class of locally solvable groups and groups with a layer-finite periodic part in more general case with Φ-groups accuracy. It is possible to find more information on layer-finite groups in [14].

The following theorem characterizes finite groups with Φ-groups accuracy.

Theorem 4.5 *(Senashov V.I. [7]). Let G be a group with involutions and i be some involution from G satisfying the following conditions:*

(1) G is generated by the involutions which are conjugate with i;

(2) almost all groups $< i, i^g >$ are finite, for all $g \in G$;

(3) normalizer of every $< i >$-invariant finite subgroup has a layer-finite periodic part.

Then G is either finite or a Φ-group.

5 Φ_0-groups

In this section we investigate the properties of the class of Φ_0-groups, which is a subclass of the class of Φ-groups. Such groups are very close to T_0-groups, but in this section we show their differences.

Definition 5.1 *Let G be a group and i be an involution of G, satisfying the following conditions:*

(1) all subgroups of the form $< i, i^g >, g \in G$, are finite;

(2) $C_G(i)$ is infinite and has a finite periodic part;

(3) $C_G(i) \neq G$ and $C_G(i)$ is not contained in any other subgroup of G with a periodic part;

(4) if K is a finite subgroup of G, which is not inside $C_G(i)$ and $V = K \cap C_G(i) \neq 1$, then K is a Frobenius group with complement V.

The group G with some involution i satisfying the above conditions (1)–(4) is called a Φ_0-group.

This class of groups has been introduced by V.P.Shunkov.

In [22], V.P. Shunkov raised the next question for discussion:

Do the classes of Φ_0-groups T_0-groups coincide or not?

In the same article, V.P. Shunkov specially emphasized that the most difficult problem is to establish the satisfiability for Φ_0-group of conditions (4) and (5) from the definition of T_0-group.

In [13] V.I. Senashov proved that Φ_0-group satisfies all conditions from the definition of T_0-group except the fourth condition. In the same article he constructed an example of Φ_0-group which is not a T_0-group, i. e. it was shown that the fourth condition does not hold in every Φ_0-gruop.

Example of Φ_0-group Let's take isomorphic copies of the T_0-groups $G = V \leftthreetimes (i)$ from [22]:

$$G_1 = V_1 \leftthreetimes (i_1), G_2 = V_2 \leftthreetimes (i_2), ..., G_n = V_n \leftthreetimes (i_n), ...$$

In the cartesian product of the groups G_n, $n = 1, 2, ...$, consider the subgroup $U = W \rightthreetimes (j)$, where W is a direct product of subgroups V_n, $n = 1, 2, ...$, and $j = i_1 \cdot i_2 \cdot ...$ is an involution from the cartesian product of G_n, $n = 1, 2, ...$ One can check that such a group U is a Φ_0-group. It is easy to see that the fourth condition from the definition of T_0-group is incorrect for the group U.

6 Groups with Finitely Embedded Involution

It is necessary to introduce the next concept, which was comprehended by V.P. Shunkov at the end of the 80th.

Let G be a group, i some of its involution and $\mathfrak{L}_i = \{i^g | g \in G\}$ the set of conjugations of i in G. We shall call the involution i *finitely embedded in* G, when for any element g of G the intersection $(\mathfrak{L}_i\mathfrak{L}_i) \cap gC_G(i)$ is finite, where $\mathfrak{L}_i\mathfrak{L}_i = \{i^{g_1}i^{g_2} | g_1, g_2 \in G\}$.

Let us give the simplest examples of the groups with a finitely embedded involution.

1. If in a group G there exists an involution i with finite centralizer $C_G(i)$, then i is a finitely embedded involution in G.

2. If in some group G the involution i is contained in a finite normal subgroup of G, then i is a finitely embedded involution in G.

3. Let G be a Frobenius group with the periodic kernel and infinite complement H, containing an involution i. Then i is a finitely embedded involution in G.

An involution of a group is called *finite* one, if it generates a finite subgroup with all of its conjugations are involution.

Now let us formulate some results, which the following is the main one.

Theorem 6.1 *(Shunkov V.P. [19]). Let G be a group, i be its finite and finitely embedded involution, $\mathfrak{L}_i = \{i^g | g \in G\}$, $B =< \mathfrak{L}_i >$, $R =< \mathfrak{L}_i\mathfrak{L}_i >$, Z be a subgroup generated by all 2-elements from R. Then B, R and Z are normal subgroups in G and one of the following statements is valid:*

(1) B is a finite subgroup;

(2) B is locally finite, $B = R \leftthreetimes (i)$ and Z is a finite extension of a complete Abelian 2-subgroup A_2 with the minimal condition and $ici = c^{-1}(c \in A_2)$.

A number of corollaries follow from the above theorem.

Corollary 6.2 *If a group has a finite involution with a finite centralizer, then it is locally finite.*

Corollary 6.3 *If a periodic group has an involution with a finite centralizer, then it is locally finite.*

Corollary 6.4 *If a finitely embedded involution exists in a group, then its closure is a periodic subgroup.*

Corollary 6.5 *A simple group with involutions is finite if and only if some of its involutions is finite and is finitely embedded.*

As in a periodic group any involution is finite, the next result follows from the last corollary.

Corollary 6.6 *A periodic simple group with involutions is finite if and only if some of its involutions are finitely embedded.*

Corollary 6.7 *Let G be a group, H a subgroup containing a finite involution i, and (G, H) is a Frobenius pair. Then G is a Frobenius group with a periodic Abelian kernel and with a complement $H = C_G(i)$ if and only if i is a finitely embedded involution in G.*

The above corollary is incorrect even for periodic groups, if the involution i is not finitely embedded.

R. Brauer proved the next result in the middle of twentieth century.

Theorem 6.8 *(Brauer R. [3]). There exists only finite number of finite simple groups with a given centralizer of involution.*

Proposition 6.9 *(Shunkov V.P.([25], [27].) For an arbitrary natural number M there is only finite number of finite simple groups G with involution τ satisfies to condition $|C_G(\tau) \cap \tau^G| \leq M$.*

Definition 6.10 *Let G be a group and τ be its involution. The number*

$$t(G, \tau) = \max_{g \in G} |gC_G(\tau) \cap (\tau^G \tau^G)|$$

is called the parameter of embedding of involution τ in the group G.

This concept is the basis of the next generalization for periodic groups of Brauer Theorem which was developed by V.P. Shunkov in 2001 (for the announcement see [25], [27]).

There is only finite number of periodic simple groups with involution and with given finite parameter of embedding of this involution, besides all the others are also finite.

Modulo the classification of finite simple groups, it is enough to verify the hypothesis for infinite families of alternating groups and for groups of Lie type.

Theorem 6.11 *(Golovanova O.V. [5]). Let M be natural number and G a finite simple group with one class of conjugate involutions. Then for any involution τ of G, $|C_G(\tau) \cap \tau^G| > M$, for large enough $|G|$.*

Theorem 6.12 *(Golovanova O.V., Levchuk V.M. [4]). Let M be an arbitrary natural number and $G = S_n,\ A_n$ or $PSL_n(q)$ with even q. Then for any involution τ of G, $|C_G(\tau) \cap \tau^G| > M$, for large enough $|G|$.*

The last two theorems verify hypothesis of V.P. Shunkov for mentioned classes of groups.

For another class of groups with similar assumptions are investigated in the next theorems, which were proved in 2006.

Theorem 6.13 *(Golovanova O.V. [4]).Let M be an arbitrary natural number and G a group of Lie type over the field of even order. If the order of G is large* *enough, then there is an involution τ in the group G with condition $|C_G(\tau) \cap \tau^G| > M$.*

Theorem 6.14 *(Golovanova O.V., Levchuk V.M. [4]).Let M be an arbitrary natural number and $G = PSL_n(q)$ with odd q. If the of G is large enough, then for any diagonalizable involution τ of G the inequality $|C_G(\tau) \cap \tau^G| > M$ is valid.*

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Mashhad R. J. Math. Sci., Vol.1(1)(2008) 41-50.

Groups with soluble minimax conjugate classes of subgroups

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Abstract

A classical result of Neumann characterizes the groups in which each subgroup has finitely many conjugates only as central-by-finite groups. If $\mathfrak{X}$ is a class of groups, a group G is said to have $\mathfrak{X}$-conjugate classes of subgroups if $G/core_G(N_G(H)) \in \mathfrak{X}$ for each subgroup H of G. Here we study groups which have soluble minimax conjugate classes of subgroups, giving a description in terms of $G/Z(G)$. We also characterize FC-groups which have soluble minimax conjugate classes of subgroups.

Keywords and phrases: Conjugacy classes; soluble minimax groups, FC-groups, polycyclic groups.

AMS Subject Classification 2000: Primary 20F24; Secondary 20F14.

1 Introduction

Following [11], the class of all *abelian minimax* groups is the class of all *max*-by-*min* abelian groups. A group G is called *soluble minimax* if it has a finite

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characteristic series $1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$ whose factors are abelian minimax groups. Moreover a soluble minimax group is said to be *reduced minimax* if it has no nontrivial normal Chernikov radicable subgroups. Fundamental properties of soluble minimax and reduced minimax groups are described in [11].

Let $\mathfrak{X}$ be a class of groups. A group G is said to be an $\mathfrak{X}C$-group, if $G/C_G(x^G) \in \mathfrak{X}$ for all $x \in G$. If $\mathfrak{X}$ is the class of all finite groups, we obtain the class of FC-groups; Baer in [1] introduced this class of groups. If $\mathfrak{X}$ is the class of all polycyclic-by-finite groups, then the class of PC-groups are obtained which are introduced in [2]. If $\mathfrak{X}$ is the class of all Chernikov groups, then one obtains the class of CC-groups and introduced in [9].

If $\mathfrak{X}$ is the class of all (soluble minimax)-by-finite groups, we obtain the class of MC-groups and when $\mathfrak{X}$ is the class of all (reduced minimax)-by-finite groups, then the class of M_rC-groups is obtained. These classes of groups are introduced in [4].

Let $\mathfrak{X}$ be a class of groups. A group G is said to be an $\mathfrak{X}CS$-group, or a group with $\mathfrak{X}$-conjugate classes of subgroups, if $G/core_G(N_G(H)) \in \mathfrak{X}$ for each subgroup H of G.

If $\mathfrak{X}$ is the class of all finite groups, we obtain the class of FCS-groups. Neumenn in [8] has investigated FCS-groups with a different approach. The current approach can be found in [6]. If $\mathfrak{X}$ is the class of all polycyclic-by-finite groups, one obtains the class of PCS-groups, which are studied in [6]. If $\mathfrak{X}$ is the class of all Chernikov groups, we obtain the class of CCS-groups, which are described in [7] and [10].

If $\mathfrak{X}$ is the class of all (soluble minimax)-by-finite groups, we obtain the class of MCS-groups. In particular, if $\mathfrak{X}$ is the class of all (reduced minimax)-by-finite groups, then the class of M_rCS-groups are obtained.

The present paper is devoted to the studying the classes of MCS and M_rCS-groups. We prove the following description of the groups with soluble minimax conjugate classes of subgroups.

2 Main Theorem

(i) *Let G be a periodic group. Then G is an MCS-group if and only if it is central-by-Chernikov*;

(ii) *Let G be an MCS-group. If $InnG$ has finite abelian subgroup rank, then G is central-by-(soluble minimax)-by-finite*;

(iii) *Let G be an MCS-group. If G contains proper maximal abelian normal subgroups, then G is (soluble minimax)-by-finite-by-abelian.*

Our group-theoretic notation is standard and refered to [11]. Section 2 contains the preparatory results, which are used in Section 3 to prove the Main Theorem. Section 3 is devoted to give the proof of Main Theorem. In section 4, we describe some special classes of MCS-groups.

3 Preliminary results

By defintion each PCS-group is an MCS-group and each CCS-group is an MCS-group. In [6] and [7] some classes of MCS-groups are studied, giving a first answer to Main Theorem.

We omit the elementary proofs of the next two results.

Lemma 2.1. *Let G be a central-by-(soluble minimax)-by-finite group. If H is a subgroup of G, then $H/core_G(H)$ is (soluble minimax)-by-finite group.*

Lemma 2.2. *Let G be an MCS-group. If $L \triangleleft H \leq G$, then H/L is an MCS-group.*

Lemma 2.3. *Let G be a periodic group. If G is an MCS-group, then G is a CCS-group.*

Proof. For each subgroup H of G, $G/core_G(N_G(H))$ is periodic (soluble minimax)-by-finite, so it is Chernikov by [11, vol.II,p.166].

The following lemma extends [6, Corollary 2.7] and [7, Lemma 2.3].

Lemma 2.4. *If G is an MCS-group, then G is an MC-group.*

Proof. If G is periodic, then the result follows by Lemma 2.3 and [7, Lemma 2.3]. If G is a PCS-group, then the result follows by [6, Corollary 2.7]. Let G be neither periodic nor a PCS-group. Take $g \in G$ and assume $H = core_G(N_G(< g >))$, $H_1 = C_H(< g >)$, $H_2 = core_G(H_1) = C_H(g^G)$. We have that G/H is (soluble minimax)-by-finite, $H \leq N_G(< g >)$ and $H/C_H(< g >)$ is finite abelian. It is sufficient to prove that G/H_2 is (soluble minimax)-by-finite.

Since

$$H_2 = \bigcap_{x \in G} (C_H(< g >))^x = \bigcap_{x \in G} C_H(< g >^x) = \bigcap_{x \in G} C_H(< g^x >)$$

and $H/(C_H(< g >))^x \simeq H/C_H(< g >)$ for every $x \in G$, we obtain the embedding

$$H/H_2 \hookrightarrow \prod_{x \in G} H/H_1^x.$$

In particular we deduce that H/H_2 is a bounded abelian group. Lemmas 2.2 and 2.3 imply that G/H_2 is an MCS-group such that H/H_2 is a periodic normal CCS-subgroup of G/H_2. H/H_2 has no nontrivial Chernikov normal subgroups, so [7, Lemma 2.5] implies that H/H_2 is central-by-finite. By definition we can find a subgroup A/H_2 of $Z(H/H_2) \leq Z(G/H_2)$ such that $(H/H_2)/(A/H_2) \simeq H/A$ is finite. Obviously G/A is (soluble minimax)-by-finite, so G/H_2 is central-by-(soluble minimax)-by-finite. By Lemma 2.1, H/H_2 is (soluble minimax)-by-finite, and so is G/H_2.

There are MC-groups which are not MCS-groups, improving [3, Proposition 2.2].

Example 2.5. Here exibit a metabelian 2-nilpotent MC-group G which is not an MCS-group. Let p be a prime number and C a nontrivial subgroup of the additive group of rational numbers, whose denominators are p-numbers. Let $Q = Dr_{n \in \mathbb{N}} < x_n >$ be a free abelian group of countably infinite rank. Denote multiplicatively the operation in C and let $C = \{c_n | n \in \mathbb{N}\} \cup \{1\}$, where $c_n \neq 1$ for all n and $c_n \neq c_m$ if $n \neq m$. A central extension $C \rightarrowtail G \twoheadrightarrow Q$ can be defined by putting $[x_{2i-1}, x_{2i}] = c_i$ for all $i \in \mathbb{N}$ and $[x_i, x_j] = 1$, otherwise. Given $z \in G \setminus C$, $z = cx_{i_1}^{k_1} \dots x_{i_t}^{k_t}$, where $c \in C$, $i_1 < \dots < i_t$ and $k_{i_1} \neq 0$. Put $y = x_{i_1 - 1}$ if i_1 is even and $y = x_{i_1+1}$ if i_1 is odd. Then $[x_{i_j}, y] = 1$ if $j > 1$, so that $[z, y] = [x_{i_1}^{k_1}, y] = [x_{i_1}, y]^{k_1} \neq 1$ and $Z(G) = G' = C$.

Moreover, $[z, G] =< [z, x_j] : i_1 - 1 \leq j \leq i_t + 1 >$, so that $[z, G]$ is finitely generated and hence it is cyclic. By construction we have that z^G is (infinite cyclic)-by-cyclic and G is an M_rC-group (precisely G is a PC-group). The subgroup $H = Dr_{i\in\mathbb{N}} < x_{2i} >$ of G has $K = N_G(H) = core_G(N_G(H)) = CH$, so that $G/K \geq Dr_{i\in\mathbb{N}} < x_{2i-1}K >$ and G/K has infinite abelian rank.

To convenience the reader, we recall two properties of MC-groups.

Lemma 2.6. *Let G be an MC-group and $x_1, \dots, x_n \in G$. If $X =< x_1, \dots, x_n >$, then X^G is (soluble minimax)-by-finite. Moreover, if G is an M_rC-group then X^G and $G/C_G(X^G)$ are reduced minimax.*

Proof. It follows by [4, Theorem 2].

Lemma 2.6 shows that an MC-group can be covered by normal (soluble minimax)-by-finite subgroups (see [4, p.161-162]). [2, Theorem 2.2] and [11, Theorem 4.36] give the corresponding condition for PC-groups and CC-groups.

Proposition 2.7. *If G is an MC-group, then it is locally-(normal and (soluble minimax)-by-finite). Moreover if G is an MC-group then G' is locally-(normal and (soluble minimax)-by-finite).*

Proof. It follows by Lemma 2.6.

4 Proof of the Main Theorem

Proof. (i) By Lemma 2.3, G is a periodic CCS-group and so [10, Main Theorem] implies that G is central-by-Chernikov. Conversely, let G be central-by-

Chernikov, H be a subgroup of G such that $H \not\leq Z(G)$ and $K = core_G(N_G(H))$. If $K \geq Z(G)$, then the result obviously is obtained. If $K \cap Z(G) = 1$, then K is isomorphic with $KZ(G)/Z(G)$, so it is Chernikov and G/K is isomorphic with $(G/Z(G))/(KZ(G)/Z(G))$, which is again Chernikov.

(ii) Since $InnG \simeq G/Z(G)$, we may suppose that $G/Z(G)$ has finite abelian subgroup rank. Lemma 2.2 implies that $G/Z(G)$ is an MCS-group, so it is an MC-group, by Lemma 2.4. Thanks to Proposition 2.7, $G/Z(G)$ can be covered by (soluble minimax)-by-finite normal subgroups $S_\lambda/Z(G)$, where λ is an ordinal, indiciated in Λ. Without loss of generality assume $Z(G) = 1$. We exibit a covering of G with subgroups T_α such that $\alpha \in A \leq \Lambda$, $T_\alpha < T_{\alpha+1}$, T_α is (soluble minimax)-by-finite and $T_\beta = T_{\beta+1} = \ldots$ for an ordinal $\beta \in A$.

$G =< S_\lambda : \lambda \in \Lambda >$ and we obviously conclude when λ is a limit ordinal, so let λ be not a limit ordinal. By induction the chain

$$T_1 = \bigcap_{\lambda \in \Lambda} S_\lambda,$$

$$T_\alpha =< T_{\alpha-1}, x_{\alpha-1} >, \quad \text{where } \ \mathrm{x}_{\alpha-1} \notin \mathrm{T}_{\alpha-1}$$

has $T_\alpha < T_{\alpha+1}$, T_α is (soluble minimax)-by-finite, $A \leq \Lambda$. $H = Dr_{\alpha \in A} < x_\alpha >$ has infinite abelian rank which is a contradiction. It follows that G can be covered by finitely many (soluble minimax)-by-finite normal subgroups T_α, so that G is (soluble minimax)-by-finite.

(iii) Let A be a proper maximal abelian normal subgroup of G. By Lemma 2.4 and [5, Corollary 3], A has finite index in G. It is enough to verify that G' is (soluble minimax)-by-finite. If G is periodic the result follows Lemma 2.4 and by [7, Lemma 3.7]. A similar situation happens when G is a PCS-group by [6, Lemma 3.1]. Let G be an MCS-group which is neither periodic nor a PCS-group. Put $X = \{x_1, \ldots, x_n\}$ a transversal to A in G, $G/A = \{x_1A, \ldots, x_nA\}$ and $G = XA$. Lemmas 2.4 and 2.6 imply that $X^G = Y$ is (soluble minimax)-by-finite, in particular $G' = [G, G] = [YA, YA] = Y'[Y, A]$. Now Y' is (soluble minimax)-by-finite and $[Y, A] \leq Y^A = (X^G)^A = Y$ is (soluble minimax)-by-finite, and so G' is.

5 Special classes of MCS-groups

The Example in [7] shows that there is a CCS-group G such that $G/Z(G)$ has infinite abelian rank. The consideration of this group does not yield to characterize an MCS-group G without restrictions on the rank of $G/Z(G)$. On the other hand, the restriction on the size of Frattini subgroup of an MCS-group gives rise the structural informations.

Corollary 4.1. *Let G be an MCS-group. If G contains a subgroup H such that $N_G(H)$ has a non-generator element g of G, then G is (soluble minimax)-by-(radicable nilpotent of class at most 2).*

Proof. By Lemma 2.4, G is an MC-group such that $FratG \geq N_G(H)$, but $FratG = core_G(FratG) \geq core_G(N_G(H))$ and [5, Theorem 4] complete the proof.

Given a group G, a subgroup H of G is said to be $\mathfrak{F}$-perfect if H has no proper subgroups of finite index (in H). The subgroup $\mathfrak{F}(G)$ of G generated by all normal $\mathfrak{F}$-perfect subgroups of G is clearly $\mathfrak{F}$-perfect. This subgroup is called the $\mathfrak{F}$-perfect part of G and if $D(G)$ is the subgroup of G generated by all periodic radicable abelian normal subgroups of G, then $D(G) \leq \mathfrak{F}(G)$.

Corollary 4.2. *If G is an $\mathfrak{F}$-perfect MCS-group, then G is metabelian.*

Proof. Put $R = \mathfrak{F}(G)$ and $D = D(G)$, then Lemma 2.4 and [5, Lemma 2] imply that the series $1 \triangleleft D \triangleleft R = G$ has abelian factors.

The notion of Fitting subgroup allows us to characterize an M_rCS-group.

Proposition 4.3. *Let G be an M_rCS-group and H a subgroup of G. Then G is central-by-polycyclic-by-finite if and only if $Fit(G/core_G(N_G(H))$ is finitely generated.*

Proof. Let $G/Z(G)$ be polycyclic-by-finite and $H \leq G$. Put $K = core_G(N_G(H))$, we may assume that $K \not\leq Z(G)$. If $K \geq Z(G)$ then the result follows immediately. If $K \cap Z(G) = 1$, then $K \simeq KZ(G)/Z(G)$ is polycyclic-by-finite, and hence so is G. It follows that $Fit(G/K)$ is finitely generated. Conversely, if G is an M_rCS-group, then $Fit(G/K)$ is nilpotent by [11, Theorem 10.33]. $Fit(G/K)$ is finitely generated so that G is a PCS-group. Now the main Theorem of [6] completes our proof.

A special situation happens for the class of FC-groups.

Proposition 4.4. *Let G be an FC-group. Then the following conditions are equivalent:*

(i) G is FCS-group;

(ii) G is CCS-group;

(iii) G is PCS-group;

(iv) G is MCS-group;

(v) G is central-by-finite.

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious. (v) $\Rightarrow$ (i) is described in [7, Proposition 2.4].

(ii) $\Rightarrow$ (iii). By [7, Proposition 2.4], the class of CCS-groups coincide with the class of FCS-groups, but each FCS-group is a PCS-group, which gives the result.

(iv) $\Rightarrow$ (v). Put U to be the maximal torsion-free subgroup of $Z(G)$ and $G/Z(G)$ is periodic (see [11, Theorem 4.32]), so it implies that G/U is also periodic. If T is the periodic part of G and G/T is torsion-free abelian, then $T \cap U = 1$ and $G \hookrightarrow G/T \times G/U$. By Lemma 2.2 and the Main Theorem of [10] implies that G/U is central-by-finite. Since G/T is abelian and $G/T \times G/U$ is central-by-finite, we conclude that G is central-by-finite.

Acknowledgment

The author is indebted to Dr. A. Scarinzi, C. Sasso and Dr. M. Sasso for the translations of the original papers. Special thanks are due to the Mathematics Department of Wuerzburg for the hospitality.

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Mashhad R. J. Math. Sci., Vol.1(1)(2008) 51-58.

A numerical solution for an inverse heat conduction problem

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Abstract

In this paper, we demonstrate the existence and uniqueness a semi-analytical solution of an inverse heat conduction problem (IHCP) in the form : $u_t = u_{xx}$ in the domain $D = \{(x,t)|\ 0 < x < 1,\ 0 < t \leq T\}$, $u(x,T) = f(x)$, $u(0,t) = g(t)$, and $u_x(0,t) = p(t)$, for any $0 \leq t \leq T$. Some numerical experiments are given in the final section.

Keywords and phrases: Inverse heat conduction problem, semi-analytical solution, finite difference method

AMS Subject Classification 2000: Primary 35R30; Secondary 18G10.

1 Introduction

The procedure to solve an IHCP is very important in determining unknown temperature histories and heat flux from known values in the body, which are usually

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measured as a function of space and time. Especially, a direct measurement of the heat flux or temperature in a boundary or initial time of hot body is almost impossible. Therefore, recent studies of IHCPs have been numerically treated and extended of multiple dimensions with the help of computing architecture. Some numerical and theoretical approaches to IHCPs are summarized in [4], [2]. Such a procedure using an exact solution are given by Burggraf [3]. It has been shown that, if an error is made in known boundary condition, then there will be some errors in unknown heat flux of other boundary. A lower bound of this error can be estimated by $\frac{1}{\sqrt{\Delta t}}\sinh(\frac{1}{\sqrt{\Delta t}})$. These results are consistent with earlier observation that small values of time Δt can produce large error in surface flux. In this paper, we apply a finite difference method of semi-implicit type for $\frac{\partial u}{\partial t}$ and use a parameter θ_M for driving a stable and convergent solution to the IHCPs.

Now, suppose that for any given t, $0 \leq t \leq T,\ u(x,t) \in C^4[0,1]$ and satisfying

$$\begin{aligned}
u_t(x,t) &= u_{xx}(x,t),\ in\ \ D = \{(x,t)|\ 0 < x < 1,\ 0 < t \leq T\}, && (1.1)\\
u(x,T) &= f(x), \qquad\qquad 0 \leq x \leq 1, && (1.2)\\
u(0,t) &= g(t), \qquad\qquad\qquad 0 \leq t \leq T, && (1.3)\\
u_x(0,t) &= p(t), \qquad\qquad\qquad 0 \leq t \leq T, && (1.4)\\
u_x(1,t) &= h(t), \qquad\qquad\qquad 0 \leq t \leq T, && (1.5)\\
u(x,0) &= \phi(x), \qquad\qquad 0 \leq x \leq 1, && (1.6)
\end{aligned}$$

where $f(x),\ g(t)$, and $p(t)$ are piecewise-continuous known functions, T is a given positive constant number and $h(t),\ \phi(x)$, and $u(x,t)$ are unknown functions, which remain to be determined.

In the next section, we discrete the variable t and reduce $(1.1)-(1.4)$ to a system of linear, nonhomogenous second order differential equations. Stability and convergency of this method is studied in section 3. Some numerical result and discussion are given in section 4.

2 A Numerical Solution

In this section, we discrete the variable t to approximate the solution of (1.1) − (1.4).

Let $M \in \mathbf{N}$, $\Delta t_M = \frac{T}{M}$, and $t_i = i\Delta t$, for $i = 0,\ 1,\ ...,\ M$. For the solution u, we define $u_i(x) = u(x, t_i)$, for $i = 0,\ 1,\ ...,\ M$. Similarly for a given sequence of functions $\{u_i(x)|\ i = 0,\ 1,\ ...,\ M\}$ we use $\hat{u}_i(x)$ instead of the approximate of $u_i(x)$. Putting

$$\hat{u}_{i+1}(x) = \hat{u}_i(x) + (\theta_M \frac{\partial \hat{u}(x,t_i)}{\partial t} - (\theta_M - 1)\frac{\partial \hat{u}(x,t_{i+1})}{\partial t})\Delta t_M, \tag{2.1}$$

for $\theta_M \geq 0$, then by using (2.1) into (1.1) − (1.4) we obtain a system of linear nonhomogenous second order differential equations with given initial conditions in the form

$$\begin{aligned} \theta_M \Delta t_M \hat{u}_i''(x) + \hat{u}_i(x) &= \hat{u}_{i+1}(x) + (\theta_M - 1)\Delta t_M \hat{u}_{i+1}''(x), && (2.2)\\ \hat{u}_i(0) &= g(t_i) = g_i, && (2.3)\\ \hat{u}_i'(0) &= p(t_i) = p_i, && (2.4) \end{aligned}$$

for $i = 0,\ 1,\ ...,\ M-1$, where $\hat{u}_0(x)$ $(0 < x \leq 1)$ and $\hat{u}_i(1), i = 0,\ 1,\ ...,\ M-1$ are unknown.

Clearly $\hat{u}_M(x) = u_M(x) = f(x)$. Using these assumptions, the problem (2.2)–(2.4) has a solution of the following form

$$\hat{u}_i(x) = g_i \cos\frac{x}{\sqrt{\theta_M \Delta t_M}} + \frac{\hat{u}_i'(1) - \hat{V}_i'(1)}{\hat{W}_i'(1)}\ p_i \sin\frac{x}{\sqrt{\theta_M \Delta t_M}} + F_i(x), \tag{2.5}$$

for $i = 0,\ 1,\ ...,\ M-1$, where

$$\begin{aligned} F_i(x) &= \frac{1}{\theta_M\sqrt{\theta_M \Delta t_M}} \int_0^x \hat{u}_{i+1}(s) \sin\frac{x-s}{\sqrt{\theta_M \Delta t_M}}\, ds \\ &- \frac{\theta_M - 1}{\theta_M}\sqrt{\Delta t_M}\ p_{i+1} \sin\frac{x}{\sqrt{\theta_M \Delta t_M}} + \frac{\theta_M - 1}{\theta_M}\hat{u}_{i+1}(x) \\ &- \frac{\theta_M - 1}{\theta_M}\ g_{i+1} \cos\frac{x}{\sqrt{\theta_M \Delta t_M}},\quad i = 0,\ 1,\ ...,\ M-1, \end{aligned} \tag{2.6}$$

$\hat{V}_i$ and $\hat{W}_i$ are the solutions of the following problems, respectively,

$$\theta_M \Delta t_M \hat{V}_i''(x) + \hat{V}_i(x) = \hat{u}_{i+1}(x) + (\theta_M - 1)\Delta t_M \hat{u}_{i+1}''(x),$$

$$\begin{aligned} \hat{V}_i(0) &= g_i, \\ \hat{V}_i'(0) &= p_i, \qquad i = 0,\ 1,\ ...,\ M-1, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \theta_M \Delta t_M \hat{W}_i''(x) + \hat{W}_i(x) &= 0, \\ \hat{W}_i(0) &= 0, \\ \hat{W}_i'(0) &= 1, \qquad i = 0,\ 1,\ ...,\ M-1. \end{aligned} \tag{2.8}$$

Clearly, the heat flux $\hat{u}_x(1, t_i)$ may be obtained from (2.6) of the form

$$\hat{u}_x(1, t_i) = p_i\ \hat{W}_i'(1) + \hat{V}_i'(1), \qquad\qquad i = 0,\ 1,\ ...,\ M-1. \tag{2.9}$$

Now, for each $n \in \mathbf{N}$, if $\theta_M \Delta t_M \neq (n\pi)^{-2}$ and $f''(x)$ is a piecewise-continuous function in $[0, 1]$, then the system of solutions (2.5) are unique [1].

The above result may be summarized in the following statement.

Theorem 2.1. *If, for each* $n \in\mathbf{N}$, $\theta_M \Delta t_M \neq (n\pi)^{-2}$, *and* f'' *is a piecewise-continuous function in* $[0, 1]$, *then the system of differential equations (2.2)-(2.4) has a unique solution.*

Proof. See the analysis preceding the of above theorem.

3 Stability and convergency of solution

In the next theorem, the convergency of the solution (2.5) to the unique solution will be shown.

Theorem 3.1. *If* θ_M, Δt_M *and* f *satisfy the assumptions of Theorem 2.1 and* $|\frac{\partial^2 u(x,t)}{\partial t^2}| \leq C < \infty$ *for any* $t \in [0, 1]$, *where* C *is a positive constant number and* $\theta_M = \beta_M \Delta t^{-\alpha_M}$ *such that* $\beta_M > 0$, $1 + \frac{\ln(2\beta_M)}{\ln(\Delta t_M)} \leq \alpha_M \leq 1$ *for any* $M \geq 3$, *then the solution of the system (2.2)-(2.4) is convergent to the unique solution of the problem (1.1)-(1.4), for any* $0 \leq x \leq 1$.

Proof. For all $i = 0,\ 1,\ ...,\ M-1$, we have

$$\theta_M \Delta t_M\ u_i''(x) +\ u_i(x) \quad = \quad u_{i+1}(x) + (\theta_M - 1)\Delta t_M\ u_{i+1}''(x)$$

$$- \quad \frac{1}{2}\Delta t_M^2(\theta_M^2 \ \frac{\partial^2 u(x,\xi_i)}{\partial t^2} - (\theta_M - 1)^2 \ \frac{\partial^2 u(x,\eta_i)}{\partial t^2}), \qquad (3.1)$$

where $t_i < \xi_i < t_i + \theta_M \Delta t_M$ and $t_{i+1} < \eta_i < t_{i+1} + (\theta_M - 1)\Delta t_M$.
Now, if we put

$$e_i(x) = \hat{u}_i(x) - u_i(x) \qquad for \quad i = \ 0, \ 1, \ ..., \ M, \qquad (3.2)$$

then $e_i(x)$ satisfies.

$$\begin{aligned} \theta_M \Delta t_M \ e_i''(x) + \ e_i(x) \ &= \ e_{i+1}(x) + (\theta_M - 1)\Delta t_M \ e_{i+1}''(x) \\ &- \ \frac{1}{2}\Delta t_M^2(\theta_M^2 \ \frac{\partial^2 u(x,\xi_i)}{\partial t^2} - (\theta_M - 1)^2 \ \frac{\partial^2 u(x,\eta_i)}{\partial t^2}), \ (3.3) \end{aligned}$$

and $e_i(0) = e_i'(0) = 0$, from which, we conclude that

$$\begin{aligned} e_i(x) \ &= \ \frac{1}{\theta_M\sqrt{\theta_M \Delta t_M}} \int_0^x e_{i+1}(s) \sin \frac{x - s}{\sqrt{\theta_M \Delta t_M}} ds + \frac{\theta_M - 1}{\theta_M} \ e_{i+1}(x) \\ &- \ \frac{1}{2}\theta_M^{-1/2}\Delta t_M^{3/2} \int_0^x (\theta_M^2 \frac{\partial^2 u(s,\xi_i)}{\partial t^2} - (\theta_M - 1)^2 \frac{\partial^2 u(s,\eta_i)}{\partial t^2}) \sin \frac{x - s}{\sqrt{\theta_M \Delta t_M}} ds \\ &= \ I_{i,1}(x) + I_{i,2}(x) + I_{i,3}(x), \qquad i = 0, \ 1, \ ..., \ M - 1. \qquad (3.4) \end{aligned}$$

Clearly, the integrand in $I_{i,3}(x)$ denotes the truncation error and the other terms $I_{i,1}(x)$ and $I_{i,2}(x)$ show that, errors of initial and boundary data for the problem (1.1) how to propagate. In remaining of the proof, we consider two cases:
Case I. If

$$\begin{aligned} e_i(0) = e_i'(0) \ &= \ 0, \qquad i = 0, \ 1, \ ..., \ M - 1, \\ e_M(x) \ &= \ 0, \qquad 0 < x < 1, \qquad (3.5) \end{aligned}$$

then clearly one may conclude that

$$|I_{i,3}(x)| \leq C(\theta_M \Delta t_M)^{3/2}, \quad for \quad i = 0, \ 1, \ ..., \ M - 1, \qquad (3.6)$$

where C is defined in Theorem 2.2. Thus, $e_{M-1}(x) = I_{M-1,3}(x)$ and one may show that

$$|e_{M-i}(x)| \ \leq \ (\frac{1}{2}\theta_M^{-2}\Delta t_M^{-1} + 1 - \theta_M^{-1})|e_{M-i-1}(x)| + I_{M-i,3}(x)$$

$$
\begin{aligned}
&= \sum_{k=0}^{i-1}(1-\theta_M^{-1}+\frac{1}{2}\theta_M^{-2}\Delta t_M^{-1})^k|I_{M-i-k,3}(x)| \\
&\leq C(\theta_M\Delta t_M)^{(3/2)}\sum_{k=0}^{i-1}(\frac{1}{2}\theta_M^{-2}\Delta t_M^{-1}+1-\theta_M^{-1})^k, \quad i=1,\ 2,\ ...,\ M. \quad (3.7)
\end{aligned}
$$

Clearly, for any fixed value θ_M, limit $|e_{M-i}(x)|$ will not be zero, when Δt_M is vanished, which implies the divergency of the solution. If we choose $\theta_M = \beta_M \Delta t_M^{-\alpha_M}$ for $0 < \alpha_M \leq 1$ and $\beta_M \geq 0$, then for each $M \geq 3$, and α_M which satisfies in the following inequality

$$1+\frac{\ln(2\beta_M)}{\ln \Delta t_M} \leq \alpha_M \leq 1,$$

we obtain that

$$|e_{M-i}(x)| \leq C_1\beta_M^{3/2}\Delta t_M^{3/2(1-\alpha_M)}, \tag{3.8}$$

where

$$C_1 = C\sum_{k=0}^{i-1}(1-\theta_M^{-1}+\frac{1}{2}\theta_M^{-2}\Delta t_M^{-1})^k.$$

Consequently, if Δt_M tends to zero, then $|e_{M-i}(x)|$ is vanished and $\hat{u}_i(x)$ for $i=1,\ 2,\ ...,\ M$, convergences to the exact unique solution of the problem (1.1)-(1.4).

Case 2. In this case, let us suppose

$$
\begin{aligned}
\hat{g}_i &= g_i+\epsilon_{i,1}, \\
\hat{p}_i &= p_i+\epsilon_{i,2}, \quad for \quad i=0,\ 1,\ ...,\ M-1,
\end{aligned}
$$

and $\hat{f}(x) = f(x)+\epsilon(x)$, then by using (2.5), (2.6), (2.7) and (2.8), we obtain

$$
\begin{aligned}
e_i(x) &= \epsilon_{i,1}\cos\frac{x}{\sqrt{\theta_M\Delta t_M}} + \epsilon_{i,2}\sqrt{\theta_M\Delta t_M}\sin\frac{x}{\sqrt{\theta_M\Delta t_M}} \\
&+ \frac{1}{\theta_M\sqrt{\theta_M\Delta t_M}}\int_0^x e_{i+1}(s)\sin\frac{x-s}{\sqrt{\theta_M\Delta t_M}}ds \\
&- \frac{\theta_M-1}{\theta_M}\sqrt{\Delta t_M}\ \epsilon_{i+1,2}\sin\frac{x}{\sqrt{\theta_M\Delta t_M}} \\
&- \frac{\theta_M-1}{\theta_M}\sqrt{\Delta t_M}\ \epsilon_{i+1,1}\cos\frac{x}{\sqrt{\theta_M\Delta t_M}} + I_{i,3}(x), \qquad (3.9)
\end{aligned}
$$

where $e_i(x)$ is a global error for $i = 0,\ 1,\ ...,\ M$, and $e_M(x) = \epsilon(x)$.
Now if $|\epsilon(x)|,\ |\epsilon_{i,1}|$, and $|\epsilon_{i,2}|$ tend to zero for each $i = 0,\ 1,\ ...,\ M$, then

$$|e_{M-i}(x)| \leq \sum_{k=0}^{i-1}(1-\theta_M^{-1}+\frac{1}{2}\theta_M^{-2}\Delta t_M^{-1})^k\ |I_{M-i+k,3}(x)|,$$

and

$$|I_{M-i+k,3}(x)| \leq C\beta_M^{3/2}\ \Delta t_M^{3/2(1-\alpha_M)} \to 0.$$

Finally, $|e_{M-i}(x)|$ vanishes for all $i = 0,\ 1,\ ...,\ M$. ■

4 Numerical Examples

In this section we will present simulated cases to evaluate the capability of the proposed robust input estimation scheme.

Example 4.1. Assume that

$$\begin{aligned} f(x) &= x^2+2, \\ g(t) &= 2t, \\ p(t) &= 2, \\ T &= 1. \end{aligned}$$

Obviously, $u(x,t) = x^2 + 2t$ is an exact solution of the problem. Now, we use our numerical method to this problem. For $x = 1,\ \Delta t_M = 0.02,\ \alpha_M = 0.8$, and $\beta_M = 0.9$, the result are given in the following table.

t	$\hat{u}(1,t)$	$u(1,t)$	$\frac{\lvert u(1,t)-\hat{u}(1,t)\rvert}{\lvert u(1,t)\rvert}$	$u_x(1,t)$	$\hat{u}_x(1,t)$	$\frac{\lvert u_x(1,t)-\hat{u}_x(1,t)\rvert}{\lvert u_x(1,t)\rvert}$
0	0.993614	1	0.0063	2	1.98329	0.0083
0.2	1.38377	1.4	0.019	2	1.96382	0.018
0.4	1.77821	1.8	0.012	2	1.95272	0.023
0.6	2.1803	2.2	0.0089	2	1.95764	0.021
0.8	2.58875	2.6	0.0043	2	1.97603	0.011

Table 1. *Exact and estimate of the temperature and heat flux in the above problem*

One can see from the data in table 1, the relation errors generated through the computation show that the approximate and the exact solutions are vanished.

Example 4.2. Suppose that

$$\begin{aligned} f(x) &= x^3 + 6x, \\ g(t) &= 0, \\ p(t) &= 6t, \\ T &= 1. \end{aligned}$$

Clearly, the exact solution to this problem is $u(x,t) = x^3 + 6xt$.
Now, for $x = 1$, $\Delta t_M = \frac{1}{30}$, $\alpha_M = 0.9$, and $\beta_M = 5$, we obtain the following result given in table 2.

t	$\hat{u}(1,t)$	$u(1,t)$	$\frac{\lvert u(1,t)-\hat{u}(1,t)\rvert}{\lvert u(1,t)\rvert}$	$u_x(1,t)$	$\hat{u}_x(1,t)$	$\frac{\lvert u_x(1,t)-\hat{u}_x(1,t)\rvert}{\lvert u_x(1,t)\rvert}$
0	0.992943	1	0.007	3	2.9757	0.008
0.2	2.17963	2.2	0.009	4.2	4.13599	0.001
0.4	3.37199	3.4	0.008	5.4	5.31314	0.001
0.6	4.57445	4.6	0.005	6.6	6.52107	0.001
0.8	5.7853	5.8	0.002	7.8	7.75477	0.005

Table 2. *Exact and estimate of the temperature and heat flux in Example 4.2*

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Mashhad R. J. Math. Sci., Vol.1(1)(2008) 59-65.

Uniformly continuous 1-1 functions on ordered fields not mapping interior to interior

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Abstract

In an earlier work we showed that for ordered fields F not isomorphic to the reals R, there are continuous 1-1 functions on $[0,1]_F$ which map some interior point to a boundary point of the image (and so are not open). Here we show that over closed bounded intervals in the rationals Q as well as in all non-Archimedean ordered fields of countable cofinality, there are *uniformly continuous* 1-1 functions not mapping interior to interior. In particular, the minimal non-Archimedean ordered field $Q(x)$, as well as ordered Laurent series fields with coefficients in an ordered field accommodate such pathological functions.

Keywords and phrases: Ordered field, gap, monotone complete, open map.

AMS Subject Classification 2000: Primary 12J15, 26A03; Secondary 26E30.

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1 Introduction

A cut C of an ordered field F is a subset satisfying $C < F \setminus C$. A nonempty proper cut is said to be a gap, whenever it fails to have a supremum in the field. A gap G in F is called regular, when for all $\epsilon \in F^{>0}$, $G + \epsilon \nsubseteq G$. An ordered field is Archimedean (has no infinitesimals) if and only if it is (isomorphic to) a subfield of real ordered field R. The latter is, up to isomorphism, the unique ordered field which is Dedekind complete, i.e. does not have any gaps. Any Dedekind incomplete ordered field F has gaps in all its non-degenerated intervals: In the Archimedean case, F is a proper subfield of R which therefore misses some points in any real interval. In the non-Archimedean case, and given any two points $a < b$ of F, downward closure of the set of points x such that $\frac{x-a}{b-a}$ is an infinitesimal, forms a gap in (a, b).[1]

Monotone complete ordered fields were introduced in [2]. They are ordered fields with no bounded strictly increasing divergent functions. From [[4] Corollary 2.7], follows that there are monotone complete ordered fields of any uncountable regular cardinality and so there exist plenty of monotone complete ordered fields not isomorphic to R. On the other hand, it is clear that there are no monotone complete ordered field of countable cofinality, except (those isomorphic to) R. For the notions of cofinality and regular cardinals, we refer to [1]. We use cf for cofinality. If F is a monotone incomplete ordered field, then any non-degenerated interval of F contains the image of a strictly increasing divergent function.

We proved in [[3], Theorem 1.2] that an ordered field F is Dedekind complete, if all continuous 1-1 functions defined on some (equivalently all) non-degenerated closed bounded interval(s) of F map interior points [of the interval(s)] to interior points [of their range(s)]. For proper subfields of R, we show here that the rather strange functions coming from above can not be uniformly continuous provided that their unique continuous extensions to R are 1-1. On the other

[1] Alternatively, one can use linear increasing functions between intervals to map a given gap (somewhere in the field) to a gap in a given interval.

hand, on closed bounded intervals in the rational ordered field, as well as in all non-Archimedean ordered fields with countable cofinality, there are uniformly continuous 1-1 functions which do not map interior to interior. We will finish by presenting some ordered fields over which this phenomenon occurs.

2 The Archimedean case

In this section, a well known property of the real ordered field is treated for its subfields.

Lemma 2.1 *Let F be an Archimedean ordered field. If $f : [0,1]_F \to [0,1]_F$ is a uniformly continuous function, then f can be extended to a unique (uniformly) continuous function $\overline{f}$ on $[0,1]_R$.*

Proof. Given $x \in [0,1]_R$, there exists a sequence $(r_n)_{n\geq 1}$ in $[0,1]_Q$ such that $\lim_R r_n = x$. As f is uniformly continuous on $[0,1]_F$ which contains $[0,1]_Q$ and the sequence $(r_n)_{n\geq 1}$ is Cauchy, $(f(r_n))_{n\geq 1}$ is Cauchy in $[0,1]_F$ and so has a limit in $[0,1]_R$. Let $\overline{f}(x) = \lim_R f(r_n)$. This is well defined, since if $(r_n)_{n\geq 1}$ and $(s_n)_{n\geq 1}$ are two Cauchy sequences in $[0,1]_Q$ such that $\lim_R r_n = \lim_R s_n$, then by uniform continuity of f, we have $\lim_R f(r_n) = \lim_R f(s_n)$. Note that $\overline{f}$ is indeed the unique such extension of f. It is also continuous on $[0,1]_R$: Let $x_0 \in [0,1]_R$ and $\epsilon \in R^{>0}$. By uniform continuity of f, there exists $\delta > 0$ such that for all $x, y \in [0,1]_F, (|x-y| < \delta \to |f(x) - f(y)| < \frac{\epsilon}{2})$ $(*)$. We claim that for this δ, $(\forall x \in [0,1]_R)(|x - x_0| < \delta \to |\overline{f}(x) - \overline{f}(x_0| < \epsilon)$. Let $x \in [0,1]_R$ be such that $|x - x_0| < \delta$. There exist sequences $(r_n)_{n\geq 1}$ and $(s_n)_{n\geq 1}$ in $[0,1]_Q$ such that $\lim_R r_n = x_0$ and $\lim_R s_n = x$ respectively. Now let N be a nonnegative integer such that $(\forall n \geq N)(|r_n - s_n| < \delta)$, so from $(*)$ we have $(\forall n \geq N)(|f(r_n) - f(s_n)| < \frac{\epsilon}{2})$. Thus $|\overline{f}(x) - \overline{f}(x_0)| = \lim_R |f(r_n) - f(s_n)| \leq \frac{\epsilon}{2} < \epsilon$.

Proposition 2.2 *Let F be an Archimedean ordered field. If $f : [0,1]_F \to [0,1]_F$ is a uniformly continuous function whose unique extension (as above) to R is one-to-one, then it maps every open subset of $[0,1]_F$ onto an open subset of $f([0,1]_F)$.*

Proof. Let U be an open subset of $[0,1]_F$. There is an open subset V of R such that $U = V \cap [0,1]_F$. We have $f(U) = \overline{f}(U) = \overline{f}(V \cap [0,1]_F) = \overline{f}(V \cap [0,1]_R) \cap \overline{f}([0,1]_F) = \overline{f}(V \cap [0,1]_R) \cap f([0,1]_F)$. But $\overline{f}$ is open, hence $\overline{f}(V \cap [0,1]_R)$ is a relatively open subset of $\overline{f}([0,1]_R)$ and so there is an open subset W of R, such that $\overline{f}(V \cap [0,1]_R) \cap f([0,1]_F) = (W \cap \overline{f}([0,1]_R)) \cap f([0,1]_F) = W \cap f([0,1]_F)$. Therefore $f(U)$ is a relatively open subset of $f([0,1]_F)$.

Proposition 2.3 *There is a uniformly continuous 1-1 function on $[0,1]_Q$, which its range has empty interior.*

Proof. Consider the function $f(x) = |\frac{1}{4}x^2 + x - \frac{1}{2}|$. By changing the variable $x = 2(u-1)$, the proof of f being 1-1 is based on the fact that the equation $r^2 + s^2 = 3$ is not solvable in Q. A further argument shows that the complement of the range of f with respect to $[0,1]_Q$ is dense in $[0,1]_Q$ and so the range, as a subspace of $[0,1]_Q$, has empty interior.

3 The non-archimedean case of countable cofinality

Theorem 3.1 *Let F be a non-Archimedean ordered field with $cf(F) = \omega$. Then for all $a < b$ in F, there exist 1-1 uniformly continuous functions $f : [a,b]_F \to [a,b]_F$ whose ranges are closed such that f maps some interior point of $[a,b]_F$ to a boundary point of its image.*

Proof. Let $[a,b]$ be a non-degenerated interval in F and $c = \frac{a+b}{2}$. Fix a strictly increasing sequence $(a_k)_{k\in\omega}$ in $[a,c)$ such that $a_0 = a$, $(\forall k \geq 1)|a_{k+1} - a_k| \leq \frac{1}{4}|a_k - a_{k-1}|$ and $\lim_k a_k = c$. Put $b_0 = b$, and for all $k \geq 1$, $b_k = b_0 - (a_k - a_0)$. Downward closure of the monad of 0 is an irregular gap G in $[0,1]_F$. Fix (an infinitesimal) $\gamma \in G^{>0}$. Note that $(\forall x \in G)(\forall y \in F \setminus G)(y - x > \gamma)$. For each $k \in \omega$, let U_k be the image of $G \cap [0,1]$ under the linear increasing function from $[0,1]$ onto $[a_k, a_{k+1}]$ and V_k the image of $G \cap [0,1]$ under the linear increasing function from $[0,1]$ onto $[b_{k+1}, b_k]$. Let $S_0 = U_0$, $T_0 = [a,b] \setminus$ (downward closure of V_0), and for all $k \geq 1$, $S_k = U_k \setminus U_{k-1}$, $T_k = V_{k-1} \setminus V_k$. For all $k \geq 1$, we have

$(\forall x \in S_k)(\forall y \in S_{k+1})(y - x > \gamma(a_{k+1} - a_k))$ $(*)$ and, from a similar observation for the T_k's, $I_k = [b_k - \gamma(b_k - b_{k+1}), b_k + \gamma(b_k - b_{k+1})] \subseteq T_k$. For $k \in \omega \setminus \{0\}$, let h_k be the linear increasing function which maps $[a_{k-1}, a_{k+1}]$ onto I_{2k}. For $k = 0$, first pick out some $d_0 \in T_0 \setminus \{b\}$ such that $b - d_0 < a_1 - a$ and put $I_0 = [d_0, b]$. Then let $h_0 : [a, a_1] \to I_0$ be the onto linear increasing function. Let f_k be the restriction of h_k to S_k; so f_k is a function which maps S_k linearly and increasingly into $I_{2k} \subseteq T_{2k}$. Similarly, for all $k \in \omega$, we can get linear increasing functions $g_k : T_k \to T_{2k+1}$. Let $f = (\bigcup_{k\in\omega} f_k) \cup (\bigcup_{k\in\omega} g_k) \cup \{(c, c)\}$. Then f is a one-to-one function from $[a, b]$ into $[c, b]$ with a closed range which maps $c \in (a, b)$ to the boundary point $c \in f([a, b]) \subseteq [c, b]$.

To prove that f is uniformly continuous, we proceed as follows. Given $\epsilon \in F^{>0}$ which we may assume without loss of generality to be less than $a_2 - a_1$, let $\delta = \frac{1}{4}\gamma\epsilon$. Take $x, y \in [a, b]$ with $|x - y| < \delta$. If either x or y (but to avoid trivialities not both) equals c or $x < c < y$, then one easily checks that $|f(x) - f(y)| < |x - y|$. So assume both x and y are strictly less or greater than c, they will be either both in two S's or both in two T's. The arguments will be similar, we only deal with the S's. Suppose $x \in S_k$ and $y \in S_l$. There are the following exclusive cases of how l is compared to k.

(A) $l = k$. Here $f = f_k$ is the restriction of h_k to S_k. If $k = 0$, then by the condition on d_0, we have $|f(x) - f(y)| = |f_0(x) - f_0(y)| < |x - y|$. Assume $k \geq 1$. Then $h_k : [a_{k-1}, a_{k+1}] \to I_{2k}$ is linear and $|I_{2k}| < (a_{k+1} - a_{k-1})$, so $|f(x) - f(y)| = |f_k(x) - f_k(y)| < |x - y| < \delta$, which is infinitely smaller than ϵ.

(B) $l = k + 1$. By the choice of δ, we must have $k \geq 2$. The reason is that we will have $\delta < \gamma(a_2 - a_1)$ (and in particular $\delta < \gamma(a_1 - a_0)$). We have $|f(x) - f(y)| = |f_k(x) - f_{k+1}(y)| < |b_{2(k+1)+1} - b_{2k-1}| \leq |b_{k+1} - b_k| = |a_{k+1} - a_k| <$ [by $(*)$] $\frac{1}{\gamma}|x - y| < \frac{1}{\gamma}\delta = \frac{1}{4}\epsilon < \epsilon$.

(C) $k \geq 2$ and $l = k + 2$. We will have $|f(x) - f(y)| \leq |f(x) - f(a_{k+1})| + |f(a_{k+1}) - f(y)|$. Both of these terms are less than $\frac{1}{4}\epsilon$, by case (B).

(D) $k \geq 2$ and $l \geq k + 3$. We will have $|f(x) - f(y)| < |b_{2l+1} - b_{2k-1}|$. Using $|a_{i+1} - a_i| \leq \frac{1}{4}|a_i - a_{i-1}|$, one gets $|b_{2l+1} - b_{2k-1}| < |a_{k+1} - a_{l-1}|$. The latter value is less than $|x - y|$.

(E) $k = 0$ and $l \geq 2$. We will have $|f(x) - f(y)| \leq |f(x) - f(a_1)| + |f(a_1) - f(a_2)| + |f(a_2) - f(y)|$. The first two terms on the right hand side are, by case (B), less than $\frac{1}{4}\epsilon$. The last one is also less than $\frac{1}{4}\epsilon$: If $l = 2$, $l = 3$, $l = 4$, $l \geq 5$, then one may use the cases (A), (B), (C) and (D) respectively.

(F) $k = 1$ and $l \geq 3$. We will have $|f(x) - f(y)| \leq |f(x) - f(a_2)| + |f(a_2) - f(y)|$. The first term on the right is, by case (B), less than $\frac{1}{4}\epsilon$. The second one is also less than $\frac{1}{4}\epsilon$: If $l = 3$, $l = 4$, $l \geq 5$, then one may use the cases (B), (C) and (D) respectively.

Finally, We present some non-Archimedean ordered fields allowing the mentioned kind of strange functions.

Example 3.2 Consider the ordered field $(Q(x), +, \cdot, <)$ with $x > Q$. It is non-Archimedean and of cofinality ω, hence monotone incomplete. So there are uniformly continuous one-to-one functions $f : [0, 1]_{Q(x)} \to [0, 1]_{Q(x)}$, which map some interior point to a boundary point of its image.

Example 3.3 Let F be an ordered field. Then the ordered field of Laurent series with coefficients in F is non-Archimedean of cofinalty ω, and so it is monotone incomplete. Once again, there will exist the mentioned kind of functions over closed bounded intervals of this field.

Note added in the proof. In 2002, Lobachevskii (J. Math.) (it is available online), in addition to the above for non-Archimedean ordered fields of countable cofinality, we constructed uniformly continuous 1-1 functions which although map interior to the interior of the image, but still are not open.

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Mashhad R. J. Math. Sci., Vol.1(1)(2008) 67-82.

Statistical inference based on k-records

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Abstract

In this paper, an extension of record models, well known as k-records, is considered. Bayesian estimation as well as prediction based on k-records are presented when the underlying distribution is assumed to have a general form. The proposed procedure is applied to the Exponential, Weibull and Pareto models in one parameter case. Also, the two-parameter Exponential distribution, when both parameters are unknown, is studied in more details. Since the ordinary record values are contained in the k-records, by putting $k = 1$, the results for usual records can be obtained as special case.

Keywords and phrases: Admissibility; Bayes prediction; Bayesian estimation; conjugate prior; weibull distribution.

AMS Subject Classification 2000: Primary 62G30, 62G32; Secondary 62C15.

1 Introduction

Let $\{X_i,\ i \geq 1\}$ be a sequence of independent and identically distributed (iid) continuous random variables each distributed according to cumulative distribution

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function (cdf) $F(t)$ and probability density function (pdf) $f(t)$. An observation X_j will be called an upper record value if its value exceeds that of all previous observations. Thus, X_j is an upper record if $X_j > X_i$ for every $i < j$. An analogous definition can be given for lower record values. Today there are over 500 papers and several books published on record-breaking data (see, for instance, Chandler [8], Resnick [19], Shorrock [21], Glick [14], Samaniego and Whitaker [20], Arnold *et al.* [5] and Nevzorov [18]).

There are several situations where the second or third largest values of special interest, insurance claims some non-life insurance can be used as an example, see Kamps [16], so the usual record models is inadequate. Also, in the ordinary record value theory, while inverse sampling considerations have given valuable insights, their practical implementation is greatly hindered by the sparsity of records. These problems caused the researchers to study the theory of k-record models. Upper k-record process is defined in terms of the k-th largest X yet seen. For a formal definition, we consider the definition in Arnold *et al.* [5], p. 43, in the continuous case, let $T_{1(k)} = k, R_{1(k)} = X_{1:k}$ and for $n \geq 2$, let

$$T_{n(k)} = \min\{j : j > T_{n-1(k)}, X_j > X_{T_{n-1(k)}-k+1:T_{n-1(k)}}\},$$

where $X_{i:m}$ denotes the i-th order statistic in a sample of size m. The sequence of *upper k-records* is then defined by $R_{n(k)} = X_{T_{n(k)}-k+1:T_{n(k)}}$ for $n \geq 1$. Arnold *et al.* [5] call this a *Type 2 k-record* sequence. For $k = 1$, note that the usual records are recovered. An analogous definition can be given for *lower k-records* as well. This sequence of k-records was introduced by Dziubdziela and Kopocinski [12] and it has found acceptance in the literature. Some work has been done on the statistical inference, based on k-records. See, for instance, Deheuvels and Nevzorov [11], Berred [7], Ali Mousa *et al.* [4], Malinowska and Szynal [17], Danielak and Raqab [9],[10], Ahmadi *et al.* [2], Fashandi and Ahmadi [13] and references therein.

We assume that this type of k-record data is available and the aim of this paper is to develop inference methods as well as prediction of future k-records based on past observed k-records. The rest of the paper is organized as follows. In Section 2, Bayesian estimation as well as prediction based on k-records are presented when the underlying distribution is assumed to have a general model. In Section 3, a two-parameter exponential distribution is considered; the maximum likelihood and Bayes estimators for the unknown parameters, are obtained. Bayesian prediction of the future k-records, either point or interval, are obtained in Section 4, when the k-records are assumed to come from the two-parameter exponential model.

2 A General Model

In this section, we consider the problems of estimation and prediction based on k-records, when the underlying distribution has a general form. In order to do this, let C be the class of all absolute continuous distribution functions F of the form

$$F_\theta(x) = 1 - e^{-\lambda_\theta(x)}, \quad x > 0, \tag{2.1}$$

where $\lambda'_\theta(x)$ (the derivative of $\lambda_\theta(x)$ w.r.t θ) exists and is a positive function of θ and x. Then

$$f_\theta(x) = \lambda'_\theta(x) e^{-\lambda_\theta(x)}, \quad x > 0. \tag{2.2}$$

This class includes several important life time families such as: Exponential, Weibull, compound Weibull, Pareto, Beta, Gompertz, compound Gompertz and Burr type XII, among others.

2.1 Estimation

Using the joint pdf of usual records, we readily have the joint density of the first m, k-records $R_{1(k)}, R_{2(k)}, ..., R_{m(k)}$ as

$$f(x_1, ..., x_m) = k^m \prod_{i=1}^{m} \frac{f(x_i)}{1 - F(x_i)} (1 - F(x_m))^k, \tag{2.3}$$

(see, Arnold *et al.* [5]). Now, suppose we observe $R_{1(k)} = x_1, \cdots, R_{m(k)} = x_m$ then by substitution of (2.1) and (2.2) in (2.3), the likelihood function $L(\theta)$ is

$$L(\theta) \propto A(\theta; \mathbf{x}) e^{-B(\theta, x_m)}, \tag{2.4}$$

where $\mathbf{x} = (x_1, \cdots, x_m)$,

$$A(\theta; \mathbf{x}) = \prod_{i=1}^{m} \lambda'_\theta(x_i) \quad \text{and} \quad B(\theta, x_m) = k\lambda_\theta(x_m).$$

There is clearly no way in which one can say that one prior is better than any other. Presumably one has own subjective prior and must live with all of its lumps and bumps. It is more frequently the case that we elect to restrict attention to a given flexible family of prior distributions and we choose one from the family which seems to the best of our match and personal believes. With this in mind, let the conjugate prior density function for θ, proposed by AL-Hussaini [3], is given by

$$\pi(\theta; \delta) \propto C(\theta; \delta) e^{-D(\theta; \delta)}, \quad \theta \in \Theta, \ \delta \in \Omega, \tag{2.5}$$

where Ω is the prior parameter(s) space. Then the posterior density function is derived as

$$\pi(\theta | \mathbf{x}) = C_1(M, N) M(\theta; \mathbf{x}, \delta) e^{-N(\theta;\ x_m, \delta)}, \tag{2.6}$$

where

$$\begin{aligned} M(\theta; \mathbf{x}, \delta) &= C(\theta; \delta) A(\theta; \mathbf{x}), \\ N(\theta; x_m, \delta) &= D(\theta; \delta) + B(\theta, x_m), \end{aligned}$$

and $C_1(M, N)$ is the normalizing constant given by

$$C_1(M, N) = \left[\int_\Theta M(\theta; \mathbf{x}, \delta) e^{-N(\theta;\ x_m,\ \delta)} d\theta \right]^{-1}. \tag{2.7}$$

If Θ is one dimensional then the Bayes estimator of θ, under squared error (SE) loss function, is

$$\hat{\theta}_{BS} = \frac{C_1(M, N)}{C_1(M^\star, N)}, \tag{2.8}$$

where $M^\star(x) = xM(x)$.

Remark. $\hat{\theta}_{BS}$ in (2.8) is the unique Bayes estimate of θ under SE loss function with respect to the above mentioned proper prior and hence is admissible.

Example 2.1 (One-Parameter Exponential Model)

Let $\lambda_\theta(x) = (x - \mu_0)/\sigma$ where μ_0 is known and $\theta = \sigma$, i.e. we have a one parameter exponential distribution. Then $A(\theta; \mathbf{x}) = 1/\sigma^m$ and $B(\theta; x_m) = k(x_m - \mu_0)/\sigma$. It can be shown that the maximum likelihood estimation of σ is $\hat{\sigma}_M = k(R_{m(k)} - \mu_0)/m$. We use Inverted Gamma with parameters a and b as the conjugate prior, i.e. $\pi(\sigma) = b^a \sigma^{-(a+1)} \exp\{-b/\sigma\}/\Gamma(a)$, where from (2.5), $C(\theta; \delta) = \sigma^{-(m+2)}$, $D(\theta; \delta) = b/\sigma$ and $\delta = (a, b)$. Therefore, $M(\theta; \mathbf{x}, \delta) = 1/\sigma^{m+a+1}$ and $N(\theta; x_m, \delta) = (b + k(x_m - \mu_0))/\sigma$. From (2.8), the Bayes estimate of σ under SE loss function is given by

$$\hat{\sigma}_{BS} = \frac{b + k(R_{m(k)} - \mu_0)}{m + a - 1}.$$

It may be noted that, from (2.6), the posterior distribution of σ^{-1} is $\Gamma(m + a, b + k(x_m - \mu_0))$.

Example 2.2 (Weibull Model)

Suppose $\lambda_\theta(x) = \alpha x^\beta$, where β is known and $\theta = \alpha$. Then $\lambda'_\theta(x) = \alpha\beta x^{\beta-1}$. It can be shown that the maximum likelihood estimate of α is $\hat{\alpha}_M = m/(kR^\beta_{m(k)})$. Assuming a Gamma conjugate prior with parameter a and b, i.e.
$\pi(\alpha) = b^a \alpha^{a-1} \exp\{-b\alpha\}/\Gamma(a)$, the Bayes estimate of α under SE loss function is given by

$$\hat{\alpha}_{BS} = (m + a)/(kR^\beta_{m(k)} + b).$$

Example 2.3 (Pareto Model)

In this model, $\lambda_\theta(x) = \alpha \ln(x/\beta)$ where β is known and $\theta = \alpha$. So, by (2.4), maximum likelihood estimate of α is $\hat{\alpha}_M = m/(k \ln[R_{m(k)}/\beta])$. Assuming a conjugate prior Gamma with parameters a and b, i.e. $\pi(\alpha) = b^a \alpha^{a-1} \exp\{-b\alpha\}/\Gamma(a)$, the Bayes estimate of α under SE loss function is given by

$$\hat{\alpha}_{BS} = \frac{m+a}{k \ln[R_{m(k)}/\beta] + b}.$$

When both of the parameters in the above examples are unknown, In [1] we have obtained similar results based on usual records ($k = 1$).

2.2 Prediction

Assume that we have the first m upper k-records $R_{1(k)} = x_1, R_{2(k)} = x_2, ..., R_{m(k)} = x_m$ from a member of class C in (2.1). Based on such a sample, prediction, either point or interval, is needed for s-th upper k-record, $1 \leq m < s$. Now, let $Y = R_{s(k)}$ be the s-th upper k-record value, $s > m$. The conditional pdf of Y for the given vector parameter θ and that the first m k-record $R_{1(k)}, \cdots, R_{m(k)}$ is given by

$$f(y|\mathbf{x}, \theta) = k^{s-m} \frac{[\lambda_\theta(y) - \lambda_\theta(x_m)]^{s-m-1}}{\Gamma(s-m)} \lambda'_\theta(y) e^{-k(\lambda_\theta(y) - \lambda_\theta(x_m))}. \tag{2.9}$$

Hence, from equations (2.6) and (2.9) we get the Bayes predictive density function of Y

$$\begin{aligned} h^*(y|\mathbf{x}) &= \int_\Theta f(y|\mathbf{x}, \theta)\pi(\theta|\mathbf{x}) d\theta \\ &= \frac{k^{s-m} C_1(M, N)}{\Gamma(s-m)} \\ & \quad \int_\Theta M(\theta; \mathbf{x}, \delta)[\lambda_\theta(y) - \lambda_\theta(x_m)]^{s-m-1} \lambda'_\theta(y) e^{-k[\lambda_\theta(y) - \lambda_\theta(x_m)] - N(\theta; x_m, \delta)} d\theta. \end{aligned} \tag{2.10}$$

The Bayes point predictor of the s-th upper k-record based on the first m ($m < s$) observed k-records is given by

$$\begin{aligned}
\hat{Y}_{BS} &= \int_{x_m}^{+\infty} y h^*(y|\mathbf{x}) dy \\
&= \frac{k^{s-m} C_1(M,N)}{\Gamma(s-m)} \int_\Theta M(\theta;\mathbf{x},\delta) e^{-N(\theta;x_m,\delta)} \\
&\quad \int_{x_m}^{+\infty} y[\lambda_\theta(y) - \lambda_\theta(x_m)]^{s-m-1} \lambda'_\theta(y) e^{-k(\lambda_\theta(y)-\lambda_\theta(x_m))} dy \; d\theta \\
&= \frac{k^{s-m} C_1(M,N)}{\Gamma(s-m)} \int_\Theta M(\theta;\mathbf{x},\delta) e^{-N(\theta;x_m,\delta)} \\
&\quad \left\{ \int_0^{+\infty} \lambda_\theta^{-1}(z + \lambda_\theta(x_m)) z^{s-m-1} e^{-kz} dz \right\} d\theta \\
&= C_1(M,N) \int_\Theta M(\theta;\mathbf{x},\delta) e^{-N(\theta;x_m,\delta)} E\left\{ \lambda_\theta^{-1}(Z + \lambda_\theta(x_m)) \right\} d\theta,
\end{aligned} \tag{2.11}$$

where $Z \sim \Gamma(s-m,k)$ and $\lambda_\theta^{-1}(x)$ is the inverse function of $\lambda_\theta(x)$.

Example 2.4 (Continuation Examples 2.1-2.3)

Using (2.11), we obtain the Bayesian point prediction of $R_{s(k)}$ for the following three models. We have

i. One parameter Exponential model:

$$\hat{Y}_{BS} = \left(\frac{s+a-1}{m+a-1}\right) R_{m(k)} + \left(\frac{s-m}{m+a-1}\right)\left(\frac{b}{k} - \mu_0\right).$$

ii. Weibull model:

$$\hat{Y}_{BS} = \left(\frac{s+a-1}{m+a-1}\right) R^{\beta}_{m(k)} + \left(\frac{s-m}{m+a-1}\right) \frac{b}{k}.$$

iii. Pareto model:

$$\hat{Y}_{BS} = \frac{[b + k\log(R_{m(k)}/\beta)]^{m+a}}{\Gamma(m+a)} I(R_{m(k)}),$$

where $I(R_{m(k)}) = \int_0^{+\infty} \frac{\alpha^{s+a-1}}{(\alpha-1/k)^{s-m}} e^{-[b+k\log(R_{m(k)}/\beta)]\alpha} d\alpha$.

Remark. It may be noted that one may use (2.10) to obtain Bayesian prediction interval for $R_{s(k)}$.

In the rest of the paper, we consider two-parameter Exponential distribution which does not belong to the class C in (2.1), where the location parameter μ is unknown. Its cdf and pdf are given by

$$F(x;\mu,\sigma) = 1 - e^{-\frac{1}{\sigma}(x-\mu)} \quad x \geq \mu, \quad \sigma > 0, \tag{2.12}$$

and

$$f(x;\mu,\sigma) = \frac{1}{\sigma} e^{-\frac{1}{\sigma}(x-\mu)} \quad x \geq \mu, \quad \sigma > 0, \tag{2.13}$$

respectively, which is denoted by $X \sim Exp(\mu,\sigma)$. Ahmadi *et al.* [2] studied the problem of estimation and prediction in $Exp(\mu,\sigma)$ under LINEX (LINear-EXPonential) loss function based on k-records from Bayesian view point.

3 Estimation in Exponential Model

As mentioned in Section 1, the usual record data are rare in practical situations. In fact, the expected waiting time is infinite for every record after the first; but, this problem will be fixed by considering k-records instead (see Theorem 2.1 of [15]). So, in this section, we shall be concerned with estimation of the two unknown parameters μ and σ of $Exp(\mu,\sigma)$ based on k-record values. Suppose, we observed the first m upper k-records $R_{1(k)} = x_1,\ R_{2(k)} = x_2, ..., R_{m(k)} = x_m$ from an $Exp(\mu,\sigma)$.Then from (2.3), (2.12) and (2.13) the likelihood function is given by

$$L(\mu,\sigma|\mathbf{x}) = (\frac{k}{\sigma})^m e^{-\frac{k}{\sigma}(x_m-\mu)}, \quad \mu \leq x_1 < x_2 < ... < x_m,\ \sigma > 0, \tag{3.1}$$

where $\mathbf{x} = (x_1, x_2, ..., x_m)$.

3.1 Maximum likelihood estimation

In the case $k = 1$, the MLE (maximum likelihood estimation) of the two-parameters of the Exponential distribution can be found in Arnold *et al.* [5], p. 123. We obtained MLE based on k-record values, by (3.1). The natural logarithm of (3.1) is given by

$$l = m\ln k - m\ln\sigma - \frac{k}{\sigma}(x_m - \mu), \quad \mu \le x_1 < x_2 < ... < x_m. \tag{3.2}$$

Assume that the parameters μ and σ are unknown, from (3.2) we readily obtain the MLE of μ and σ as follows:

$$\hat{\mu}_M = R_{1(k)}, \tag{3.3}$$

and

$$\hat{\sigma}_M = \frac{k}{m}(R_{m(k)} - R_{1(k)}). \tag{3.4}$$

It is easy to verify that

- $R_{1(k)} \sim Exp(\mu, \sigma/k)$,
- $R_{m(k)} - R_{1(k)}$ and $R_{1(k)}$ are independent random variables,
- $R_{m(k)} - R_{1(k)}$ has gamma distribution with parameters $m-1$ and k/σ.

Then by (3.3) and (3.4) we have

- $E(\hat{\mu}_M) = \mu + \frac{\sigma}{k}$,
- $MSE(\hat{\mu}_M) = 2\frac{\sigma^2}{k^2}$.

Also,

- $E(\hat{\sigma}_M) = \frac{m-1}{m}\sigma$,
- $MSE(\hat{\sigma}_M) = \frac{\sigma^2}{m}$, do not depend on k.
- $Cov(\hat{\mu}_M, \hat{\sigma}_M) = 0$.

Notice that $\hat{\mu}_M$ is a biased estimator μ, while an unbiased estimator for μ is given by

$$\tilde{\mu} = \frac{m+k-1}{m-1}R_{1(k)} - \frac{k}{m-1}R_{m(k)}.$$

3.2 Bayes estimation

Our aim is to obtain Bayes estimate of the unknown parameters based on $x_1, ..., x_m$ under SE loss function. We consider the following two cases for our Bayesian estimation problem.

a) σ *is known.*

Without loss of generality, we may assume $\sigma = 1$ then by (3.1), we have

$$f(\mathbf{x}|\mu) = k^m e^{-k(x_m-\mu)}, \quad \mu < x_1 < x_2 < ... < x_m. \tag{3.5}$$

Assume the Jeffreys non-informative prior distribution (see [6]) of the parameter μ in the form

$$\pi(\mu) \propto 1. \tag{3.6}$$

Hence the posterior distribution of μ is

$$\pi(\mu|\mathbf{x}) \propto f(\mathbf{x}|\mu)\pi(\mu),$$

where $f(\mathbf{x}|\mu)$ is the joint density function given by (3.5) and $\pi(\mu)$ is the prior density given by (3.6). So, we have

$$\pi(\mu|\mathbf{x}) = ke^{k(\mu-x_1)}, \quad \mu < x_1. \tag{3.7}$$

Suppose an SE loss function, the Bayes estimate of a parameter is its posterior mean. Therefore, by (3.7), the Bayes estimate of the parameter μ is given by

$$\hat{\mu}_{1BS} = R_{1(k)} - \frac{1}{k}. \tag{3.8}$$

From Eq. (3.8) we get

- $E(\hat{\mu}_{1BS}) = \mu$,
- $MSE(\hat{\mu}_{1BS}) = \frac{1}{k^2}$.

b) σ *is unknown.*

Under the assumption that both of the parameters μ and σ are unknown, we may consider the joint density as a product of the conditional density of μ for given σ and a two parameter inverted gamma density for σ. So, we have

$$\pi(\mu,\sigma) \propto \frac{\beta^{\alpha}}{\Gamma(\alpha)}\frac{1}{\sigma^{\alpha+2}}e^{-\frac{\beta}{\sigma}}. \tag{3.9}$$

In fact $\sigma^{-1} \sim \Gamma(\alpha,\beta)$, which is the conjugate prior distribution of the parameter σ for the fixed value of μ, and $\pi_1(\mu|\sigma) \propto \sigma^{-1}$ which is the Jeffreys non-informative prior distribution (see [6]) of the parameter μ for fixed value of the parameter σ. Thus, the joint posterior density is given by

$$\pi(\mu,\sigma|\mathbf{x}) = \frac{k[\beta + k(x_m - x_1)]^{m+\alpha}}{\Gamma(m+\alpha)}\frac{1}{\sigma^{m+\alpha+2}}e^{-\frac{1}{\sigma}[\beta+k(x_m-\mu)]}. \tag{3.10}$$

Therefore, by (3.10) under SE loss function the Bayes estimate of the parameter σ is given by

$$\hat{\sigma}_{2BS} = \frac{\beta + k(R_{m(k)} - R_{1(k)})}{m+\alpha-1}. \tag{3.11}$$

Notice that, as $\beta \to 0$ and $\alpha \to 1$, $\hat{\sigma}_{2BS} \to \hat{\sigma}_{ML}$. By (3.11) we have

- $E(\hat{\sigma}_{2BS}) = \frac{\beta+\sigma(m-1)}{m+\alpha-1}$,
- $MSE(\hat{\sigma}_{2BS}) = \frac{m-1}{(m+\alpha-1)^2}\sigma^2 + \frac{(\beta-\sigma\alpha)^2}{(m+\alpha-1)^2}$.

Also, the Bayes estimate of the parameter μ is given by

$$\hat{\mu}_{2BS} = R_{m(k)} + \beta - \frac{m+\alpha}{k(m+\alpha-1)}[\beta + k(R_{m(k)} - R_{1(k)})]. \tag{3.12}$$

By (3.12), we have

- $E(\hat{\mu}_{2BS}) = \mu + \alpha\frac{\sigma}{k} + [1 - \frac{(m+\alpha)}{k(m+\alpha-1)}]\beta$,
- $Var(\hat{\mu}_{2BS}) = \frac{\sigma^2}{k^2}[1 + \frac{m-1}{(m+\alpha-1)^2}]$,
- $Cov(\hat{\mu}_{2BS}, \hat{\sigma}_{2BS}) = -\frac{(m-1)\sigma^2}{(m+\alpha-1)k}$.

Remark. It may be noted that one may use (3.7) and (3.10) to obtain Bayesian estimation interval for the parameters μ and σ.

4 Prediction in Exponential Model

In this section, we consider the problem of prediction, either point or interval, for future k-record values by Bayesian approach. Assume that we have the first m upper k-records $R_{1(k)} = x_1, R_{2(k)} = x_2, ..., R_{m(k)} = x_m$ from the $Exp(\mu, \sigma)-$distribution. Based on such a sample, prediction, either point or interval, is needed for s-th upper k-record, $1 \leq m < s$. We consider the following two cases:

a) *σ is known.*

Without loss of generality, we may assume $\sigma = 1$, then by (2.12), (2.13) and (2.9), we have

$$f^*(y|x_m, \mu) = \frac{k^{s-m}}{\Gamma(s-m)}(y - x_m)^{s-m-1}e^{-k(y-x_m)}, \quad y > x_m, \tag{4.1}$$

which is independent of μ. So by (2.10) and (4.1) we have

$$\begin{aligned} h^*(y|\mathbf{x}) &= \int_{-\infty}^{x_1} f^*(y|\mathbf{x}, \mu)\pi(\mu|\mathbf{x})d\mu \\ &= \frac{k^{s-m}}{\Gamma(s-m)}(y - x_m)^{s-m-1}e^{-k(y-x_m)}, \quad y > x_m, \end{aligned} \tag{4.2}$$

for any posterior distribution (therefore, for any prior distribution) $\pi(\mu|\mathbf{x})$. By (4.2), we have

$$Y - x_m|\mathbf{x} \sim \Gamma(s - m, k).$$

So,

$$\hat{Y}_1 = R_{m(k)} + \frac{s-m}{k}. \tag{4.3}$$

By (4.3) we have

- $E(\hat{Y}_1) = \mu + \frac{s}{k}$,
- $MSE(\hat{Y}_1) = \frac{s-m}{k^2}$.

Ahmadi *et al.* [2] obtained the $100(1-\gamma)\%$ Bayesian prediction interval for $R_{s(k)}$, with equal tail as $(L_1,\ U_1)$, where L_1 and U_1 are the lower and upper bounds, respectively which are given by

$$L_1 = R_{m(k)} + \frac{\chi^2_{\frac{\gamma}{2}}}{2k},$$

and

$$U_1 = R_{m(k)} + \frac{\chi_{1-\frac{\gamma}{2}}{}^2}{2k}.$$

where χ^2_γ stands for the γ-th percentage of Chi-square distribution with $2(s-m)$ degrees of freedom.

b) *σ is unknown*

Let $Y = R_{s(k)}$ be the s-th upper k-record value, $1 \leq m < s$. So, by (2.12) and (2.13), we have

$$f^*(y|\mathbf{x}, \mu, \sigma) = (\frac{k}{\sigma})^{s-m} \frac{(y-x_m)^{s-m-1}}{\Gamma(s-m)} e^{-\frac{k}{\sigma}(y-x_m)}. \tag{4.4}$$

By (2.13), (3.10) and (4.4) Bayesian predictive density function of $Y = R_{s(k)}$, for the given past m records, is given by

$$\begin{aligned} h(y|\mathbf{x}) &= \int_{-\infty}^{x_1} \int_0^\infty f^*(y|\mathbf{x}, \mu, \sigma) \pi(\mu, \sigma|\mathbf{x}) d\sigma d\mu \\ &= \frac{1}{B(m+\alpha, s-m)} (\frac{k(x_m - x_1) + \beta}{k(y-x_1)+\beta})^{m+\alpha} \\ &\quad \times (1 - \frac{k(x_m - x_1) + \beta}{k(y-x_1)+\beta})^{s-m} \frac{1}{y-x_m}, \ y > x_m. \end{aligned} \tag{4.5}$$

Now, by (4.5) the Bayes point predictor of the s-th upper k-record is given by

$$\hat{Y}_2 = \frac{s+\alpha-1}{m+\alpha-1} R_{m(k)} + \frac{s-m}{m+\alpha-1}(\frac{\beta}{k} - R_{1(k)}). \tag{4.6}$$

By (4.6) we have

- $E(\hat{Y}_2) = \frac{k\mu(\alpha+m-1)+s\sigma(m-1)+\beta(s-m)}{k(m+\alpha-1)}$,
- $MSE(\hat{Y}_2) = (s-m)\{\frac{\sigma^2}{k^2}(1 + \frac{(m-1)(s-m)}{(m+\alpha-1)^2}) + (s-m)(\frac{\alpha}{m+\alpha-1} - \frac{\beta}{k})^2\}$.

In this case, Ahmadi *et al.* [2] also derived a Bayesian prediction interval for $R_{s(k)}$ as follow: The $100(1-\gamma)\%$ Bayesian prediction interval for $R_{s(k)}$ is given by

$$(L_2, \ U_2),$$

where

$$L_2 = \frac{R_{m(k)} - R_{1(k)}}{b_{1-\frac{\gamma}{2}}} + \frac{\beta}{k}(\frac{1}{b_{1-\frac{\gamma}{2}}} - 1) + R_{1(k)},$$

and

$$U_2 = \frac{R_{m(k)} - R_{1(k)}}{b_{\frac{\gamma}{2}}} + \frac{\beta}{k}(\frac{1}{b_{\frac{\gamma}{2}}} - 1) + R_{1(k)},$$

where b_γ is the γ-th percentage of $Beta(m+\alpha, s-m)$-distribution.

5 Conclusion

In this paper, we have tackled the problems of estimation and prediction based on k-record data while the underlying distribution is assumed to have a general form. This family contains several life distribution such as Exponential, Weibull and Pareto and so on. A general form of conjugate prior was considered to obtain Bayesian estimation of unknown parameters and prediction of future k-record values. The proposed procedure was applied to the Exponential, Weibull and Pareto models in one parameter case. Moreover, we have developed the proposed procedure for two-parameter Exponential distribution in details.

Acknowledgments

The authors are grateful to the referees for their careful reading and useful comments. Partial support from "Ordered and Spatial Data Center of Excellence of Ferdowsi University of Mashhad" is acknowledged.

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Mashhad R. J. Math. Sci., Vol.1(1)(2008) 83-100.

Shrinkage estimation of the regression parameters with multivariate normal errors

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Abstract

In the linear model $y = X\beta + e$ with the errors distributed as normal, we obtain generalized least square (GLS) , restricted GLS (RGLS), preliminary test (PT), Stein-type shrinkage (S) and positive-rule shrinkage (PRS) estimators for regression vector parameter β when the covariance structure in known. We compare the quadratic risks of the underlying estimators and propose the dominance orders of the five estimators.

Keywords and phrases: GLS estimator, preliminary test estimator, stein-type shrinkage estimator, positive-rule shrinkage estimator.

AMS Subject Classification 2000: Primary 62J05; Secondary 62H12.

1 Introduction

The most important model belonging to the class of general linear hypotheses is the *multiple regression* model. The general purpose of *multiple regression* is

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to learn more about the relationship between several independent or predictor variables and a dependent or criterion variable.

To deal with a common *multiple regression* equation, consider the linear model

$$y = X\beta + e, \tag{1.1}$$

where y is an n-vector of response, X is an $n \times p$ design matrix with full rank p, $\beta = (\beta_1, \cdots, \beta_p)'$ is a p-vector of regression coefficients and $e = (e_1, \cdots, e_n)'$ is the n-vector of errors distributed as multivariate normal with location parameter zero and positive definite (p.d.) covariance matrix Σ, denoted by $e \sim N_n(0, \Sigma)$. Then directly

$$y \sim N_n(X\beta, \Sigma). \tag{1.2}$$

Let us assume that in addition to the sample information y in the model (1.1), that information also exists in the form of q independent linear hypothesis about the unknown vector parameter β where $q \leq p$. These general restrictions can be shown as

$$H\beta = h, \tag{1.3}$$

where H is a $q \times p$ known hypothesis design matrix of rank q and h is a $q \times 1$ vector of prespecified, hypothetical values.

The estimation of parameters of the multiple regression model is a common interest to many users. Often the properties of the estimators are of prime concern. Selection of any specific statistical property of any estimator often depends on the objective of the study. The choice of any particular estimator may very well be determined by the aim of the end users. It is well known that the ordinary least squares estimators are best linear unbiased. However, if the objective of any study is to minimize some specific risk function then other types of estimators perform better than the ordinary least squares estimator. Our primary object of this paper is to estimate β when the p.d. covariance matrix Σ is known under the subspace restriction (1.3); and then obtain shrinkage estimators of β using the likelihood ratio test (LRT) statistic of (1.3). For complete review of underlying study in the special case $\Sigma = \sigma^2 I_n$ for both known and unknown σ^2 and may σ^2 have inverse gamma distribution, see Saleh and Han [9], Tabatabaey [13], Khan [5, 6], Srivastava and Saleh [12] and Saleh [10].

2 Estimation

Given classical conditions (see Kuan [7]), it is well known that for known p.d. covariance matrix Σ, the generalized least square (GLS) estimator of β is

$$\hat{\beta} \;=\; (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y. \tag{2.1}$$

Obtaining GLS estimator of β under the constraint $H_0 : H\beta = h$, using method of Lagrangian multipliers, the restricted GLS estimator of β subject to the linear restriction $H_0 : H\beta = h$ as $\widetilde{\beta}$ is given by

$$\widetilde{\beta} \;=\; \hat{\beta} - (X'\Sigma^{-1}X)^{-1}H'[H(X'\Sigma^{-1}X)^{-1}H']^{-1}(H\hat{\beta} - h). \tag{2.2}$$

See Ravishanker and Dey [8].
Let $G_1 = (X'\Sigma^{-1}X)^{-1}$ and $G_2 = [HG_1H']^{-1}$, then simplifying (2.2) we obtain

$$\widetilde{\beta} \;=\; \hat{\beta} - G_1H'G_2(H\hat{\beta} - h). \tag{2.3}$$

Now we consider the linear hypothesis $H\beta = h$ in (1.3) and obtain the test statistic for the null hypothesis $H_0 : H\beta = h$.
Now let $\omega = \{\beta : \beta \in \Re^p,\ H\beta = h,\ \Sigma > 0\}$ and $\Omega = \{\beta : \beta \in \Re^p,\ \Sigma > 0\}$, then the likelihood test statistic for underlying hypothesis is

$$\begin{aligned}
\lambda &= \frac{\max_{\beta\in\omega} L(\beta,\Sigma)}{\max_{\beta\in\Omega} L(\beta,\Sigma)} \\
&= \frac{\exp\{\frac{-1}{2}\ [(y - X\widetilde{\beta})'\Sigma^{-1}(y - X\widetilde{\beta})]\}}{\exp\{\frac{-1}{2}\ [(y - X\hat{\beta})'\Sigma^{-1}(y - X\hat{\beta})]\}} \\
&= \exp\{\frac{-1}{2}\ [(H\hat{\beta} - h)'G_2(H\hat{\beta} - h)]\},
\end{aligned}$$

which is a decreasing function with respect to $\chi = (H\hat{\beta} - h)' G_2 (H\hat{\beta} - h)$.
Let $u = G_2^{1/2}(H\hat{\beta} - h)$; then using (1.2), $\chi = u'u$ has non-central chi-square distribution with q degrees of freedom and noncentrality parameter $\mu'\mu/2$, where $\mu = G_2^{1/2}(H\beta - h)$.
Bancroft [2] defined the preliminary test estimator (PTE) of β as a convex combination of $\hat{\beta}$ and $\widetilde{\beta}$ by

$$\hat{\beta}^{PT} = \widetilde{\beta} + [1 - I(\chi \leq \chi^2(\alpha))](\hat{\beta} - \widetilde{\beta}), \tag{2.4}$$

where $I(A)$ is the indicator of the set A and $\chi^2(\alpha)$ is the upper 100α percentile of the central χ^2 distribution with q degrees of freedom.
The PTE has the disadvantage that it depends on α $(0 < \alpha < 1)$, the level of significance and also it yields the extreme results, namely $\hat{\beta}$ and $\widetilde{\beta}$ depending on the outcome of the test. Therefore we define an intermediate value as Stein-type shrinkage estimator (SE) of β, by

$$\hat{\beta}^{S} = \widetilde{\beta} + (1 - \rho\chi^{-1})(\hat{\beta} - \widetilde{\beta}), \tag{2.5}$$

where

$$\rho = \frac{(q-2)(n-p)}{q(n-p+2)} \quad \text{and} \quad q \geq 3. \tag{2.6}$$

The SE has the disadvantage that it has strange behavior for small values of χ. Also, the shrinkage factor $(1 - \rho\chi^{-1})$ becomes negative for $\chi < \rho$. Hence we define a better estimator by positive-rule shrinkage estimator (PRSE) of β as

$$\begin{aligned} \hat{\beta}^{S+} &= \widetilde{\beta} + (1 - \rho\chi^{-1}) I[\chi > \rho](\hat{\beta} - \widetilde{\beta}) \\ &= \hat{\beta}^{S} - (1 - \rho\chi^{-1}) I[\chi \leq \rho](\hat{\beta} - \widetilde{\beta}). \end{aligned} \tag{2.7}$$

Note that this estimator is a convex combination of $\hat{\beta}$ and $\widetilde{\beta}$.

The quadratic risk functions of the estimators are given in the following section and the dominance properties are studied in section 4.

3 Risk Evaluations

For a given non-singular matrix W, consider the *weighted quadratic error loss function* of the form

$$L(\beta^*;\beta) \;=\; (\beta^*-\beta)'W(\beta^*-\beta), \tag{3.1}$$

where β^* is any estimator of β. Then the weighted quadratic risk function associated with (3.1) is defined as

$$R(\beta^*;\beta) \;=\; E[(\beta^*-\beta)'W(\beta^*-\beta)]. \tag{3.2}$$

In this section, using the risk function (3.2), we evaluate the quadratic risks of the five different estimators under study.
Direct computations using (1.2), (2.1) and (3.2) lead to

$$R(\hat{\beta};\beta) \;=\; tr(G_1W). \tag{3.3}$$

Let $\delta = G_1H'G_2(H\beta - h)$, then using (2.4) we have

$$\begin{aligned} R(\widetilde{\beta};\beta) &= tr\{W[G_1(I_p - H'G_2HG_1) + \delta\delta']\} \\ &= tr(G_1W) - tr\{W[G_1(H'G_2HG_1)]\} + \delta'W\delta. \end{aligned} \tag{3.4}$$

Note that $R = G_1^{1/2}H'G_2HG_1^{1/2}$ is a symmetric idempotent matrix of rank $q \le p$. Thus, there exists an orthogonal matrix Q $(Q'Q = I_p)$ (see Judge and Bock [4]) such that

$$QRQ' \;=\; \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.5}$$

$$\begin{aligned} QG_1^{1/2}WG_1^{1/2}Q' &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ &= A. \end{aligned} \tag{3.6}$$

The matrices A_{11} and A_{22} are of orders q and $p-q$, respectively.
Define the random variable

$$w \;=\; QG_1^{-1/2}\hat{\beta} - QG_1^{1/2}H'G_2h, \tag{3.7}$$

then

$$w \quad \sim \quad N_q(\eta, I_p). \tag{3.8}$$

Also

$$\eta \quad = \quad QG_1^{-1/2}\beta - QG_1^{1/2}H'G_2h. \tag{3.9}$$

Partitioning the vectors $w = (w_1', w_2')'$ and $\eta = (\eta_1', \eta_2')'$, where w_1 and w_2 are independent sub-vectors of orders q and $p-q$ respectively, we obtain

$$\hat{\beta} - \beta \quad = \quad G_1^{1/2}Q'(w - \eta). \tag{3.10}$$

Using (3.7) we can obtain

$$\chi = w_1'w_1,\ \theta = \eta_1'\eta_1 = (H\beta - h)'G_2(H\beta - h). \tag{3.11}$$

Now, we may write

$$\begin{aligned} tr\{W[G_1H'G_2HG_1]\} &= tr\{QG_1^{1/2}WG_1^{1/2}Q'QRQ'\} \\ &= tr\left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \right\} \\ &= tr(A_{11}). \end{aligned} \tag{3.12}$$

Using (3.11) we have

$$\begin{aligned} \delta'W\delta &= (H\beta - h)'G_2HG_1WG_1H'G_2(H\beta - h) \\ &= \eta_1'A_{11}\eta_1. \end{aligned} \tag{3.13}$$

Therefore, we obtain

$$R(\widetilde{\beta}; \beta) \quad = \quad tr(G_1W) - tr(A_{11}) + \eta_1'A_{11}\eta_1. \tag{3.14}$$

Using (2.5)

$$\begin{aligned} R(\hat{\beta}^{PT}; \beta) &= E[(\hat{\beta}^{PT} - \beta)'W(\hat{\beta}^{PT} - \beta)] \\ &= E\{[(\hat{\beta} - \beta) - I(\chi \le \chi_\alpha^2)(\hat{\beta} - \widetilde{\beta})]'W[(\hat{\beta} - \beta) \\ &\quad -I(\chi \le \chi_\alpha^2)(\hat{\beta} - \widetilde{\beta})]\} \\ &= E[(\hat{\beta} - \beta)'(\hat{\beta} - \beta)] - 2E[I(\chi \le \chi_\alpha^2)(\hat{\beta} - \beta)'W(\hat{\beta} - \widetilde{\beta})] \\ &\quad +E[I(\chi \le \chi_\alpha^2)(\hat{\beta} - \widetilde{\beta})'W(\hat{\beta} - \widetilde{\beta})]. \end{aligned} \tag{3.15}$$

Using (3.7)-(3.11) and (3.15)

$$\begin{aligned} R(\hat{\beta}^{PT};\beta) &= tr(G_1W) - E[w_1'A_{11}w_1 I(\chi \le \chi^2_\alpha)] \\ &\quad -2E[w_2'A_{21}w_1 I(\chi \le \chi^2_\alpha)] + 2\eta_1' A_{11} E[w_1 I(\chi \le \chi^2_\alpha)] \\ &\quad +2\eta_2' A_{21} E[w_1 I(\chi \le \chi^2_\alpha)], \end{aligned} \tag{3.16}$$

because w_1 and w_2 are independent

$$E[w_2'A_{21}w_1 I(\chi \le \chi^2_\alpha)] = \eta_2' A_{21} E[w_1 I(\chi \le \chi^2_\alpha)], \tag{3.17}$$

using Lemma1 in Appendix with $\phi(\chi)$ as indicator function of χ, we get

$$\begin{aligned} R(\hat{\beta}^{PT};\beta) &= tr(G_1W) - \chi^2_{q+2,\theta}(\alpha) tr(A_{11}) \\ &\quad +[2\chi^2_{q+2,\theta}(\alpha) - \chi^2_{q+4,\theta}(\alpha)]\eta_1' A_{11}\eta_1. \end{aligned} \tag{3.18}$$

Using (2.6) and (2.7)

$$\begin{aligned} R(\hat{\beta}^{S};\beta) &= E[(\hat{\beta}^S - \beta)'W(\hat{\beta}^S - \beta)] \\ &= E\{[(\hat{\beta} - \beta) - \rho\chi^{-1}(\hat{\beta} - \widetilde{\beta})]'W[(\hat{\beta} - \beta) - \rho\chi^{-1}(\hat{\beta} - \widetilde{\beta})] \\ &= E[(\hat{\beta} - \beta)'W(\hat{\beta} - \beta)] - 2\rho E[\chi^{-1}(\hat{\beta} - \beta)'W(\hat{\beta} - \widetilde{\beta})] \\ &\quad +\rho^2 E[\chi^{-2}(\hat{\beta} - \widetilde{\beta})'W(\hat{\beta} - \widetilde{\beta})], \end{aligned} \tag{3.19}$$

using (3.7)-(3.11) and (3.15)

$$\begin{aligned} R(\hat{\beta}^{S};\beta) &= tr(G_1W) - 2\rho E[\chi^{-1}(w_1'A_{11}w_1 - \eta_1'A_{11}w_1 + w_2'A_{21}w_1 \\ &\quad -\eta_2'A_{21}w_1)] + \rho^2 E[\chi^{-2}(w_1'A_{11}w_1)]. \end{aligned} \tag{3.20}$$

Using Lemma1 in Appendix for $\phi(\chi) = \chi^{-1}$, we have

$$E[\chi^{-1}\eta_1'A_{11}w_1] = \eta_1'A_{11}\eta_1 E\left[\frac{1}{\chi^2_{q+2,\theta}}\right], \tag{3.21}$$

$$E[\chi^{-1}w_1'A_{11}w_1] = E\left[\frac{1}{\chi^2_{q+2,\theta}}\right] tr(A_{11}) + E\left[\frac{1}{\chi^2_{q+4,\theta}}\right]\eta_1'A_{11}\eta_1. \tag{3.22}$$

Using Lemma1 in Appendix for $\phi(\chi) = \chi^{-2}$, we have

$$E[\chi^{-2}w_1'A_{11}w_1] = E\left[\frac{1}{\chi^2_{q+2,\theta}}\right]^2 tr(A_{11}) + E\left[\frac{1}{\chi^2_{q+4,\theta}}\right]^2 \eta_1'A_{11}\eta_1. \tag{3.23}$$

Using (3.21)-(3.23) one can obtain

$$\begin{aligned} R(\hat{\beta}^S;\beta) &= tr(G_1W) - \rho\Big\{2E\left[\frac{1}{\chi^2_{q+2,\theta}}\right] - \rho E\left[\frac{1}{\chi^2_{q+2,\theta}}\right]^2\Big\} tr(A_{11}) \\ &\quad +\rho\Big\{2E\left[\frac{1}{\chi^2_{q+2,\theta}}\right] - 2E\left[\frac{1}{\chi^2_{q+4,\theta}}\right] \\ &\quad +\rho E\left[\frac{1}{\chi^2_{q+4,\theta}}\right]^2\Big\}\eta_1' A_{11}\eta_1. \end{aligned} \tag{3.24}$$

Finally the risk of PRSE is given by

$$\begin{aligned} R(\hat{\beta}^{S+};\beta) &= E[(\hat{\beta}^{S+}-\beta)'W(\hat{\beta}^{S+}-\beta)] \\ &= E\{[(\hat{\beta}^S-\beta) - (1-\rho\chi^{-1})I(\chi\leq\rho)(\hat{\beta}-\widetilde{\beta})]'W[(\hat{\beta}^S-\beta) \\ &\quad -(1-\rho\chi^{-1})I(\chi\leq\rho)(\hat{\beta}-\widetilde{\beta})]\} \\ &= R(\hat{\beta}^S;\beta) + E[(1-\rho\chi^{-1})^2 I(\chi\leq\rho)(\hat{\beta}-\widetilde{\beta})'W(\hat{\beta}-\widetilde{\beta})] \\ &\quad -2E[(\hat{\beta}^S-\beta)'W(1-\rho\chi^{-1})I(\chi\leq\rho)(\hat{\beta}-\widetilde{\beta})]. \end{aligned} \tag{3.25}$$

But using (2.6)

$$\begin{aligned} &E[(\hat{\beta}^S-\beta)'W(1-\rho\chi^{-1})I(\chi\leq\rho)(\hat{\beta}-\widetilde{\beta})] \\ = \;& E[(\widetilde{\beta}-\beta) + (1-\rho\chi^{-1})(\hat{\beta}-\widetilde{\beta})]'W(1-\rho\chi^{-1})I(\chi\leq\rho)(\hat{\beta}-\widetilde{\beta})] \\ = \;& E\{(\widetilde{\beta}-\beta)'W[(1-\rho\chi^{-1})I(\chi\leq\rho)(\hat{\beta}-\widetilde{\beta})]\} \\ & +E[(1-\rho\chi^{-1})^2 I(\chi\leq\rho)(\hat{\beta}-\widetilde{\beta})'W(\hat{\beta}-\widetilde{\beta})]. \end{aligned} \tag{3.26}$$

Thus, we obtain

$$\begin{aligned} R(\hat{\beta}^{S+};\beta) &= R(\hat{\beta}^S;\beta) - E[(1-\rho\chi^{-1})^2 I(\chi\leq\rho)(\hat{\beta}-\widetilde{\beta})'W(\hat{\beta}-\widetilde{\beta})] \\ &\quad -2E\{(\widetilde{\beta}-\beta)'W[(1-\rho\chi^{-1})I(\chi\leq\rho)(\hat{\beta}-\widetilde{\beta})]\}. \end{aligned} \tag{3.27}$$

Using (3.21)-(3.23), and Lemma1 in Appendix for $\phi(\chi) = (1-\rho\chi^{-1})^i I(\chi \leq \rho)$, $(i = 1, 2)$ we get

$$\begin{aligned} R(\hat{\beta}^{S+};\beta) &= R(\hat{\beta}^{S};\beta) - E\left[\left(1-\frac{\rho}{\chi^2_{q+2,\theta}}\right)^2 I(\chi^2_{q+2,\theta} \leq \rho)\right] tr(A_{11}) \\ &\quad + E\left[\left(1-\frac{\rho}{\chi^2_{q+4,\theta}}\right)^2 I(\chi^2_{q+4,\theta} \leq \rho)\right] \eta_1' A_{11}\eta_1 \\ &\quad - 2E\left[\left(\frac{\rho}{\chi^2_{q+2,\theta}}-1\right) I(\chi^2_{q+2,\theta} \leq \rho)\right] \eta_1' A_{11}\eta_1 \qquad (3.28) \end{aligned}$$

4 Comparison

Providing risk analysis of the underlying estimators with the weight matrix W, we have (see e.g. Searle [11])

$$\theta ch_1(A_{11}) \leq \eta_1' A_{11}\eta_1 \leq \theta ch_q(A_{11}), \qquad (4.1)$$

where $ch_1(A_{11})$ and $ch_q(A_{11})$ are the minimum and maximum eigenvalues of A_{11} respectively. Then by (3.3) and (3.14) one may easily see that

$$R(\hat{\beta};\beta) - tr(A_{11}) + \theta ch_1(A_{11}) \leq R(\widetilde{\beta};\beta) \leq R(\hat{\beta};\beta) - tr(A_{11}) + \theta ch_q(A_{11}).$$

By (3.11) and (3.30), under the null hypothesis $H_0 : H\beta = h$, we conclude

$$R(\widetilde{\beta};\beta) \leq R(\hat{\beta};\beta).$$

In general by (3.30), $\widetilde{\beta}$ performs better then $\hat{\beta}$ whenever

$$\begin{aligned} \theta &\leq \frac{tr(A_{11})}{ch_q(A_{11})} \\ &= \frac{\sum_{i=1}^{q} ch_i(A_{11})}{ch_q(A_{11})} \\ &\leq q. \end{aligned}$$

Using (3.3) and (3.18) we have

$$\begin{aligned} R(\hat{\beta}^{PT};\beta) - R(\hat{\beta};\beta) &= [2\chi^2_{q+2,\theta}(\alpha) - \chi^2_{q+4,\theta}(\alpha)]\eta_1' A_{11}\eta_1 \\ &\quad - \chi^2_{q+2,\theta}(\alpha) tr(A_{11}). \qquad (4.2) \end{aligned}$$

Therefore $\hat{\beta}^{PT}$ performs better than $\hat{\beta}$ whenever

$$\theta \ \leq \ \frac{tr(A_{11})}{ch_q(A_{11})} \times \frac{\chi^2_{q+2,\theta}(\alpha)}{[2\chi^2_{q+2,\theta}(\alpha) - \chi^2_{q+4,\theta}(\alpha)]}. \tag{4.3}$$

Because $tr(A_{11}) = q$, (3.33) satisfies for $W = X'\Sigma^{-1}X$.
Also under the null hypothesis H_0, since (3.32) is negative for all α, $\hat{\beta}^{PT}$ performs better than $\hat{\beta}$.
Using (3.14), (3.18) and the risks difference we can conclude that $\hat{\beta}^{PT}$ performs better than $\widetilde{\beta}$ whenever

$$\theta \ \geq \ \frac{[1 - \chi^2_{q+2,\theta}(\alpha)]tr(A_{11})}{[1 - 2\chi^2_{q+2,\theta}(\alpha) + \chi^2_{q+4,\theta/2}(\alpha)]ch_q(A_{11})}. \tag{4.4}$$

Thus, the dominance order of the three estimator $\hat{\beta}$, $\widetilde{\beta}$ and $\hat{\beta}^{PT}$, under the null hypothesis H_0 is given by

$$\widetilde{\beta} \succ \hat{\beta}^{PT} \succ \hat{\beta},$$

where the notation $\succ$ means dominate.

Under the null hypothesis,

$$R(\hat{\beta}^S;\beta) - R(\hat{\beta};\beta) = -\rho tr(A_{11})\frac{2(q-2)-\rho}{q(q-2)}.$$

By the direct computations using the fact $n \geq p$, we get $\rho \leq 2(q-2)$. Therefore, the risk difference $R(\hat{\beta}^S;\beta) - R(\hat{\beta};\beta)$ is negative and $\hat{\beta}^S \succ \hat{\beta}$ uniformly.

Under the null hypothesis H_0, we have

$$R(\hat{\beta}^S;\beta) \ = \ R(\widetilde{\beta};\beta) + tr(A_{11})f(n,q,p),$$

where $f(n,q,p) = \frac{\rho^2 - 2\rho(q-2) + q(q-2)}{q(q-2)}$.
The function $f(n,q,p)$ is positive for $q \geq 3$. Thus $R(\hat{\beta}^S;\beta) > R(\widetilde{\beta};\beta)$. However, as η_1 moves away from 0, $\eta_1' A_{11}\eta_1$ increases and the risk of $\widetilde{\beta}$ becomes unbounded while the risk of $\hat{\beta}^S$ remains below the risk of $\hat{\beta}$; thus $\hat{\beta}^S$ dominates $\widetilde{\beta}$ outside an interval around the origin.

Comparing $\hat{\beta}^S$ and $\hat{\beta}^{PT}$, under H_0, we get

$$\begin{aligned} R(\hat{\beta}^S;\beta) &= R(\hat{\beta}^{PT}) + \left[\chi^2_{q+2,0}(\alpha) - 2\rho E[\frac{1}{\chi^2_{q+2,0}}] + \rho^2 E[\frac{1}{\chi^2_{q+2,0}}]^2\right] tr(A_{11}) \\ &= R(\hat{\beta}^{PT}) + \left[\chi^2_{q+2,0}(\alpha) - \frac{2\rho}{q} + \frac{\rho^2}{q(q-2)}\right] tr(A_{11}) \\ &\geq R(\hat{\beta}^{PT}), \end{aligned}$$

for all α such that $l = \chi^2_{q+2,0}(\alpha) - \frac{2\rho}{q} + \frac{\rho^2}{q(q-2)} \geq 0$ and $R(\hat{\beta}^S;\beta) \leq R(\hat{\beta}^{PT})$ for all α such that $l \leq 0$.

Because w_1 is independent of w_2, we get

$$\begin{aligned} R(\hat{\beta}^{S+};\beta) - R(\hat{\beta}^S;\beta) &= -E[(1-\rho\chi^{-1})^2 I(\chi \leq \rho) w_1' A_{11} w_1] \\ &\quad -2E[(1-\rho\chi^{-1}) I(\chi \leq \rho)(w_1' A_{11} w_1 - \eta_1' A_{11} w_1)]. \end{aligned} \tag{4.5}$$

Note that for such θ under which $\chi^2_{q+2,\theta} \leq \rho$ we have

$$E[(1 - \frac{\rho}{\chi^2_{q+2,\theta}}) I(\chi^2_{q+2,\theta}) \leq \rho)] \leq 0.$$

Moreover, the expectation of a positive random variable, is positive, then one can obtain the risk difference in (4.5) is negative. Therefore, for all β, $\hat{\beta}^{S+} \succ \hat{\beta}^S$ and under H_0, $\widetilde{\beta} \succ \hat{\beta}^{S+}$.

However, as η_1 moves away from 0, $\eta_1' A_{11} \eta_1$ increases and the risk of $\widetilde{\beta}$ becomes unbounded while the risk of $\hat{\beta}^{S+}$ remains below the risk of $\hat{\beta}$; thus $\hat{\beta}^{S+}$ dominates $\widetilde{\beta}$ outside an interval around the origin.

Under the conditions are given above, it can be found that the dominance order of five estimators of β can be categorized in the following two orders:

1. $$\widetilde{\beta} \succ \hat{\beta}^{PT} \succ \hat{\beta}^{S+} \succ \hat{\beta}^S \succ \hat{\beta} \tag{4.6}$$

and

2. $$\widetilde{\beta} \succ \hat{\beta}^{S+} \succ \hat{\beta}^S \succ \hat{\beta}^{PT} \succ \hat{\beta}. \tag{4.7}$$

5 Illustrative Example

For an illustrative example of domination orders of five estimators under study, we proceed with numerical and graphical examples.

Numerical Example Now for an illustrative example of domination orders given in the previous section, we accomplish with a numerical example from Searle [11]. Suppose we have the following five sets of observations (including $x_{i0} = 1$ for $i = 1, \cdots, 5$).

i	y_i	x_{i0}	x_{i1}	x_{i2}
1	62	1	2	6
2	60	1	9	10
3	57	1	6	4
4	48	1	3	13
5	23	1	5	2

Then the model can be represented as

$$\begin{bmatrix} 62 \\ 60 \\ 57 \\ 48 \\ 23 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 1 & 9 & 10 \\ 1 & 6 & 4 \\ 1 & 3 & 13 \\ 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix},$$

where the covariance structure of the error term has the form $\Sigma = \sigma^2 R$ for $R = (1-\rho)I_5 + \rho J_5$, when $\rho = \frac{1}{2}$ and $\sigma^2 = 2$, which satisfies the condition under which Σ^{-1} exists. Then $\Sigma^{-1} = I_5 - \frac{1}{6}J_5$.
Moreover, assume that we want to test the null hypothesis

$$H_0 : \begin{cases} \beta_2 = 0.5 \\ 2\beta_1 - \beta_2 + 3\beta_3 = 2 \\ \beta_1 = -1 \end{cases}.$$

In this approach we have

$$H = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 3 \\ 1 & 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0.5 \\ 2 \\ -1 \end{bmatrix}.$$

Direct algebraic computations lead to

$$G_1 = \begin{bmatrix} 2.64583 & -0.16667 & -0.0875 \\ -0.16667 & 0.03333 & 0.0000 \\ -0.08750 & 0.00000 & 0.0125 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 83.7037 & 23.1481 & -40.1852 \\ 23.1481 & 13.4259 & -24.9074 \\ -40.1852 & -24.9074 & 46.7593 \end{bmatrix}.$$

Using (2.1) and (2.2) we obtain

$$\hat{\beta} = \begin{bmatrix} 37.0 \\ 0.5 \\ 1.5 \end{bmatrix}, \ \widetilde{\beta} = \begin{bmatrix} 0.5 \\ 2 \\ -1 \end{bmatrix} \ \text{and } \chi = 1203.\bar{3}.$$

Consider in this example $n = 5$, $p = 3$ and $q = 3$. Therefore using (2.6) we get $\rho = \frac{1}{6}$. Then using (2.4), (2.5) and (2.7) we have

$$\hat{\beta}^{PT} = \begin{bmatrix} 0.5 \\ 2 \\ -1 \end{bmatrix} + [1 - I(1203 \leq \chi_3^2(\alpha))] \begin{bmatrix} 38 \\ 0 \\ 0 \end{bmatrix},$$

$$\hat{\beta}^{S+} = \hat{\beta}^{S} = \begin{bmatrix} 0.5 \\ 2 \\ -1 \end{bmatrix} + 0.9998 \begin{bmatrix} 38 \\ 0 \\ 0 \end{bmatrix},$$

In order to compare the risks of the above five estimators, suppose the weight matrix is given by

$$W = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then from (3.12) we get $tr(A_{11}) = 0.0125$.
Using (3.3) and (3.14) we have $R(\hat{\beta}; \beta) = 0.0125$ and $R(\widetilde{\beta}; \beta) = \theta$. Clearly $\widetilde{\beta}$

performs better than $\hat{\beta}$ whenever $\theta < 0.0125$. Using Lemma2 from Appendix we can determine the risk functions for different values α and θ. We will continue with large values of θ, to do better comparisons, which result in large unreasonable risks' values. The results are given in Table1.

Table1: Risks' comparison

α	θ	$R(\hat{\beta};\beta)$	$R(\widetilde{\beta};\beta)$	$R(\hat{\beta}^{PT};\beta)$	$R(\hat{\beta}^{S};\beta)$	$R(\hat{\beta}^{S+};\beta)$
0.05	0	0.0125	0	0.0044	0.0113	0.0113
	0.001	0.0125	0.001	0.0045	0.0110	0.0110
	0.1	0.0125	0.1	0.0187	0.0221	0.0221
	1	0.0125	1	0.3041	0.0694	0.0694
	10	0.0125	10	37.1286	0.0995	0.0995

From the Table1, it can be easily seen that

1. Under H_0 ($\theta = 0$), the domination order given in (4.6) satisfies.

2. For $\theta \leq 0.001$, the risks of PRSE and SE have decreasing trends and for $\theta \geq 0.1$ those change to increasing.

3. For $\theta \geq 0.1$, GLSE performs better than both RGLSE and PTE, and PTE performs better than RGLSE.

Graphical Example Some graphical perspectives of the risks of estimators $\hat{\beta}$, $\widetilde{\beta}$, $\hat{\beta}^{PT}$, $\hat{\beta}^{S}$ and $\hat{\beta}^{S+}$ can be shown using approximations of (3.24) and (3.28). In this approach, we use lemma 2 in Appendix to compute (3.34) and (3.35) Then substituting suitable expression in (3.24) and (3.285), we compute underlying risks approximately using packages MATLAB release 7.2 and MAPLE release 9.5.

For special case $n = 20$, $p = 5$ and $q = 3$, when $W = X'\Sigma^{-1}X$, the graphical displays are as follow (Because changing values α in (3.18), does not clear graphically we use just $\alpha = 0.3$. Note that when α increases $R(\hat{\beta}^{PT};\beta)$ decreases).

In Figure 1, the horizontal axis are the values of θ and

$R1 = R(\hat{\beta};\beta)$, $R2 = R(\widetilde{\beta};\beta)$, $R3 = R(\hat{\beta}^{PT};\beta)$, $R4 = R(\hat{\beta}^{S};\beta)$, $R5 = R(\hat{\beta}^{S+};\beta)$.

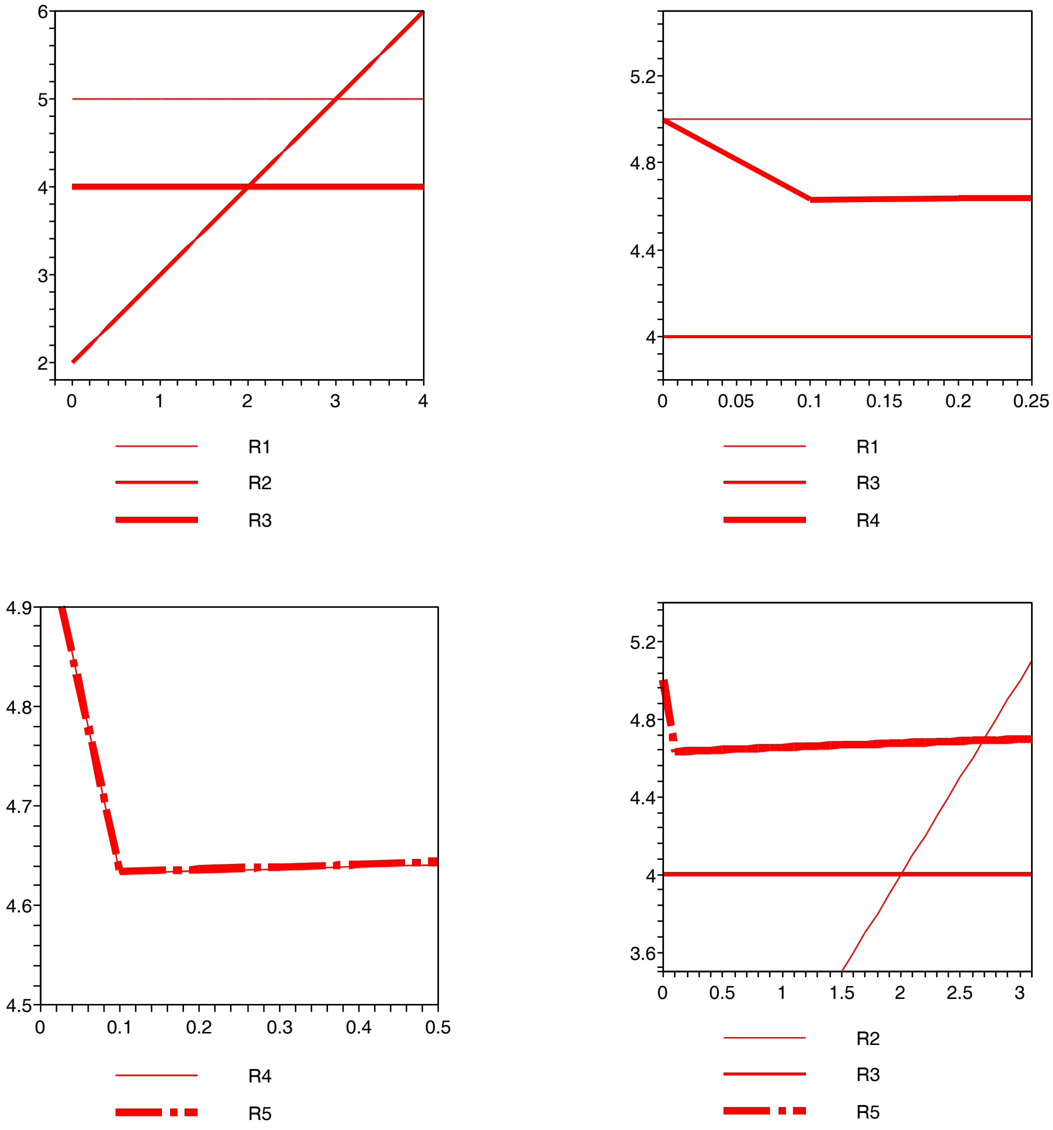

Figure 1: **Risks Comparison**

6 Appendix

Lemma 6.1 *Assume the random variable w is normally distributed with mean vector τ and covariance matrix I_j and A is any p.d. symmetric matrix. Also assume $\phi(.)$ is a Borel measurable function, then*

$$\begin{aligned} E[\phi(w'w)w] &= E[\phi(\chi^2_{j+2,\tau'\tau/2})]\tau, \\ E[\phi(w'w)w'Aw] &= E[\phi(\chi^2_{j+2,\tau'\tau/2})]tr(A) + E[\phi(\chi^2_{j+4,\tau'\tau/2})]\tau'A\tau. \end{aligned}$$

Proof. For the proof see Appendix B.2. in Judge and Bock [4].

Lemma 6.2 *Let p be an integer greater that $2m$ $(p > 2m)$ then*

$$E[(1-\frac{\rho}{\chi^2_{q,\theta/2}})^2 I(\chi^2_{q,\theta/2} \le \rho)] = \chi^2_{q,\theta/2}(\rho) + \Upsilon,$$

where

$$\Upsilon = \sum_{r=0}^{\infty} \frac{\rho[\rho - 2q - 4r + 8)]e^{-\theta/4}(\theta/4)^r \chi^2_{q+2r,0}(\rho)}{r!(q+2r-2)(q+2r-4)}.$$

Proof. Using the series expansion for inverse non-central chi-square distribution (see Johnson and Kotz [3]), we have

$$\begin{aligned} E[(\frac{1}{\chi^2_{q,\theta}})^m] &= \sum_{r=0}^{\infty} \frac{e^{-\theta/2}(\theta/2)^r}{r!} E[(\frac{1}{\chi^2_{q+2r,0}})^m] \\ &= \sum_{r=0}^{\infty} \frac{e^{-\theta/2}(\theta/2)^r}{2^m r!} \times \frac{\Gamma(q/2+r-m)}{\Gamma(q/2+r)}. \end{aligned}$$

Thus we can obtain

$$\begin{aligned} E[(\frac{1}{\chi^2_{q,\theta}})^m I(\chi^2_{q,\theta} \le \rho)] &= \sum_{r=0}^{\infty} \frac{e^{-\theta/2}(\theta/2)^r}{r!} E[(\frac{1}{\chi^2_{q+2r,0}})^m I(\chi^2_{q+2r,0} \le \rho)] \\ &= \sum_{r=0}^{\infty} \frac{e^{-\theta/2}(\theta/2)^r}{2^m r!} \times \frac{\Gamma(q/2+r-m)}{\Gamma(q/2+r)} \times \chi^2_{q+2r,0}(\rho) \end{aligned}$$

Therefore

$$
\begin{aligned}
E[(1-\frac{\rho}{\chi^2_{q,\theta/2}})^2 I(\chi^2_{q,\theta/2}\le\rho)] &= \chi^2_{q,\theta/2}(\rho) - 2\rho E[(\frac{1}{\chi^2_{q,\theta}})I(\chi^2_{q,\theta}\le\rho)] \\
&\quad +\rho^2 E[(\frac{1}{\chi^2_{q,\theta}})^2 I(\chi^2_{q,\theta}\le\rho)] \\
&= \chi^2_{q,\theta/2}(\rho) + \sum_{r=0}^{\infty}\frac{e^{-\theta/2}(\theta/2)^r}{r!\Gamma(q/2+r)}\times\chi^2_{q+2r,0}(\rho) \\
&\quad \times\left[\frac{\rho^2\Gamma(q/2+r-2)}{4} - \rho\Gamma(q/2+r-1)\right] \\
&= \chi^2_{q,\theta/2}(\rho) + \sum_{r=0}^{\infty}\frac{\rho[\rho-2q-4r+8)]e^{-\theta/2}(\theta/2)^r\chi^2_{q+2r,0}(\rho)}{r!(q+2r-2)(q+2r-4)}.
\end{aligned}
$$

Acknowledgments

We would like to thank the referees for their comments and suggestions which substantially improved the presentation and the quality of the paper.

Partial support from the "Ordered and Spatial Data Center of Excellence" of Ferdowsi University of Mashhad, Iran is acknowledged.

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Abstracts

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مجله پژوهش های علوم ریاضی مشهد، جلد ۱، شماره ۱، (۱۳۸۶)

درباره مشخصه‌سازی‌های جبری برای متناهی بودن بعد EG

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چکیده

کوهمولوژی فارل – تت قبلا برای گروههای از بعد کوهمولوژی متناهی تعریف شده است. در سال ۱۹۸۴ ایکاناگا و در سال ۱۹۸۷ گدریچ و گرونبرگ برای تعمیم این مفهوم تلاش کردند و برای این منظور پایاهای جبری معینی از یک گروه را در نظر گرفتند و نشان دادند که اگر این پایاها متناهی باشند، آنگاه تعمیم کوهمولوژی تت برای گروه قابل تعریف است. در این مقاله ما مروری بر حقایق و نتایج معلوم داریم و سپس در مورد بعضی از نتایج جدید خود بحث خواهیم کرد.

واژه های کلیدی: کوهمولوژی فارل – تت، بعد کوهمولوژی، سوپریمم طول تصویر.
رده بندی موضوعی ریاضیات AMS (۲۰۰۰): اولیه ۲۰G۱۰، ۲۰J۰۶؛ ثانویه ۱۸G۱۰.

مجله پژوهش های علوم ریاضی مشهد، جلد ۱، شماره ۱، (۱۳۸۶)

رده‌های جدید گروه های نامتناهی

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چکیده

در این مقاله رده‌های تزویجی از M_p-گروهها، T_0-گروهها، ϕ-گروهها، ϕ_0-گروهها و گروههای با تعداد متناهی از عناصر نشاندن مورد بررسی قرار می‌دهیم که در اواخر قرن بیستم پا به عرصه وجود گذاشته‌اند. این رده‌ها از گروههای نامتناهی با شرایط متناهی توسط وی. پی. شانکف معرفی شده‌اند. در این جا مروری بر نتایج جدیدی روی این رده‌های گروهها خواهیم داشت.

واژه های کلیدی: ϕ-گروهها، عناصر از مرتبه ۲، گروههای فرابنیوس، گروههای چرنیکف و گروههای شبه دوری.

رده بندی موضوعی ریاضیات AMS (۲۰۰۰): اولیه ۲۰E۳۴؛ ثانویه ۲۰C۹۹.

مجله پژوهش های علوم ریاضی مشهد، جلد ۱، شماره ۱، (۱۳۸۶)

گروه های با رده‌های تزویجی کم بیشینه حل‌پذیر از زیرگروه ها

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چکیده

یک نتیجه کلاسیک از بی. اچ. نویمن گروه هایی را رده‌بندی می‌کند، که در آنها هر زیرگروه داری تعداد متناهی مزدوج تنها به عنوان گروه های مرکزی-با-متناهی است. اگر $\mathcal{X}$ رده‌ای از گروه ها باشد، یک گروه G داری رده‌های $\mathcal{X}$-مزدوج از زیرگروه هاست، هرگاه به ازای هر زیرگروه H از G، $\frac{G}{Core_G(N_G(H))} \in \mathcal{X}$. در این مقاله گروه هایی را مورد بررسی قرار می‌دهیم که دارای رده‌های تزویجی مینیمم ماکزیمم حل‌پذیر از زیرگروهاست که توصیفی بر حسب $\frac{G}{Z(G)}$ به دست می‌دهند. همچنین FC-گروه هایی را رده‌بندی می‌کنیم که دارای رده‌های تزویجی مینیمم ماکزیمم حل‌پذیر از زیرگروه ها می‌باشند.

واژه های کلیدی: رده‌های تزویجی، گروه های کم بیشینه حل‌پذیر، FC-گروه ها و گروه های چند دوری.

رده بندی موضوعی ریاضیات AMS (۲۰۰۰): اولیه ۲۰F۲۴؛ ثانویه ۲۰F۱۴.

مجله پژوهش های علوم ریاضی مشهد، جلد ۱، شماره ۱، (۱۳۸۶)

یک حل عددی برای یک مساله معکوس انتقال حرارت

عبدا... شیدفر و علی ذاکری
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تهران — ایران

چکیده

در این مقاله وجود و یکتایی یک حل تحلیلی از یک مساله معکوس انتقال حرارت (IHCP) به صورت $u_t = u_{xx}$ در دامنه

$$D = \{(x,t)|\ 0 < x < 1,\ 0 < t \leq T\}\ ,\ u(x,T) = f(x)\ ,\ u(0,t) = g(t)$$

نشان داده می‌شود، که $0 \leq t \leq T$. در بخش آخر چند آزمون عددی نیز ارائه خواهد شد.

واژه های کلیدی: مساله معکوس انتقال حرارت، حل نیم تحلیلی، روش تفاضل متناهی.

رده بندی موضوعی ریاضیات AMS (۲۰۰۰): اولیه ۳۵R۳۰؛ ثانویه ۱۸G۱۰.

مجله پژوهش های علوم ریاضی مشهد، جلد ۱، شماره ۱، (۱۳۸۶)

توابع ۱—۱ به طور یکنوا پیوسته بر روی میدان های مرتب که درون را به درون نقش می‌کنند

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چکیده

اخیراً در مقاله‌ای نشان داده‌ایم که میدانهای مرتب F که با میدان اعداد حقیقی $\mathbb{R}$ یکریخت نیستند، توابع ۱—۱ پیوسته بر روی [۰, ۱] موجود است که یک نقطه درونی را به یک نقطه مرز از تصویر می نگارد و از این رو باز نیستند. در اینجا نشان می‌دهیم که روی بازه‌های کراندار بسته در میدان اعداد گویا $\mathbb{Q}$ و همچنین در تمام میدان های مرتب غیر ارشمیدسی هم‌پایان شمارش‌پذیر، توابع ۱—۱ به طور یکنوا پیوسته وجود دارد که درون را به درون نمی‌برند. به ویژه میدان مرتب غیر ارشمیدسی کمین $\mathbb{Q}(x)$ و همچنین میدان های مرتب لوران، با ضرایب در یک میدان، مرتب چنین توابع آسیب شناسی را به دست می‌دهد.

واژه های کلیدی: میدان مرتب، میدان های غیر ارشمیدسی، یکنوای کامل، نگاشت باز.
رده بندی موضوعی ریاضیات AMS (۲۰۰۰): اولیه ۱۲J۱۵، ۲۶A۰۳؛ ثانویه ۲۶E۳۰.

مجله پژوهش های علوم ریاضی مشهد، جلد ۱، شماره ۱، (۱۳۸۶)

استنباط آماری مبتنی بر k—رکوردها

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چکیده

در مسائل کاربردی، براساس داده های رکوردی معمولی به علت نادر بودن این نوع داده ها مشکلاتی در استنباط آماری بروز می کند. اما با در نظرگرفتن داده های k—رکوردی، که تعمیمی از رکورد معمولی است، این مشکل تا حدودی مرتفع می‌شود. در این مقاله با در نظرگرفتن یک مدل کلی شامل کلاس وسیعی از توزیع های آماری نظیر نمایی، وایبل و پارتو، برآوردگرهای بیزی برای پارامترهای مجهول و پیش‌بینی کننده بیزی برای مقدار k—رکوردهای آینده بدست آمده است. در ادامه، با در نظرگرفتن مدل نمایی دو پارامتری، نتایج بدست آمده با جزئیات بیشتر مورد مطالعه قرار گرفته است.

واژه های کلیدی: مجاز بودن، پیش بینی بیزی، برآورد بیزی، توزیع پیشین مزدوج، توزیع پیشین جفری.

رده بندی موضوعی ریاضیات AMS (۲۰۰۰): اولیه ۶۲G۳۰، ۶۲G۳۲؛ ثانویه ۶۲C۱۵.

مجله پژوهش های علوم ریاضی مشهد، جلد ۱، شماره ۱، (۱۳۸۶)

برآورد انقباضی پارامترهای مدل رگرسیون با خطاهای نرمال چند متغیره

محمد آرشی و سید محمد مهدی طباطبایی
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مشهد — ایران

چکیده

در مدل رگرسیون چندگانه $y = X\beta + e$ با خطاهای دارای توزیع نرمال چند متغیره، برآوردگرهای حداقل مربعات تعمیم یافته (GLS)، GLS محدود شده، آزمون مقدماتی (PT)، انقباضی نوع اشتاین (S) و انقباضی نوع مثبت (PRS) را برای بردار پارامتر رگرسیون به دست می آوریم، هنگامی که ماتریس کوواریانس σ معلوم است. همچنین ریسک برآوردگرهای مورد نظر را با یکدیگر مقایسه می کنیم. در نهایت یک مثال عددی و نموداری برای روشن شدن موضوع مورد بررسی ارائه می دهیم.

واژه های کلیدی: برآوردگر GLS، برآوردگر آزمون مقدماتی، برآوردگر انقباضی نوع اشتاین، برآوردگر انقباضی نوع مثبت.
رده بندی موضوعی ریاضیات AMS (۲۰۰۰): اولیه ۶۲J۰۵؛ ثانویه ۶۲H۱۲.

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[1] Ahlfors, L.V., Finitely generated Kleinian group, Amer. J. Math., **86** (1964), 413-424.

[2] Lehmann, E.L. and Casella, G., Theory of point estimation, Second edition, Springer-Verlag, New York, 1998.

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