Iranaian J. Numerical Analysis and Optimization
Vol. 3, No. 1, (2013), 31-44

Accelerated normal and skew-Hermitian splitting methods for positive definite linear systems

F. Toutounian and D. Hezari

Abstract

For solving large sparse non-Hermitian positive definite linear equations, Bai et al. proposed the Hermitian and skew-Hermitian splitting methods (HSS). They recently generalized this technique to the normal and skew-Hermitian splitting methods (NSS). In this paper, we present an accelerated normal and skew-Hermitian splitting methods (ANSS) which involve two parameters for the NSS iteration. We theoretically study the convergence properties of the ANSS method. Moreover, the contraction factor of the ANSS iteration is derived. Numerical examples illustrating the effectiveness of ANSS iteration are presented.

Keywords: Non-Hermitian matrix; Normal matrix; Hermitian matrix; Skew-Hermitian matrix; Splitting iteration method.

1 Introduction

Many problems in scientific computation give rise to solving the linear system

$$Ax = b, \tag{1}$$

with $A \in \mathbb{C}^{n\times n}$ a large non-Hermitian positive definite matrix and $x, b \in \mathbb{C}^n$. We observe that the coefficient matrix A naturally possesses the Hermitian/skew-Hermitian (HS) splitting

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*) \quad \text{and} \quad \mathrm{S} = \frac{1}{2}(\mathrm{A} - \mathrm{A}^*),$$

with A^* being the conjugate transpose of A. Bai et al. [2] presented the HSS iteration method: Given an initial guess $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \ldots,$ until

F. Toutounian
Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran. e-mail: toutouni@math.um.ac.ir

Davood Hezari
Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran. e-mail: hezari_ h@yahoo.com

$\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{2}$$

where α is a given positive constant. They have also proved that for any positive α, the HSS method converges unconditionally to the unique solution of the system of linear equations.

Moreover, based on the HS splitting, Li et al. [5] presented the asymmetric Hermitian/skew-Hermitian splitting (AHSS) iteration method: Given an initial guess $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \ldots$, until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\beta I + S)x^{(k+1)} = (\beta I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{3}$$

where α is a given nonnegative constant and β is a given positive constant. They proved that if the coefficient matrix A is positive definite (Hermitian or non-Hermitian) the AHSS iteration converges to the unique solution of linear system (1) with any given nonnegative α, if β is restricted to an appropriate region.

Bai et al. [1] recently generalized the HS splitting to the normal/skew-Hermitian (NS) splitting

$$A = N + S, \tag{4}$$

where $N \in \mathbb{C}^{n\times n}$ is a normal matrix and $S \in \mathbb{C}^{n\times n}$ is a skew-Hermitian matrix, and obtained the following normal/skew-Hermitian splitting (NSS) method to iteratively compute a reliable and accurate approximate solution for the system of linear equations (1):

The NSS iteration method: Given an initial guess $x^{(0)} \in \mathbb{C}^n$. For $k = 0, 1, 2 \ldots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + N)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - N)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{5}$$

where α is a given positive constant. They have also proved that for any positive α the NSS method converges unconditionally to the unique solution of the system of linear equations.

In this paper, we introduce two constants for the NSS iteration and present a different approach to solve Eq. (1), called the *accelerated normal and skew-Hermitian splitting iteration*, shortened to the ANSS iteration. Moreover, theoretical analysis shows that if the coefficient matrix A is positive definite (Hermitian or non-Hermitian) the ANSS method can converge to the unique solution of the linear system (1) with any given nonnegative α, if β is restricted to an appropriate region. In addition the upper bound of the contraction factor of the ANSS iteration is dependent on the choice of α and β, the spectrum of the normal matrix N and the singular-values of the skew-

Hermitian, but it is not dependent on the eigenvectors of the matrices N, S and A.

The organization of this paper is as follows. In section 2, we establish the ANSS iteration and study its convergence properties. Numerical experiments are presented in section 3 to show the effectiveness of our method. Finally, in section 4, some concluding remarks are given.

2 The ANSS Method

Throughout the paper, the non-Hermitian matrix $A \in \mathbb{C}^{n\times n}$ (i.e. $A \neq A^*$) is positive definite if its Hermitian part is Hermitian positive definite.

Based on the NSS iteration (5), in this paper we present a new approach to solve the system of linear equations (1), called the ANSS iteration, and it is as follows.

The ANSS iteration method: Given an initial guess $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2\dots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + N)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\beta I + S)x^{(k+1)} = (\beta I - N)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{6}$$

where α is a given nonnegative constant and β is a given positive constant.

The ANSS iteration alternates between the normal matrix N and the skew-Hermitian matrix S. In fact, we can reverse the roles of the matrices N and S in the above ANSS iteration so that we may first solve the system of linear equations with coefficient matrix $\beta I + S$ and then solve the system of linear equations with coefficient matrix $\alpha I + N$.

Note that both $\alpha I + N$ and $\beta I + S$ are normal matrices. Therefore, the linear systems with the coefficient matrices $\alpha I + N$ and $\beta I + S$ may be solved accurately and efficiently by some Krylov subspace iteration methods, e.g. GMRES. It is known that the GMRES method naturally reduces to an iterative process of the three-term recurrence. See [4, 3] for other iteration methods about solving large sparse normal system of linear equations.

In matrix-vector form, the ANSS iteration method can be equivalently rewritten as

$$x^{(k+1)} = M(\alpha,\beta)x^{(k)} + G(\alpha,\beta)b, \qquad k = 0, 1, 2, \dots, \tag{7}$$

where

$$M(\alpha,\beta) = (\beta I + S)^{-1}(\beta I - N)(\alpha I + N)^{-1}(\alpha I - S) \tag{8}$$

and

$$G(\alpha,\beta) = (\alpha+\beta)(\beta I + S)^{-1}(\alpha I + N)^{-1}.$$

Here, $M(\alpha,\beta)$ is the iteration matrix of the ANSS iteration. In fact, (7) may also result from the splitting

$$A = B(\alpha,\beta) - C(\alpha,\beta)$$

of the coefficient matrix A, with

$$\begin{cases} B(\alpha,\beta) = \dfrac{1}{\alpha+\beta}(\alpha I + N)(\beta I + S) \\ \\ C(\alpha,\beta) = \dfrac{1}{\alpha+\beta}(\beta I - N)(\alpha I - S). \end{cases} \tag{9}$$

Obviously

$$M(\alpha,\beta) = B(\alpha,\beta)^{-1}C(\alpha,\beta) \quad \text{and} \quad G(\alpha,\beta) = B(\alpha,\beta)^{-1}.$$

To study the convergence properties of the ANSS iteration and derive the upper bound of the contraction factor, we first represent the following lemmas.

Lemma 2.1. *Let α be a nonnegative constant and β be a positive constant. If $(\gamma,\eta) \in \Omega$, where $\Omega = [\gamma_{\min}, \gamma_{\max}] \times [\eta_{\min}, \eta_{\max}]$, $\gamma_{\min} > 0$ and $\eta_{\min} \geq 0$, then*

$$f(\alpha,\beta) \equiv \max_{(\gamma,\eta)\in\Omega} \left\{ \frac{(\beta-\gamma)^2+\eta^2}{(\alpha+\gamma)^2+\eta^2} \right\}$$

$$= \begin{cases} \max\limits_{\gamma_{\min}\leq\gamma\leq\frac{\beta-\alpha}{2}} \left\{ \dfrac{(\beta-\gamma)^2+\eta_{\min}^2}{(\alpha+\gamma)^2+\eta_{\min}^2} \right\} & \text{for} \quad \gamma_{\min} \leq \dfrac{\beta-\alpha}{2} \\ \\ \max\limits_{\gamma_{\min}\leq\gamma\leq\gamma_{\max}} \left\{ \dfrac{(\beta-\gamma)^2+\eta_{\max}^2}{(\alpha+\gamma)^2+\eta_{\max}^2} \right\} & \text{for} \quad \dfrac{\beta-\alpha}{2} \leq \gamma_{\min}. \end{cases} \tag{10}$$

Proof. Let us define the function $g(\eta)$ by

$$g(\eta) = \frac{(\beta-\gamma)^2+\eta^2}{(\alpha+\gamma)^2+\eta^2}.$$

Differentiation gives

$$g'(\eta) = \frac{2\eta(\alpha+\beta)(\alpha-\beta+2\gamma)}{[(\alpha+\gamma)^2+\eta^2]^2}.$$

Since $(\alpha+\beta) > 0$, it follows that the function $g(\eta)$ is an increasing function if $\gamma \geq \dfrac{\beta-\alpha}{2}$ and is a decreasing function if $\gamma \leq \dfrac{\beta-\alpha}{2}$.

If $\dfrac{\beta-\alpha}{2} \leq \gamma_{\min}$, then for all γ satisfying $\gamma_{\min} \leq \gamma \leq \gamma_{\max}$, we have $\dfrac{\beta-\alpha}{2} \leq \gamma$. So, for $\gamma_{\min} \leq \gamma \leq \gamma_{\max}$, the function $g(\eta)$ is an increasing function, and

$$f(\alpha,\beta) = \max_{\gamma_{\min}\leq\gamma\leq\gamma_{\max}} \left\{ \frac{(\beta-\gamma)^2+\eta_{\max}^2}{(\alpha+\gamma)^2+\eta_{\max}^2} \right\} \quad \text{if} \quad \gamma_{\min} \leq \frac{\beta-\alpha}{2}. \tag{11}$$

If $\gamma_{\min} \leq \dfrac{\beta-\alpha}{2}$, then, by using $\beta+\alpha > 0$, for all γ satisfying $\gamma_{\min} \leq \dfrac{\beta-\alpha}{2} \leq \gamma$, we obtain $(\beta-\gamma)^2 \leq (\alpha+\gamma)^2$, which implies that

$$\frac{(\beta-\gamma)^2+\eta^2}{(\alpha+\gamma)^2+\eta^2} \leq 1.$$

Similarly, for all γ satisfying $\gamma_{\min} \leq \gamma \leq \dfrac{\beta-\alpha}{2}$, we obtain $(\beta-\gamma)^2 \geq (\alpha+\gamma)^2$ and

$$\frac{(\beta-\gamma)^2+\eta^2}{(\alpha+\gamma)^2+\eta^2} \geq 1.$$

Therefore,

$$f(\alpha,\beta) = \max_{\gamma_{\min}\leq\gamma\leq\frac{\beta-\alpha}{2},\eta_{\min}\leq\eta\leq\eta_{\max}} \left\{ \frac{(\beta-\gamma)^2+\eta^2}{(\alpha+\gamma)^2+\eta^2} \right\}, \quad \text{if} \quad \gamma_{\min} \leq \frac{\beta-\alpha}{2}$$

From the fact that, for $\gamma_{\min} \leq \gamma \leq \dfrac{\beta-\alpha}{2}$, the function $g(\eta)$ is a decreasing function, we can conclude that

$$f(\alpha,\beta) = \max_{\gamma_{\min}\leq\gamma\leq\frac{\beta-\alpha}{2}} \left\{ \frac{(\beta-\gamma)^2+\eta_{\min}^2}{(\alpha+\gamma)^2+\eta_{\min}^2} \right\}, \quad \text{if} \quad \gamma_{\min} \leq \frac{\beta-\alpha}{2} \tag{12}$$

Therefore (11) and (12) immediately result relation (10). □

Lemma 2.2. *Let α be a nonnegative constant and β be a positive constant. If $\dfrac{\beta-\alpha}{2} \leq \gamma_{\min}$, where $0 < \gamma_{\min}$, then*

$$\max_{\gamma_{\min}\leq\gamma\leq\gamma_{\max}} \left\{ \frac{(\beta-\gamma)^2+\eta_{\max}^2}{(\alpha+\gamma)^2+\eta_{\max}^2} \right\} = \max \left\{ \frac{(\beta-\gamma_{\min})^2+\eta_{\max}^2}{(\alpha+\gamma_{\min})^2+\eta_{\max}^2}, \frac{(\beta-\gamma_{\max})^2+\eta_{\max}^2}{(\alpha+\gamma_{\max})^2+\eta_{\max}^2} \right\} \tag{13}$$

Proof. Let us define the function $g(\gamma)$ by

$$g(\gamma) = \frac{(\beta-\gamma)^2+\eta_{\max}^2}{(\alpha+\gamma)^2+\eta_{\max}^2}.$$

Differentiation gives

$$g'(\gamma) = \frac{-2(\alpha+\beta)\left[-\gamma^2 + (\beta-\alpha)\gamma + \beta\alpha + \eta_{\max}^2\right]}{\left[(\alpha+\gamma)^2 + \eta_{\max}^2\right]^2}.$$

The smallest root of $g'(\gamma)$ is negative and is not in the interval $[\gamma_{\min}, \gamma_{\max}]$. The largest root of $g'(\gamma)$ is

$$\gamma_1 = \frac{(\beta-\alpha) + \sqrt{(\beta-\alpha)^2 + 4(\beta\alpha + \eta_{\max}^2)}}{2}.$$

By simple computation, we can show that this root is a minimum point for the function $g(\gamma)$. Hence (13) holds and the proof of Lemma is completed. □

Lemma 2.3. *Let α be a nonnegative constant and β be a positive constant. If $0 < \gamma_{\min} \leq \dfrac{\beta-\alpha}{2}$, then*

$$f(\alpha,\beta) = \max_{\gamma_{\min} \leq \gamma \leq \frac{\beta-\alpha}{2}} \left\{ \frac{(\beta-\gamma)^2 + \eta_{\min}^2}{(\alpha+\gamma)^2 + \eta_{\min}^2} \right\} = \frac{(\beta-\gamma_{\min})^2 + \eta_{\min}^2}{(\alpha+\gamma_{\min})^2 + \eta_{\min}^2}.$$

Proof. Let us define the function $h(\gamma)$ by

$$h(\gamma) = \frac{(\beta-\gamma)^2 + \eta_{\min}^2}{(\alpha+\gamma)^2 + \eta_{\min}^2}.$$

Differentiation gives

$$h'(\gamma) = \frac{-2(\alpha+\beta)\left[(\beta-\gamma)(\alpha+\gamma) + \eta_{\min}^2\right]}{\left[(\alpha+\gamma)^2 + \eta_{\min}^2\right]^2}$$

Since $(\alpha+\beta) > 0$ and $\gamma \leq \beta$, for all γ satisfying $\gamma_{\min} \leq \gamma \leq \gamma_{\max}$ and $\gamma \leq \dfrac{\beta-\alpha}{2}$, we have $h'(\gamma) < 0$. Thus

$$f(\alpha,\beta) = \frac{(\beta-\gamma_{\min})^2 + \eta_{min}^2}{(\alpha+\gamma_{\min})^2 + \eta_{\min}^2}.$$

□

The following theorem describes the convergence property of the ANSS iteration.

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix, $N \in \mathbb{C}^{n\times n}$ be a normal matrix and $S \in \mathbb{C}^{n\times n}$ be a skew-Hermitian matrix such that $A = N + S$, and α be a nonnegative constant and β be a positive constant. Then the spectral radius $\rho(M(\alpha,\beta))$ of the iteration matrix $M(\alpha,\beta)$ of the ANSS iteration is bounded by*

$$\delta(\alpha,\beta) \equiv \max_{\sigma_j \in \sigma(S)} \frac{\sqrt{\alpha^2+\sigma_j^2}}{\sqrt{\beta^2+\sigma_j^2}} \max_{\gamma_j + i\eta_j \in \lambda(N)} \sqrt{\frac{(\beta-\gamma_j)^2+\eta_j^2}{(\alpha+\gamma_j)^2+\eta_j^2}} \tag{14}$$

where $\lambda(N)$ is the spectral set of N and $\sigma(S)$ is the singular-value set of S. Let $\gamma_{\min}$ and $\gamma_{\max}$, $\eta_{\min}$ and $\eta_{\max}$ be the lower and the upper bound of the real, the absolute values of the imaginary parts of the eigenvalues of the matrix N, respectively, and $\sigma_{\min}$, $\sigma_{\max}$ be the lower and the upper bound of the singular-value set of the matrix S, respectively. Then $\delta(\alpha,\beta) < 1$ if one of the following conditions holds:

(a) Any given parameter α and β satisfies

$$\max\left\{\frac{\alpha(\gamma_{\min}^2+\eta_{\max}^2)}{2\alpha\gamma_{\min}+\gamma_{\min}^2+\eta_{\max}^2}, \frac{\alpha(\gamma_{\max}^2+\eta_{\max}^2)}{2\alpha\gamma_{\max}+\gamma_{\max}^2+\eta_{\max}^2}\right\} < \beta \le \alpha + 2\gamma_{\min}$$

(b) Any given parameter α and β satisfies

$$\alpha + 2\gamma_{\min} < \beta$$

if $\sigma_{\max} \le \sqrt{\gamma_{\min}+\eta_{\min}+2\gamma_{\min}\alpha}$.

(c) Any given parameter α and β satisfies

$$\alpha + 2\gamma_{\min} < \beta \le \frac{\alpha(\gamma_{\min}^2+\eta_{\min}^2-\sigma_{\max}^2)-2\sigma_{\max}^2\gamma_{\min}}{\gamma_{\min}^2+\eta_{\min}^2-\sigma_{\max}^2+2\alpha\gamma_{\min}}$$

if $\sigma_{\max} \ge \sqrt{\gamma_{\min}+\eta_{\min}+2\gamma_{\min}\alpha}$.

Proof. By the similarity invariance of the matrix spectrum, we have

$$\begin{aligned}\rho(M(\alpha,\beta)) &= \rho((\beta I - N)(\alpha I + N)^{-1}(\alpha I - S)(\beta I + S)^{-1}) \\ &\le \|(\beta I - N)(\alpha I + N)^{-1}\|_2 \|(\alpha I - S)(\beta I + S)^{-1}\|_2.\end{aligned}$$

Letting $Q(\alpha,\beta) = (\alpha I - S)(\beta I + S)^{-1}$ and noting that $S^* = -S$, we have

$$\begin{aligned}Q(\alpha,\beta)^* Q(\alpha,\beta) &= \left[(\alpha I - S)(\beta I + S)^{-1}\right]^* \left[(\alpha I - S)(\beta I + S)^{-1}\right] \\ &= (\beta I - S)^{-1}(\alpha I + S)(\alpha I - S)(\beta I + S)^{-1} \\ &= (\alpha I - S)(\beta I + S)^{-1}(\beta I - S)^{-1}(\alpha I + S) \\ &= Q(\alpha,\beta)Q(\alpha,\beta)^*\end{aligned}$$

That is to say, $Q(\alpha,\beta)$ is a normal matrix. Therefore, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a complex diagonal matrix $\bigwedge_q = \mathrm{diag}(\tilde\lambda_1, \tilde\lambda_2, \ldots, \tilde\lambda_n) \in \mathbb{C}^{n\times n}$ such that $Q(\alpha,\beta) = U^* \bigwedge_q U$. Suppose that $\tilde\lambda$ be an eigenvalue of $Q(\alpha,\beta)$ and x be an associated eigenvector, we have

$$
\begin{aligned}
Q(\alpha,\beta)x &= \tilde{\lambda}x \\
(\alpha I - S)(\beta I + S)^{-1}x &= \tilde{\lambda}x \\
(\beta I + S)^{-1}(\alpha I - S)x &= \tilde{\lambda}x \\
(\alpha I - S)x &= \tilde{\lambda}(\beta I + S)x
\end{aligned}
$$

If $\tilde{\lambda} \neq -1$, then

$$Sx = \frac{\alpha - \tilde{\lambda}\beta}{1 + \tilde{\lambda}}x \tag{15}$$

If $\tilde{\lambda} = -1$, then

$$(\alpha + \beta)x = 0.$$

Since $\alpha + \beta > 0$, it implies $x = 0$, and this contradicts the definition of eigenvector. Therefore $\tilde{\lambda} = -1$ can not be an eigenvalue of $Q(\alpha,\beta)$.

From (15), $\dfrac{\alpha - \tilde{\lambda}\beta}{1 + \tilde{\lambda}}$ is an eigenvalue of S and x is an associated eigenvector. Since S is a skew-Hermitian matrix, its eigenvalues are pure imaginary and thus of the form $i\tau_j$, $j = 1, \ldots, n$, where $\tau_j \in \mathbb{R}$, So

$$\tilde{\lambda_j} = \frac{\alpha - i\tau_j}{\beta + i\tau_j},$$

where $i\tau_j$ is an eigenvalue of S. Therefore

$$\|Q(\alpha,\beta)\|_2 = \|U^* \wedge_q U\|_2 = \| \wedge_q \|_2 = \max_{\sigma_j \in \ \sigma(S)} \frac{\sqrt{\alpha^2 + \sigma_j^2}}{\sqrt{\beta^2 + \sigma_j^2}}. \tag{16}$$

Because N is a normal matrix, there exists a unitary matrix $V \in \mathbb{C}^{n\times n}$ and a complex diagonal matrix $\bigwedge_N = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^{n\times n}$ such that $N = V^* \bigwedge_N V$. Hence, we have

$$
\begin{aligned}
\|(\alpha I + N)^{-1}(\beta I - N)\|_2 &= \max_{\lambda_j \in \ \lambda(N)} \frac{\mid \beta - \lambda_j \mid}{\mid \alpha + \lambda_j \mid} \\
&= \max_{\lambda_j = \gamma_j + i\eta_j \in \ \lambda(N)} \sqrt{\frac{(\beta - \gamma_j)^2 + \eta_j^2}{(\alpha + \gamma_j)^2 + \eta_j^2}}
\end{aligned} \tag{17}
$$

Now, from (16) and (17), we see that

$$\rho(M(\alpha,\beta)) \leq \max_{\sigma_j \in \ \sigma(S)} \frac{\sqrt{\alpha^2 + \sigma_j^2}}{\sqrt{\beta^2 + \sigma_j^2}} \max_{\lambda_j = \gamma_j + i\eta_j \in \ \lambda(N)} \sqrt{\frac{(\beta - \gamma_j)^2 + \eta_j^2}{(\alpha + \gamma_j)^2 + \eta_j^2}}.$$

Then the bound for $\rho(M(\alpha,\beta))$ is given by (14).

To prove (a), we note that, if $\dfrac{\beta-\alpha}{2} \leq \gamma_{\min}$, then $\beta - \alpha \leq 2\gamma$ for $\gamma_{\min} \leq \gamma \leq \gamma_{\max}$. By using $0 < (\beta+\alpha)$, we obtain $(\beta-\gamma)^2 \leq (\alpha+\gamma)^2$. Thus, by Lemma 2.1, we have

$$f(\alpha,\beta) = \max_{\gamma_{\min}\leq\gamma\leq\gamma_{\max}} \left\{ \frac{(\beta-\gamma)^2+\eta_{\max}^2}{(\alpha+\gamma)^2+\eta_{\max}^2} \right\} \leq 1, \quad \text{for} \quad \frac{\beta-\alpha}{2} \leq \gamma_{\min} \tag{18}$$

Moreover, if $\beta > \alpha$, then $\displaystyle\max_{\sigma_j \in\, \sigma(S)} \frac{\sqrt{\alpha^2+\sigma_j^2}}{\sqrt{\beta^2+\sigma_j^2}} < 1$, and therefore

(i) if $\alpha < \beta \leq \alpha + 2\gamma_{\min}$ then $\delta(\alpha,\beta) < 1$.

If $\beta \leq \alpha$, then $\displaystyle\max_{\sigma_j \in\, \sigma(S)} \frac{\sqrt{\alpha^2+\sigma_j^2}}{\sqrt{\beta^2+\sigma_j^2}} \leq \frac{\alpha}{\beta}$. By using (18), we have

$$\delta(\alpha,\beta) \leq \frac{\alpha}{\beta} \max_{\gamma_{\min}\leq\gamma\leq\gamma_{\max}} \sqrt{\frac{(\beta-\gamma)^2+\eta_{\max}^2}{(\alpha+\gamma)^2+\eta_{\max}^2}}.$$

So, in order to have the bound $\delta(\alpha,\beta) < 1$, the following inequality must hold

$$\max_{\gamma_{\min}\leq\gamma\leq\gamma_{\max}} \left\{ \frac{(\beta-\gamma)^2+\eta_{\max}^2}{(\alpha+\gamma)^2+\eta_{\max}^2} \right\} < \frac{\beta^2}{\alpha^2}. \tag{19}$$

By using the results of Lemma 2.2, the following inequalities must hold

$$\frac{(\beta-\gamma_{\min})^2+\eta_{\max}^2}{(\alpha+\gamma_{\min})^2+\eta_{\max}^2} < \frac{\beta^2}{\alpha^2} \quad \text{and} \quad \frac{(\beta-\gamma_{\max})^2+\eta_{\max}^2}{(\alpha+\gamma_{\max})^2+\eta_{\max}^2} < \frac{\beta^2}{\alpha^2} \tag{20}$$

By simple computation, we can show that, for $\alpha+\beta > 0$, these two inequalities hold if β satisfies the following inequalities.

$$\frac{\alpha(\gamma_{\min}^2+\eta_{\max}^2)}{2\alpha\gamma_{\min}+\gamma_{\min}^2+\eta_{\max}^2} < \beta \quad \text{and} \quad \frac{\alpha(\gamma_{\max}^2+\eta_{\max}^2)}{2\alpha\gamma_{\max}+\gamma_{\max}^2+\eta_{\max}^2} < \beta.$$

Therefore

(ii) if $\max\left\{ \dfrac{\alpha(\gamma_{\min}^2+\eta_{\max}^2)}{2\alpha\gamma_{\min}+\gamma_{\min}^2+\eta_{\max}^2}, \dfrac{\alpha(\gamma_{\max}^2+\eta_{\max}^2)}{2\alpha\gamma_{\max}+\gamma_{\max}^2+\eta_{\max}^2} \right\} < \beta \leq \alpha$ then $\delta(\alpha,\beta) < 1$.

Combining (i) and (ii), we have

(iii) if $\max\left\{ \dfrac{\alpha(\gamma_{\min}^2+\eta_{\max}^2)}{2\alpha\gamma_{\min}+\gamma_{\min}^2+\eta_{\max}^2}, \dfrac{\alpha(\gamma_{\max}^2+\eta_{\max}^2)}{2\alpha\gamma_{\max}+\gamma_{\max}^2+\eta_{\max}^2} \right\} < \beta \leq \alpha + 2\gamma_{\min}$ then $\delta(\alpha,\beta) < 1$

To prove parts (b) and (c), we note that, if $\gamma_{\min} \le \dfrac{\beta-\alpha}{2}$, by Lemmas 2.1 and 2.3, we have

$$f(\alpha,\beta) = \max_{\gamma_{\min}\le\gamma\le\frac{\beta-\alpha}{2}} \left\{ \frac{(\beta-\gamma)^2+\eta_{\min}^2}{(\alpha+\gamma)^2+\eta_{\min}^2} \right\} = \frac{(\beta-\gamma_{\min})^2+\eta_{\min}^2}{(\alpha+\gamma_{\min})^2+\eta_{\min}^2} \ge 1,$$

since $(\alpha+\gamma_{\min}) \le (\beta-\gamma_{\min})$.

On the other hand,

$$\max_{\sigma_j\in\ \sigma(S)} \frac{\sqrt{\alpha^2+\sigma_j^2}}{\sqrt{\beta^2+\sigma_j^2}} = \frac{\sqrt{\alpha^2+\sigma_{\max}^2}}{\sqrt{\beta^2+\sigma_{\max}^2}} < 1,$$

since $\alpha<\beta$. So, the relation

$$\delta(\alpha,\beta) = \frac{\sqrt{\alpha^2+\sigma_{\max}^2}}{\sqrt{\beta^2+\sigma_{\max}^2}} \sqrt{\frac{(\beta-\gamma_{\min})^2+\eta_{\max}^2}{(\alpha+\gamma_{\min})^2+\eta_{\max}^2}} < 1$$

will hold if α and β satisfy the following inequality,

$$\frac{(\beta-\gamma_{\min})^2+\eta_{\max}^2}{(\alpha+\gamma_{\min})^2+\eta_{\max}^2} < \frac{\beta^2+\sigma_{\max}^2}{\alpha^2+\sigma_{\max}^2}. \tag{21}$$

For $\alpha+\beta>0$, this inequality is equivalent to

$$0 < (\beta-\alpha)(\gamma_{\min}^2+\eta_{\min}^2-\sigma_{\max}^2+2\alpha\gamma_{\min}) + 2\gamma_{\min}(\alpha^2+\sigma_{\max}^2). \tag{22}$$

Since $(\beta-\alpha)>0$, (22) holds if $\sigma_{\max} \le \sqrt{\gamma_{\min}^2+\eta_{\min}^2+2\alpha\gamma_{\min}}$. Thus

(iv) if $\sigma_{\max} \le \sqrt{\gamma_{\min}^2+\eta_{\min}^2+2\alpha\gamma_{\min}}$ and $\alpha+2\gamma_{\min}\le\beta$, then $\delta(\alpha,\beta)<1$.

If $\sigma_{\max} > \sqrt{\gamma_{\min}^2+\eta_{\min}^2+2\alpha\gamma_{\min}}$, then (22) holds if β satisfies the following inequality

$$\beta < \frac{\alpha(\gamma_{\min}^2+\eta_{\min}^2-\sigma_{\max}^2+2\alpha\gamma_{\min}) - 2\gamma_{\min}(\alpha^2+\sigma_{\max}^2)}{\gamma_{\min}^2+\eta_{\min}^2-\sigma_{\max}^2+2\alpha\gamma_{\min}}.$$

Thus

(v) if $\sigma_{\max} > \sqrt{\gamma_{\min}^2+\eta_{\min}^2+2\alpha\gamma_{\min}}$ and

$$\alpha+2\gamma_{\min} \le \beta \le \frac{\alpha(\gamma_{\min}^2+\eta_{\min}^2-\sigma_{\max}^2+2\alpha\gamma_{\min}) - 2\gamma_{\min}(\alpha^2+\sigma_{\max}^2)}{\gamma_{\min}^2+\eta_{\min}^2-\sigma_{\max}^2+2\alpha\gamma_{\min}}$$

then $\delta(\alpha,\beta)\le 1$.

□

Theorem 2.1 mainly discusses the available β for a convergent ANSS iteration for any given nonnegative α. It also shows that the choice of β is dependent on the choice of α, the spectrum of the matrix N, the singular-values of S, but is not dependent on the spectrum of A. Notice that

$$\alpha + 2\gamma_{\min} - \frac{\alpha(\gamma_{\min}^2 + \eta_{\max}^2)}{2\alpha\gamma_{\min} + \gamma_{\min}^2 + \eta_{\max}^2} = \frac{2\alpha\gamma_{\min}^2 + 2\gamma_{\min}^3 + 2\gamma_{\min}\eta_{\max}^2 + 2\alpha^2\gamma_{\min}}{2\alpha\gamma_{\min} + \gamma_{\min}^2 + \eta_{\max}^2} > 0$$

and

$$\alpha + 2\gamma_{\min} - \frac{\alpha(\gamma_{\max}^2 + \eta_{\max}^2)}{2\alpha\gamma_{\max} + \gamma_{\max}^2 + \eta_{\max}^2} = \frac{2\alpha\gamma_{\min}\gamma_{\max} + 2\gamma_{\min}\gamma_{\max}^2 + 2\gamma_{\min}\eta_{\max}^2 + 2\alpha^2\gamma_{\max}}{2\alpha\gamma_{\max} + \gamma_{\max}^2 + \eta_{\max}^2} > 0,$$

we remark that for any given nonnegative α the available β always exists. The bound $\delta(\alpha, \beta)$ of the convergence rate depends on the spectrum of N and S and the choice of α and β. Moreover, $\delta(\alpha, \beta)$ is also an upper bound of the contraction factor of the ANSS iteration.

3 Numerical Example

In this section, we give a numerical example to illustrate the effectiveness of ANSS iteration.

We consider the differential equation

$$-u^{''} + qu^{'} = f,$$

on the inteval $[0, 1]$, with the constant coefficient q and the homogeneous boundary condition. When the finite difference discretization, for example, the centered difference is applied to the above equation, we get the system of linear equations (1) with the coefficient matrix

$$A = \text{tridiag}(-1 - \frac{qh}{2}, 2, -1 + \frac{qh}{2})$$

where the equidistant step-size $h = \dfrac{1}{n+1}$ is used.

Let $H = \frac{1}{2}(A + A^*)$ and $S_0 = \frac{1}{2}(A - A^*)$ be Hermitian and skew-Hermitian parts of A, respectively. We consider a NS splitting

$$A = N + S$$

where

$$N = H + icI \quad \text{and} \quad S = S_0 - icI$$

and c is a real number. We test the spectral radius of the iteration matrix $M(\alpha, \beta)$ (8) with different values of qh. All the tested matrices are 64×64.

In Figs. 1 and 2, we show the spectral radius of the iteration matrix of the ANSS method and the NSS method with different values of α. ANSS represents the spectral radius of the iteration matrix of the ANSS method, where parameter β is tested to be the optimal one, and NSS represents that of the NSS method.

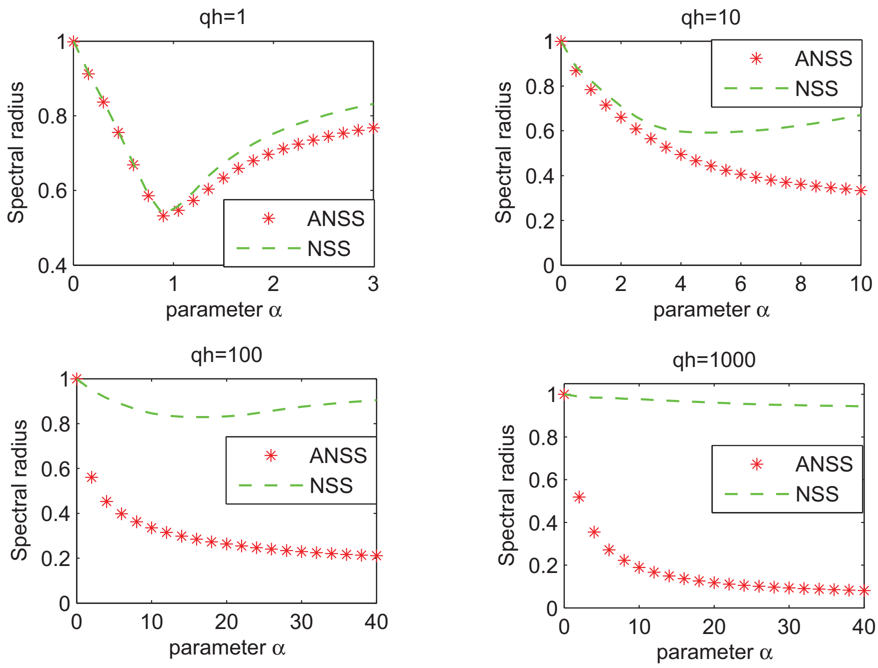

Fig. 1: Spectral radius of iteration matrices of ANSS and NSS methods for $c = .1$

We find that if $c = 0.1$ is used, the spectral radius of the iteration matrix of the ANSS method is always smaller than that of the NSS method, and when qh is large, the spectral radius of the iteration matrix of the ANSS method is much smaller than that of the NSS method, but if $c = 10$ is used, these two spectral radius of the iteration matrices are almost the same.

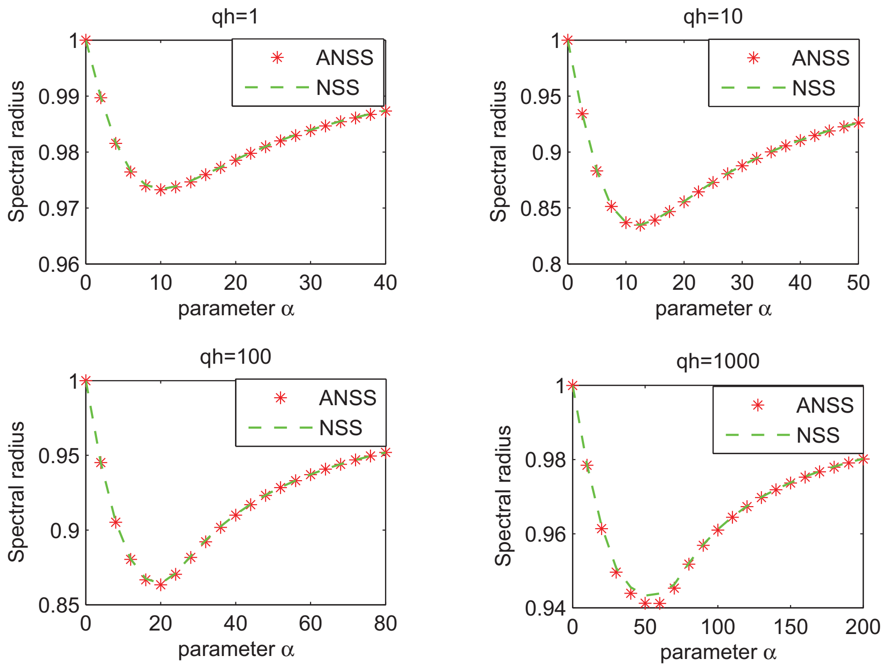

Fig. 2: Spectral radius of iteration matrices of ANSS and NSS methods for $c = 10$

4 Conclusion

In this paper, we have introduced two constants for the NSS iteration and presented a different approach to solve the system of linear equations (1), called ANSS method.

Theoretical analysis showed that if the coefficient matrix A is positive definite (Hermitian or non-Hermitian) the ANSS method can converge to the unique solution of the linear system (1) with any given nonnegative α, if β is restricted to an appropriate region. In addition the upper bound of the contraction factor of the ANSS iteration is dependent on the choice of α and β, the spectrum of the normal matrix N and the singular-values of the skew-Hermitian, but is not dependent on the eigenvectors of the matrices N, S and A. Numerical examples illustrated the effectiveness of ANSS iteration and showed that the spectral radius of the iteration matrix of the ANSS method is always smaller than or equal to that of the NSS method.

Acknowledgment

The authors wish to thank the referee for valuable comments and suggestions.

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