Iranian Journal of Numerical Analysis and Optimization
Vol. 12, No. 1, (2022), pp 55-72
DOI:10.22067/ijnao.2021.70731.1038
https://ijnao.um.ac.ir/

How to cite this article
Research Article

Asymptotic and numerical methods for solving singularly perturbed differential difference equations with mixed shifts

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Abstract

This article deals with an efficient approximation method named successive complementary expansion method (SCEM) for solving singularly perturbed differential-difference equations with mixed shifts. It is compared with the method of matched asymptotic expansion (MMAE) and the parameter uniform upwind finite difference scheme for solving such a model. The comparison shows, unlike the MMAE, the SCEM method requires no matching procedure. It requires less computation when compared to the upwind finite difference scheme on the Shishkin mesh. The error analysis is carried out to prove the robustness of the method. Some numerical experiments are provided, which show the effectiveness of the proposed method.

AMS subject classifications (2020): 65L10; 65L11.

Keywords: Singular perturbation; Mixed shifts; Asymptotic expansion; MMAE; SCEM; Upwind scheme.

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Received 31 May 2021; revised 21 July 2021; accepted 3 August 2021
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1 Introduction

It is noteworthy to observe the wide use of differential equations involving small parameters in a variety of disciplines from physics, chemistry, economics to biology. A differential equation, in general, is said to be singularly perturbed if its higher-order derivative gets multiplied with a small parameter ε $(0 < \varepsilon \ll 1)$ known as the perturbation parameter. In many mathematical models, these types of equations contain some small delay and small advance terms, which make them singularly perturbed differential-difference equations (SPDDE). Due to the presence of the perturbation parameter, these problems exhibit boundary or interior layers. The development in solutions to these problems both analytically and numerically is not a settled direction in mathematics. The massive use of these equations in many fields of science and engineering leads many researchers to solve such problems more efficiently. These equations usually arise in control theory [12], generation of action potential in nerve cells by random synaptic inputs in dendrites, modeling of activation of a neuron, and many more [25, 27].

Over the last few decades, many researches have been done on singularly perturbed differential equations. One may refer to [1, 6, 5, 13, 14, 24] and the references therein for different kinds of numerical methods. To mention a few, Kadalbajoo and Sharma [10, 9] surveyed various asymptotic and numerical methods for the solution of such problems. Lange and Miura [15, 16] gave an asymptotic approach to solve SPDDE along with the turning point problems. Sirisha, Phaneendra, and Reddy [26] used the idea of domain decomposition and introduced the mixed finite difference method. Using post-processing and grid distribution techniques, Mohapatra and Natesan [19, 20, 21] formulated uniformly convergent numerical methods. Duressa and Reddy [8] constructed a domain decomposition method and used a terminal boundary point to the domain for decomposing it into two regions. Melesse, Tiruneh, and Derese [18] used an initial value method for solving these problems. A collocation method was proposed to solve these boundary value problems using a modified B-spline basis function by Arora and Kaur [2]. Rao and Chakravarthy [23] used an exponentially fitted tridiagonal finite difference method. Recently, Cengizci, Natesan, and Atay [4] used an asymptotic numerical hybrid method for the singularly perturbed system of two-point reaction-diffusion boundary-value problems. Mushahary, Sahu, and Mohapatra [22] used a finite difference scheme for solving SPDDE.

Our main objective in this paper is to prove a successive complementary expansion method (SCEM) as an efficient alternative to the method of matched asymptotic expansion (MMAE) for solving SPDDE. This method, in general, applies to all the singularly perturbed problems that can be approximated by the MMAE. The main idea of SCEM is to use a correction term for the approximation of the inner layer to complement the solution for the entire domain. The quality of the approximation is improved by iteratively using corrections to the new terms for the inner layer. The ability to provide a uniformly valid approximation without any matching procedure is the main advantage of using SCEM over MMAE.

In this work, some SPDDEs are taken into consideration, which exhibit boundary layer behavior at the left-end of the interval. These problems contain mixed shifts, that is, both the delay and the advance terms. The complete description of the problem is given in section 2. At first, the model is approximated asymptotically by the MMAE, and then we have used an efficient asymptotic method as an alternative of widely used MMAE, that is, the SCEM. This method was first introduced by Cousteix and Mauss in [7, 17]. We have also approximated these problems by an upwind finite difference scheme on the Shishkin mesh. The complete working principle of the parameter uniform numerical scheme, MMAE, and SCEM, along with the use of SCEM over MMAE, has been described in section 3. In the support of the above-said methods, some numerical examples with their results are discussed in section 4. The complete discussion with the conclusion is given in the final section.

2 Description of the problem

Consider the following SPDDE with mixed shifts terms:

$$\begin{cases} \varepsilon y''(x) + a(x)y'(x) + b(x)y(x-\delta) \\ \quad +c(x)y(x) + d(x)y(x+\eta) = f(x), & 0 < x < 1, \\ y(x) = \phi(x), & on \ -\delta \le x \le 0, \\ y(x) = \gamma(x), & on \ 1 \le x \le 1+\eta, \end{cases} \tag{1}$$

where $a(x), b(x), c(x), d(x), f(x)$,$\phi(x)$,$\gamma(x)$ are bounded and continuously differentiable functions on $(0,1)$. Here, ε is the singular perturbation parameter with $0 < \varepsilon \ll 1$. The parameters δ and η are the delay and advance terms, respectively, with $0 < \delta = o(\varepsilon); 0 < \eta = o(\varepsilon)$. Now for the terms of (1) containing the delay and the advance operators, we shall use the Taylor's series approximation in the neighborhood of the point x, that is,

$$\begin{cases} y(x-\delta) \approx y(x) - \delta y'(x), \\ y(x+\eta) \approx y(x) + \eta y'(x). \end{cases} \tag{2}$$

The application of (2) converts (1) to the following form:

$$\begin{cases} \varepsilon y''(x) + \alpha(x)y'(x) + \beta(x)y(x) = f(x), & 0 < x < 1, \\ y(0) = \phi(0), \qquad y(1) = \gamma(1), \end{cases} \tag{3}$$

where $\alpha(x) = a(x) - \delta b(x) + \eta d(x)$ and $\beta(x) = b(x) + c(x) + d(x)$. Assume $\alpha(x) \geq \lambda > 0$ and $\beta(x) > 0$ for some constant λ. Since the parameters δ and η are of $o(\varepsilon)$, so the use of Taylor's series approximation and the conversion of (1) to (3) is valid.

3 Description of the methods

3.1 Asymptotic approximation

Any two functions $\psi(x,\varepsilon)$ and $\xi(x,\varepsilon)$ are said to be asymptotically identical to $\rho(\varepsilon)$ if their difference is less than $\rho(\varepsilon)$ in the domain Ω,

$$\psi(x,\varepsilon) - \xi(x,\varepsilon) = o(\rho(\varepsilon)). \tag{4}$$

Here, $\rho(\varepsilon)$ is an order function and $\xi(x,\varepsilon)$ is the asymptotic approximation of $\psi(x,\varepsilon)$. The general form of asymptotic expansion is

$$\psi(x,\varepsilon) = \sum_{i=0}^{n} \delta_i(\varepsilon)\psi_i(x,\varepsilon), \tag{5}$$

where $\delta_i(\varepsilon)$, $i = 0, 1, \ldots$, is an asymptotic sequence and $\psi_i(x,\varepsilon)$ is of $o(\rho(\varepsilon))$. If $\delta_i(\varepsilon)$ satisfies $\delta_{i+1}(\varepsilon)= o(\delta_i(\varepsilon))$, then the expansion is called generalized asymptotic expansion. If the expansion is of the form

$$\psi(x,\varepsilon) = E_0\psi = \sum_{i=0}^{n} \delta_i^{(0)}(\varepsilon)\psi_i^{(0)}(x,\varepsilon), \tag{6}$$

then it is said to be regular asymptotic expansion. Here, E_0 is an outer expansion operator. So, we conclude $\psi(x,\varepsilon) - E_0\psi(x,\varepsilon) = o(\rho(\varepsilon))$.

3.2 Method of matched asymptotic expansion (MMAE)

In many cases, the functions are not regular in the domain of consideration Ω. So, (6) is only valid upon one finite part of the region Ω. Let that restricted region be named $\Omega_0 \subset \Omega$. Now, the other part of the region (inner region), that is, Ω_1 is located near the boundary layer where the solution undergoes a rapid change to satisfy the boundary conditions. For the inner region, we need to introduce a new variable(stretching variable or boundary layer variable) $\overline{x}=\dfrac{x - x_0}{\xi(\varepsilon)}$. Here, x_0 is the point where the rapid change of the solution occurs and $\xi(\varepsilon)$ is the order of the thickness of the boundary layer.

Now, similarly as in (6), we can construct an asymptotic expansion for the function in Ω_1,

$$\psi(x,\varepsilon) = E_1\psi = \sum_{i=0}^{n} \delta_i^{(1)}(\varepsilon)\psi_i^{(1)}(x,\varepsilon). \tag{7}$$

Both the expansion operators E_0 and E_1 have the same order, that is, $o(\rho(\varepsilon))$. So, we have $\psi(x,\varepsilon) - E_1\psi(\overline{x},\varepsilon) = o(\rho(\varepsilon))$. The valid approximation of the solution thus can be written as

$$\psi_{mmae}(x,\varepsilon) = E_0\psi(x,\varepsilon) + E_1\psi(\overline{x},\varepsilon) - E_0E_1\psi(x,\varepsilon). \tag{8}$$

In MMAE, we calculate two different results separately with two different variables for both the inner and outer layers. Then, to obtain the uniformly valid approximation for the entire domain, we need to subtract the region where both the approximations overlap. For this purpose, we need to develop a matching procedure by the limiting process [3].

3.3 Successive complementary expansion method (SCEM)

As the functions that we consider are not regular in the entire domain Ω, after calculating the solution for the outer region Ω_0 through the expansion E_0 as in (6), we need to extend the solution through a complementary function for the entire domain. This complementary function depends upon the variable $\overline{x}$. The general form of the asymptotic expansion in SCEM can be formulated as

$$\psi_{scem}(x,\overline{x},\varepsilon) = \sum_{i=0}^{n} \delta_i(\varepsilon)(\psi_i^{out}(x,\varepsilon) + \psi_i^{in}(\overline{x},\varepsilon)). \tag{9}$$

Here, $\psi_i^{out}(x,\varepsilon)$ is the same as the outer region solution that has been obtained by MMAE before. The outer solution depends upon x but not upon ε but the complementary function depends both upon x and ε. As in the MMAE method, SCEM does not require any matching procedure.

3.4 Finite difference scheme

In this section, we first construct a piecewise uniform Shishkin mesh. The Shishkin mesh is distinguished from other piecewise uniform meshes in terms of its choice of the transition parameter τ defined as $\tau = \min\left(\frac{1}{2}, \frac{\varepsilon}{\alpha}\ln N\right)$.

Now, we divide each of the intervals $[0,\tau]$ and $[\tau,1]$ into $N/2$ equal sub-intervals. The theory of Shishkin mesh demands as $N\to\infty$ the number of sub-intervals in each interval should be bounded below.

$$\Omega^N=\begin{cases} x(i)=h_i, & i\le N/2,\\ x(i)=H_i, & i>N/2,\end{cases} \tag{10}$$

The coarse part of Shishkin mesh has spacing $H=\dfrac{2(1-\tau)}{N}$ and the fine part has spacing $h=\dfrac{2\tau}{N}$. Now, let us denote $h_i=x_i-x_{i-1}$. The forward and central difference operators are denoted as

$$D^+=\frac{Y_{i+1}-Y_i}{h_{i+1}},\quad D^+D^-=\frac{2}{h_i+h_{i+1}}\left[\frac{Y_{i+1}-Y_i}{h_{i+1}}-\frac{Y_i-Y_{i-1}}{h_i}\right].$$

Now (3) takes the form

$$\begin{aligned}\varepsilon D^+D^-Y_i+\alpha(x)D^+Y_i-\beta(x)Y_i&=f(x),\\ y(0)=Y(0),\qquad y(1)&=Y(1),\end{aligned} \tag{11}$$

for any mesh function $Y(x_i)=Y_i$. So, for the domain, in (10), we get the following system of equations:

$$P_i^-Y_{i-1}-P_i^0Y_i+P_i^+Y_{i+1}=P_i^N, \tag{12}$$

where

$$\begin{aligned}P_i^-&=\frac{2\varepsilon}{h_i(h_i+h_{i+1})},\\ P_i^0&=\frac{2\varepsilon}{h_i(h_i+h_{i+1})}+\frac{2\varepsilon}{h_{i+1}(h_i+h_{i+1})}+\frac{\alpha(x)}{h_{i+1}}+\beta(x_i),\\ P_i^-&=\frac{2\varepsilon}{h_{i+1}(h_i+h_{i+1})}+\frac{\alpha(x)}{h_{i+1}}.\end{aligned} \tag{13}$$

The matrix associated here is an M-matrix. For detailed analysis of the proposed scheme one can refer [11].

4 Results and discussion

Even though much work has been done by the MMAE method, there is no specific theory that can describe the method in a proper way. So, we need to discuss the working principle of both MMAE and SCEM by some illustrative examples. The proposed SCEM is found efficient in solving examples of the type (1), which possess the left-end boundary layer. The approximated exact solution of SPDDE of the form (1) with constant coefficients is given by

$$y(x) = C_1 e^{M_1 x} + C_2 e^{M_2 x}, \tag{14}$$

where

$$C_1 = \frac{1 - e^{M2}}{e^{M1} - e^{M2}}, \qquad C_2 = 1 - C_1,$$

$$M_1 = \frac{-\alpha(x) + \sqrt{\alpha(x)^2 - 4\beta(x)}}{2\varepsilon}, \quad M_2 = \frac{-\alpha(x) - \sqrt{\alpha(x)^2 - 4\beta(x)}}{2\varepsilon}.$$

Example 1. Consider the following problem with a delay term:

$$\begin{cases} \varepsilon y''(x) + y'(x) + 2y(x-\delta) - 3y(x) = 0, \\ y(x) = 1, \quad -\delta \le x \le 0, \quad y(1) = 1. \end{cases}$$

The solution of this problem possesses a rapid change near the point $x = 0$ when $\varepsilon \to 0^+$. This region is said to be the boundary layer or the inner layer. So, this possesses a left-end boundary layer. The other region, which is far away from the boundary layer, is said to be the outer region. In case of the outer region, there is no record of any unusual behavior of the solution. We will solve this problem employing MMAE and SCEM and the proposed numerical scheme. For the given example using Taylor's series expansion of the form (2) in the neighborhood of x, we have

$$\varepsilon^* y''(x) + y'(x) - \frac{1}{1-2\delta} y(x) = 0, \qquad \textit{where} \ \ \varepsilon^* = \frac{\varepsilon}{1-2\delta}. \tag{15}$$

Now, the equation has converted to a second order singularly perturbed problem, which can be solved by both MMAE and SCEM.

4.1 For MMAE

In MMAE, for the outer region solution, which is far away from $x = 0$, we take $x = 1$. Now, the asymptotic approximation for the outer region can be written as

$$y(x) \approx y_0(x) + \varepsilon^* y_1(x) + (\varepsilon^*)^2 y_2(x) + \cdots. \tag{16}$$

Substituting (16) in (15), we reach at

$$\varepsilon^*[y_0''(x) + \varepsilon^* y_1''(x) + (\varepsilon^*)^2 y_2''(x) + \cdots] + [y_0'(x) + \varepsilon^* y_1'(x) + \tag{17}$$
$$(\varepsilon^*)^2 y_2'(x) + \cdots] - \left(\frac{1}{1-2\delta}\right)[y_0(x) + \varepsilon^* y_1(x) + (\varepsilon^*)^2 y_2(x) + \cdots] = 0.$$

For $\varepsilon = 0$, model (17) is reduced to

$$y_0'(x) - \left(\frac{1}{1-2\delta}\right)y_0(x) = 0. \tag{18}$$

The solution of (18) is easily found to be $y_0(x){=}A\exp\left(\frac{x}{1-2\delta}\right)$. Here, $A \in \mathbf{R}$ and imposing the outer boundary layer condition (at $x = 1$) we get our required outer solution as

$$y_0(x) = \exp\left(\frac{x-1}{1-2\delta}\right). \tag{19}$$

To obtain the solution for the inner layer, we need to introduce the stretching variable $\overline{x} = \frac{x}{\varepsilon}$. The solution obtained in the inner region that depends upon the variable $\overline{x}$, is denoted by $Y(\overline{x})$. By the use of this stretching variable, we are able to stretch the thin layer near the boundary. Now by applying the chain rule, we get

$$\frac{d}{dx} = \frac{1}{\varepsilon}\frac{d}{d\overline{x}}. \tag{20}$$

Using (20) in (15), we have

$$(\varepsilon^*)^{-1}\frac{d^2Y(\overline{x})}{d\overline{x}^2} + (\varepsilon^*)^{-1}\frac{dY(\overline{x})}{d\overline{x}} - \frac{1}{(1-2\delta)}Y(\overline{x}) = 0. \tag{21}$$

Now, multiplying (21) with ε^*, we have

$$\frac{d^2Y(\overline{x})}{d\overline{x}^2} + \frac{dY(\overline{x})}{d\overline{x}} - \frac{\varepsilon^*}{(1-2\delta)}Y(\overline{x}) = 0. \tag{22}$$

Equation (22) is a regularly perturbed differential equation, and to obtain the solution for the inner region, we need to use the asymptotic approximation of the form

$$Y(\overline{x}) \approx Y_0(\overline{x}) + \varepsilon^*Y_1(\overline{x}) + (\varepsilon^*)^2Y_2(\overline{x}) + \cdots . \tag{23}$$

We are only doing the expansion for the first term. So, we can make our calculations for $\varepsilon = 0$. Now, we have

$$Y_0''(\overline{x}) + Y_0'(\overline{x}) = 0. \tag{24}$$

On solving (24) and using the boundary condition for the inner layer (at $x = 0$), we arrive at

$$Y_0(\overline{x}) = 1 + C(e^{-\overline{x}} - 1). \tag{25}$$

Clearly, there are no other conditions, so that we can obtain the value of the constant C. Here is the situation where the matching procedure of MMAE works. We have already found out two different solutions, that is, (19) and (25) for two different regions, but, it is obvious that they belong to the same approximation. Followed by this matching idea [3], we conclude

$$\lim_{x\to 0} y_0(x) = \lim_{\overline{x}\to\infty} Y_0(\overline{x}). \tag{26}$$

Thus, we get $C = 1 - \exp\left(\frac{1}{2\delta - 1}\right)$. So, the solution of our inner region becomes

$$Y_0(\overline{x}) = \exp\Big(-\overline{x}\Big) - \exp\left(\frac{1}{2\delta-1} - \overline{x}\right) + \exp\left(\frac{1}{2\delta-1}\right). \tag{27}$$

As we have calculated the solutions for both the outer and inner regions, now we can find out the composite solution to our equation. For this, we need to add the solutions of both the regions and subtract the common limit between them. As per (8), we have

$$\begin{aligned} y &\approx y_0(x) + Y_0(\overline{x}) - y_0(0^+), \\ y &\approx y_0(x) + Y_0(\overline{x}) - Y_0(\infty). \end{aligned} \tag{28}$$

Using (28), we obtain the composite MMAE solution as

$$y_{mmae} \approx \exp\left(\frac{x-1}{1-2\delta}\right) + \exp\left(\frac{-x}{\varepsilon}\right) - \exp\left(\frac{1}{2\delta-1}\right) - \frac{x}{\varepsilon}. \tag{29}$$

4.2 For SCEM

The uniformly valid composite expansion in general form of SCEM can be written as

$$y_{scem}(x, \overline{x}, \varepsilon) = \sum_{i=0}^{n} \delta_i(\varepsilon)[y_i(x) + Y_i(\overline{x})]. \tag{30}$$

As we are only interested in solving our problem for $x = 0$, so we get

$$y_{0scem}(x, \overline{x}, \varepsilon) = [y_0(x) + Y_0(\overline{x})]. \tag{31}$$

Here, $y_0(x)$ is the same outer region solution that has been obtained for MMAE in (20). The outer region solution remains the same as the solution here showing no unusual behavior in accordance with the boundary conditions. Using (20) in (31), we get

$$y_{scem} = \exp\left(\frac{x-1}{1-2\delta}\right) + Y_0(\overline{x}). \tag{32}$$

Substituting value of y_{scem} in (15), we have

$$\varepsilon^* y''_{0scem}(x,\overline{x},\varepsilon) + y'_{0scem}(x,\overline{x},\varepsilon) - \frac{1}{1-2\delta} y_{0scem}(x,\overline{x},\varepsilon) = 0. \tag{33}$$

This follows

$$\varepsilon^* \frac{d^2}{dx^2}\left[\exp\left(\frac{x-1}{1-2\delta}\right) + Y_0(\overline{x})\right] + \frac{d}{dx}\left[\exp\left(\frac{x-1}{1-2\delta}\right) + Y_0(\overline{x})\right] - \frac{1}{1-2\delta}\left[\exp\left(\frac{x-1}{1-2\delta}\right) + Y_0(\overline{x})\right] = 0. \tag{34}$$

Using the chain rule for the solution of $Y_0(\overline{x})$ in terms of x, we have

$$\varepsilon^*\left[\left(\frac{1}{1-2\delta}\right)^2 \exp\left(\frac{x-1}{1-2\delta}\right) + \frac{1}{(\varepsilon^*)^2} Y''_0(\overline{x})\right] + \left[\left(\frac{1}{1-2\delta}\right) \exp\left(\frac{x-1}{1-2\delta}\right) + \frac{1}{\varepsilon^*} Y'_0(\overline{x})\right] - \frac{1}{1-2\delta}\left[\exp\left(\frac{x-1}{1-2\delta}\right) + Y_0(\overline{x})\right] = 0. \tag{35}$$

Multiplying ε^* in (35), the resulting equation will be

$$\varepsilon^{*2}\left[\frac{1}{1-2\delta}\right]^2 \exp\left(\frac{x-1}{1-2\delta}\right) + Y''_0(\overline{x}) + \varepsilon^*\left[\frac{1}{1-2\delta}\right]\exp\left(\frac{x-1}{1-2\delta}\right) + Y'_0(\overline{x}) - \varepsilon^*\left[\frac{1}{1-2\delta}\right]\left[\exp\left(\frac{x-1}{1-2\delta}\right) + Y_0(\overline{x})\right] = 0, \tag{36}$$

which is regularly perturbed ordinary differential equation, and for $\varepsilon^* = 0$, it will be reduced to

$$Y''_0(\overline{x}) + Y'_0(\overline{x}) = 0. \tag{37}$$

Now, the solution of the above equation is given by

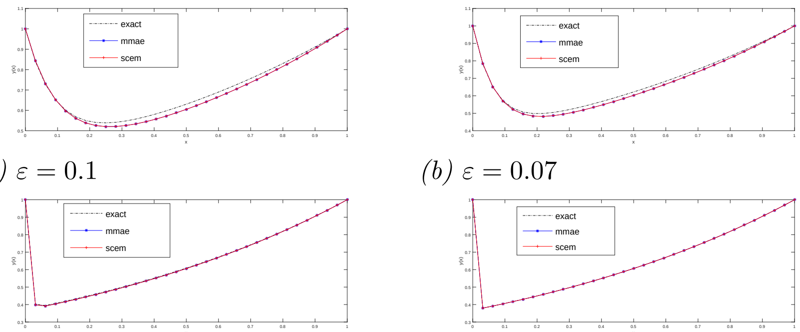
$$Y_0(\overline{x}) = C_1 + C_2 e^{-\overline{x}}. \tag{38}$$

For a uniformly valid SCEM solution, y_{0scem} must satisfy the following boundary conditions:

$$x = 0 \Rightarrow \overline{x} = \frac{x}{\varepsilon} = 0 \Rightarrow y_{0scem}(0,0,\varepsilon) = y_0(0,\varepsilon) + Y_0(0,0,\varepsilon) = 1 \Rightarrow Y_0(0,0,\varepsilon) = 1 - y_0(0,\varepsilon), \tag{39}$$

$$x = 1 \Rightarrow \overline{x} = \frac{x}{\varepsilon} = 1 \Rightarrow y_{0scem}(1,\frac{1}{\varepsilon},\varepsilon) = y_0(1,\varepsilon) + Y_0(1,\frac{1}{\varepsilon},\varepsilon) = 1 \Rightarrow Y_0(1,\frac{1}{\varepsilon},\varepsilon) = 1 - y_0(1,\varepsilon).$$

Using the boundary conditions in (38), we get



(a) $\varepsilon = 0.1$ *(b)* $\varepsilon = 0.07$

(c) $\varepsilon = 0.009$ *(d)* $\varepsilon = 0.0004$

Figure 1: Comparison of solutions with different values of ε for Example 1.

$$C_2 = \frac{\exp\left(\frac{-1}{1-2\delta}\right) - 1}{\exp\left(\frac{-1}{\varepsilon^*}\right) - 1}, \qquad C_1 = -C_2 \exp\left(\frac{-1}{\varepsilon^*}\right). \tag{40}$$

Now, using (40) and (39) in (32), we get the composite SCEM solution as

$$\begin{aligned} y_{scem} =& \exp\left(\frac{x-1}{1-2\delta}\right) - \left[\frac{\exp(\frac{-1}{1-2\delta}) - 1}{\exp(\frac{-1}{\varepsilon^*}) - 1}\right] \exp\left(\frac{-1}{\varepsilon^*}\right) \\ &+ \left[\frac{\exp(\frac{-1}{1-2\delta}) - 1}{\exp(\frac{-1}{\varepsilon^*}) - 1}\right] \exp\left(\frac{-x}{\varepsilon^*}\right). \end{aligned} \tag{41}$$

After getting the approximation to the exact solution by both SCEM and MMAE, it is quite obvious that SCEM is more adaptable. The immense advantage of SCEM is that it gives the uniformly valid approximation to the exact solution without any matching procedure. The error in comparison with the approximated exact solution for both SCEM and MMAE is given in Table 1 for different values of ε. The result is proved to be valid for any value of N. Here for the computational purposes, the nodal points and the values of η and δ are kept fixed to $N = 32$ and $\eta = 0.1\varepsilon = \delta$.

Example 2. Consider the following problem having an advance term:

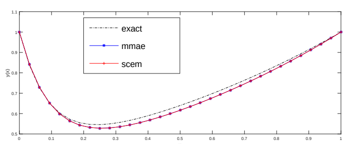
$$\begin{cases} \varepsilon y''(x) + y'(x) - 3y(x) + 2y(x+\eta) = 0, \\ y(0) = 1, \qquad y(x) = 1, 1 \le x \le 1+\eta. \end{cases}$$

This problem has a rapid change near $x = 0$. Now using Taylor's Series expansion in the neighborhood of x, the above problem reduces to

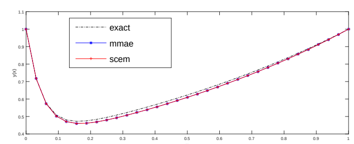
$$\varepsilon^* y''(x) + y'(x) - \frac{1}{1+2\eta} y(x) = 0, \qquad where\ \varepsilon^* = \frac{\varepsilon}{1+2\eta}. \tag{42}$$

ε	$L^{\infty} error\, in\, MMAE$	$L^{\infty} error\, in\, SCEM$	Error in Upwind
0.0001	0.00003676	0.00003676	0.04450943
0.0005	0.00018374	0.00018374	0.04428418
0.0010	0.00036728	0.00036728	0.04401059
0.0050	0.00182533	0.00182533	0.04209958
0.0100	0.00361593	0.00361593	0.04025378
0.0500	0.01598307	0.01598307	0.03472166
0.1000	0.02552850	0.02556347	0.03481582
0.3000	0.02853413	0.03217078	0.03427254
0.4000	0.06644741	0.03539967	0.03541439
0.6000	0.15664433	0.04561360	0.03723168
0.8000	0.24352992	0.05726515	0.03825982
1.0000	0.32059406	0.06915099	0.03877738

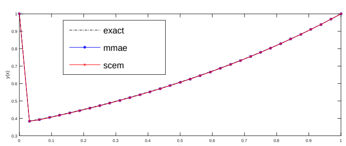
Table 1: Numerical results with different values of ε for Example 1.



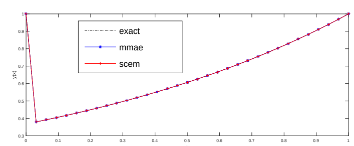
(a) $\varepsilon = 0.1$



(b) $\varepsilon = 0.05$



(c) $\varepsilon = 0.006$



(d) $\varepsilon = 0.0009$

Figure 2: Comparison of solutions with different values of ε for Example 2.

The one-term SCEM and MMAE approximations for (42) are given as follows:

$$y_{mmae} = \left[1 - \exp\left(\frac{-1}{1+2\eta}\right)\right]\exp\left(\frac{-x}{\varepsilon^*}\right) + \exp\left(\frac{x-1}{1+2\eta}\right),$$

$$y_{scem} = -\left[\frac{\exp(\frac{-1}{1+2\eta}) - 1}{\exp(\frac{-1}{\varepsilon^*}) - 1}\right]\exp\left(\frac{-1}{\varepsilon^*}\right) + \left[\frac{\exp(\frac{-1}{1+2\eta}) - 1}{\exp(\frac{-1}{\varepsilon^*}) - 1}\right]\exp\left(\frac{-x}{\varepsilon^*}\right)$$
$$+ \exp\left(\frac{x-1}{1+2\eta}\right).$$

Example 3. Consider the following problem having mixed shifts:

ε	$L^{\infty}error\,in\,MMAE$	$L^{\infty}error\,in\,SCEM$	Error in Upwind
0.0001	0.00003676	0.00003676	0.04448748
0.0005	0.00018367	0.00018367	0.04417452
0.0010	0.00036699	0.00036699	0.04379147
0.0050	0.00181786	0.00181786	0.04101098
0.0100	0.00358632	0.00358632	0.03808839
0.0500	0.01533655	0.01533655	0.02519028
0.1000	0.02377291	0.02379587	0.01688285
0.3000	0.01783685	0.02658575	0.01728136
0.4000	0.04058107	0.02660582	0.02334674
0.6000	0.09131635	0.02829255	0.03160623
0.8000	0.13551419	0.03009785	0.03671486
1.0000	0.17029574	0.03133850	0.04028387

Table 2: Numerical results with different values of ε for Example 2.

$$\begin{cases} \varepsilon y''(x) + y'(x) - 2y(x-\delta) - 5y(x) + y(x+\eta) = 0, \\ y(x) = 1, \quad -\delta \le x \le 0 \quad y(x) = 1, \quad 1 \le x \le 1+\eta. \end{cases}$$

This problem has a rapid change near $x = 0$. By the use of Taylor's Series expansion for both the delay and shift terms as in (2), we have

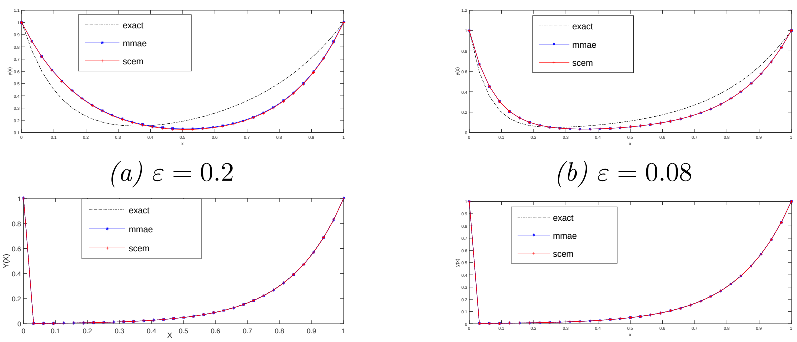
$$\varepsilon^* y''(x) + y'(x) - \left[\frac{6}{1+2\delta+\eta}\right] y(x) = 0, \qquad where\ \varepsilon^* = \frac{\varepsilon}{1+2\delta+\eta}.$$

The one-term SCEM and MMAE approximations are given by

$$y_{mmae} = \left[1 - \exp\left(\frac{-6}{1+2\delta+\eta}\right)\right] \exp\left(\frac{-x}{\varepsilon^*}\right) + \exp\left(\frac{6(x-1)}{1+2\delta+\eta}\right),$$

$$y_{scem} = -\left[\frac{\exp(\frac{-6}{1+2\delta+\eta}) - 1}{\exp(\frac{-1}{\varepsilon^*}) - 1}\right] \exp\left(\frac{-1}{\varepsilon^*}\right) + \left[\frac{\exp(\frac{-6}{1+2\delta+\eta}) - 1}{\exp(\frac{-1}{\varepsilon^*}) - 1}\right] \exp\left(\frac{-x}{\varepsilon^*}\right) + \exp\left[\frac{6(x-1)}{1+2\delta+\eta}\right].$$

The comparison of errors of this example for both the proposed methods and the method prescribed in [10] for $N = 8$ and $N = 256$ are given in Tables 4 and 5, respectively. These comparison results confirm the efficiency of the proposed method.



(a) $\varepsilon = 0.2$ *(b)* $\varepsilon = 0.08$

(c) $\varepsilon = 0.001$ *(d)* $\varepsilon = 0.0007$

Figure 3: *Comparison of solutions with different values of* ε *for Example 3.*

ε	$L^\infty error\, in\, MMAE$	$L^\infty error\, in\, SCEM$	Error in Upwind
0.0001	0.00022006	0.00022006	0.05991807
0.0005	0.00109602	0.00109602	0.05951004
0.0010	0.00218145	0.00218145	0.05900561
0.0050	0.01050670	0.01050670	0.05517982
0.0100	0.02012147	0.02012147	0.05086176
0.0500	0.07770075	0.07770075	0.03679054
0.1000	0.12342561	0.12345914	0.03402622
0.3000	0.16404633	0.18962499	0.01593674
0.4000	0.16021248	0.19825547	0.01255964
0.6000	0.15178975	0.20931461	0.00894834
0.8000	0.21056755	0.21768491	0.00694484
1.0000	0.26983417	0.22335443	0.00562079

Table 3: Numerical results with different values of ε for Example 3.

ε	$L^\infty error\, in\, MMAE$	$L^\infty error\, in\, SCEM$	Error in [10]
10^{-1}	0.10596702	0.10597709	0.12011566
10^{-2}	0.01882187	0.01882187	0.18727108
10^{-3}	0.00209815	0.00209815	0.20429729
10^{-4}	0.00021228	0.00021228	0.20614146
10^{-5}	0.00002125	0.00002125	0.20632746
10^{-6}	0.00000212	0.00000212	0.20634608

Table 4: Comparison of maximum error with $N = 8$ for Example 3.

ε	$L^{\infty} error\, in\, MMAE$	$L^{\infty} error\, in\, SCEM$	Error in [10]
10^{-1}	0.10685019	0.10685019	0.00775036
10^{-2}	0.01976044	0.01976044	0.00799076
10^{-3}	0.00218116	0.00218116	0.00963304
10^{-4}	0.00022045	0.00022045	0.00984236
10^{-5}	0.00002206	0.00002206	0.00986365
10^{-6}	0.00000220	0.00000220	0.00986578

Table 5: Comparison of maximum error with $N = 256$ for Example 3.

5 Conclusion

In this work, the well-known MMAE and a parameter uniform numerical scheme, that is, the upwind scheme in Shishkin mesh were compared with the proposed relatively new method, that is, SCEM, for solving singularly perturbed differential-difference equation with mixed shifts. It was observed that both MMAE and SCEM provide highly accurate approximations to the exact solution for comparatively small values ε. Similarly, comparison results confirmed that the numerical scheme, which gives better results for higher values of N, at the same time, MMAE and SCEM give higher-order accuracy for comparatively smaller values of N. It is observed that we do not require any matching procedure in SCEM as required for MMAE, and the boundary conditions are satisfied exactly for all values of the perturbation parameters. These properties make SCEM more flexible and a better alternative while solving singularly perturbed differential equations.

Acknowledgment

The first author expresses her sincere gratitude to DST, Govt. of India for providing INSPIRE fellowship (IF 180938).

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How to cite this article

Priyadarshana, S., Sahu, S., Mohapatra, J. Asymptotic and numerical methods for solving singularly perturbed differential difference equations with mixed shifts. *Iranian Journal of Numerical Analysis and Optimization*, 2022; 12(1): 55-72. doi:10.22067/ijnao.2021.70731.1038