Iranian Journal of Numerical Analysis and Optimization
Vol. 11, No. 1, (2021), pp 135–157
DOI:10.22067/ijnao.2021.11300.0

Research Article

Using approximate endpoint property on existing solutions for two inclusion problems of the fractional q-differential†

G.K. Ranjbar* and M.E. Samei

Abstract

Using the approximate endpoint property, we describe a technique for existing solutions of the fractional q-differential inclusion with boundary value conditions on multifunctions. For this, we use an approximate endpoint result on multifunctions. Also, we give an example to elaborate on our results and to present the obtained results by fractional calculus.

AMS subject classifications (2020): 34A08, 39A13, 34A60.

Keywords: Approximate endpoint; Fractional q-differential inclusion; Boundary value conditions.

1 Introduction

Fractional calculus and q-calculus are some of the significant branches in mathematical analysis. The field of fractional calculus has countless applications (for instance, consider [2, 10, 32]). Similarly, the subject of fractional differential equations ranges from the theoretical views of the existence and uniqueness of solutions to analytical methods (for more details, see [4, 5]). There has been intensive development in fractional differential equations and

*Corresponding author
Received 1 January 2020; revised 16 November 2020; accepted 15 January 2021
Ghorban Kalilzadeh Ranjbar
Department of Mathematics, Faculty of Science, Bu-Ali Sina University, Hamedan, Iran.
e-mail: ghkhalilzadeg@yahoo.com

Mohammad Esmael Samei
Department of Mathematics, Faculty of Science, Bu-Ali Sina University, Hamedan, Iran.
e-mail: mesamei@basu.ac.ir; mesamei@gmail.com

† This article was suggested by the Scientific Committee of the "Third Iranian Seminar on Control and Optimization" for publication in IJNAO, which was accepted after independent review.

inclusion (for example, see [9, 13, 38, 36, 35, 37, 29, 20, 34]). Also, great attention was devoted to the fractional differential inclusions (for more information, see [4, 7, 12, 30, 40]). For finding more details about elementary notions and definitions of fractional differential equations one can study, for instance, [6, 26, 31].

In this article, motivated by [8, 33] and among these achievements, we try to stretch out the problem in a sense for the fractional q-differential inclusions with integral boundary conditions, in conformity with the definition of the fractional Caputo type q-derivative of order α and the fractional Riemann–Liouville type q-integral. We state the basic definitions and some properties of fractional q-differential inclusion from [18]. For this purpose, we consider and discuss the inclusion problem

$$ {}^{c}\mathbb{D}_q^{\alpha} u(t) \in \mathcal{W}\left(t, u(t), u'(t), u''(t)\right), \tag{1} $$

under conditions $u(0) + u(p) + u(1) = \int_0^1 f_0(s, u(s))\,\mathrm{d}s$ and

$$ \begin{cases} {}^{c}\mathbb{D}_q^{\beta} u(0) + {}^{c}\mathbb{D}_q^{\beta} u(p) + {}^{c}\mathbb{D}_q^{\beta} u(1) = \int_0^1 f_1(s, u(s))\,\mathrm{d}s, \\ {}^{c}\mathbb{D}_q^{\gamma} u(0) + {}^{c}\mathbb{D}_q^{\gamma} u(p) + {}^{c}\mathbb{D}_q^{\gamma} u(1) = \int_0^1 f_2(s, u(s))\,\mathrm{d}s, \end{cases} \tag{2} $$

where $\alpha \in (2,3]$, $0 < q, p, \beta < 1$, $\gamma \in (1,2)$, $f_i : J \times \mathbb{R} \to \mathbb{R}$ for $i = 1,2,3$, are continuous functions, $\mathcal{W} : J \times \mathbb{R}^3 \to P_{cp}(\mathbb{R})$ is a multifunction, and ${}^{c}\mathbb{D}_q^{\beta}$ is the fractional Caputo type q-derivative for $t \in J = [0,1]$. The set of all compact subsets of $\mathbb{R}$ is denoted by $P_{cp}(\mathbb{R})$. Also, we look into the existence of solutions for the fractional q-differential inclusion problem on the multifunction $\mathcal{W} : J \times \mathbb{R}^{n+1} \to P(\mathbb{R})$,

$$ {}^{c}\mathbb{D}_q^{\alpha} u(t) \in \mathcal{W}\left(t, u(t), {}^{c}\mathbb{D}_q^{\gamma_1} u(t), \dots, {}^{c}\mathbb{D}_q^{\gamma_n} u(t)\right), \tag{3} $$

with conditions

$$ \begin{cases} u'(0) + a_1 u'(1) = \sum_{i=1}^{n} {}^{c}\mathbb{D}_q^{\gamma_i} u(p), \\ u(0) + a_2 u(1) = \sum_{i=1}^{n} \mathbb{I}_q^{\gamma_i} u(p), \end{cases} \tag{4} $$

where $\alpha \in (1,2]$, $0 < q, p, \gamma_i < 1$, $\alpha - \gamma_i \in [1, \infty)$ for all $1 \leq i \leq n$,

$$ a_1 > \sum_{i=1}^{n} \frac{p^{1-\gamma_i}}{\Gamma_q(2-\gamma_i)}, \quad a_2 > \sum_{i=1}^{n} \frac{p^{\gamma_i+1}}{\Gamma_q(\gamma_i+2)}, $$

here $n \geq 1$ and $t \in J = [0,1]$.

As before, we remind some of the previous works briefly. In 1910, the subject of q-difference equations was introduced by Jackson [22, 24, 23]. After that, at the beginning of the last century, studies on q-difference equations have been appeared in many works, especially in [1, 3, 15, 28, 39]. An excellent account in the study of fractional differential and q-differential equations can be found in [21, 25, 26].

2 Preliminaries

We recall from [14, 25] some basic definitions, notation, and results of q-fractional calculus, which are needed throughout this article. In fact, we consider the fractional q-calculus on the specific time scale $\mathbb{T} = \mathbb{R}$, where $\mathbb{T}_{s_0} = \{0\} \cup \{s : s = s_0 q^n\}$, for a nonnegative integer n, $s_0 \in \mathbb{R}$, and $q \in J_0$. Let $a \in \mathbb{R}$. Define $[a]_q = (1-q^a)/(1-q)$ [24]. The power function $(y-z)_q^n$ with $n \in \mathbb{N}_0$ is defined by $(y-z)_q^{(n)} = \prod_{k=0}^{n-1}(y-zq^k)$, for $n \geq 1$ and $(y-z)_q^{(0)} = 1$, where y and z are real numbers and $\mathbb{N}_0 := \{0\}\cup\mathbb{N}$ [1]. Also, for $\sigma \in \mathbb{R}$ and $q \neq 0$, we have $(y-z)_q^{(\sigma)} = v^\sigma \prod_{k=0}^{\infty}(y-zq^k)/(y-zq^{\sigma+k})$. If $z = 0$, then it is clear that $y^{(\sigma)} = y^\sigma$ [11] (Algorithm 1). The q-Gamma function is given by $\Gamma_q(\sigma) = (1-q)^{(\sigma-1)}/(1-q)^{\sigma-1}$, where $\sigma \in \mathbb{R}\backslash\{0, -1, -2, \ldots\}$ [24] (Algorithm 2). Note that, $\Gamma_q(\sigma+1) = [\sigma]_q\Gamma_q(\sigma)$. The q-derivative of function $\mathfrak{g}$ is defined by

$$\mathbb{D}_q[\mathfrak{g}](\tau) = \frac{\mathfrak{g}(\tau) - \mathfrak{g}(q\tau)}{(1-q)\tau},$$

and $\mathbb{D}_q[\mathfrak{g}](0) = \lim_{\tau\to 0} \mathbb{D}_q[\mathfrak{g}](\tau)$, which is shown in Algorithm 3 [1]. Furthermore, the higher order q-derivative of a function $\mathfrak{g}$ is defined by $\mathbb{D}_q^n[\mathfrak{g}](\tau) = \mathbb{D}_q[\mathbb{D}_q^{n-1}[\mathfrak{g}]](\tau)$, for $n \geq 1$, where $\mathbb{D}_q^0[\mathfrak{g}](\tau) = \mathfrak{g}(\tau)$ [1]. The q-integral of a function $\mathfrak{g}$ is defined on $[0, b]$ by

$$\mathbb{I}_q[\mathfrak{g}](\tau) = \int_0^\tau \mathfrak{g}(\xi)\,\mathrm{d}_q\xi = \tau(1-q)\sum_{k=0}^{\infty} q^k \mathfrak{g}(\tau q^k),$$

for $0 \leq \tau \leq b$, provided the series is absolutely converges [1]. If τ in $[0, T]$, then

$$\int_\tau^T \mathfrak{g}(\xi)\,\mathrm{d}_q\xi = \mathbb{I}_q[\mathfrak{g}](T) - \mathbb{I}_q[\mathfrak{g}](\tau) = (1-q)\sum_{k=0}^{\infty} q^k \left[T\mathfrak{g}(Tq^k) - \tau\mathfrak{g}(\tau q^k)\right],$$

whenever the series exists. The operator $\mathbb{I}_q^n$ is given by $\mathbb{I}_q^0[\mathfrak{g}](\tau) = \mathfrak{g}(\tau)$ and $\mathbb{I}_q^n[\mathfrak{g}](\tau) = \mathbb{I}_q[\mathbb{I}_q^{n-1}[\mathfrak{g}]](\tau)$ for $n \geq 1$ and $\mathfrak{g} \in C([0, T])$ [1]. It has been proved that $\mathbb{D}_q[\mathbb{I}_q[\mathfrak{g}]](\tau) = \mathfrak{g}(\tau)$ and $\mathbb{I}_q[\mathbb{D}_q[\mathfrak{g}]](\tau) = \mathfrak{g}(\tau) - \mathfrak{g}(0)$ whenever $\mathfrak{g}$ is continuous at $\tau = 0$ [1]. The fractional Riemann–Liouville type q-integral of the function $\mathfrak{g}$ on $J_0 = (0, 1)$ for $\sigma \geq 0$ is defined by $\mathbb{I}_q^0[\mathfrak{g}](\tau) = \mathfrak{g}(\tau)$ and

$$\begin{aligned}\mathbb{I}_q^\sigma[\mathfrak{g}](\tau) &= \frac{1}{\Gamma_q(\sigma)}\int_0^\tau (\tau - q\xi)^{(\sigma-1)}\mathfrak{g}(\xi)\,\mathrm{d}_q\xi \\ &= \tau^\sigma(1-q)^\sigma \sum_{k=0}^{\infty} q^k \frac{\prod_{i=1}^{k-1}\left(1-q^{\sigma+i}\right)}{\prod_{i=1}^{k-1}\left(1-q^{i+1}\right)}\mathfrak{g}(\tau q^k), \end{aligned} \tag{5}$$

for $t \in \overline{J}_0$ [17, 10](Algorithm 4). The Caputo fractional q-derivative of a function $\mathfrak{g}$ is defined by

$$
\begin{aligned}
{}^{c}\mathbb{D}_q^{\sigma}[\mathfrak{g}](\tau) &= \mathbb{I}_q^{[\sigma]-\sigma}[\mathbb{D}_q^{[\sigma]}[\mathfrak{g}]](\tau) \\
&= \frac{1}{\Gamma_q\left([\sigma]-\sigma\right)} \int_0^{\tau} (\tau - q\xi)^{([\sigma]-\sigma-1)} \mathbb{D}_q^{[\sigma]}[\mathfrak{g}](\xi)\, \mathrm{d}_q q\xi,
\end{aligned} \tag{6}
$$

where $t \in \overline{J}_0$ and $\sigma > 0$ [17]. It has been proved that $\mathbb{I}_q^{\nu}[\mathbb{I}_q^{\sigma}[\mathfrak{g}]](\tau) = \mathbb{I}_q^{\sigma+\nu}[\mathfrak{g}](\tau)$, and $\mathcal{D}_q^{\sigma}[\mathbb{I}_q^{\sigma}[\mathfrak{g}]](\tau) = \mathfrak{g}(\tau)$, where $\sigma, \nu \geq 0$ [17]. Algorithm 4 shows pesudo-code $\mathbb{I}_q^{\sigma}[\mathfrak{g}](\tau)$.

We say a multifunction $G : J \to P_{cl}(\mathbb{R})$ is measurable whenever for each real number y, the function $t \mapsto d(y, G(t))$ is measurable [16]. The Pompeiu–Hausdorff metric $H_d \colon 2^X \times 2^X \to [0, \infty)$ on a metric space (X, ρ) is defined by

$$
H_\rho(A, B) = \max\Big\{ \sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(A, b) \Big\},
$$

where $\rho(A, b) = \inf_{a \in A} \rho(a, b)$ [19]. The set of closed and bounded of X, and the set of closed subsets of X are denoted by $CB(X)$ and $C(X)$, respectively. In this case, $(CB(X), H_\rho)$ and $(C(X), H_\rho)$ are a metric space and a generalized metric space, respectively [27]. An element $z \in X$ is called an endpoint of multifunction $\mathcal{W} : X \to 2^X$ whenever $\mathcal{W}z = \{z\}$ [8]. Also, the multifunction $\mathcal{W}$ has the approximate endpoint property whenever $\inf_{x \in X} \sup_{y \in \mathcal{W}x} \rho(x, y) = 0$ [8]. A function $\theta : \mathbb{R} \to \mathbb{R}$ is called upper semi-continuous whenever $\limsup_{n \to \infty} \theta(\lambda_n) \leq \theta(\lambda)$ for all sequence $\{\lambda_n\}_{n \geq 1}$ with $\lambda_n \to \lambda$ [8].

Lemma 1. [8] Consider an upper semi-continuous function $\theta : [0, \infty) \to [0, \infty)$ such that $\theta(t) < t$ and $\liminf_{t \to \infty}(t - \theta(t)) > 0$, for all $t > 0$. Also, Assume that (X, ρ) a complete metric space and that $\mathcal{W} : X \to CB(X)$ is a multifunction such that $H_d(\mathcal{W}(x), \mathcal{W}() \leq \theta(\rho(x, y))$, for all $x, y \in X$. Then $\mathcal{W}$ has a unique endpoint if and only if $\mathcal{W}$ has the approximate endpoint property.

3 Main results

Right away, we are ready to state and prove our main results. Foremost, we make the adjacent one.

Lemma 2. Suppose that $v \in C(J, \mathbb{R})$, that $\alpha \in (2, 3]$, that $0 < \beta, q, p < 1$, that $\gamma \in (1, 2)$, and that $f_i : J \times \mathbb{R} \to \mathbb{R}$, for $i = 0, 1, 2$, are continuous functions. The unique solution of the fractional q-differential problem

$$
{}^{c}\mathbb{D}_q^{\alpha} u(t) = v(t), \tag{7}
$$

with conditions (2) is given by

$$
\begin{aligned}
u(t) = {} & \mathbb{I}_q^{\alpha} v(t) + \frac{1}{3}\int_0^1 f_0(s,u(s))\,\mathrm{d}s - \frac{1}{3}\left[\mathbb{I}_q^{\alpha} v(1) + \mathbb{I}_q^{\alpha} v(p)\right] \\
& + a_1(t)\int_0^1 f_1(s,u(s))\,\mathrm{d}s + a_2(t)\left[\mathbb{I}_q^{\alpha-\beta} v(1) + \mathbb{I}_q^{\alpha-\beta} v(p)\right] \\
& + (b_1 + a_3(t))\int_0^1 g_2(s,u(s))\,\mathrm{d}s \\
& + (b_2 + a_4(t))\left[\mathbb{I}_q^{\alpha-\gamma} v(1) + \mathbb{I}_q^{\alpha-\gamma} v(p)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
a_1(t) &= \frac{3t\Gamma_q(2-\beta) - (p+1)\Gamma_q(2-\beta)}{3(p^{1-\beta}+1)}, \\
a_2(t) &= \frac{(p+1)\Gamma_q(2-\beta) - 3\Gamma_q(2-\beta)t}{3(p^{1-\beta}+1)}, \\
a_3(t) &= \frac{-6(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)t}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} \\
&\quad + \frac{3(p^{1-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(3-\beta)t^2}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}, \\
a_4(t) &= \frac{6(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)t}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} \\
&\quad - \frac{3\Gamma_q(3-\gamma)\Gamma_q(3-\beta)(p^{1-\beta}+1)t^2}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}, \\
b_1 &= \frac{2(p+1)(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} \\
&\quad - \frac{(p^2+1)\Gamma_q(3-\gamma)(p^{1-\beta}+1)\Gamma_q(3-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}, \\
b_2 &= \frac{(p^2+1)\Gamma_q(3-\gamma)(p^{1-\beta}+1)\Gamma_q(3-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} \\
&\quad - \frac{2(p+1)(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}.
\end{aligned}
\tag{8}
$$

Proof. As we have known that the general solution of (7) is

$$
u(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} v(s)\,\mathrm{d}_q s + c_0 + c_1 t + c_2 t^2, \tag{9}
$$

where c_i $(i=0,1,2)$ are arbitrary real constants [26, 31, 38]. Thus,

$$
\begin{aligned}
{}^c\mathbb{D}_q^{\beta} u(t) &= \mathbb{I}_q^{\alpha-\beta} v(t) + \frac{c_1 t^{1-\beta}}{\Gamma_q(2-\beta)} + \frac{2c_2 t^{2-\beta}}{\Gamma_q(3-\beta)},\\
{}^c\mathbb{D}_q^{\gamma} u(t) &= \mathbb{I}_q^{\alpha-\gamma} v(t) + \frac{2c_2 t^{2-\gamma}}{\Gamma_q(3-\gamma)}.
\end{aligned}
$$

Hence, we get

$$
u(0) + u(p) + u(1) = 3c_0 + (1+p)c_1 + (1+p^2)c_2 + \mathbb{I}_q^{\alpha} v(1) + \mathbb{I}_q^{\alpha} v(p)
$$

and

$$
\begin{aligned}
{}^c\mathbb{D}_q^{\beta} u(0) + {}^c\mathbb{D}_q^{\beta} u(p) + {}^c\mathbb{D}_q^{\beta} u(1) &= c_1 \frac{p^{1-\beta}+1}{\Gamma_q(2-\beta)} + c_2 \frac{2(p^{2-\beta}+1)}{\Gamma_q(3-\beta)}\\
&\quad + \mathbb{I}_q^{\alpha-\beta} v(1) + \mathbb{I}_q^{\alpha-\beta} v(p),\\
{}^c\mathbb{D}_q^{\gamma} u(0) + {}^c\mathbb{D}_q^{\gamma} u(p) + {}^c\mathbb{D}_q^{\gamma} x(1) &= c_2 \frac{2(p^{2-\gamma}+1)}{\Gamma_q(3-\gamma)}\\
&\quad + \mathbb{I}_q^{\alpha-\gamma} v(1) + \mathbb{I}_q^{\alpha-\gamma} v(p).
\end{aligned}
$$

By employing the boundary conditions, we have

$$
\begin{aligned}
3c_0 + (1+p)c_1 + (1+p^2)c_2 &= \int_0^1 f_0(s, u(s))\, \mathrm{d}s\\
&\quad - \mathbb{I}_q^{\alpha} v(1) - \mathbb{I}_q^{\alpha} v(p),\\
c_1 \frac{p^{1-\beta}+1}{\Gamma_q(2-\beta)} + c_2 \frac{2(p^{2-\beta}+1)}{\Gamma_q(3-\beta)} &= \int_0^1 f_1(s, x(s))\, \mathrm{d}s\\
&\quad - \mathbb{I}_q^{\alpha-\beta} v(1) - \mathbb{I}_q^{\alpha-\beta} v(p),\\
c_2 \frac{2(p^{2-\gamma}+1)}{\Gamma_q(3-\gamma)} &= \int_0^1 f_2(s, x(s))\, \mathrm{d}s\\
&\quad - \mathbb{I}_q^{\alpha-\gamma} v(1) - \mathbb{I}_q^{\alpha-\gamma} v(p).
\end{aligned}
$$

This is a linear system of equations of triangular shape, having c_0, c_1, and c_2 as unknowns. By a back substitution, we obtain

$$\begin{aligned}
c_0 = {} & \frac{1}{3}\int_0^1 f_0(s,u(s))\,\mathrm{d}s - \frac{1}{3}\left[\mathbb{I}_q^{\alpha}v(1) + \mathbb{I}_q^{\alpha}v(p)\right] \\
& - \frac{\Gamma_q(2-\beta)(p+1)}{3(p^{1-\beta}+1)}\int_0^1 f_1(s,u(s))\,\mathrm{d}s \\
& + \frac{(p+1)\Gamma_q(2-\beta)}{3(p^{1-\beta}+1)}\left[\mathbb{I}_q^{\alpha-\beta}v(1) + \mathbb{I}_q^{\alpha-\beta}v(p)\right] \\
& + b_1\int_0^1 f_2(s,u(s))\,\mathrm{d}s + b_2\left[\mathbb{I}_q^{\alpha-\gamma}v(1) + \mathbb{I}_q^{\alpha-\gamma}v(p)\right], \\
c_1 = {} & \frac{\Gamma_q(2-\beta)}{(p^{1-\beta}+1)}\int_0^1 f_1(s,u(s))\,\mathrm{d}s \\
& - \frac{\Gamma_q(2-\beta)}{(p^{1-\beta}+1)}\left[\mathbb{I}_q^{\alpha-\beta}v(1) + \mathbb{I}_q^{\alpha-\beta}v(p)\right] \\
& - \frac{(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}\int_0^1 f_2(s,u(s))\,\mathrm{d}s \\
& + \frac{(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}\left[\mathbb{I}_q^{\alpha-\gamma}v(1) + \mathbb{I}_q^{\alpha-\gamma}v(p)\right], \\
c_2 = {} & \frac{\Gamma_q(3-\gamma)}{2(p^{2-\gamma}+1)}\int_0^1 f_2(s,u(s))\,\mathrm{d}s \\
& - \frac{\Gamma_q(3-\gamma)}{2(p^{2-\gamma}+1)}\left[\mathbb{I}_q^{\alpha-\gamma}v(1) + \mathbb{I}_q^{\alpha-\gamma}v(p)\right].
\end{aligned}$$

At once, we replace c_0, c_1, and c_2 in (9) and find the solution $u(t)$ as we stated. □

Assume that $\mathcal{X} = C^2(J)$ endowed with the norm

$$\|u\| = \sup_{t\in J}|u(t)| + \sup_{t\in J}|u'(t)| + \sup_{t\in J}|u''(t)|.$$

Then $(\mathcal{X}, \|\cdot\|)$ is a Banach space (see [16]). For $u \in \mathcal{X}$, we define the selection set $S_{\mathcal{W},u}$ by the set of all $v \in L^1(J)$ such that $v(t) \in \mathcal{W}(t, u(t), u'(t), u''(t))$ for all $t \in J$. For the study of problem (1) and (2), we shall consider the following conditions:

(C1) The multifunction $\mathcal{W} : J \times \mathbb{R}^3 \to P_{cp}(\mathbb{R})$ be is an integrable and bounded such that $\mathcal{W}(\cdot, x_1, x_2, x_3) : J \to P_{cp}(\mathbb{R})$ is measurable for all $x_i \in \mathbb{R}$;

(C2) The functions $f_i : J \times \mathbb{R} \to \mathbb{R}$ be are continuous and map $\theta : [0,\infty) \to [0,\infty)$ be a nondecreasing upper semi-continuous such that

$$\liminf_{t\to\infty}(t - \theta(t)) > 0$$

and $\theta(t) < t$ for all $t > 0$;

(C3) There exist $m, m_0, m_1, m_2 \in C(J, [0, \infty))$ such that

$$H_d\left(\mathcal{W}(t, x_1, x_2, x_3), \mathcal{W}(t, x'_1, x'_2, x'_3)\right) \leq \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3} m(t)\theta\biggl(\sum_{k=1}^{3} |x_i - x'_i|\biggr)$$

and

$$|f_j(t, x) - f_j(t, x')| \leq \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3} m_j(t)\psi(|x - x'|),$$

for all $t \in J$, $x, x', x_i, x'_i \in \mathbb{R}$, where

$$\begin{aligned}
\Lambda_1 = \Biggl[&\frac{\|m\|_\infty}{\Gamma_q(\alpha+1)} + \frac{\|m_0\|_\infty}{3} + \frac{2\|m\|_\infty}{3\Gamma_q(\alpha+1)} \\
&+ \frac{5\Gamma_q(2-\beta)\|m_1\|_\infty}{3} + \frac{10\Gamma_q(2-\beta)\|m\|_\infty}{3\Gamma_q(\alpha-\beta+1)} \\
&+ 10\bigl(2\Gamma_q(2-\beta) + \Gamma_q(3-\beta)\bigr) \\
&\times \left(\frac{\Gamma_q(3-\gamma)\left(\|m_2\|_\infty \Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty\right)}{3\Gamma_q(3-\beta)\Gamma_q(\alpha-\gamma+1)}\right)\Biggr], \\
\Lambda_2 = \Biggl[&\frac{\|m\|_\infty}{\Gamma_q(\alpha)} + \frac{2\Gamma_q(2-\beta)\|m\|_\infty}{\Gamma_q(\alpha-\beta+1)} + \bigl(2\Gamma_q(2-\beta) + \Gamma_q(3-\beta)\bigr) \\
&\times \left(\frac{\Gamma_q(3-\gamma)\left(\|m_2\|_\infty \Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty\right)}{\Gamma(3-\beta)\Gamma_q(\alpha-\gamma+1)}\right)\Biggr], \\
\Lambda_3 = \Biggl[&\frac{\|m\|_\infty}{\Gamma_q(\alpha-1)} + \frac{\Gamma_q(3-\gamma)\left(\|m_2\|_\infty \Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty\right)}{\Gamma_q(\alpha-\gamma+1)}\Biggr];
\end{aligned}$$

(C4) Multifunction $N : \mathcal{X} \to 2^{\mathcal{X}}$ is given by

$$N(u) = \left\{h \in \mathcal{X} \mid \exists\, v \in S_{\mathcal{W},u} \;:\; h(t) = w(t)\right\},$$

for each $t \in J$, where by applying the notation in (8), we have

$$\begin{aligned}
w(t) = \mathbb{I}_q^\alpha v(t) &+ \frac{1}{3}\int_0^1 f_0(s, u(s))\,\mathrm{d}s - \frac{1}{3}\left[\mathbb{I}_q^\alpha v(1) + \mathbb{I}_q^\alpha v(p)\right] \\
&+ a_1(t)\int_0^1 f_1(s, u(s))\,\mathrm{d}s + a_2(t)\left[\mathbb{I}_q^{\alpha-\beta} v(1) + \mathbb{I}_q^{\alpha-\beta} v(p)\right] \\
&+ (b_1 + a_3(t))\int_0^1 f_2(s, u(s))\,\mathrm{d}s \\
&+ (b_2 + a_4(t))\left[\mathbb{I}_q^{\alpha-\gamma} v(1) + \mathbb{I}_q^{\alpha-\gamma} v(p)\right].
\end{aligned}$$

Theorem 1. The boundary value q-differential inclusion problem (1) and (2) has a solution, whenever the multifunction $N : \mathcal{X} \to P(\mathcal{X})$ has the approximate endpoint property and conditions (C1)–(C4) are hold.

Proof. We demonstrate that the multifunction N has an endpoint, which is a solution of the problem (1) and (2). Because the multivalued map $t \vdash \mathcal{W}(t, u(t), u'(t), u''(t))$ is measurable and so has closed values, it has measurable selection and so $S_{\mathcal{W},u}$ is nonempty for all $u \in \mathcal{X}$. At present, we show that $N(u) \subset \mathcal{X}$ is closed for $u \in \mathcal{X}$. Let $u \in \mathcal{X}$ and let $\{x_n\}_{n\geq 1}$ be a sequence in $N(u)$ with $u_n \to x$. For each $n \in \mathbb{N}$, choose $v_n \in S_{\mathcal{W},u}$ such that

$$\begin{aligned} x_n(t) = \mathbb{I}_q^{\alpha} v_n(t) + \frac{1}{3}\int_0^1 f_0(s, u(s))\,\mathrm{d}s - \frac{1}{3}\left[\mathbb{I}_q^{\alpha} v_n(1) + \mathbb{I}_q^{\alpha} v_n(p)\right] \\ + a_1(t)\int_0^1 f_1(s, u(s))\,\mathrm{d}s + a_2(t)\left[\mathbb{I}_q^{\alpha-\beta} v_n(1) + \mathbb{I}_q^{\alpha-\beta} v_n(p)\right] \\ + (b_1 + a_3(t))\int_0^1 f_2(s, u(s))\,\mathrm{d}s \\ + (b_2 + a_4(t))\left[\mathbb{I}_q^{\alpha-\gamma} v_n(1) + \mathbb{I}_q^{\alpha-\gamma} v_n(p)\right]. \end{aligned}$$

It is noteworthy that $\{v_n\}_{n\geq 1}$ has a subsequence that converges to some $v \in L^1(J)$, because $\mathcal{W}$ has compact values. Again, we denote this subsequence by $\{v_n\}_{n\geq 1}$. It is easy to go over that $v \in S_{\mathcal{W},u}$ and $x_n(t)$ tends to $x(t)$, where

$$\begin{aligned} x(t) = \mathbb{I}_q^{\alpha} v(t) + \frac{1}{3}\int_0^1 f_0(s, u(s))\,\mathrm{d}s - \frac{1}{3}\left[\mathbb{I}_q^{\alpha} v(1) + \mathbb{I}_q^{\alpha} v(p)\right] \\ + a_1(t)\int_0^1 f_1(s, u(s))\,\mathrm{d}s + a_2(t)\left[\mathbb{I}_q^{\alpha-\beta} v(1) + \mathbb{I}_q^{\alpha-\beta} v(p)\right] \\ + (b_1 + a_3(t))\int_0^1 f_2(s, u(s))\,\mathrm{d}s \\ + (b_2 + a_4(t))\left[\mathbb{I}_q^{\alpha-\gamma} v(1)\mathbb{I}_q^{\alpha-\gamma} v(p)\right], \end{aligned}$$

for each $t \in J$. This implies that $x \in N(u)$ and so N has closed values. Since $\mathcal{W}$ is a compact multivalued map, it is easy to check that $N(u)$ is a bounded set for all $u \in \mathcal{X}$. Now, we show that

$$H_d(N(u), N(v)) \leq \theta(\|u - v\|).$$

Let $u, v \in \mathcal{X}$ and let $h_1 \in N(v)$. Choose $w_1 \in S_{\mathcal{W},v}$ such that

$$\begin{aligned} h_1(t) = \mathbb{I}_q^{\alpha} w_1(t) + \frac{1}{3}\int_0^1 f_0(s, v(s))\,\mathrm{d}s - \frac{1}{3}\left[\mathbb{I}_q^{\alpha} w_1(1) + \mathbb{I}_q^{\alpha} w_1(p)\right] \\ + a_1(t)\int_0^1 f_1(s, v(s))\,\mathrm{d}s + a_2(t)\left[\mathbb{I}_q^{\alpha-\beta} w_1(1) + \mathbb{I}_q^{\alpha-\beta} w_1(p)\right] \end{aligned}$$

$$+ (b_1 + a_3(t)) \int_0^1 f_2(s, v(s))\, \mathrm{d}s$$
$$+ (b_2 + a_4(t)) \left[\mathbb{I}_q^{\alpha-\gamma} w_1(1) + \mathbb{I}_q^{\alpha-\gamma} w_1(p)\right],$$

for almost all $t \in J$. Put

$$\tilde{\mathcal{W}}_{u(t)} = \mathcal{W}\left(t, u(t), u'(t), u''(t)\right), \quad \tilde{\mathcal{W}}_{v(t)} = \mathcal{W}\left(t, v(t), v'(t), v''(t)\right).$$

Since

$$H_d\left(\tilde{\mathcal{W}}_{u(t)}, \tilde{\mathcal{W}}_{v(t)}\right) \le \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3} m(t) \times \theta\left(|u(t) - n(t)| + |u'(t) - v'(t)| + |u''(t) - v''(t)|\right),$$

for all $t \in J$, there exists $w \in \tilde{\mathcal{W}}_{u(t)}$ such that

$$|w_1(t) - w| \le \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3} m(t) \times \theta\left(|u(t) - v(t)| + |u'(t) - v'(t)| + |u''(t) - v''(t)|\right), \tag{10}$$

for all $t \in J$. Consider the multivalued map $G : J \to P(\mathbb{R})$, which defines the set of all $w \in \mathbb{R}$ such that w satisfies in (10). Since w_1 and

$$\varphi = m\theta\big(|u - v| + |u' - v'| + |u'' - v''|\big)\left[\frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3}\right],$$

are measurable, the multifunction

$$G(.) \bigcap \mathcal{W}\left(\cdot, u(\cdot), u'(\cdot), u''(\cdot)\right),$$

is measurable. Choose $w_2(t) \in \mathcal{W}\left(t, u(t), u'(t), u''(t)\right)$ such that

$$|w_1(t) - w_2(t)| \le \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3} m(t) \times \psi\left(|u(t) - v(t)| + |u'(t) - v'(t)| + |u''(t) - v''(t)|\right),$$

for all $t \in J$. Now, Consider the element $h_2 \in N(u)$, which is defined by

$$
\begin{aligned}
h_2(t) = {} & \mathbb{I}_q^{\alpha} w_2(t) + \frac{1}{3}\int_0^1 f_0(s,u(s))\,\mathrm{d}s - \frac{1}{3}\left[\mathbb{I}_q^{\alpha} w_2(1) + \mathbb{I}_q^{\alpha} w_2(p)\right] \\
& + a_1(t)\int_0^1 f_1(s,u(s))\,\mathrm{d}s + a_2(t)\left[\mathbb{I}_q^{\alpha-\beta} w_2(1) + \mathbb{I}_q^{\alpha-\beta} w_2(p)\right] \\
& + (b_1 + a_3(t))\int_0^1 f_2(s,u(s))\,\mathrm{d}s \\
& + (b_2 + a_4(t))\left[\mathbb{I}_q^{\alpha-\gamma} w_2(1) + \mathbb{I}_q^{\alpha-\gamma} w_2(p)\right],
\end{aligned}
$$

for all $t \in J$. Thus,

$$
\begin{aligned}
|h_1(t) - h_2(t)| \le {} & \mathbb{I}_q^{\alpha}|w_1(t) - w_2(t)| \\
& + \frac{1}{3}\int_0^1 |f_0(s,v(s)) - f_0(s,u(s))|\,\mathrm{d}s \\
& + \frac{1}{3}\left[\mathbb{I}_q^{\alpha}|w_1(1) - w_2(1)|\mathbb{I}_q^{\alpha-1}|w_1(p) - w_2(p)|\,\mathrm{d}s\right] \\
& + |a_1(t)|\int_0^1 |f_1(s,v(s)) - f_1(s,u(s))|\,\mathrm{d}s \\
& + |a_2(t)|\left[\mathbb{I}_q^{\alpha-\beta}|w_1(1) - w_2(1)|\right. \\
& \left. + \mathbb{I}_q^{\alpha-\beta}|w_1(p) - w_2(p)|\right] \\
& + |b_1 + a_3(t)|\int_0^1 |f_2(s,v(s)) - f_2(s,u(s))|\,\mathrm{d}s \\
& + |b_2 + a_4(t)|\left[\mathbb{I}_q^{\alpha-\gamma}|w_1(1) - w_2(1)|\,\mathrm{d}s\right. \\
& \left. + \mathbb{I}_q^{\alpha-\gamma}|w_1(p) - w_2(p)|\,\mathrm{d}s\right] \\
\le {} & \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3}\theta\left(\|u - v\|\right)\left[\frac{\|m\|_\infty}{\Gamma_q(\alpha+1)} + \frac{\|m_0\|_\infty}{3}\right. \\
& + \frac{2\|m\|_\infty}{3\Gamma_q(\alpha+1)} + \frac{5\Gamma_q(2-\beta)\|m_1\|_\infty}{3} \\
& + \frac{10\Gamma_q(2-\beta)\|m\|_\infty}{3\Gamma_q(\alpha-\beta+1)} \\
& + 10\big(2\Gamma_q(2-\beta) + \Gamma_q(3-\beta)\big) \\
& \left. \times\left(\frac{\Gamma_q(3-\gamma)(\|m_2\|_\infty\Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty)}{3\Gamma_q(3-\beta)\Gamma_q(\alpha-\gamma+1)}\right)\right] \\
= {} & \frac{\Lambda_1}{\Lambda_1 + \Lambda_2 + \Lambda_3}\psi(\|u - v\|),
\end{aligned}
$$

$$
\begin{aligned}
|h_1'(t) - h_2'(t)| \le {} & \mathbb{I}_q^{\alpha-1}|w_1(t) - w_2(t)| \\
& + \frac{\Gamma(2-\beta)}{(p^{1-\beta}+1)}\left[\mathbb{I}_q^{\alpha-\beta}|w_1(1) - w_2(1)|\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{I}_q^{\alpha-\beta}|w_1(p)-w_2(p)|\big] \\
&+|a_5(t)|\int_0^1|f_2(s,v(s))-f_2(s,u(s))|\,\mathrm{d}s \\
&+|a_6(t)|\left[\mathbb{I}_q^{\alpha-\gamma}|w_1(1)-w_2(1)|\right. \\
&\left.+\mathbb{I}_q^{\alpha-\gamma-1}|w_1(p)-w_2(p)|\right] \\
\leq{}& \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}\theta(\|u-v\|)\left[\frac{\|m\|_\infty}{\Gamma_q(\alpha)}+\frac{2\Gamma_q(2-\beta)\|m\|_\infty}{\Gamma_q(\alpha-\beta+1)}\right. \\
&+(2\Gamma_q(2-\beta)+\Gamma_q(3-\beta)) \\
&\left.\times\left(\frac{\Gamma_q(3-\gamma)\left(\|m_2\|_\infty\Gamma_q(\alpha-\gamma+1)+2\|m\|_\infty\right)}{\Gamma_q(3-\beta)\Gamma_q(\alpha-\gamma+1)}\right)\right] \\
={}& \frac{\Lambda_2}{\Lambda_1+\Lambda_2+\Lambda_3}\theta(\|u-v\|),
\end{aligned}
$$

and

$$
\begin{aligned}
|h_1''(t)-h_2''(t)| \leq{}& \mathbb{I}_q^{\alpha-2}|w_1(t)-w_2(t)| \\
&+\frac{\Gamma_q(3-\gamma)}{(p^{2-\gamma}+1)}\int_0^1|f_2(s,v(s))-f_2(s,u(s))|\,\mathrm{d}s \\
&+\frac{\Gamma_q(3-\gamma)}{(p^{2-\gamma}+1)}\left[\mathbb{I}_q^{\alpha-\gamma}|w_1(1)-w_2(1)|\right. \\
&\left.+\mathbb{I}_q^{\alpha-\gamma}|w_1(p)-w_2(p)|\right] \\
\leq{}& \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}\psi(\|u-v\|)\left[\frac{\|m\|_\infty}{\Gamma_q(\alpha-1)}\right. \\
&\left.+\frac{\Gamma_q(3-\gamma)\left(\|m_2\|_\infty\Gamma_q(\alpha-\gamma+1)+2\|m\|_\infty\right)}{\Gamma_q(\alpha-\gamma+1)}\right] \\
={}& \frac{\Lambda_3}{\Lambda_1+\Lambda_2+\Lambda_3}\psi(\|u-v\|),
\end{aligned}
$$

where

$$
\begin{aligned}
a_5(t) ={}& \frac{(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} \\
&+\frac{\Gamma_q(3-\gamma)(p^{1-\beta}+1)\Gamma_q(3-\beta)t}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}, \\
a_6(t) ={}& \frac{(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)\Gamma_q(\alpha-\gamma)} \\
&-\frac{\Gamma_q(3-\gamma)\Gamma_q(3-\beta)(p^{1-\beta}+1)t}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)\Gamma_q(\alpha-\gamma)}.
\end{aligned}
$$

Hence,

$$\begin{aligned}\|h_1 - h_2\| &= \sup_{t\in J} |h_1(t) - h_2(t)| + \sup_{t\in J} |h_1'(t) - h_2'(t)| \\ &\quad + \sup_{t\in J} |h_1''(t) - h_2''(t)| \\ &\le \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3}\theta(\|x-y\|)(\Lambda_1 + \Lambda_2 + \Lambda_3) = \theta(\|x-y\|).\end{aligned}$$

Therefore, it is easy to get that

$$H_d(N(u), N(v)) \le \theta(\|u - v\|),$$

for all $u, v \in \mathcal{X}$. On the other hand, the multifunction N has the approximate endpoint property. By using Lemma 1, there exists $u^* \in \mathcal{X}$ such that $N(u^*) = \{u^*\}$. Thus, by employing Lemma 2, u^* is a solution of problem (1) and (2). □

At present, we investigate the existence of a solution for the fractional q-differential inclusion problem with integral boundary value conditions

$$\begin{cases} {}^c\mathbb{D}_q^{\alpha} u(t) \in \mathcal{W}\left(t, u(t), {}^c\mathbb{D}_q^{\gamma_1} u(t), \dots, {}^c\mathbb{D}_q^{\gamma_n} u(t)\right), \\ u'(0) + a_1 u'(1) = \sum_{i=1}^{n} {}^c\mathbb{D}_q^{\gamma_i} u(p), \\ u(0) + a_2 u(1) = \sum_{i=1}^{n} \mathbb{I}_q^{\gamma_i} u(p), \end{cases} \tag{11}$$

for the multifunction $\mathcal{W} : J \times \mathbb{R}^{n+1} \to P(\mathbb{R})$, where $t \in J$, $\alpha \in (1, 2]$, $n \ge 2$, $0 < q, p, \gamma_i < 1$, $\alpha - \gamma_i \ge 1$ for all $1 \le i \le n$, and

$$a_1 > \sum_{i=1}^{n} \frac{p^{1-\gamma_i}}{\Gamma_q(2 - \gamma_i)}, \quad a_2 > \sum_{i=1}^{n} \frac{p^{\gamma_i + 1}}{\Gamma_q(\gamma_i + 2)}.$$

4 Examples and algorithms for the problems

In this part, we give a complete computational technique for solving problems (1) and (2), such that it covers all the problems and presents numerical examples solving perfect. To this aim, we consider a pseudo-code description of the method for the calculated q-Gamma function of order n in Algorithm 2 (for more details, see the link https://en.wikipedia.org/wiki/Q-gamma_function).

Table 1 shows that when q is constant, the q-Gamma function is an increasing function. Also, for smaller values of x, an approximate result is obtained with a fewer values of n. It has been shown by underlined rows. Table 2 shows that the q-Gamma function for values q near one is obtained with more values of n in comparison with other columns. They have been underlined in line 8 of the first column, line 17 of the second column, and line 29 of the third column of Table 2. Also, Table 3 is the same as Table 2, but x values increase in Table 3. Similarly, the q-Gamma function for values q near to one is obtained with more values of n in comparison with other columns. Furthermore, we provide Algorithm 3 that calculates $(\mathbb{D}_q^{\alpha} f)(x)$. Here, we give an example to illustrate our first main result, which applies to the different values q in Theorem 1.

Example 1. Consider the fractional q-differential inclusion problem

$$\begin{aligned} {}^{c}\mathbb{D}_q^{\frac{9}{4}} u(t) \in \Bigg[0, \frac{t^2}{100} \sin u(t) + \frac{1}{100} \cos u'(t) \\ + \frac{1}{100} \left(\frac{|u''(t)|}{1 + |u''(t)|} \right) \Bigg], \end{aligned} \tag{12}$$

under the integral boundary conditions

$$u(0) + u(\frac{3}{4}) + u(1) = \int_0^1 \frac{s^2}{20} \cos u(s)\, \mathrm{d}s$$

and

$$\begin{cases} {}^{c}\mathbb{D}_q^{\frac{2}{3}} u(0) + {}^{c}\mathbb{D}_q^{\frac{2}{3}} u(\frac{3}{4}) + {}^{c}\mathbb{D}_q^{\frac{2}{3}} u(1) = \int_0^1 \frac{e^{s^2-1}}{20} \cos u(s)\, \mathrm{d}s, \\ {}^{c}\mathbb{D}_q^{\frac{5}{3}} u(0) + {}^{c}\mathbb{D}_q^{\frac{5}{3}} u(\frac{3}{4}) + {}^{c}\mathbb{D}_q^{\frac{5}{3}} u(1) = \int_0^1 \frac{2s^3+1}{20\pi} \cos u(s)\, \mathrm{d}s, \end{cases} \tag{13}$$

where $t \in J = [0, 1]$, $\alpha = \frac{9}{4}$, $\beta = \frac{2}{3}$, $\gamma = \frac{5}{3}$, and $p = \frac{3}{4}$ in (1) and (2). Consider the map $\mathcal{W} : J \times \mathbb{R}^3 \to P(\mathbb{R})$ defined by

$$\mathcal{W}(t, x_1, x_2, x_3) = \left[0, \frac{t^2}{100} \sin x_1 + \frac{1}{100} \cos x_2 + \frac{1}{100} \left(\frac{|x_3|}{1 + |x_3|} \right) \right].$$

Also, $f_i : J \times \mathbb{R} \to \mathbb{R}$ are define by

$$f_0(t, x) = \frac{t^2}{20} \cos x, \quad f_1(t, x) = \frac{e^{t^2-1}}{20} \cos x, \quad f_2(t, x) = \frac{2t^3 + 1}{300\pi} \cos x,$$

and $N : C^2(J) \to 2^{C^2(J)}$ is defined by

$$N(u) = \Big\{ h \in C^2(J) \,\Big|\, \exists v \in S_{\mathcal{W},u} : h(t) = w(t) \Big\},$$

for all $t \in J$ such that

$$\begin{aligned} w(t) = \mathbb{I}_q^{\frac{9}{4}} v(t) + \frac{1}{3} \int_0^1 \frac{s^2}{20} \cos u(s)\, \mathrm{d}s - \frac{1}{3} \left[\mathbb{I}_q^{\frac{9}{4}} v(1) + \mathbb{I}_q^{\frac{9}{4}} v(\frac{3}{4}) \right] \\ + a_1(t) \int_0^1 \frac{e^{s^2-1}}{20} \cos u(s)\, \mathrm{d}s + a_2(t) \left[\mathbb{I}_q^{\frac{19}{12}} v(1) + \mathbb{I}_q^{\frac{19}{12}} v(\frac{3}{4}) \right] \\ + (b_1 + a_3(t)) \int_0^1 \frac{2s^3 + 1}{300\pi} \cos u(s)\, \mathrm{d}s \end{aligned}$$

$$+ (b_2 + a_4(t)) \left[\mathbb{I}_q^{\frac{7}{12}} v(1) + \mathbb{I}_q^{\frac{7}{12}} v(\frac{3}{4}) \right],$$

where

$$\begin{aligned}
a_1(t) &= \frac{3\Gamma_q(\frac{4}{3})t - \frac{7}{4}\Gamma_q(\frac{4}{3})}{3((\frac{3}{4})^{\frac{1}{3}} + 1)}, \\
a_2(t) &= \frac{\frac{7}{4}\Gamma_q(\frac{4}{3}) - 3\Gamma_q(\frac{4}{3})t}{3((\frac{3}{4})^{\frac{1}{3}} + 1)}, \\
a_3(t) &= \frac{-6((\frac{3}{4})^{\frac{4}{3}} + 1)\Gamma_q(\frac{4}{3})\Gamma_q(\frac{4}{3})t + 3((\frac{3}{4})^{\frac{1}{3}} + 1)\Gamma_q(\frac{4}{3})\Gamma_q(\frac{7}{3})t^2}{6((\frac{3}{4})^{\frac{1}{3}} + 1)((\frac{3}{4})^{\frac{1}{3}} + 1)\Gamma_q(\frac{7}{3})}, \\
a_4(t) &= \frac{6((\frac{3}{4})^{\frac{4}{3}} + 1)\Gamma_q(\frac{4}{3})\Gamma_q(\frac{4}{3})t - 3\Gamma_q(\frac{4}{3})\Gamma_q(\frac{7}{3})((\frac{3}{4})^{\frac{1}{3}} + 1)t^2}{6((\frac{1}{3})^{\frac{1}{3}} + 1)((\frac{3}{4})^{\frac{1}{3}} + 1)\Gamma_q(\frac{7}{3})}, \\
b_1 &= \frac{\frac{7}{2}((\frac{3}{4})^{\frac{4}{3}} + 1)\Gamma_q(\frac{4}{3})\Gamma_q(\frac{4}{3}) - ((\frac{3}{4})^2 + 1)\Gamma_q(\frac{4}{3})((\frac{3}{4})^{\frac{1}{3}} + 1)\Gamma_q(\frac{7}{3})}{6((\frac{3}{4})^{\frac{1}{3}} + 1)((\frac{3}{4})^{\frac{1}{3}} + 1)\Gamma_q(\frac{7}{3})}, \\
b_2 &= \frac{((\frac{3}{4})^2 + 1)\Gamma_q(\frac{4}{3})((\frac{3}{4})^{\frac{1}{3}} + 1)\Gamma_q(\frac{7}{3}) - \frac{8}{3}((\frac{3}{4})^{\frac{4}{3}} + 1)\Gamma_q(\frac{4}{3})\Gamma_q(\frac{4}{3})}{6((\frac{3}{4})^{\frac{1}{3}} + 1)((\frac{3}{4})^{\frac{1}{3}} + 1)\Gamma_q(\frac{7}{3})}.
\end{aligned} \tag{14}$$

Put $m(t) = \frac{3t}{20}$, $m_0(t) = \frac{t^2}{20}$, $m_1(t) = \frac{e^{t^2-1}}{20}$, $m_2(t) = \frac{2t^3+1}{300\pi}$, and $\psi(t) = \frac{t}{5}$. Then, we have

$$\begin{aligned}
\Lambda_1 &= \left[\frac{\|m\|_\infty}{\Gamma_q(\frac{13}{4})} + \frac{\|m_0\|_\infty}{3} + \frac{2\|m\|_\infty}{3\Gamma_q(\frac{13}{4})} + \frac{5\Gamma_q(\frac{4}{3})\|m_1\|_\infty}{3} + \frac{10\Gamma_q(\frac{4}{3})\|m\|_\infty}{3\Gamma_q(\frac{31}{12})} \cdot \right. \\
&\quad \left. + \frac{10(2\Gamma_q(\frac{4}{3}) + \Gamma_q(\frac{7}{3}))\Gamma_q(\frac{4}{3})\left(\|m_2\|_\infty \Gamma_q(\frac{19}{12}) + 2\|m\|_\infty\right)}{3\Gamma_q(\frac{7}{3})\Gamma_q(\frac{19}{12})} \right], \\
\Lambda_2 &= \left[\frac{\|m\|_\infty}{\Gamma_q(\frac{9}{4})} + \frac{2\Gamma_q(\frac{4}{3})\|m\|_\infty}{\Gamma_q(\frac{31}{12})} \right. \\
&\quad \left. + \frac{(2\Gamma_q(\frac{4}{3}) + \Gamma_q(\frac{7}{3}))\Gamma_q(\frac{4}{3})\left(\|m_2\|_\infty \Gamma_q(\frac{19}{12}) + 2\|m\|_\infty\right)}{\Gamma(\frac{7}{3})\Gamma_q(\frac{19}{12})} \right], \\
\Lambda_3 &= \left[\frac{\|m\|_\infty}{\Gamma_q(\frac{5}{4})} + \frac{\Gamma_q(\frac{4}{3})\left(\|m_2\|_\infty \Gamma_q(\frac{19}{12}) + 2\|m\|_\infty\right)}{\Gamma_q(\frac{19}{12})} \right].
\end{aligned}$$

In following data of Tables 4 and 5, it is easy to check that

$$H_d(\mathcal{W}(t, u_1, u_2, u_3), F(t, v_1, v_2, v_3)) \leq \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3} m(t)\theta(\sum_{k=1}^{3} |u_k - v_k|)$$

and that

$$|f_j(t,u) - f_j(t,v)| \leq \frac{1}{\Lambda_1 + \Lambda_2 + \Lambda_3} m_j(t)\psi(|u-v|),$$

for $t \in J$, $j = 0, 1, 2$. Because $\sup_{u \in N(0)} \|u\| = 0$, we have

$$\inf_{u \in C^2(J)} \left[\sup_{v \in N(u)} \|u - v\| \right] = 0.$$

Thus, N has the approximate endpoint property. At present, by applying Theorem 1, the system of fractional q-differential inclusions (12) and (13) has at least one solution.

5 Conclusion

The q-differential boundary equations and their applications represent a matter of high interest in the area of fractional q-calculus and its applications in various areas of science and technology. q-differential boundary value problems occur in the mathematical modeling of a variety of physical operations. The end of this article was to investigate a complicated case by utilizing an appropriate basic theory. In this manner, we proved the existence of a solution for familiar problems of q-differential equations under three boundary conditions (1)–(2) and (3)–(4) on a time scale and showed the perfect numerical effects for the problem, which confirm our results.

Acknowledgements

Authors are grateful to the anonymous referees and editor for their constructive comments. The authors were supported by Bu-Ali Sina University.

References

1. Adams, C.R. *The general theory of a class of linear partial q- difference equations*, Trans. Am. Math. Soc. 26 (1924), 283–312.

2. Adams, C.R. *Note on the integro-q-difference equations*, Trans. Am. Math. Soc. 31 (4) (1929), 861–867.

3. Agarwal, R.P. *Certain fractional q-integrals and q-derivatives*, Proceedings of the Cambridge Philosophical Society 66 (1969), 365–370.

4. Agarwal, R.P., Belmekki, M. and Benchohra, M. *A survey on semi- linear differential equations and inclusions involving Riemann-Liouville fractional derivative*, Adv. Differ. Equ. 2009, 981728 (2009).

5. Ahmad, B. and Nieto, J.J. *Riemann-Liouville fractional integro- differential equations with fractional nonlocal integral boundary conditions*, Bound. Value Probl. 2011, 2011:36, 9 pp.

6. Ahmad B. and Ntouyas, S.K. *Existence of solutions for nonlinear fractional q-difference inclusions with nonlocal robin (separated) conditions*, Mediterr. J. Math. 10 (2013), 1333–1351.

7. Ahmad B. and Ntouyas, S.K. *Boundary value problem for fractional differential inclusions with four-point integral boundary conditions*, Surv. Math. Appl. 6 (2011), 175–193.

8. Amini-Harandi, A. *Endpoints of set-valued contractions in metric spaces*, Nonlinear Anal. 72 (2010), 132–134.

9. Anastassiou, G.A. *Principles of delta fractional calculus on time scales and inequalities*, Math. Comput. Modelling 52 (2010), 556–566.

10. Annaby, M.H. and Mansour, Z.S. *q-fractional calculus and equations*, With a foreword by Mourad Ismail. Lecture Notes in Mathematics, 2056. Springer, Heidelberg, 2012.

11. Atici, F. and Eloe, P.W. *Fractional q-calculus on a time scale*, J. Nonlinear Math. Phys. 14 (2007), 333–344.

12. Aubin, J. and Ceuina, A. *Differential inclusions: set-valued maps and viability theory*, Springer-Verlag, 1984.

13. Baleanu, D., Agarwal, R. P., Mohammadi, H. and Rezapour, S. *Some existence results for a nonlinear fractional differential equation on partially ordered banach spaces*, Bound. Value Probl. 2013 (2013), 112 pp.

14. Bohner M. and Peteson, A. *Dynamic equations on time scales*, Birkhauser, Boston, 2001.

15. Carmichael, R.D. *The general theory of linear q-difference equations*, Amer. J. Math. 34 (1912), 147–168.

16. Deimling, K. *Multi-valued differential equations*, Walter de Gruyter, Berlin, 1992.

17. Ferreira, R.A.C. *Nontrivials solutions for fractional q-difference boundary value problems*, Electron. J. Qual. Theory Differ. Equ. 2010, No. 70, 10 pp.

18. Gasper, G.and Rahman, M. *Basi hypergeometric series*, University Press, Cambridge, 1990.

19. Granas, A. and Dugundji, J. *Fixed point theory*, Springer-Verlag, 2005.

20. Hedayati, V. and Samei, M.E. *Positive solutions of fractional differential equation with two pieces in chain interval and simultaneous Dirichlet boundary conditions*, Bound. Value Probl. 2019 (2019), 163 pp.

21. Hilfer, R. *Applications of fractional calculus in physics*, World Scientific, 2000.

22. Jackson, F.H. *On q-functions and a certain difference operator*, Transactions of the Royal Society of Edinburgh 46 (1909), 253–281.

23. Jackson, F.H. *On q-definite integrals*, Pure Appl. Math. Q. 41 (1910), 193–203.

24. Jackson, F.H. *q-difference equations*, Am. J. Math. 32 (1910), 305–314.

25. Kac, V. and Cheung, P. *Quantum calculus*, Universitext, Springer, New York, 2002.

26. Kilbas, A.A., Srivastava, H. M. and Trujillo, J.J. *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier Science, North-Holland, 2006.

27. Kisielewicz, M. *Differential inclusions and optimal control*, Kluwer, Dordrecht, 1991.

28. Mason, T.E. *On properties of the solution of linear q-difference equations with entire function coefficients*, Am. J. Math. 37 (1915), 439–444.

29. Ntouyas, S.K. and Samei, M.E. *Existence and uniqueness of solutions for multi-term fractional q-integro-differential equations via quantum calculus* Adv. Difference Equ. 2019, Paper No. 475, 20 pp.

30. Phung, P. D. and Truong, L. X. *On a fractional differential inclusion with integral boundary conditions in Banach space*, Fract. Calc. Appl. Anal. 16 (2013), 538–558.

31. Podlubny, I. *Fractional differential equations*, Academic Press, 1999.

32. Rajković, P. M., Marinković, S. D. and Stanković, M. S. *Fractional integrals and derivatives in q-calculus*,Appl. Anal. Discrete Math. 1 (2007), 311–323.

33. Rezapour, Sh. and Hedayati, V. *On a Caputo fractional differential inclusion with integral boundary condition for convex-compact and nonconvex- compact valued multifunctions*,Kragujev. J. Math. 41 (1) (2017), 143–158.

34. Samei, M.E. *Existence of solutions for a system of singular sum fractional q-differential equations via quantum calculus*, Adv. Differ. Equ. 2020 (2020), 23.

35. Samei, M.E., Hedayati, V. and Rezapour, Sh. *Existence results for a fraction hybrid differential inclusion with Caputo–Hadamard type fractional derivative*, Adv. Difference Equ. 2019, Paper No. 163, 15 pp.

36. Samei, M.E. and Khalilzadeh Ranjbar, G. *Some theorems of existence of solutions for fractional hybrid q-difference inclusion*, J. Adv. Math. Stud. 12 (2019), 63–76.

37. Samei, M.E., Ranjbar,G.K. and Hedayati, V. *Existence of solutions for equations and inclusions of multi-term fractional q-integro-differential with non-separated and initial boundary conditions*, J. Inequal. Appl. 2019 (2019), 273.

38. Samko, S.G. ,Kilbas, A.A. and Marichev, O.I. *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach Science Publishers, Switzerland; Philadelphia, Pa., USA, 1993.

39. Trjitzinsky, W.J. *Analytic theory of linear q-difference equations,* Acta Math. 61 (1933), 1–38.

40. Wang, J. and Ibrahim, A.G. *Existence and controllability results for nonlocal fractional impulsive differential inclusions in Banach spaces,* Journal of Function Spaces and Applications, 2013, Article ID 518306, 16 pp.

Supporting informations

Table 1: Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}$ that is constant, $x = 4.5$, 8.4, 12.7 and $n = 1, 2, \ldots, 15$ of Algorithm 2.

n	$x = 4.5$	$x = 8.4$	$x = 12.7$	n	$x = 4.5$	$x = 8.4$	$x = 12.7$
1	2.472950	11.909360	68.080769	9	<u>2.340263</u>	11.257158	64.351366
2	2.383247	11.468397	65.559266	10	2.340250	<u>11.257095</u>	64.351003
3	2.354446	11.326853	64.749894	11	2.340245	11.257074	<u>64.350881</u>
4	2.344963	11.280255	64.483434	12	2.340244	11.257066	64.350841
5	2.341815	11.264786	64.394980	13	2.340243	11.257064	64.350828
6	2.340767	11.259636	64.365536	14	2.340243	11.257063	64.350823
7	2.340418	11.257921	64.355725	15	2.340243	11.257063	64.350822
8	2.340301	11.257349	64.352456				

Table 2: Some numerical results for calculation of $\Gamma_q(x)$ with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $x = 5$ and $n = 1, 2, \ldots, 35$ of Algorithm 2.

n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$	n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$
1	3.016535	6.291859	18.937427	18	2.853224	4.921884	8.476643
2	2.906140	5.548726	14.154784	19	2.853224	4.921879	8.474597
3	2.870699	5.222330	11.819974	20	2.853224	4.921877	8.473234
4	2.859031	5.069033	10.537540	21	2.853224	4.921876	8.472325
5	2.855157	4.994707	9.782069	22	2.853224	4.921876	8.471719
6	2.853868	4.958107	9.317265	23	2.853224	4.921875	8.471315
7	2.853438	4.939945	9.023265	24	2.853224	4.921875	8.471046
8	2.853295	4.930899	8.833940	25	2.853224	4.921875	8.470866
9	2.853247	4.926384	8.710584	26	2.853224	4.921875	8.470747
10	2.853232	4.924129	8.629588	27	2.853224	4.921875	8.470667
11	2.853226	4.923002	8.576133	28	2.853224	4.921875	8.470614
12	2.853224	4.922438	8.540736	29	2.853224	4.921875	8.470578
13	2.853224	4.922157	8.517243	30	2.853224	4.921875	8.470555
14	2.853224	4.922016	8.501627	31	2.853224	4.921875	8.470539
15	2.853224	4.921945	8.491237	32	2.853224	4.921875	8.470529
16	2.853224	4.921910	8.484320	33	2.853224	4.921875	8.470522
17	2.853224	4.921893	8.479713	34	2.853224	4.921875	8.470517

Table 3: Some numerical results for calculation of $\Gamma_q(x)$ with $x = 8.4$, $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \ldots, 40$ of Algorithm 2.

n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$	n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$
1	11.909360	63.618604	664.767669	21	11.257063	49.065390	260.033372
2	11.468397	55.707508	474.800503	22	11.257063	49.065384	260.011354
3	11.326853	52.245122	384.795341	23	11.257063	49.065381	259.996678
4	11.280255	50.621828	336.326796	24	11.257063	49.065380	259.986893
5	11.264786	49.835472	308.146441	25	11.257063	49.065379	259.980371
6	11.259636	49.448420	290.958806	26	11.257063	49.065379	259.976023
7	11.257921	49.256401	280.150029	27	11.257063	49.065379	259.973124
8	11.257349	49.160766	273.216364	28	11.257063	49.065378	259.971192
9	11.257158	49.113041	268.710272	29	11.257063	49.065378	259.969903
10	11.257095	49.089202	265.756606	30	11.257063	49.065378	259.969044
11	11.257074	49.077288	263.809514	31	11.257063	49.065378	259.968472
12	11.257066	49.071333	262.521127	32	11.257063	49.065378	259.968090
13	11.257064	49.068355	261.666471	33	11.257063	49.065378	259.967836
14	11.257063	49.066867	261.098587	34	11.257063	49.065378	259.967666
15	11.257063	49.066123	260.720833	35	11.257063	49.065378	259.967553
16	11.257063	49.065751	260.469369	36	11.257063	49.065378	259.967478
17	11.257063	49.065564	260.301890	37	11.257063	49.065378	259.967427
18	11.257063	49.065471	260.190310	38	11.257063	49.065378	259.967394
19	11.257063	49.065425	260.115957	39	11.257063	49.065378	259.967371
20	11.257063	49.065402	260.066402	40	11.257063	49.065378	259.967357

Table 4: Some numerical results of $\Gamma_q(\alpha)$ in Example 1 with different values of q by Algorithm 2.

n	$\alpha=\frac{3}{4}$	$\alpha=\frac{5}{4}$	$\alpha=\frac{4}{3}$	$\alpha=\frac{19}{12}$	$\alpha=\frac{9}{4}$	$\alpha=\frac{7}{3}$	$\alpha=\frac{31}{12}$	$\alpha=\frac{13}{4}$
	$q=\frac{1}{3}$							
1	1.1174	0.9592	0.9559	0.9673	1.1055	1.1315	1.2199	1.5327
2	1.1311	0.9505	0.9448	0.9500	1.0747	1.0990	1.1824	1.4805
3	1.1356	0.9476	0.9411	0.9444	1.0647	1.0886	1.1703	1.4638
4	1.1371	0.9467	0.9400	0.9425	1.0615	1.0851	1.1664	1.4583
5	1.1376	0.9464	0.9396	0.9419	1.0604	1.0840	1.1651	1.4564
6	1.1377	0.9463	0.9394	0.9417	1.0600	1.0836	1.1646	1.4558
7	1.1378	0.9462	0.9394	0.9417	1.0599	1.0835	1.1645	1.4556
8	1.1378	0.9462	0.9394	0.9416	1.0599	1.0834	1.1644	1.4556
9	1.1378	0.9462	0.9394	0.9416	1.0598	1.0834	1.1644	1.4555
10	1.1378	0.9462	0.9394	0.9416	1.0598	1.0834	1.1644	1.4555
	$q=\frac{1}{2}$							
1	1.1069	0.9743	0.9772	1.0122	1.2620	1.3087	1.4715	2.1039
2	1.1377	0.9526	0.9493	0.9662	1.1655	1.2049	1.3437	1.8906
3	1.1522	0.9426	0.9364	0.9453	1.1221	1.1583	1.2865	1.7960
4	1.1593	0.9378	0.9302	0.9353	1.1015	1.1362	1.2594	1.7514
5	1.1628	0.9355	0.9272	0.9303	1.0915	1.1254	1.2463	1.7297
6	1.1645	0.9343	0.9257	0.9279	1.0865	1.1201	1.2398	1.7190
7	1.1654	0.9337	0.9249	0.9267	1.0841	1.1175	1.2365	1.7137
8	1.1658	0.9334	0.9245	0.9261	1.0828	1.1161	1.2349	1.7111
9	1.1660	0.9333	0.9244	0.9258	1.0822	1.1155	1.2341	1.7098
10	1.1662	0.9332	0.9243	0.9257	1.0819	1.1152	1.2337	1.7091
	$q=\frac{4}{5}$							
1	0.9665	1.1206	1.1787	1.4118	2.6441	2.8906	3.8168	8.5184
2	1.0284	1.0602	1.0963	1.2516	2.1063	2.2761	2.9085	6.0237
3	1.0710	1.0218	1.0443	1.1539	1.8020	1.9312	2.4107	4.7288
4	1.1018	0.9954	1.0091	1.0891	1.6109	1.7160	2.1053	3.9658
5	1.1248	0.9766	0.9840	1.0438	1.4826	1.5723	1.9040	3.4780
6	1.1421	0.9628	0.9657	1.0111	1.3927	1.4718	1.7646	3.1483
7	1.1555	0.9523	0.9519	0.9868	1.3275	1.3992	1.6647	2.9162
8	1.1659	0.9444	0.9414	0.9685	1.2793	1.3455	1.5913	2.7481
9	1.1740	0.9383	0.9334	0.9545	1.2429	1.3051	1.5363	2.6235
10	1.1803	0.9335	0.9271	0.9436	1.2151	1.2743	1.4945	2.5297

Table 5: Some numerical results of Λ_1, Λ_2, and Λ_3 in Example 1 with different values of q.

n	Λ_1	Λ_2	Λ_3	$\frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}$
		$q=\frac{1}{3}$		
1	3.3364	1.1763	0.4559	0.2013
2	3.3955	1.1988	0.4592	0.1979
3	3.4147	1.2061	0.4602	0.1968
4	3.4219	1.2088	0.4606	0.1964
5	3.4240	1.2096	0.4608	0.1963
6	3.4244	1.2098	0.4608	0.1963
7	3.4246	1.2099	0.4608	0.1963
8	3.4251	1.2101	0.4608	0.1962
9	3.4251	1.2101	0.4608	0.1962
10	3.4251	1.2101	0.4608	0.1962
		$q=\frac{1}{2}$		
1	2.9820	1.0480	0.4467	0.2234
2	3.1379	1.1076	0.4552	0.2127
3	3.2160	1.1375	0.4593	0.2078
4	3.2553	1.1525	0.4613	0.2054
5	3.2754	1.1601	0.4623	0.2042
6	3.2852	1.1639	0.4628	0.2036
7	3.2898	1.1656	0.4630	0.2033
8	3.2923	1.1666	0.4631	0.2032
9	3.2939	1.1671	0.4632	0.2031
10	3.2943	1.1673	0.4632	0.2031
		$q=\frac{4}{5}$		
1	1.8371	0.6109	0.3881	0.3526
2	2.0805	0.7071	0.4077	0.3130
3	2.2800	0.7853	0.4216	0.2868
4	2.4430	0.8488	0.4319	0.2686
5	2.5751	0.9001	0.4395	0.2554
6	2.6823	0.9415	0.4454	0.2457
7	2.7686	0.9748	0.4499	0.2385
8	2.8380	1.0016	0.4534	0.2329
9	2.8943	1.0232	0.4562	0.2286
10	2.9392	1.0404	0.4584	0.2253

Algorithm 1 The proposed method for calculated $(a-b)_q^{(\alpha)}$

Input: a, b, α, n, q
$s \leftarrow 1$
if $n = 0$ **then**
  $p \leftarrow 1$
else
  for $k = 0$ to n **do**
    $s \leftarrow s * (a - b * a^k)/(a - b * q^{\alpha+k})$
  end for
  $p \leftarrow a^\alpha * s$
end if
Output: $(a-b)^{(\alpha)}$

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

Input: n, $q \in (0,1)$, $x \in \mathbb{R} \backslash \{0, -1, 2, \ldots\}$
$p \leftarrow 1$
for $k = 0$ to n **do**
  $p \leftarrow p(1 - q^{k+1})(1 - q^{x+k})$
end for
$\Gamma_q(x) \leftarrow p/(1-q)^{x-1}$
Output: $\Gamma_q(x)$

Algorithm 3 The proposed method for calculated $(\mathbb{D}_q f)(x)$

Input: $q \in (0,1)$, $f(x)$, x
syms z
if $x = 0$ **then**
  $g \leftarrow \lim((f(z) - f(q * z))/((1-q)z), z, 0)$
else
  $g \leftarrow (f(x) - f(q * x))/((1-q)x)$
end if
Output: $(\mathbb{D}_q f)(x)$

Algorithm 4 The proposed method for calculated $(\mathbb{I}_q^\alpha f)(x)$

Input: $q \in (0,1)$, α, n, $f(x)$, x
$s \leftarrow 0$
for $i = 0$ to n **do**
  $pf \leftarrow (1 - q^{i+1})^{\alpha-1}$
  $s \leftarrow s + pf * q^i * f(x * q^i)$
end for
$g \leftarrow (x^\alpha * (1-q) * s)/(\Gamma_q(x))$
Output: $(\mathbb{I}_q^\alpha f)(x)$