Iranian Journal of Numerical Analysis and Optimization
Vol. 12, No. 1, (2022), pp 73-109
DOI:10.22067/ijnao.2021.70478.1036
https://ijnao.um.ac.ir/

How to cite this article
Research Article

Numerical approximation for inverse problem of the Ostrovsky–Burgers equation

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Abstract

This article considers a nonlinear inverse problem of the Ostrovsky–Burgers equation by using noisy data. Two B-Splines with different levels, the quintic B-spline and septic B-spline, are used to study this problem. For both B-splines, the stability and convergence analysis are calculated, and results show that an excellent estimation of the unknown functions of the nonlinear inverse problem.

AMS subject classifications (2020): 35Q53; 35B35, 68W25, 35R30, 65M70.

Keywords: Ostrovsky equation; Quintic B-spline collocation; Septic B-spline collocation; Convergence analysis; Stability analysis; Noisy data.

1 Introduction

In this paper, we consider the Ostrovsky–Burgers equation

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Received 18 May 2021; revised 3 August 2021; accepted 8 August 2021
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$$(u_t + bu_{xxx} + uu_x - au_{xx})_x = \gamma u + f(x,t). \tag{1}$$

Equation (1) was appeared in modeling internal waves in the ocean or surface waves in a shallow channel with an uneven bottom under the effects of the interfacial friction (see [11, Chapter 1] and [16, 22]). In (1), the positive constants γ and a are the rotation and friction coefficients, respectively. The function f denotes the external force and b is the dispersion coefficient, which its sign is related to the type of dispersion. Ignoring the dissipation term u_{xx}, (1) leads to the Ostrovsky equation

$$(u_t + bu_{xxx} + uu_x)_x = \gamma u, \tag{2}$$

which was derived by Ostrovsky in 1978 [18] to model weakly nonlinear surface and internal waves in a rotating ocean; see also [9, 10]. It was also demonstrated in [17] that the nonlinear oblique magneto-acoustic waves in a rotating plasma can be described by (2). A model of the propagation of long internal waves in a deep rotating fluid can be found in [7]. If one considers the limit of no high-frequency dispersion $b = 0$, the resulting equation is called the Ostrovsky–Hunter equation [4]. When $\gamma = 0$, (1) turns into the Korteweg-de Vries-Burgers equation

$$u_t + bu_{xxx} + uu_x - au_{xx} = f(x,t), \tag{3}$$

which is the dissipated version of the KdV equation ($a = 0$) modeling the propagation of weakly nonlinear dispersive long waves in some contexts [19]. It is worth noting that despite the similarity of structures of (2) and the KdV equation, the Ostrovsky equation, unlike the KdV equation, is nonintegrable by the method of the inverse scattering transform [10].

In the present paper, we study numerically equation (1) in the domain $(x,t) \in [0,1] \times [0,T]$ with the final time T, the initial condition

$$u(x,0) = p(x), \qquad x \in [0,1] \tag{4}$$

and boundary conditions

$$u(0,t) = f_1(t), \quad u_x(0,t) = f_2(t), \quad t \in [0,T], \tag{5}$$

$$u(1,t) = g_1(t), \quad u_x(1,t) = g_2(t), \quad u_{xx}(1,t) = g_3(t), \quad t \in [0,T], \tag{6}$$

where $p(x)$, $g_1(t)$, $g_2(t)$, $g_3(t)$, and $f(x,t)$ are continuous known functions, while $f_1(t)$, $f_2(t)$, and the wave amplitude $u(x,t)$ are unknown, which remain to be determined. There are different methods to obtain numerical solutions of the dissipative KdV-type equations.

There are different methods to obtain numerical solutions of the dissipative KdV-type equations. Bhatta [2] found numerical solutions of (3) using the modified Bernstein polynomials (B-polynomials). In [14], soliton solutions of the KdVB equation were found by using two numerical methods, finite difference and a semi-analytic method with the Adomian decomposition. A spectral method based on Lagrange polynomials was used in [24] to approximate solutions of boundary-value problems associated with (3). The classical radial basis functions (RBFs) collocation (Kansa) method for the numerical solution of (3) was formulated in [12]. A spectral collocation method based on differentiated Chebyshev polynomials was elaborated in [15] to obtain numerical solutions of (3). Chen and Wu [5] presented the MQ quasi-interpolation method to find the numerical solutions of (3). The local discontinuous Galerkin method was tested in [23] to study the Kuramoto–Sivashinsky equation.

Here, we consider two numerical methods to find the solutions of (1), the collocation method based on septic B-spline basis functions and quintic B-spline basis functions.

It is known that the use of B-splines has many different features and is effective in numerical works. One of the most important features is that the conditions on the continuity of functions are built-in and have smooth interpolation functions. On the other hand, as the support of each B-spline is embedded only on a few sub-intervals, the resulting matrix related to the discretized equation will be tightly banded.

Moreover, if one combines with collocation, then the solution procedure will be clear and short. In the present work, we provide the combination of the finite difference in t and the quintic B-spline collocation method in x, and employ the Tikhonov regularization method to solve the associated ill-conditioned system, which gives an efficient explicit solution that has high accuracy with the minimal computational effort for the inverse problem (1) and (4)–(6).

This paper is arranged as follows. In Section 2, a description of the septic B-splines collocation method, the uniform convergence, and the stability are explained. The quintic B-splines collocation method and its convergence and the stability are explained in Section 3. The numerical results are presented in Section 4, and finally, Section 5 completes this paper with some concluding remarks.

2 Septic *B*-spline collection method

To apply the Septic B-spline collection method, region of the solution of the problem is restrained over $0 \leq x \leq 1$. Space interval $[0, 1]$ is separated into uniformly sized finite elements of length h by the knots x_m like that $0 = x_0 < x_1 < \cdots < x_N = 1$. Lengths of these finite elements are $h = \frac{1-0}{N} = (x_{m+1} - x_m)$ for $m = 0, 1, 2, \ldots, N-1$.
Problem (1)–(6) will be solved with the over-specified conditions

$$u(\kappa, t) = h_1(t), \quad u_x(\kappa, t) = h_2(t), \quad u_{xx}(\kappa, t) = h_3(t), \tag{7}$$

where $t \in [0, T]$ and $0 < \kappa < 1$ is a fixed number. A spline is a function that is piecewise-defined by polynomial functions and possesses a high degree of smoothness at the places, where polynomial pieces connect. A B-spline is a spline function that has minimal support to the given degree, smoothness, and domain partition. We define the septic B-spline $B_j(x)$ for $j = -3(1)N+3$ by the following relations:

$$B_j(x) = h^{-7} \begin{cases} (x-x_{j-4})^7, & x \in [x_{j-4}, x_{j-3}), \\ (x-x_{j-4})^7 - 8(x-x_{j-3})^7, & x \in [x_{j-3}, x_{j-2}), \\ (x-x_{j-4})^7 - 8(x-x_{j-3})^7 & \\ +28(x-x_{j-2})^7, & x \in [x_{j-2}, x_{j-1}), \\ (x-x_{j-4})^7 - 8(x-x_{j-3})^7 & \\ +28(x-x_{j-2})^7 - 56(x-x_{j-1})^7, & x \in [x_{j-1}, x_j), \\ (x_{j+4}-x)^7 - 8(x_{j+3}-x)^7 & \\ +28(x_{j+2}-x)^7 - 56(x_{j+1}-x)^7, & x \in [x_j, x_{j+1}), \\ (x_{j+4}-x)^7 - 8(x_{j+3}-x)^7 & \\ +28(x_{j+2}-x)^7, & x \in [x_{j+1}, x_{j+2}), \\ (x_{j+4}-x)^7 - 8(x_{j+3}-x)^7, & x \in [x_{j+2}, x_{j+3}), \\ (x_{j+4}-x)^7, & x \in [x_{j+3}, x_{j+4}), \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

The set

$$\Omega = \{B_{-3}(x), B_{-2}(x), B_{-1}(x), B_0(x), \ldots, B_N(x), \\ B_{N+1}(x), B_{N+2}(x), B_{N+3}(x)\}$$

forms a basis for the approximate solution, which will be defined over the interval $[0, 1]$. Thus $\varpi = \text{Span}(\Omega)$ is a subspace of $C^2[0, 1]$ and ϖ is $N + 7$-dimensional. The values of B_j and its derivatives may be tabulated as in Table 1.

Now let $U(x, t) \in \varpi$ be the B-spline approximation to the exact solution $u(x, t)$ in the form

$$U(x,t) = \sum_{j=-3}^{N+3} c_j(t) B_j(x), \tag{9}$$

where $c_j(t)$ are time-dependent parameters determined from boundary conditions and collocation condition.

By substituting the trial function (8) into equation (9), the nodal values of $U, U^{'}, U^{''}, U^{'''}, U^{(4)}$, and $U^{(5)}$ are obtained in terms of the element parameters c_m by

Table 1: Values of $B_j(x)$ and its derivatives at the nodal points.

x	x_{j-4}	x_{j-3}	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}	x_{j+3}	x_{j+4}
$B_j(x)$	0	1	120	1191	2416	1191	120	1	0
$B'_j(x)$	0	$\frac{7}{h}$	$\frac{392}{h}$	$\frac{1715}{h}$	0	$\frac{-1715}{h}$	$\frac{-392}{h}$	$\frac{-7}{h}$	0
$B''_j(x)$	0	$\frac{42}{h^2}$	$\frac{1008}{h^2}$	$\frac{630}{h^2}$	$\frac{-3560}{h^2}$	$\frac{630}{h^2}$	$\frac{1008}{h^2}$	$\frac{42}{h^2}$	0
$B'''_j(x)$	0	$\frac{210}{h^3}$	$\frac{1680}{h^3}$	$\frac{-3990}{h^3}$	0	$\frac{3990}{h^3}$	$\frac{-1680}{h^3}$	$\frac{-210}{h^3}$	0
$B^{(4)}_j(x)$	0	$\frac{840}{h^4}$	0	$\frac{-7560}{h^4}$	$\frac{13440}{h^4}$	$\frac{-7560}{h^4}$	0	$\frac{840}{h^4}$	0
$B^{(5)}_j(x)$	0	$\frac{2520}{h^5}$	$\frac{-10080}{h^5}$	$\frac{12600}{h^5}$	0	$\frac{-12600}{h^5}$	$\frac{10080}{h^5}$	$\frac{-2520}{h^5}$	0

$$
\begin{aligned}
U_m &= c_{m-3} + 120c_{m-2} + 1191c_{m-1} + 2416c_m + 1191c_{m+1} + 120c_{m+2} + c_{m+3},\\
U'_m &= \frac{7}{h}(-c_{m-3} - 56c_{m-2} - 245c_{m-1} + 2451c_{m+1} + 56c_{m+2} + c_{m+3}),\\
U''_m &= \frac{42}{h^2}(c_{m-3} + 24c_{m-2} + 15c_{m-1} - 80c_m + 15c_{m+1} + 24c_{m+2} + c_{m+3}),\\
U'''_m &= \frac{210}{h^3}(-c_{m-3} - 8c_{m-2} + 19c_{m-1} - 19c_{m+1} + 8c_{m+2} + c_{m+3}),\\
U^{(4)}_m &= \frac{840}{h^4}(c_{m-3} - 9c_{m-1} + 16c_m - 9c_{m+1} + c_{m+3}),\\
U^{(5)}_m &= \frac{2520}{h^5}(-c_{m-3} + 4c_{m-2} - 5c_{m-1} + 5c_{m+1} - 4c_{m+2} + c_{m+3}).
\end{aligned}
\tag{10}
$$

2.1 Numerical discretization

Let us consider a uniform mesh (x_i, t_j) to discretize the region $[0, 1] \times [0, T]$. More precisely, each (x_i, t_j) is defined by $x_i = ih$, $i = 0, 1, 2, \ldots, N$ and $t_j = jk$ for $j = 0, 1, 2, \ldots,$ where h and k are mesh sizes in the space and time directions, respectively (see [8]). Suppose that u^n is linearly interpolated between two time levels n and $n+1$ by using the Crank–Nicolson formula and then its time derivative is discretized by the forward finite difference formula:

$$
u = \frac{u^{n+1} + u^n}{2}, \quad u_t^n = \frac{u^{n+1} - u^n}{k}, \tag{11}
$$

where $u^n = u(x, t_n)$ and $u^0 = u(x, 0) = p(x)$. Substituting the above approximation into equation (1) and discretizing in time variable, we have

$$
u_{xt} = \gamma u + au_{xxx} - bu_{xxxx} - uu_{xx} - u_x^2 + f(x, t),
$$

so

$$u_x^{n+1} - u_x^n = \frac{k}{2}\left[\gamma u^{n+1} + \gamma u^n + a u_{xxx}^{n+1} + a u_{xxx}^n - (u u_{xx})^{n+1} - (u u_{xx})^n \right.$$
$$- \left(u_x^2\right)^{n+1} - \left(u_x^2\right)^n - b u_{xxxx}^{n+1} - b u_{xxxx}^n$$
$$\left. + f(x, t_{n+1}) + f(x, t_n)\right].$$

The nonlinear term is linearized by using the quasi-linearization formula as given below:

$$f\left(u^{n+1}, u_x^{n+1}\right) = f(u^n, u_x^n) + (u^{n+1} - u^n)\frac{\partial f^n}{\partial u} + (u_x^{n+1} - u_x^n)\frac{\partial f^n}{\partial u_x}.$$

Thus, we have

$$(u u_{xx})^{n+1} = u^{n+1} u_{xx}^n + u^n u_{xx}^{n+1} - (u u_{xx})^n$$

and

$$\left(u_x^2\right)^{n+1} = 2 {u_x}^{n+1} {u_x}^n - \left(u_x^2\right)^n.$$

Now, we obtain by rearranging the terms that

$$\begin{aligned}
&\theta_1 u^{n+1} + \theta_2 u_x^{n+1} + \theta_3 u_{xx}^{n+1} - \theta_4 u_{xxx}^{n+1} + \theta_5 u_{xxxx}^{n+1} \\
&\quad = \theta_6 u^n + u_x^n + \theta_4 u_{xxx}^n - \theta_5 u_{xxxx}^n + \theta_7 (f(x, t_{n+1}) + f(x, t_n)),
\end{aligned} \tag{12}$$

where

$$\theta_1 = \frac{k}{2}\left(u_{xx}^n - \gamma\right), \quad \theta_2 = \frac{7}{h}\left(1 + k u_x^n\right), \quad \theta_3 = \frac{42}{h^2}\left(\frac{k}{2} u^n\right)$$
$$\theta_4 = \frac{210}{h^3}\left(\frac{k}{2} a\right), \quad \theta_5 = \frac{840}{h^4}\left(\frac{k}{2} b\right), \quad \theta_6 = \frac{k}{2}\gamma, \quad \theta_7 = \frac{k}{2}.$$

By replacing the approximate solution U with u, and using the nodal values U and the derivatives of U, we deduce from equation (10) at the knots in equation (12) the following difference equation in the variable c:

$$\begin{aligned}
&A^* c_{i-3}^{n+1} + B^* c_{i-2}^{n+1} + C^* c_{i-1}^{n+1} + D^* c_i^{n+1} + E^* c_{i+1}^{n+1} + F^* c_{i+2}^{n+1} + G^* c_{i+3}^{n+1} \\
&\quad = H(x_i, t_n) + H(x_i, t_{n+1}),
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
A^* &= \theta_1 - \theta_2 + \theta_3 + \theta_4 + \theta_5, \\
B^* &= 120\theta_1 - 56\theta_2 + 24\theta_3 + 8\theta_4, \\
C^* &= 1191\theta_1 - 245\theta_2 + 15\theta_3 - 19\theta_4 - 9\theta_5, \\
D^* &= 2416\theta_1 - 80\theta_3 + 16\theta_5, \\
E^* &= 1191\theta_1 + 245\theta_2 + 15\theta_3 + 19\theta_4 - 9\theta_5,
\end{aligned}$$

$$F^* = 120\theta_1 + 56\theta_2 + 24\theta_3 - 8\theta_4,$$
$$G^* = \theta_1 + \theta_2 + \theta_3 - \theta_4 + \theta_5,$$

$$\begin{aligned}H(x_i,t_n) = {} & \theta_6(c_{i-3} + 120c_{i-2} + 1191c_{i-1} + 2416c_i + 1191c_{i+1} + 120c_{i+2} + c_{i+3}) \\ & + \left(\frac{7}{h}\right)(-c_{i-3} - 56c_{i-2} - 245c_{i-1} + 2451c_{i+1} + 56c_{i+2} + c_{i+3}) \\ & + \theta_4\left(\frac{210}{h^3}\right)(-c_{i-3} - 8c_{i-2} + 19c_{i-1} - 19c_{i+1} + 8c_{i+2} + c_{i+3}) \\ & - \theta_5\left(\frac{840}{h^4}\right)(c_{i-3} - 9c_{i-1} + 16c_i - 9c_{i+1} + c_{i+3}) \\ & + \theta_7 f(x_i,t_n),\end{aligned}$$

and

$$H(x_i,t_{n+1}) = \theta_7 f(x_i,t_{n+1}), \quad 0 \le i \le N.$$

System (13) consists of $(N+1)$ linear equations with $(N+7)$ unknowns,

$$(c_{-3}, c_{-2}, c_{-1}, c_0, \ldots, c_N, c_{N+1}, c_{N+2}, c_{N+3})^T.$$

To have a unique solution to the above system, we are required the over-specified conditions (7). Suppose that $\kappa = x_s$, $1 \le s \le N-1$, then

$$u(x_s,t) = h_1(t), \quad u_x(x_s,t) = h_2(t), \quad u_{xx}(x_s,t) = h_3(t), \tag{14}$$

where $t \in [0,T]$. If we consider $m = s$ in (10), then we have

$$\begin{aligned}h_1(t_{n+1}) &= c_{s-3}^{n+1} + 120c_{s-2}^{n+1} + 1191c_{s-1}^{n+1} + 2416c_s^{n+1} + 1191c_{s+1}^{n+1} + 120c_{s+2}^{n+1} + c_{s+3}^{n+1}, \\ h_2(t_{n+1}) &= \frac{7}{h}\left(-c_{s-3}^{n+1} - 56c_{s-2}^{n+1} - 245c_{s-1}^{n+1} + 2451c_{s+1}^{n+1} + 56c_{s+2}^{n+1} + c_{s+3}^{n+1}\right), \\ h_3(t_{n+1}) &= \frac{42}{h^2}\left(c_{s-3}^{n+1} + 24c_{s-2}^{n+1} + 15c_{s-1}^{n+1} - 80c_s^{n+1} + 15c_{s+1}^{n+1} + 24c_{s+2}^{n+1} + c_{s+3}^{n+1}\right), \\ g_1(t_{n+1}) &= c_{N-3}^{n+1} + 120c_{N-2}^{n+1} + 1191c_{N-1}^{n+1} + 2416c_N^{n+1} + 1191c_{N+1}^{n+1} + 120c_{N+2}^{n+1} + c_{N+3}^{n+1}, \\ g_2(t_{n+1}) &= \frac{7}{h}\left(-c_{N-3}^{n+1} - 56c_{N-2}^{n+1} - 245c_{N-1}^{n+1} + 2451c_{N+1}^{n+1} + 56c_{N+2}^{n+1} + c_{N+3}^{n+1}\right), \\ g_3(t_{n+1}) &= \frac{42}{h^2}\left(c_{N-3}^{n+1} + 24c_{N-2}^{n+1} + 15c_{N-1}^{n+1} - 80c_{m-1}^{n+1} + 15c_{N+1}^{n+1} + 24c_{N+2}^{n+1} + c_{N+3}^{n+1}\right).\end{aligned}$$

Hence, we derive that

$$AC = B, \tag{15}$$

is a system of $(N+7)$ linear equations with $(N+7)$ unknowns, where

$$\begin{aligned}A[1,s+1] &= A[1,s+7] = A[3,s+1] = A[2,s+7] = A[N+5,N+7] \\ &= A[3,s+7] = A[N+5,N+1] = A[N+6,N+7] \\ &= A[N+7,N+1] = A[N+7,N+7] = -A[N+6,N+1] = 1,\end{aligned}$$

and

$$\begin{aligned}
&A[N+7,N+2] = A[N+7,N+6] = A[1,s+2] = A[1,s+6] = 120,\\
&A[N+7,N+3] = A[N+7,N+5] = A[1,s+3] = A[1,s+5] = 1191,\\
&A[N+7,N+4] = A[1,s+4] = 2416,\\
&A[2,s+6] = -A[2,s+2] = -A[N+6,N+2] = A[N+6,N+6] = 56,\\
&A[N+6,N+5] = -A[N+6,N+3] = -A[2,s+3] = A[2,s+5] = 245,\\
&A[3,s+2] = A[3,s+6] = A[N+5,N+2] = A[N+5,N+6] = 24,\\
&A[3,s+3] = A[3,s+5] = A[N+5,N+3] = A[N+5,N+5] = 15,\\
&A[3,s+4] = A[N+5,N+4] = -80.
\end{aligned}$$

Therefore, we have

$$A = \begin{pmatrix}
0 & \dots & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & \dots \\
0 & \dots & -1 & -56 & -245 & 0 & 245 & 56 & 1 & 0 & \dots \\
0 & \dots & 1 & 24 & 15 & -80 & 15 & 24 & 1 & 0 & \dots \\
A^* & B^* & C^* & D^* & E^* & F^* & G^* & 0 & \dots & & \\
0 & A^* & B^* & C^* & D^* & E^* & F^* & G^* & 0 & \dots & \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & & \dots & A^* & B^* & C^* & D^* & E^* & F^* & G^* & 0 \\
0 & & \dots & & A^* & B^* & C^* & D^* & E^* & F^* & G^* \\
0 & & \dots & & 1 & 24 & 15 & -80 & 15 & 24 & 1 \\
0 & & \dots & & -1 & -56 & -245 & 0 & 245 & 56 & 1 \\
0 & & \dots & & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1
\end{pmatrix}_{(N+7)\times(N+7)},$$

with

$$C = \Big(c_{-3}^{(n+1)}, c_{-2}^{(n+1)}, c_{-1}^{(n+1)}, c_{0}^{(n+1)}, \dots, c_{s}^{(n+1)}, \\ \dots, c_{N}^{(n+1)}, c_{N+1}^{(n+1)}, c_{N+2}^{(n+1)}, c_{N+3}^{(n+1)}\Big)^T,$$

$$B = \left(B_{-3}^{(n)}, B_{-2}^{(n)}, B_{-1}^{(n)}, B_{0}^{(n)}, \dots, B_{s}^{(n)}, \dots, B_{N}^{(n)}, B_{N+1}^{(n)}, B_{N+2}^{(n)}, B_{N+3}^{(n)}\right)^T,$$

and

$$\begin{aligned}
&B_{-3}^{(n)} = h_1(t_{n+1}), \quad B_{-2}^{(n)} = \left(\frac{h}{7}\right) h_2(t_{n+1}), \quad B_{-1}^{(n)} = \left(\frac{h^2}{42}\right) h_3(t_{n+1}),\\
&B_{i}^{(n)} = H(x_i,t_n) + H(x_i,t_{n+1}), \quad B_{N+1}^{(n)} = \left(\frac{h^2}{42}\right) g_3(t_{n+1}),\\
&B_{N+2}^{(n)} = \left(\frac{h}{7}\right) g_2(t_{n+1}), \quad B_{N+3}^{(n)} = g_1(t_{n+1}),
\end{aligned}$$

where $0 \le i \le N$. We notice that the matrix A is ill-conditioned, so we obtain the solution of system (15) by using the Tikhonov regularization method.

2.2 The initial vector c^0

By the initial condition (4) combined with the boundary and over-specified conditions (7), the initial vector c^0 can be written as the following expressions:

$$\begin{aligned} u(x_s,t_0) &= c^0_{s-3} + 120c^0_{s-2} + 1191c^0_{s-1} + 2416c^0_s + 1191c^0_{s+1} + 120c^0_{s+2} + c^0_{s+3} \\ &= h_1(t_0), \\ u_x(x_s,t_0) &= \frac{7}{h}\left(-c^0_{s-3} - 56c^0_{s-2} - 245c^0_{s-1} + 2451c^0_{s+1} + 56c^0_{s+2} + c^0_s\right) \\ &= h_2(t_0), \\ u_{xx}(x_s,t_0) &= \frac{42}{h^2}\left(c^0_{s-3} + 24c^0_{s-2} + 15c^0_{s-1} - 80c^0_s + 15c^0_{s+1} + 24c^0_{s+2} + c^0_{s+3}\right) \\ &= h_3(t_0), \\ u(x_i,t_0) &= c^0_{i-3} + 120c^0_{i-2} + 1191c^0_{i-1} + 2416c^0_i + 1191c^0_{i+1} + 120c^0_{i+2} + c^0_{i+3} \\ &= p(x_i), \end{aligned}$$

where $0 \le i \le N$ and

$$\begin{aligned} u_{xx}(x_N,t_0) &= \frac{42}{h^2}\left(c^0_{N-3} + 24c^0_{N-2} + 15c^0_{N-1} - 80c^0_{m-1} + 15c^0_{N+1} + 24c^0_{N+2} + c^0_{N+3}\right) \\ &= g_3(t_0), \\ u_x(x_N,t_0) &= \frac{7}{h}\left(-c^0_{N-3} - 56c^0_{N-2} - 245c^0_{N-1} + 2451c^0_{N+1} + 56c^0_{N+2} + c^0_{N+3}\right) \\ &= g_2(t_0), \\ u(x_N,t_0) &= c^0_{N-3} + 120c^0_{N-2} + 1191c^0_{N-1} + 2416c^0_N + 1191c^0_{N+1} + 120c^0_{N+2} + c^0_{N+3} \\ &= g_1(t_0). \end{aligned}$$

This yields an $(N+7)\times(N+7)$-system of equations of the form of

$$AC^0 = \beta, \tag{16}$$

where

$$\begin{aligned} A[1,s+1] &= A[1,s+7] = A[3,s+1] = A[2,s+7] = A[N+5,N+7] \\ &= A[3,s+7] = A[N+5,N+1] = A[N+6,N+7] \\ &= A[N+7,N+1] = A[N+7,N+7] = -A[N+6,N+1] = 1, \end{aligned}$$

and

$$
\begin{aligned}
&A[N+7,N+2]=A[N+7,N+6]=A[1,s+2]=A[1,s+6]=120,\\
&A[N+7,N+3]=A[N+7,N+5]=A[1,s+3]=A[1,s+5]=1191,\\
&A[N+7,N+4]=A[1,s+4]=2416,\\
&A[2,s+6]=-A[2,s+2]=-A[N+6,N+2]=A[N+6,N+6]=56,\\
&A[N+6,N+5]=-A[N+6,N+3]=-A[2,s+3]=A[2,s+5]=245,\\
&A[3,s+2]=A[3,s+6]=A[N+5,N+2]=A[N+5,N+6]=24,\\
&A[3,s+3]=A[3,s+5]=A[N+5,N+3]=A[N+5,N+5]=15,\\
&A[3,s+4]=A[N+5,N+4]=-80.
\end{aligned}
$$

Thus, we have

$$
A=\begin{pmatrix}
0 & \dots & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & \dots \\
0 & \dots & -1 & -56 & -245 & 0 & 245 & 56 & 1 & 0 & \dots \\
0 & \dots & 1 & 24 & 15 & -80 & 15 & 24 & 1 & 0 & \dots \\
1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & \dots & & \vdots \\
0 & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & 0 & \dots & \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & & \dots & & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\
0 & & \dots & & 1 & 24 & 15 & -80 & 15 & 24 & 1 \\
0 & & \dots & & -1 & -56 & -245 & 0 & 245 & 56 & 1 \\
0 & & \dots & & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1
\end{pmatrix}_{(N+7)\times(N+7)},
$$

$$
C^0=\begin{pmatrix}
c^0_{-3}\\ c^0_{-2}\\ c^0_{-1}\\ c^0_{0}\\ \vdots\\ c^0_{N}\\ c^0_{N+1}\\ c^0_{N+2}\\ c^0_{N+3}
\end{pmatrix}_{(N+7)\times 1},\quad
\beta=\begin{pmatrix}
h_1(t_0)\\ (\frac{h}{7})h_2(t_0)\\ (\frac{h^2}{42})h_3(t_0)\\ p(x_0)\\ vdots\\ p(x_N)\\ (\frac{h^2}{42})g_3(t_0)\\ (\frac{h}{7})g_2(t_0)\\ g_1(t_0)
\end{pmatrix}_{(N+7)\times 1}.
$$

Finally, the solution of (16) can be obtained by using the Tikhonov regularization method.

2.3 Convergence analysis

Here, we are going to check the convergence of our algorithm. Suppose that $U(x)=\sum_{j=-3}^{N+3}c_jB_j(x)$ is the B-spline collocation approximation of $u(x)$, where $u(x)$ is the exact solution of (1) with the over-specific conditions (7) and initial condition (4). Due to round off errors in computations, we can assume that $\widehat{U}(x)$ is the computed spline for $U(x)$ in such a way that $\widehat{U}(x) = \sum_{j=-2}^{N+1} \hat{c}_j B_j(x)$, where

$$\widehat{C} = (\hat{c}_{-3}, \hat{c}_{-2}, \hat{c}_{-1}, \hat{c}_0, \ldots, \hat{c}_N, \hat{c}_{N+1}, \hat{c}_{N+2}, \hat{c}_{N+3}).$$

Following (15) for $\widehat{U}$, we have

$$A\widehat{C} = \widehat{B},$$

where

$$\widehat{B} = (h_1(t_{n+1}), h_2(t_{n+1}), h_3(t_{n+1}), \hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_N, g_3(t_{n+1}), g_2(t_{n+1}), g_1(t_{n+1})),$$

and

$$\begin{aligned}
\hat{\tau}_i = {} & \theta_6(c_{i-3} + 120c_{i-2} + 1191c_{i-1} + 2416c_i + 1191c_{i+1} + 120c_{i+2} + c_{i+3}) \\
& + \left(\frac{7}{h}\right)(-c_{i-3} - 56c_{i-2} - 245c_{i-1} + 2451c_{i+1} + 56c_{i+2} + c_{i+3}) \\
& + \theta_4\left(\frac{210}{h^3}\right)(-c_{i-3} - 8c_{i-2} + 19c_{i-1} - 19c_{i+1} + 8c_{i+2} + c_{i+3}) \\
& - \theta_5\left(\frac{840}{h^4}\right)(c_{i-3} - 9c_{i-1} + 16c_i - 9c_{i+1} + c_{i+3}) \\
& + \theta_7\left(f(x_i, t_n) + f(x_i, t_{n+1})\right).
\end{aligned}$$

Consequently, we have

$$A(C - \widehat{C}) = (B - \widehat{B}).$$

The following lemma will be important in our analysis.

Lemma 1. If $\{B_{-3}, B_{-2}, B_{-1}, B_0, \ldots, B_N, B_{N+1}, B_{N+2}, B_{N+3}\}$ is the septic B-spline, then

$$\left|\sum_{j=-3}^{N+3} B_j(x)\right| \leq 7456, \quad x \in [0, 1]. \tag{17}$$

Proof. At any node x_i, we have from Table 1 that

$$\left|\sum_{j=-3}^{N+3} B_j\right| \leq \sum_{j=-3}^{N+3} |B_j| = 1 + 120 + 1191 + 2416 + 1191 + 120 + 1 = 5040.$$

For any point x in each sub-interval $[x_{j-1}, x_j]$, we have analogously that

$$\sum_{j=-3}^{N+3} |B_j| \leq 7456.$$

This proves the lemma. □

To obtain a suitable estimate for the error

$$\|u(x) - U(x)\|_{\infty},$$

we need to bound the error terms $\left\|u(x) - \widehat{U}\right\|_{\infty}$ and $\left\|\widehat{U} - U(x)\right\|_{\infty}$ separately. To do so, we recall the following theorem.

Theorem 1. Suppose that $g(x) \in C^8[0,1]$ and $|g^8(x)| \leq L$, for all $x \in [0,1]$. Assume that

$$\Delta = \{0 = x_0 < x_1 < \cdots < x_N = 1\},$$

is the partition of $[0,1]$ of step size h. If $S_\Delta(x)$ is the unique spline function interpolate $g(x)$ at nodes $x_0, x_1, \ldots, x_N \in \Delta$, then there exist a constant $\lambda_j \leq 2$ such that we have for all $x \in [0,1]$, that

$$\|g^j(x) - S_\Delta^j(x)\|_\infty \leq \lambda_j L h^{8-j}, \qquad j = 0, \ldots, 7,$$

where $\|\cdot\|$ represents the L^∞-norm.

Proof. See [1] for a proof. □

Now first we find a bound on $\|B - \widehat{B}\|_\infty$. Following [20, Theorem 7] and applying Theorem 1, there exists $W = W_L > 0$ such that

$$\begin{aligned}\left\|B - \widehat{B}\right\|_\infty \leq W \Big(&\left|U(x) - \widehat{U}(x)\right| + \left|U^{'}(x) - \widehat{U}^{'}(x)\right| + \left|U^{''}(x) - \widehat{U}^{''}(x)\right| \\ &+ \left|U^{'''}(x) - \widehat{U}^{'''}(x)\right| + \left|U^{(4)}(x) - \widehat{U}^{(4)}(x)\right|\Big) \\ \leq\ & WL\lambda_0 h^8 + WL\lambda_1 h^7 + WL\lambda_2 h^6 + WL\lambda_3 h^5 + WL\lambda_4 h^4.\end{aligned}$$

Thus we can get

$$\|B - \widehat{B}\|_\infty \leq W_1 h^4, \tag{18}$$

where

$$W_1 = WL(\lambda_0 h^4 + \lambda_1 h^3 + \lambda_2 h^2 + \lambda_3 h + \lambda_4).$$

Since the matrix A is ill-conditioned, we have from the Tikhonov regularization solution that

$$\left(C - \widehat{C}\right) = \left[A^T A + \alpha \left(R^{(2)}\right)^T R^2\right]^{-1} A^T \left(B - \widehat{B}\right),$$

after taking the L^∞-norm,

$$\left\| C-\widehat{C} \right\|_{\infty} \leq \left\| \left[A^T A + \alpha \left(R^{(2)} \right)^T R^2 \right]^{-1} A^T \right\|_{\infty} \left\| B-\widehat{B} \right\|_{\infty} \leq W_2 h^4, \tag{19}$$

where

$$W_2 = W_1 \left\| \left[A^T A + \alpha \left(R^{(2)} \right)^T R^2 \right]^{-1} A^T \right\|_{\infty}.$$

Now we observe that

$$U(x) - \widehat{U}(x) = \sum_{i=-3}^{N+3} (c_i - \hat{c}_i) B_i(x).$$

By taking the L^{∞}-norm and using (17) and (19), we have

$$\begin{aligned} \left\| U(x) - \widehat{U}(x) \right\|_{\infty} &= \left\| \sum_{i=-3}^{N+3} (c_i - \hat{c}_i) B_i(x) \right\|_{\infty} \leq \|(c_i - \hat{c}_i)\|_{\infty} \left| \sum_{i=-3}^{N+3} B_i(x) \right| \\ &\leq 7456 W_2 h^4. \end{aligned} \tag{20}$$

Theorem 2. The time discretization process (11) that we use to discretize equation (1) in time, is convergent of the first order.

Proof. See [21] □

Theorem 3. Let $u \in C^8[0,1]$ be an exact solution of (1) such that

$$\left| \frac{\partial^8 u(x,t)}{\partial x^8} \right| \leq L,$$

for all x and t. If $U(x,t)$ is the numerical approximation by our method of u, then

$$\|u(x) - U(x)\|_{\infty} \leq O(k + h^4).$$

Proof. We have from Theorem 1 that

$$\|u(x) - U(x)\|_{\infty} \leq \lambda_0 L h^8. \tag{21}$$

Equations (20) and (21) imply that

$$\begin{aligned} \|u(x) - U(x)\|_{\infty} &\leq \left\| u(x) - \widehat{U}(x) \right\|_{\infty} + \left\| \widehat{U}(x) - U(x) \right\|_{\infty} \\ &\leq \lambda_0 L h^8 + 7456 W_2 h^4 = \Upsilon h^4, \end{aligned}$$

where $\Upsilon = \lambda_0 L h^4 + 7456 W_2$. Therefore we deduce from Theorem 2 that

$$\|u(x) - U(x)\|_\infty \leq \rho(k + h^4),$$

where ρ is a finite positive constant independent of k and h. This completes the proof.

□

2.4 Stability analysis

In this section, we will investigate the stability by applying the Von-Neuman stability analysis. Without loss of generality and for the sake of simplicity, we assume in equation (1) that $f(x, y) = 0$. On the other hand, as we have linearized the nonlinear term uu_x by considering u as a constant k_1 in equation (1), then the equation can be rewritten by

$$(u_t + bu_{xxx} + k_1u_x - au_{xx})_x = \gamma u. \tag{22}$$

Substituting the approximation (11) into equation (22), by discretizing in time we have that

$$\begin{aligned} -P_1u^{n+1} + u_x^{n+1} + P_2u_{xx}^{n+1} - P_3u_{xxx}^{n+1} + P_4u_{xxxx}^{n+1} \\ = P_1u^n + u_x^n - P_2u_{xx}^n + P_3u_{xxx}^n - P_4u_{xxxx}^n. \end{aligned}$$

If $\frac{k}{2} = \tau$, then

$$P_1 = \tau\gamma, \quad P_2 = \frac{42}{h^2}\tau k_1 \quad P_3 = \frac{210}{h^3}\tau a, , \quad P_4 = \frac{840}{h^4}\tau b. \tag{23}$$

In terms of unknown time parameters c_i, the equation can be written from (10) by

$$\begin{aligned} x_1c_{i-3}^{n+1} + x_2c_{i-2}^{n+1} + x_3c_{i-1}^{n+1} + x_4c_i^{n+1} + x_5c_{i+1}^{n+1} + x_6c_{i+2}^{n+1} + x_7c_{i+3}^{n+1} \\ = x_8c_{i-3}^n + x_9c_{i-2}^n + x_{10}c_{i-1}^n + x_{11}c_i^n + x_{12}c_{i+1}^{n+1} + x_{13}c_{i+2}^{n+1} + x_{14}c_{i+3}^{n+1} \end{aligned}$$

where

$$\begin{aligned} x_1 &= -P_1 - 1 + P_2 + P_3 + P_4, \\ x_2 &= -120P_1 - 56 + 24P_2 + 8P_3, \\ x_3 &= -1191P_1 - 245 + 15P_2 - 19P_3 - 9P_4, \\ x_4 &= -2416P_1 - 80P_2 + 16\theta_4, \\ x_5 &= -1191P_1 + 245 + 15P_2 + 19P_3 - 9P_4, \\ x_6 &= -120P_1 + 56 + 24P_2 - 8P_3, \\ x_7 &= -P_1 + 1 + P_2 - P_3 + P_4, \end{aligned} \tag{24}$$

$$
\begin{aligned}
x_8 &= P_1 - 1 - P_2 - P_3 - P_4,\\
x_9 &= 120P_1 - 56 - 24P_2 - 8P_3,\\
x_{10} &= 1191P_1 - 245 - 15P_2 + 19P_3 + 9P_4,\\
x_{11} &= 2416P_1 + 80P_2 - 16\theta_4,\\
x_{12} &= 1191P_1 + 245 - 15P_2 - 19P_3 + 9P_4,\\
x_{13} &= 120P_1 + 56 - 24P_2 + 8P_3,\\
x_{14} &= P_1 + 1 - P_2 + P_3 - P_4.
\end{aligned}
$$

Now we substitute $c_m^n = A\zeta^n \exp(\mathrm{i}m\beta h)$, where $i = \sqrt{-1}$, A is amplitude, h is the step length and β is the mode number. We get

$$\zeta = \frac{X_1 + iY_1}{X_2 + iY_2},$$

where

$$
\begin{aligned}
X_1 &= (x_8 + x_{14})\cos(3\beta h) + (x_9 + x_{13})\cos(2\beta h) + (x_{10} + x_{12})\cos(\beta h) + x_6,\\
X_2 &= (x_7 + x_1)\cos(3\beta h) + (x_2 + x_6)\cos(2\beta h) + (x_3 + x_5)\cos(\beta h) + x_4,\\
Y_1 &= (x_{14} - x_8)\sin(3\beta h) + (x_{13} - x_9)\sin(2\beta h) + (x_{12} - x_{10})\sin(\beta h),\\
Y_2 &= (x_7 - x_1)\sin(3\beta h) + (x_6 - x_2)\sin(2\beta h) + (x_5 - x_3)\sin(\beta h).
\end{aligned}
\tag{25}
$$

Hence, we have from (25) and (24) that $X_1 = X_2$. Setting

$$
\begin{aligned}
\rho &= 2\sin(3\beta h) + 112\sin(2\beta h) + 490\sin(\beta h),\\
\varphi &= P_3(2\sin(3\beta h) + 16\sin(2\beta h) - 38\sin(\beta h)),
\end{aligned}
\tag{26}
$$

we have that

$$Y_1 = \rho + \varphi, \quad Y_2 = \rho - \varphi.$$

To get the stability, we need to prove $|\zeta| < 1$, that is,

$$X_1^2 + (\rho + \varphi)^2 < X_1^2 + (\rho - \varphi)^2. \tag{27}$$

To this end, it is enough to show that $4\rho\varphi < 0$. If we rewrite (26) by

$$
\begin{aligned}
\rho &= 8\sin(\beta h)(\cos^2(\beta h) + 28\cos(\beta h) + 61),\\
\varphi &= 8P_3\sin(\beta h)(\cos(\beta h) + 5)(\cos(\beta h) - 1),
\end{aligned}
$$

then we obtain

$$4\rho\varphi = 256P_3\sin^2(\beta h)(\cos^2(\beta h) + 28\cos(\beta h) + 61)(\cos(\beta h) + 5)(\cos(\beta h) - 1).$$

By the positivity of the constants, $P_3 > 0$. The functions $\sin^2(\beta h)$, $\cos^2(\beta h) + 28\cos(\beta h) + 61$, and $\cos(\beta h) + 5$ are positive, but $\cos(\beta h) - 1$ is negative, so $4\rho\varphi < 0$. Hence, we obtain that the seventh order B-spline collocation method is unconditionally stable.

3 Quintic B-splines collocation method

In this section, we are going to solve the inverse problem (1) by a new modification of the quintic B-splines collocation method with the over-specified conditions

$$\begin{aligned} u(\kappa,t) &= h_1(t),\\ u_x(\kappa,t) &= h_2(t), \end{aligned} \tag{28}$$

where $t \in [0,t_f]$ and $0 < \kappa < 1$ is a fixed number.

We define the quintic B-spline $B_i(x)$ for $j = -2(1)N+2$ by the following relation:

$$B_j(x) = h^{-5}\begin{cases} (x-x_{j-3})^5, & x\in[x_{j-3},x_{j-2}),\\ (x-x_{j-3})^5-6(x-x_{j-2})^5, & x\in[x_{j-2},x_{j-1}),\\ (x-x_{j-3})^5-6(x-x_{j-2})^5+15(x-x_{j-1})^5, & x\in[x_{j-1},x_{j}),\\ (x_{j+3}-x)^5-6(x_{j+2}-x)^5+15(x_{j+1}-x)^5, & x\in[x_{j},x_{j+1}),\\ (x_{j+3}-x)^5-6(x_{j+2}-x)^5, & x\in[x_{j+1},x_{j+2}),\\ (x_{j+3}-x)^5, & x\in[x_{j+2},x_{j+3}),\\ 0, & \text{otherwise}. \end{cases} \tag{29}$$

We note that the set

$$\Omega = \{B_{-2}(x), B_{-1}(x), B_0(x), \ldots, B_N(x), B_{N+1}(x), B_{N+2}(x)\},$$

forms a basis for an approximate solution that will be defined over the interval $[0,1]$. Hence, the set $\varpi = \mathrm{Span}(\Omega)$ is a subspace of $C^2[0,1]$ and ϖ is $N+5$-dimensional. The values of B_j and its derivatives are tabulated in Table 2. Let $U(x,t) \in \varpi$ be the B-spline approximation to the exact solution $u(x,t)$ in the form

$$U(x,t) = \sum_{j=-2}^{N+2} c_j(t)B_j(x), \tag{30}$$

where $c_j(t)$ are time-dependent parameters determined by the boundary and collocation conditions.

Substituting the trial function (29) into (30), the nodal values of U, U', U'', U''', and $U^{(4)}$ are obtained in terms of the element parameters c_m by

Table 2: Values of $B_j(x)$ and its derivatives at the nodal points.

x	x_{j-3}	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}	x_{j+3}
$B_j(x)$	0	1	26	66	26	1	0
$B'_j(x)$	0	$\frac{5}{h}$	$\frac{50}{h}$	0	$\frac{-50}{h}$	$\frac{5}{h}$	0
$B''_j(x)$	0	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$\frac{-120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0
$B'''_j(x)$	0	$\frac{60}{h^3}$	$\frac{-120}{h^3}$	0	$\frac{120}{h^3}$	$\frac{-60}{h^3}$	0
$B^{(4)}_j(x)$	0	$\frac{120}{h^4}$	$\frac{-480}{h^4}$	$\frac{720}{h^4}$	$\frac{-480}{h^4}$	$\frac{120}{h^4}$	0

$$\begin{aligned}
U_m &= c_{m-2} + 26c_{m-1} + 66c_m + 26c_{m+1} + c_{m+2},\\
U'_m &= \frac{5}{h}(-c_{m-2} - 10c_{m-1} + 10c_{m+1} + c_{m+2}),\\
U''_m &= \frac{20}{h^2}(c_{m-2} + 2c_{m-1} - 6c_m + 2c_{m+1} + c_{m+2}),\\
U'''_m &= \frac{60}{h^3}(-c_{m-2} + 2c_{m-1} - 2c_{m+1} + c_{m+2}),\\
U^{(4)}_m &= \frac{120}{h^4}(c_{m-2} - 4c_{m-1} + 6c_m - 4c_{m+1} + c_{m+2}).
\end{aligned} \tag{31}$$

3.1 Numerical discretization

To discretize the equation, similar to the previous section, we use (31) in

$$u_{xt} = \gamma u + a u_{xxx} - b u_{xxxx} - u u_{xx} - {u_x}^2 + f(x,t),$$

and obtain

$$\begin{aligned}
u_x^{n+1} - u_x^n = \frac{k}{2}\big[&\gamma u^{n+1} + \gamma u^n + a u_{xxx}^{n+1} + a u_{xxx}^n - (u u_{xx})^{n+1} - (u u_{xx})^n\\
&- \left(u_x^2\right)^{n+1} - \left(u_x^2\right)^n - b u_{xxxx}^{n+1} - b u_{xxxx}^n\\
&+ f(x,t_{n+1}) + f(x,t_n)\big].
\end{aligned}$$

The nonlinear term is also linearized by rearranging terms as given below:

$$\begin{aligned}
&\theta_1 u^{n+1} + \theta_2 u_x^{n+1} + \theta_3 u_{xx}^{n+1} - \theta_4 u_{xxx}^{n+1} + \theta_5 u_{xxxx}^{n+1} =\\
&\theta_6 u^n + u_x^n + \theta_4 u_{xxx}^n - \theta_5 u_{xxxx}^n + \theta_7 (f(x,t_{n+1}) + f(x,t_n)),
\end{aligned} \tag{32}$$

where

$$\theta_1 = \frac{k}{2}\left(u_{xx}^n - \gamma\right), \quad \theta_2 = \frac{5}{h}(1 + ku_x^n), \quad \theta_3 = \frac{20}{h^2}\left(\frac{k}{2}u^n\right), \quad \theta_4 = \frac{60}{h^3}\left(\frac{k}{2}a\right),$$
$$\theta_5 = \frac{120}{h^4}\left(\frac{k}{2}b\right), \quad \theta_6 = \frac{k}{2}\gamma, \quad \theta_7 = \frac{k}{2}.$$

Again, by replacing the approximate solution U with u, and using the nodal values U and the derivatives of U, we obtain from equation (31) at the knots in equation (32) the following difference equation in the variable c:

$$A^* c_{i-2}^{n+1} + B^* c_{i-1}^{n+1} + C^* c_i^{n+1} + D^* c_{i+1}^{n+1} + E^* c_{i+2}^{n+1} = H(x_i, t_n) + H(x_i, t_{n+1}), \tag{33}$$

where

$$\begin{aligned}
A^* &= \theta_1 - \theta_2 + \theta_3 + \theta_4 + \theta_5,\\
B^* &= 26\theta_1 - 10\theta_2 + 2\theta_3 - 2\theta_4 - 4\theta_5,\\
C^* &= 66\theta_1 - 6\theta_3 + 6\theta_5,\\
D^* &= 26\theta_1 + 2\theta_2 + 2\theta_3 + 2\theta_4 - 4\theta_5,\\
E^* &= \theta_1 + \theta_2 + \theta_3 - \theta_4 + \theta_5,
\end{aligned}$$

$$\begin{aligned}
H(x_i, t_n) = {} & \theta_6\left(c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}\right)\\
& + \left(\frac{5}{h}\right)\left(-c_{i-2} - 10c_{i-1} + 10c_{i+1} + c_{i+2}\right)\\
& + \theta_4\left(\frac{60}{h^3}\right)\left(-c_{i-2} + 2c_{i-1} - 2c_{i+1} + c_{i+2}\right)\\
& - \theta_5\left(\frac{120}{h^4}\right)\left(c_{i-2} - 4c_{i-1} + 6c_i - 4c_{i+1} + c_{i+2}\right)\\
& + \theta_7 f(x_i, t_n),
\end{aligned}$$

and

$$H(x_i, t_{n+1}) = \theta_7 f(x_i, t_{n+1}), \qquad 0 \le i \le N.$$

System (33) contains $(N+1)$ linear equations with $(N+5)$ unknowns,

$$\left(c_{-2}, c_{-1}, c_0, \ldots, c_N, c_{N+1}, c_{N+2}\right)^T.$$

To have a unique solution to the above system, we are required the over-specified condition (28). Assume that $\kappa = x_s$, $1 \le s \le N-1$. Equation (6) holds and moreover

$$u(x_s, t) = h_1(t), \quad u_x(x_s, t) = h_2(t),$$

where $t \in [0, T]$. If we consider $m = s$ in (31), then we have

$$\begin{aligned}
h_1(t_{n+1}) &= c_{s-2}^{n+1} + 26c_{s-1}^{n+1} + 66c_{s}^{n+1} + 26c_{s+1}^{n+1} + c_{s+2}^{n+1},\\
h_2(t_{n+1}) &= \frac{5}{h}\left(c_{s-2}^{n+1} + 10c_{s-1}^{n+1} - 10c_{s+1}^{n+1} - c_{s+2}^{n+1}\right),\\
g_1(t_{n+1}) &= c_{N-2}^{n+1} + 26c_{N-1}^{n+1} + 66c_{N}^{n+1} + 26c_{N+1}^{n+1} + c_{N+2}^{n+1},\\
g_2(t_{n+1}) &= \frac{5}{h}\left(c_{N-2}^{n+1} + 10c_{N-1}^{n+1} - 10c_{N+1}^{n+1} + c_{N+2}^{n+1}\right).
\end{aligned}$$

Consequently,

$$AC = B, \tag{34}$$

is a system of $(N+5)$ linear equations with $(N+5)$ unknown functions where

$$\begin{aligned}
&A[1, s+1] = A[1, s+5] = A[2, s+1] = A[N+4, N+5] = A[2, s+5]\\
&= A[N+4, N+1] = A[N+5, N+5] = A[N+5, N+1] = 1,\\
&A[N+5, N+2] = A[N+5, N+4] = A[1, s+2] = A[1, s+4] = 26,\\
&A[N+5, N+3] = A[1, s+3] = 66,\\
&A[N+4, N+2] = A[2, s+2] = -A[N+4, N+4] = -A[2, s+4] = 10,\\
&A[2, s+3] = A[N+4, N+3] = 0.
\end{aligned}$$

Thus, we can write the matrix A as

$$A = \begin{pmatrix}
0 & \dots & 1 & 26 & 66 & 26 & 1 & 0 & \dots\\
0 & \dots & 1 & 10 & 0 & -10 & 1 & 0 & \dots\\
A^* & B^* & C^* & D^* & E^* & 0 & \dots & \dots & \vdots\\
0 & A^* & B^* & C^* & D^* & E^* & 0 & \dots & \\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
\vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & \dots & & & A^* & B^* & C^* & D^* & E^*\\
0 & \dots & & & 1 & 10 & 0 & -10 & 1\\
0 & \dots & & & 1 & 26 & 66 & 26 & 1
\end{pmatrix}_{(N+5)\times(N+5)},$$

with

$$C = \left(c_{-2}^{(n+1)}, c_{-1}^{(n+1)}, c_{0}^{(n+1)}, \dots, c_{s}^{(n+1)}, \dots, c_{N}^{(n+1)}, c_{N+1}^{(n+1)}, c_{N+2}^{(n+1)}\right)^T,$$

$$B = \left(B_{-2}^{(n)}, B_{-1}^{(n)}, B_{0}^{(n)}, \dots, B_{s}^{(n)}, \dots, B_{N}^{(n)}, B_{N+1}^{(n)}, B_{N+2}^{(n)}\right)^T$$

$$B_{-2}^{(n)} = h_1(t_{n+1}), \quad B_{-1}^{(n)} = \left(\frac{h}{5}\right) h_2(t_{n+1}), \quad B_{i}^{(n)} = H(x_i, t_n) + H(x_i, t_{n+1}),$$

$$B_{N+1}^{(n)} = \left(\frac{h}{5}\right) g_2(t_{n+1}), \quad B_{N+2}^{(n)} = g_1(t_{n+1}),$$

where $0 \le i \le N$. We note that A is an ill-conditioned matrix, so we obtain the solution of system (34) by using the Tikhonov regularization method.

3.2 The initial vector c^0

From the initial condition (4) the boundary and over-specified conditions (28) the initial vector c^0 can be obtained as the following expressions:

$$
\begin{aligned}
u(x_s,t_0) &= c^0_{s-2} + 26c^0_{s-1} + 66c^0_{s} + 26c^0_{s+1} + c^0_{s+2},\\
u_x(x_s,t_0) &= \frac{5}{h}\left(c^0_{s-2} + 10c^0_{s-1} - 10c^0_{s+1} + c^0_{s+2}\right) = h_2(t_0),\\
u(x_i,t_0) &= c^0_{i-2} + 26c^0_{i-1} + 66c^0_{i} + 26c^0_{i+1} + c^0_{i+2} = p(x_i), \qquad 0 \le i \le N,\\
u_x(x_N,t_0) &= \frac{5}{h}\left(c^0_{N-2} + 10c^0_{N-1} - 10c^0_{N+1} + c^0_{N+2}\right) = g_2(t_0),\\
u(x_N,t_0) &= c^0_{N-2} + 26c^0_{N-1} + 66c^0_{N} + 26c^0_{N+1} + c^0_{N+2} = g_1(t_0).
\end{aligned}
$$

This gives an $(N+5)\times(N+5)$-system of equations of the form of

$$AC^0 = \beta, \tag{35}$$

where

$$
\begin{aligned}
&A[1,s+1] = A[1,s+5] = A[2,s+1] = A[N+4,N+5] = A[2,s+5],\\
&= A[N+4,N+1] = A[N+5,N+5] = A[N+5,N+1] = 1,\\
&A[N+5,N+2] = A[N+5,N+4] = A[1,s+2] = A[1,s+4] = 26,\\
&A[N+5,N+3] = A[1,s+3] = 66,\\
&A[N+4,N+2] = A[2,s+2] = -A[N+4,N+4] = -A[2,s+4] = 10,\\
&A[2,s+3] = A[N+4,N+3] = 0,
\end{aligned}
$$

Thus, we have

$$
A = \begin{pmatrix}
0 \dots & 1 & 26 & 66 & 26 & 1 & 0 & \dots\\
0 \dots & 1 & 10 & 0 & -10 & -1 & 0 & \dots\\
1 & 26 & 66 & 26 & 1 & 0 & \dots & \dots & \vdots\\
0 & 1 & 26 & 66 & 26 & 1 & 0 & \dots & \vdots\\
\vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & &\\
\vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots &\\
0 & & \dots & & 1 & 26 & 66 & 26 & 1\\
0 & & \dots & & 1 & 10 & 0 & -10 & 1\\
0 & & \dots & & 1 & 26 & 66 & 26 & 1
\end{pmatrix}_{(N+5)\times(N+5)},
$$

$$C^0 = \begin{pmatrix} c^0_{-2} \\ c^0_{-1} \\ c^0_0 \\ \vdots \\ c^0_N \\ c^0_{N+1} \\ c^0_{N+2} \end{pmatrix}_{(N+5)\times 1}, \quad \beta = \begin{pmatrix} h_1(t_0) \\ (\frac{h}{5})h_2(t_0) \\ p(x_0) \\ \vdots \\ p(x_N) \\ (\frac{h}{5})g_2(t_0) \\ g_1(t_0) \end{pmatrix}_{(N+5)\times 1}.$$

Finally the solution of (35) can be obtained by using the Tikhonov regularization method.

3.3 Convergence analysis

Similar to the Convergence of the previous part, first we need to recall the following lemma.

Lemma 2. The B-splines $\{B_{-2}, B_{-1}, B_0, \dots, B_N, B_{N+1}, B_{N+2}\}$ satisfy the following inequality:

$$\left| \sum_{j=-2}^{N+2} B_j(x) \right| \leq 186, \qquad (0 \leq x \leq 1). \tag{36}$$

Proof. We have from Table 2 at any node x_i, that

$$\left| \sum_{i=-2}^{N+2} B_i \right| \leq \sum_{i=-2}^{N+2} |B_i| = 120.$$

Similarly, we have for any point x in each sub-interval $[x_{i-1}, x_i]$ that

$$\sum_{j=-2}^{N+2} |B_j| \leq 186.$$

□

Theorem 4 ([6, 13])**.** Suppose that $f(x) \in C^6[0,1]$ and $|f^6(x)| \leq L$, for all $x \in [0,1]$. Assume that

$$\Delta = \{0 = x_0 < x_1 < \cdots < x_N = 1\},$$

is the partition of $[0,1]$ of step size h. If $S_\Delta(x)$ is the unique spline function interpolates $f(x)$ at nodes $x_0, x_1, \dots, x_N \in \Delta$, then there exist a constant λ_j

such that we have for all $x \in [0,1]$ that

$$\left\| f^j(x) - S_\Delta^j(x) \right\|_\infty \leq \lambda_j L h^{6-j}, \qquad j = 0, 1, \ldots, 5.$$

where $\| \cdot \|$ represents the L^∞-norm.

By using the ideas of [20] and Theorem 4, there exists $W = W_L > 0$ such that

$$\begin{aligned} \left\| B - \widehat{B} \right\|_\infty \leq W \Big(& \left| U(x) - \widehat{U}(x) \right| + \left| U^{'}(x) - \widehat{U}^{'}(x) \right| + \left| U^{''}(x) - \widehat{U}^{''}(x) \right| \\ & + \left| U^{'''}(x) - \widehat{U}^{'''}(x) \right| + \left| U^{(4)}(x) - \widehat{U}^{(4)}(x) \right| \Big) \\ & \leq WL\lambda_0 h^6 + WL\lambda_1 h^5 + WL\lambda_2 h^4 + WL\lambda_3 h^3 + WL\lambda_4 h^2. \end{aligned}$$

Thus we can get

$$\left\| B - \widehat{B} \right\|_\infty \leq W_1 h^2, \tag{37}$$

where $W_1 = WL\left(\lambda_0 h^4 + \lambda_1 h^3 + \lambda_2 h^2 + \lambda_3 h + \lambda_4\right)$. Since the matrix A is an ill-conditioned matrix, we have from the Tikhonov regularization solution that

$$\left(C - \widehat{C} \right) = \left[A^T A + \alpha \left(R^{(2)} \right)^T R^2 \right]^{-1} A^T \left(B - \widehat{B} \right).$$

After taking the infinity-norm we find from (37) that

$$\left\| C - \widehat{C} \right\|_\infty \leq \left\| \left[A^T A + \alpha \left(R^{(2)} \right)^T R^2 \right]^{-1} A^T \right\|_\infty \left\| B - \widehat{B} \right\|_\infty \leq W_2 h^2, \tag{38}$$

where

$$W_2 = W_1 \left\| [A^T A + \alpha (R^{(2)})^T R^2]^{-1} A^T \right\|_\infty.$$

Now we see that

$$U(x) - \widehat{U}(x) = \sum_{i=-2}^{N+2} (c_i - \hat{c}_i) B_i(x).$$

Using (36) and (38), we have after taking the L^∞-norm that

$$\begin{aligned} \left\| U(x) - \widehat{U}(x) \right\|_\infty = \left\| \sum_{i=-2}^{N+2} (c_i - \hat{c}_i) B_i(x) \right\|_\infty & \leq \| (c_i - \hat{c}_i) \|_\infty \left| \sum_{i=-2}^{N+2} B_i(x) \right| \\ & \leq 186 W_2 h^2, \end{aligned} \tag{39}$$

Theorem 5. Let $u(x,t) \in C^6[0,1]$ be the exact solution of (1) such that

$$\left|\frac{\partial^6 u(x,t)}{\partial x^6}\right| \leq L.$$

Assume that $U(x,t)$ is the numerical approximation by our methods, then

$$\|u(x) - U(x)\|_\infty \leq O(k + h^2).$$

Proof. We have from Theorem 4 that

$$\|u(x) - U(x)\|_\infty \leq \lambda_0 L h^6, \tag{40}$$

Equations (39) and (40) imply that

$$\begin{aligned}\|u(x) - U(x)\|_\infty &\leq \left\|u(x) - \widehat{U}(x)\right\|_\infty + \left\|\widehat{U}(x) - U(x)\right\|_\infty \\ &\leq \lambda_0 L h^6 + 186 W_2 h^2 = \Upsilon h^2,\end{aligned}$$

where $\Upsilon = \lambda_0 L h^4 + 186 W_2$ and $U(x,t)$ is the approximate solution by our present method. Then by Theorem 2 and the above considerations, we have

$$\|u(x) - U(x)\|_\infty \leq \rho(k + h^2),$$

where ρ is a finite positive constant independent of k and h. This completes the proof.

□

3.4 Stability analysis

Similar to the previous section, we use the Von-Neuman stability analysis. We assume again in equation (1) that $f(x,y) = 0$. On the other hand, as we have linearized the nonlinear term uu_x by considering u as a constant k_1 in the equation (1), then the equation can be rewritten by

$$(u_t + b u_{xxx} + k_1 u_x - a u_{xx})_x = \gamma u. \tag{41}$$

If we substitute the approximation (11) into the equation (41) and discretize it in time, we obtain that

$$\begin{aligned}-P_1 u^{n+1} + u_x^{n+1} + P_2 u_{xx}^{n+1} - P_3 u_{xxx}^{n+1} + P_4 u_{xxxx}^{n+1}& \\ = P_1 u^n + u_x^n - P_2 u_{xx}^n + P_3 u_{xxx}^n - P_4 u_{xxxx}^n&.\end{aligned}$$

If $\frac{k}{2} = \tau$, then we have

$$P_1 = \tau\gamma, \quad P_3 = \frac{60}{h^3}\tau a, \quad P_2 = \frac{20}{h^2}\tau k_1, \quad P_4 = \frac{120}{h^4}\tau b. \tag{42}$$

In terms of unknown time parameters c_i, the equation can be written from (31) by

$$\begin{aligned}
&A^* c_{i-2}^{n+1} + B^* c_{i-1}^{n+1} + C^* c_i^{n+1} + D^* c_{i+1}^{n+1} + E^* c_{i+2}^{n+1} \\
&= F^* c_{i-2}^{n} + G^* c_{i-1}^{n} + H^* c_i^{n} + I^* c_{i+1}^{n+1} + J^* c_{i+2}^{n+1},
\end{aligned}$$

where

$$\begin{aligned}
A^* &= -P_1 - 1 + P_2 - P_3 + P_4, \\
B^* &= -26P_1 + 10 + 2P_2 + 2P_3 - 4p_4, \\
C^* &= -66P_1 - 6P_2 + 6P_4, \\
D^* &= -26P_1 - 10 + 2P_2 - 2P_3 - 4P_4, \\
E^* &= -P_1 - 1 + P_2 + P_3 + P_4, \\
F^* &= P_1 + 1 - P_2 + P_3 - P_4, \\
G^* &= 26P_1 + 10 - 2P_2 - 2P_3 + 4P_4, \\
H^* &= 66P_1 + 6P_2 - 6P_4, \\
I^* &= 26P_1 - 10 - 2P_2 + 2P_3 + 4P_4, \\
J^* &= P_1 - 1 - P_2 - P_3 - P_4.
\end{aligned} \tag{43}$$

If we substitute $c_m^n = A\zeta^n \exp(im\beta h)$, where h is step length, A is amplitude, $i = \sqrt{-1}$, and β is mode number, then we get that

$$\zeta = \frac{X_1 + iY_1}{X_2 + iY_2},$$

where

$$\begin{aligned}
X_1 &= (F^* + J^*)\cos(2\beta h) + (G^* + I^*)\cos(\beta h) + H^*, \\
X_2 &= (A^* + E^*)\cos(2\beta h) + (B^* + D^*)\cos(\beta h) + C^*, \\
Y_1 &= (J^* - F^*)\sin(2\beta h) + (I^* - G^*)\sin(\beta h), \\
Y_2 &= (E^* - A^*)\sin(2\beta h) + (D^* - B^*)\sin(\beta h).
\end{aligned}$$

We have from (43) that $X_2 = -X_1$ and

$$\begin{aligned}
Y_1 &= \sin(2\beta h)(-2 - 2P_3) + \sin(\beta h)(-20 + 4P_3), \\
Y_2 &= \sin(2\beta h)(-2 + 2P_3) + \sin(\beta h)(-20 - 4P_3).
\end{aligned}$$

To show the stability, we need to prove $|\zeta| < 1$, that is,

$$X_1^2 + Y_1^2 < X_1^2 + Y_2^2 \tag{44}$$

We prove that $Y_1^2 - Y_2^2 < 0$, but we see that

$$Y_1^2 - Y_2^2 = 256P_3 \sin^2(\beta h)(\cos(\beta h) - 1)(\cos(\beta h) + 5).$$

As all arbitrary constants are positive, then $P_3 > 0$. Moreover, the functions $\sin^2(\beta h)$ and $\cos(\beta h) + 5$ are positive but $\cos(\beta h) - 1$ is negative, so $Y_1^2 - Y_2^2 < 0$. Hence, we obtain that the quintic B-spline collocation method is unconditionally stable.

4 Numerical result and illustrations

In this section, we will give some examples to investigate the applicability of our methods that we described in the previous sections. In general, there are two error sources for an inverse problem. The first one is the unavoidable bias deviation, known as deterministic error, and the second one is the variance caused by the enhancement of error estimation, known as stochastic error. The global impact of deterministic and stochastic errors is studied in the mean squared error or the total error as in [3]. Next, by considering the total error S defined by

$$S_{f_1} = \left[\frac{1}{N-1} \sum_{s=1}^{N} (f_1(t_{s+1}) - f_1^*(t_{s+1}))^2 \right]^{\frac{1}{2}},$$

$$S_{f_2} = \left[\frac{1}{N-1} \sum_{s=1}^{N} (f_2(t_{s+1}) - f_2^*(t_{s+1}))^2 \right]^{\frac{1}{2}},$$

we will compare the exact and the approximate solutions, where f_1 and f_2 are exact values, f_1^* and f_2^* are the estimated values, and the number of estimated values is denoted by N.

We should remark that there are no previous numerical results related to the nonlinear inverse problem (1) and (4)–(6) in the literature to compare our results, and one may use the other numerical methods to study this problem.

Example 1. In our first example, we consider the nonlinear inverse problem (1) and (4)–(6), where $a = 1$, $b = 5$, and $\gamma = 3$ with the initial data

$$u(x, 0) = \sin(x),$$

and the external force

$$f(x, t) = \cos(t + x) + \frac{3}{2} \sin(t + x) + \cos^2(t + x) - \sin^2(t + x).$$

An exact solutions of this problem is $u(x, t) = \sin(x + t)$ with

$$u(0,t) = f_1(t) = \sin(t), \quad u_x(0,t) = f_2(t) = \cos(t).$$

Table 3: Comparison between exact and numerical solutions of Example 1 for $u(0.1,t)$, $|u(0.1,t) - u^*(0.1,t)|$, by the Quintic and Septic B-spline methods with $N = 30, 50, 100$.

	N=30		N=50		N=100	
t	Quintic	Septic	Quintic	Septic	Quintic	Septic
0.1	$4.8e-04$	$4.2e-06$	$8.7e-04$	$6.5e-06$	$1.4e-03$	$8.5e-06$
0.2	$7.3e-04$	$4.4e-06$	$9.0e-04$	$4.9e-06$	$1.8e-03$	$3.7e-08$
0.3	$1.0-03$	$1.6e-06$	$1.7e-03$	$1.9e-06$	$2.3e-03$	$2.0e-06$
0.4	$1.4e-03$	$1.5e-05$	$2.2e-03$	$6.9e-06$	$3.0e-03$	$5.5e-06$
0.5	$1.7e-03$	$1.0e-05$	$2.7e-03$	$1.8e-08$	$3.5e-03$	$5.3e-08$
0.6	$1.9e-03$	$1.1e-05$	$2.9e-03$	$3.2e-06$	$3.8e-03$	$3.3e-06$
0.7	$2.3e-03$	$1.8e-05$	$3.3e-03$	$4.1e-05$	$4.3e-03$	$2.2e-06$
0.8	$2.6e-03$	$3.0e-06$	$3.7e-03$	$2.0e-06$	$4.7e-03$	$3.6e-06$
0.9	$2.8e-03$	$2.4e-06$	$3.9e-03$	$3.6e-06$	$4.9e-03$	$1.4e-07$
1	$3.0e-03$	$1.3e-06$	$4.2e-03$	$2.0e-06$	$5.2e-03$	$2.1e-06$
S_u	$5.9758e-05$	$1.0042e-06$	$8.7071e-05$	$1.0093e-06$	$1.1311e-04$	$1.0143e-06$

Table 4: Comparison between exact and numerical solutions of Example 1 for $f_1(t)$, $|f_1(t) - f_1^*(t)|$, by employing the Quintic and Septic B-spline methods with $N = 30, 50, 100$.

	N=30		N=50		N=100	
t	Quintic	Septic	Quintic	Septic	Quintic	Septic
0.1	$0.9e-04$	$1.9e-05$	$1.1e-04$	$5.9e-05$	$1.2e-04$	$1.9e-06$
0.2	$2.4e-04$	$2.2e-05$	$2.4e-04$	$3.5e-06$	$2.8e-04$	$7.3e-08$
0.3	$3.9e-04$	$3.1e-05$	$3.4e-04$	$8.5e-08$	$4.6e-04$	$5.0e-08$
0.4	$6.4e-04$	$3.4e-05$	$5.1e-04$	$9.4e-06$	$6.6e-04$	$1.4e-07$
0.5	$3.2e-04$	$2.8e-05$	$6.2e-04$	$1.3e-05$	$4.0e-04$	$2.0e-07$
0.6	$1.0e-03$	$2.7e-05$	$7.2e-04$	$3.6e-06$	$4.6e-04$	$1.8e-07$
0.7	$1.2e-03$	$2.3e-05$	$8.7e-04$	$4.7e-06$	$5.2e-04$	$4.5e-08$
0.8	$1.3e-03$	$2.2e-05$	$9.6e-04$	$2.4e-07$	$5.6e-04$	$3.3e-08$
0.9	$1.4e-03$	$1.4e-05$	$1.0e-03$	$3.7e-06$	$6.1e-04$	$4.3e-08$
1	$1.6e-03$	$1.5e-06$	$1.1e-03$	$3.7e-06$	$6.5e-04$	$4.6e-07$
S_{f_1}	$3.1833e-05$	$1.1633e-06$	$2.3106e-05$	$1.1575e-06$	$1.3381e-05$	$9.8944e-07$

Tables 3–5 show the total error S for some values of N for each method. In Figures 1 and 2, we plotted the total error S as a function of N. Also we took $\kappa = N^{-1}$, $T = 1$, $\Delta t = 0.001$, and the noisy data (input data + $0.001 \times \text{rand}(1)$).

Table 5: Comparison between exact and numerical solutions of Example 1 for $f_2(t)$, $|f_2(t) - f_2^*(t)|$, by the Quintic and Septic B-spline methods with $N = 30, 50, 100$.

	N=30		N=50		N=100	
t	Quintic	Septic	Quintic	Septic	Quintic	Septic
0.1	$4.9e-03$	$1.9e-03$	$3.6e-04$	$6.3e-03$	$1.2e-02$	$7.9e-06$
0.2	$1.2e-02$	$2.3e-03$	$8.1e-03$	$4.9e-03$	$1.9e-02$	$5.2e-04$
0.3	$1.7e-02$	$3.5e-03$	$1.3e-02$	$3.7e-03$	$2.6e-02$	$4.2e-04$
0.4	$2.3e-02$	$4.1e-03$	$2.2e-02$	$2.5e-03$	$3.3e-02$	$1.0e-04$
0.5	$2.8e-02$	$3.7e-03$	$2.8e-02$	$2.4e-03$	$4.0e-02$	$1.5e-04$
0.6	$3.4e-02$	$4.0e-03$	$3.4e-02$	$9.1e-04$	$4.6e-02$	$8.3e-05$
0.7	$3.9e-02$	$3.6e-03$	$4.1e-02$	$1.2e-03$	$5.2e-02$	$4.6e-04$
0.8	$4.3e-02$	$3.9e-03$	$4.6e-02$	$3.7e-04$	$5.6e-02$	$1.7e-05$
0.9	$4.7e-02$	$3.3e-03$	$5.1e-02$	$3.1e-04$	$6.1e-02$	$5.1e-05$
1	$5.2e-02$	$3.5e-03$	$5.5e-02$	$3.0e-04$	$6.5e-02$	$2.3e-05$
S_{f_2}	$9.6783e-04$	$7.7698e-05$	0.0012	$8.3921e-05$	0.0013	$1.5842e-06$

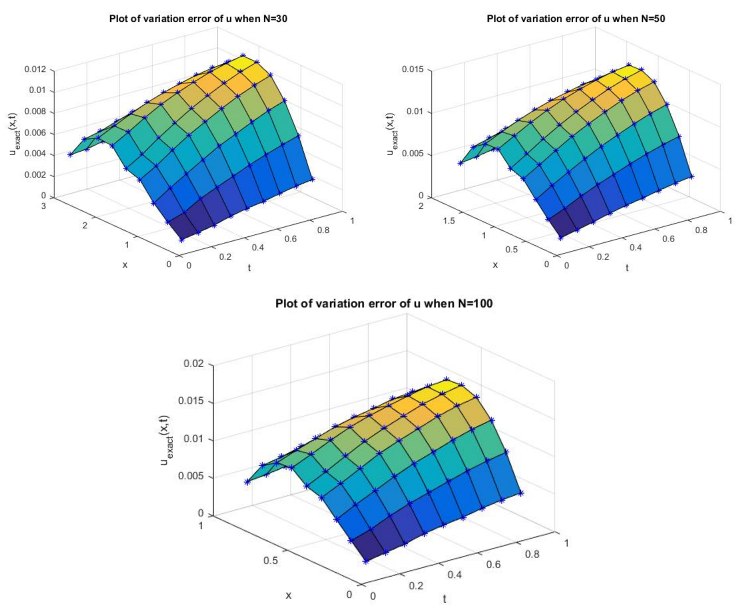

Figure 1: Plots of variation $u(x,t)$ of Example 1 by using the Quintic B-spline method when $N = 30, 50, 100$.

Example 2. In this example, we consider the inverse problem (1) and (4)–(6), where $a = 1$, $b = 3$ and $\gamma = 0.5$. The initial data is

$$u(x,0) = \operatorname{sech}(x),$$

while

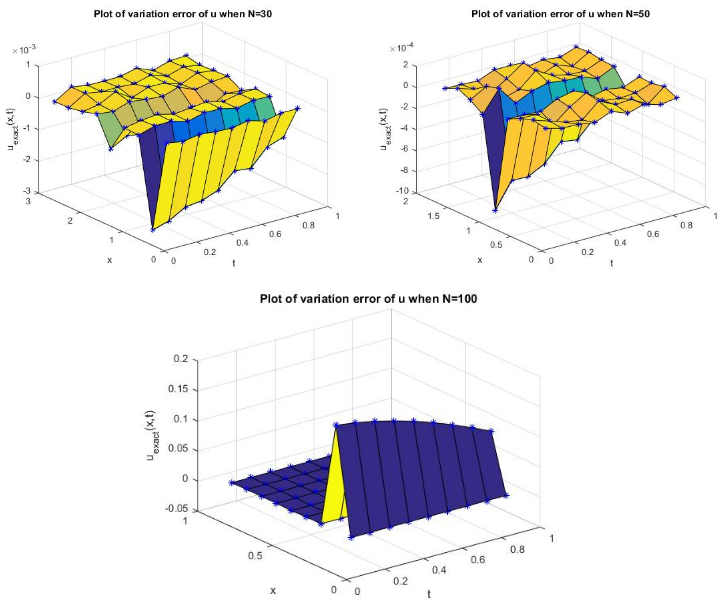

Figure 2: Plots of variation $u(x,t)$ of Example 1 by using the septic B-spline method when $N = 30, 50, 100$.

$$
\begin{aligned}
f(x,t) = &-\frac{142\sinh(t-x)^2}{\cosh(t-x)^3} + \frac{\sinh(t-x)^2 - 6\sinh(t-x)^3 + 2\sinh(t-x)^2}{\cosh(t-x)^4} \\
&+ \frac{120\sinh(t-x)^4}{\cosh(t-x)^5} + \frac{23}{\cosh(t-x)} + \frac{5\sinh(t-x)-1}{\cosh(t-x)^2},
\end{aligned}
$$

is the external force. An exact solution of this problem is $u(x,t) = \operatorname{sech}(x-t)$ such that

$$
\begin{aligned}
u(0,t) &= f_1(t) = \operatorname{sech}(-t), \\
u_x(0,t) &= f_2(t) = \frac{\sinh(t)}{\cosh(t^2)}.
\end{aligned}
$$

Tables 6 and 7 give the numerical results for conditions $u(0,t)$ and $u_x(0,t)$, respectively. To explain the accuracy of our methods, some illustrations are presented in Figures 3–6. Table 8 shows the numerical values of $u(x,t)$ at point $x = 0.1$. Figures 7 and 8 also present the comparison between the numerical and exact values of $u(x,t)$. Also we take $\kappa = 0.03, T = 1, \Delta t = 0.001$ and the noisy data (input data $+\ 0.001 \times \text{rand}(1)$).

Table 7: Comparison between exact and numerical solutions of Example 2 for $f_2(t)$, $|f_2(t) - f_2^*(t)|$, by the Quintic and Septic B-spline methods with $N = 100$.

	Quintic B-spline			Septic B-spline		
t	$f_2(t)$	$f_2^*(t)$	$\|f_2(t) - f_2^*(t)\|$	$f_2(t)$	$f_2^*(t)$	$\|f_2(t) - f_2^*(t)\|$
0.1	0.099172	0.021154	$7.801727e-02$	0.099172	0.102857	$3.685425e-03$
0.2	0.193493	0.118409	$7.508375e-02$	0.193493	0.197758	$4.265252e-03$
0.3	0.278678	0.208712	$6.996601e-02$	0.278678	0.282586	$3.907888e-03$
0.4	0.351456	0.292363	$5.909310e-02$	0.351456	0.355666	$4.210337e-03$
0.5	0.409814	0.356937	$5.287678e-02$	0.409814	0.413920	$4.105471e-03$
0.6	0.453029	0.414228	$3.880057e-02$	0.453029	0.457373	$4.344355e-03$
0.7	0.481503	0.454315	$2.718771e-02$	0.481503	0.486087	$4.583750e-03$
0.8	0.496500	0.479423	$1.707765e-02$	0.496500	0.500676	$4.175531e-03$
0.9	0.499829	0.495907	$3.922220e-03$	0.499829	0.500025	$1.958806e-04$
1	0.493554	0.499845	$6.290468e-03$	0.493554	0.492724	$8.303327e-04$
S_{f_2}	-	-	0.0018	-	-	$1.3344e-04$
Execution time (second)	40.079962			45.367252		
Condition Number of Matrix A	$1.4332e+20$			$4.1816e+19$		

Table 6: Comparison between exact and numerical solutions of Example 2 for $f_1(t)$, $|f_1(t) - f_1^*(t)|$, by the Quintic and Septic B-spline methods with $N = 100$.

	Quintic B-spline			Septic B-spline		
t	$f_1(t)$	$f_1^*(t)$	$\|f_1(t) - f_1^*(t)\|$	$f_1(t)$	$f_1^*(t)$	$\|f_1(t) - f_1^*(t)\|$
0.1	0.995021	0.997416	$2.394818e-03$	0.995021	0.995007	$1.369526e-05$
0.2	0.980328	0.982549	$2.220868e-03$	0.980328	0.980312	$1.635728e-05$
0.3	0.956628	0.958612	$1.984447e-03$	0.956628	0.956613	$1.456806e-05$
0.4	0.925007	0.926788	$1.780935e-03$	0.925007	0.924991	$1.596319e-05$
0.5	0.886819	0.888307	$1.488184e-03$	0.886819	0.886804	$1.534899e-05$
0.6	0.843551	0.844734	$1.183566e-03$	0.843551	0.843534	$1.641813e-05$
0.7	0.796705	0.797546	$8.406108e-04$	0.796705	0.796696	$9.782520e-06$
0.8	0.747700	0.748193	$4.927829e-04$	0.747700	0.747694	$6.334825e-06$
0.9	0.697795	0.698057	$2.627784e-04$	0.697795	0.697795	$3.326556e-08$
1	0.648054	0.647956	$9.857496e-05$	0.648054	0.648057	$2.488186e-06$
S_{f_1}	-	-	$5.0435e-05$	-	-	$5.0706e-07$
Execution time(second)	40.079962			45.367252		
Condition Number of Matrix A	$1.4332e+20$			$4.1816e+19$		

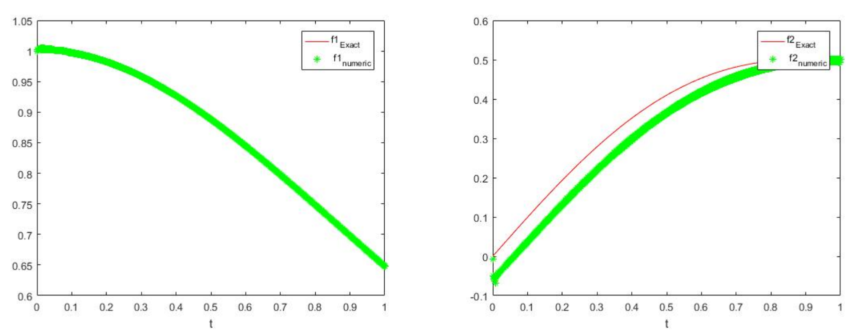

Figure 3: Comparison between exact and numerical values of (a) $u(0,t)$ and (b) $u_x(0,t)$ of Example 2 by the Quintic B-spline method.

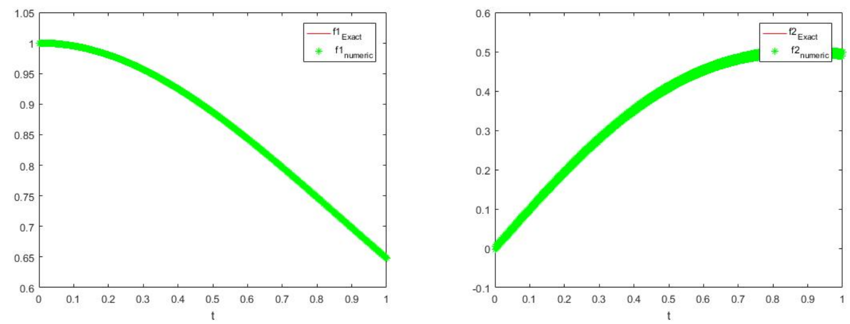

Figure 4: Comparison between exact and numerical values of (a) $u(0,t)$ and (b) $u_x(0,t)$ of Example 2 by the septic B-spline method.

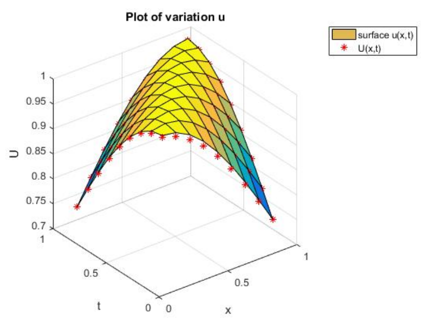

Figure 5: Comparison between exact and numerical values of $u(x,t)$ of Example 2 by the quintic B-spline method.

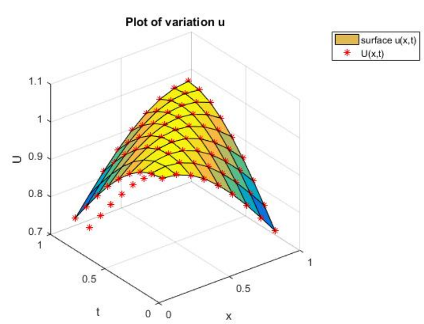

Figure 6: Comparison between exact and numerical values of $u(x,t)$ of Example 2 by the septic B-spline method.

Table 8: Comparison between exact and numerical values of $u(0.1, t)$ of Example 2 by the Quintic and Septic B-spline methods with $N = 100$.

	Quintic B-spline			Septic B-spline		
t	$u(0.1,t)$	$u^*(0.1,t)$	$\lvert u(0.1,t) - u^*(0.1,t)\rvert$	$u(0.1,t)$	$u^*(0.1,t)$	$\lvert u(0.1,t) - u^*(0.1,t)\rvert$
0.1	1.000000	0.996042	$3.958350e-03$	1.000000	1.000008	$8.493153e-06$
0.2	0.995021	0.991131	$3.889610e-03$	0.995021	0.994971	$4.942389e-05$
0.3	0.980328	0.976786	$3.542139e-03$	0.980328	0.980355	$2.729571e-05$
0.4	0.956628	0.953011	$3.616572e-03$	0.956628	0.956594	$3.413866e-05$
0.5	0.925007	0.922351	$2.656163e-03$	0.925007	0.925047	$3.991245e-05$
0.6	0.886819	0.884390	$2.429210e-03$	0.886819	0.886839	$1.991189e-05$
0.7	0.843551	0.841742	$1.808390e-03$	0.843551	0.843521	$2.948822e-05$
0.8	0.796705	0.795713	$9.927146e-04$	0.796705	0.796733	$2.729211e-05$
0.9	0.747700	0.747000	$7.002802e-04$	0.747700	0.747675	$2.541404e-05$
1	0.697795	0.697668	$1.267156e-04$	0.697795	0.697777	$1.739320e-05$
S_u	-	-	$9.9114e-05$	-	-	$8.2254e-07$

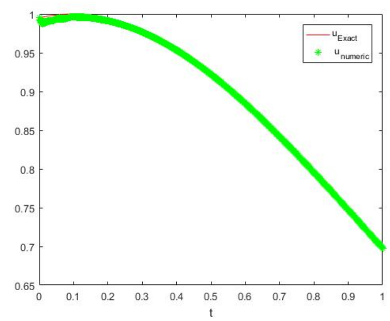

Figure 7: Plot of variation $u(0.1, t)$ of Example 2 by the Quintic B-spline method with $N = 100$.

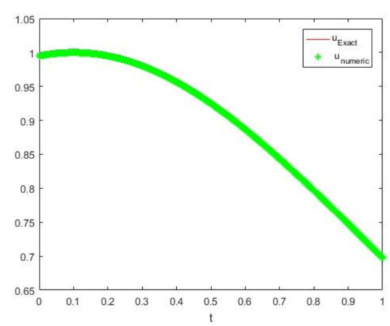

Figure 8: Plot of variation $u(0.1, t)$ of Example 2 by the septic B-spline method with $N = 100$.

Example 3. In our last example, we solve the inverse problem (4) and (4)–(6), where $a = 1$, $b = 2$, $\gamma = 0.5$ and the initial data is given by

$$u(x,0) = \cos(x),$$

The force function is

$$f(x,t) = \frac{5\cos(t-x)}{2} + \sin(t-x) - \cos(t-x)^2 + \sin(t-x)^2,$$

The function $u(x,t) = \cos(x-t)$, is an exact solution of this problem. Thus $u(0,t) = f_1(t) = \cos(-t)$ and $u(1,t) = f_2(t) = -\sin(-t)$. Tables 9 and 10 give the numerical results for $u(0,t)$ and $u_x(0,t)$, respectively. To explain the accuracy of our methods, some illustrations are presented in Figures 9–12. Table 11 shows the numerical values of $u(x,t)$ at point $x = 0.1$. Figures 13 and 14 also present the comparison between the numerical and exact values of solution $u(x,t)$. Also we take $\kappa = 0.03$, $T = 1$, $\Delta t = 0.001$, and the noisy data (input data$+0.001 \times \text{rand}(1)$).

Table 9: Comparison between exact and numerical values for $f_1(t)$ of Example 3 by applying the Quintic and Septic B-spline methods with $N = 100$.

	Quintic B-spline			Septic B-spline		
t	$f_1(t)$	$f_1^*(t)$	$\lvert f_1(t) - f_1^*(t)\rvert$	$f_1(t)$	$f_1^*(t)$	$\lvert f_1(t) - f_1^*(t)\rvert$
0.1	0.995004	0.996818	$1.813881e-03$	0.995004	0.994960	$4.446296e-05$
0.2	0.980067	0.981816	$1.749680e-03$	0.980067	0.980097	$3.086339e-05$
0.3	0.955336	0.957039	$1.702169e-03$	0.955336	0.955366	$2.917672e-05$
0.4	0.921061	0.922675	$1.613838e-03$	0.921061	0.921150	$8.917805e-05$
0.5	0.877583	0.879143	$1.560911e-03$	0.877583	0.877252	$3.307205e-04$
0.6	0.825336	0.826760	$1.424698e-03$	0.825336	0.824996	$3.394589e-04$
0.7	0.764842	0.766191	$1.349309e-03$	0.764842	0.764432	$4.104127e-04$
0.8	0.696707	0.697931	$1.224282e-03$	0.696707	0.696210	$4.966064e-04$
0.9	0.621610	0.622693	$1.082930e-03$	0.621610	0.621069	$5.406920e-04$
1	0.540302	0.541248	$9.459846e-04$	0.540302	0.539825	$4.770495e-04$
S_{f_1}	-	-	$4.6373e-05$	-	-	$3.8840e-06$
Execution time(second)	40.562310			41.030530		
Condition Number of Matrix A	$1.4332e+20$			$1.3648e+19$		

Table 10: Comparison between exact and numerical values for $f_2(t)$ of Example 3 by applying the Quintic and Septic B-spline methods with $N = 100$.

	Quintic B-spline			Septic B-spline		
t	$f_2(t)$	$f_2^*(t)$	$\lvert f_2(t) - f_2^*(t)\rvert$	$f_2(t)$	$f_2^*(t)$	$\lvert f_2(t) - f_2^*(t)\rvert$
0.1	0.099833	0.038846	$6.098711e-02$	0.099833	0.103929	$4.095972e-03$
0.2	0.198669	0.139510	$5.915962e-02$	0.198669	0.194482	$4.186958e-03$
0.3	0.295520	0.238522	$5.699784e-02$	0.295520	0.292183	$3.337169e-03$
0.4	0.389418	0.335087	$5.433175e-02$	0.389418	0.378289	$1.112973e-02$
0.5	0.479426	0.427911	$5.151492e-02$	0.479426	0.466016	$1.340928e-02$
0.6	0.564642	0.516943	$4.769963e-02$	0.564642	0.546380	$1.826272e-02$
0.7	0.644218	0.599703	$4.451437e-02$	0.644218	0.692126	$4.790832e-02$
0.8	0.717356	0.677664	$3.969196e-02$	0.717356	0.774241	$5.688534e-02$
0.9	0.783327	0.748561	$3.476550e-02$	0.783327	0.846131	$6.280393e-02$
1	0.841471	0.810553	$3.091790e-02$	0.841471	0.902065	$6.059395e-02$
S_{f_2}	-	-	0.0016	-	-	$5.1323e-04$
Execution time (second)	40.562310			41.030530		
Condition Number of Matrix A	$1.4332e+20$			$1.3648e+19$		

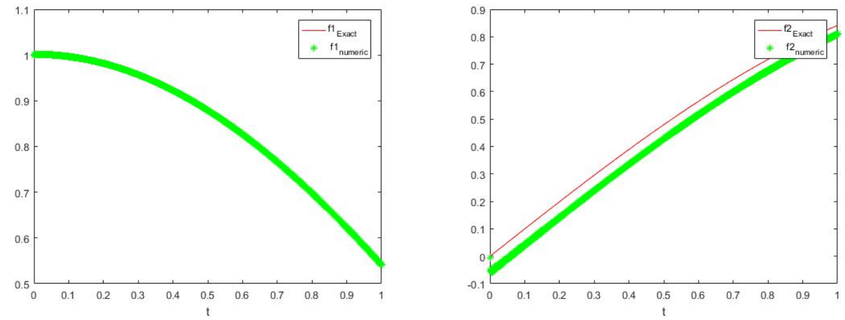

Figure 9: Comparison between exact and numerical values of (a) $u(0,t)$ and (b) $u_x(0,t)$ of Example 3 by the Quintic B-spline method.

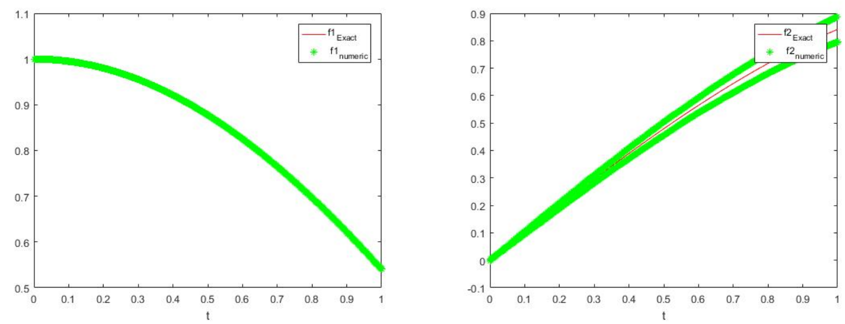

Figure 10: Comparison between exact and numerical values of (a) $u(0,t)$ and (b) $u_x(0,t)$ of Example 3 by the septic B-spline method.

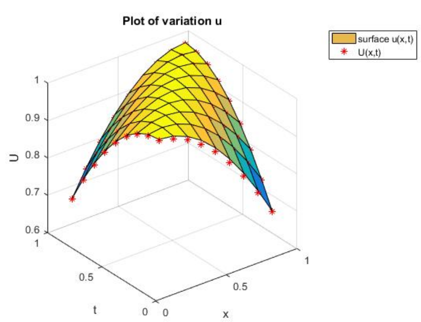

Figure 11: Comparison between exact and numerical values of $u(x,t)$ of Example 3 by the Quintic B-spline method.

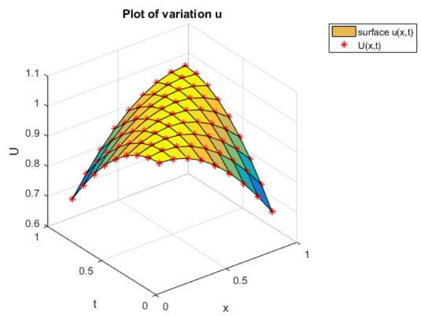

Figure 12: Comparison between exact and numerical values of $u(x,t)$ of Example 3 by the septic B-spline method.

Table 11: Comparison between exact and numerical values of $u(0.1,t)$ of Example 3 by the Quintic and Septic B-spline methods with $N = 100$.

	Quintic B-spline			Septic B-spline		
t	$u(0.1,t)$	$u^*(0.1,t)$	$\|u(0.1,t)-u^*(0.1,t)\|$	$u(0.1,t)$	$u^*(0.1,t)$	$\|u(0.1,t)-u^*(0.1,t)\|$
0.1	1.000000	0.996544	$3.455663e-03$	1.000000	0.999902	$9.781401e-05$
0.2	0.995004	0.991605	$3.399369e-03$	0.995004	0.995019	$1.462144e-05$
0.3	0.980067	0.976735	$3.331920e-03$	0.980067	0.980016	$5.030518e-05$
0.4	0.955336	0.952079	$3.257457e-03$	0.955336	0.955342	$5.215045e-06$
0.5	0.921061	0.917956	$3.104844e-03$	0.921061	0.921182	$1.214016e-04$
0.6	0.877583	0.874601	$2.981736e-03$	0.877583	0.877510	$7.290696e-05$
0.7	0.825336	0.822554	$2.781252e-03$	0.825336	0.825410	$7.484661e-05$
0.8	0.764842	0.762264	$2.577809e-03$	0.764842	0.764753	$8.910966e-05$
0.9	0.696707	0.694359	$2.347544e-03$	0.696707	0.696628	$7.884439e-05$
1	0.621610	0.619528	$2.082224e-03$	0.621610	0.621652	$4.249717e-05$
S_u	-	-	$9.5839e-05$	-	-	$2.8097e-06$

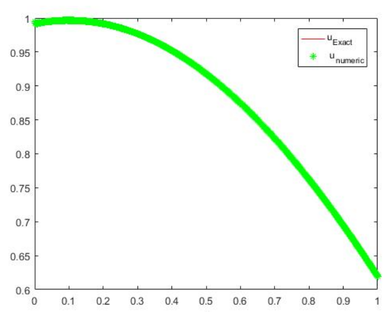

Figure 13: Plot of variation $u(0.1,t)$ in Example 3 by the Quintic B-spline method with $N = 100$.

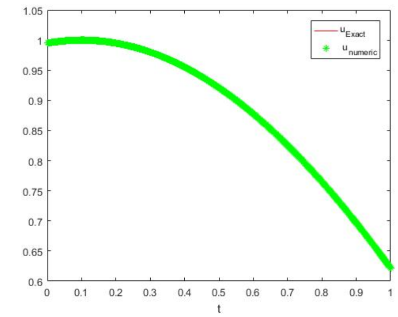

Figure 14: Plot of variation $u(0.1, t)$ in Example 3 by the Septic B-spline method with $N = 100$.

5 Conclusion

In this paper, we have successfully employed the septic B-spline and the quintic B-spline method to estimate unknown boundary conditions in an inverse problem related to the Ostrovsky–Burgers equation (1) and (4)–(6). We have discussed the convergence rate of our methods and have shown the rate of the septic B-spline method is $O(k + h^4)$, while $O(k + h^2)$ is the convergence rate for the quintic B-spline method. The stability of both methods was investigated. By comparing the numerical results, we showed that the accuracy and stability of the septic B-spline method are more than the ones for the quintic B-spline method. Since the associated coefficient matrix in the septic B-spline method and quintic B-spline method are usually ill-conditioned, we have used the Tikhonov regularization method to obtain a stable numerical approximation of the solution. The results are collected by using the MATLAB 7.10(R2016a), tested on a personal computer with Intel(R) Core(TM)2 Duo-CPU and 4-GB RAM.

References

1. Akram, G. and Siddiqi, S.S. *End conditions for interpolatory septic splines*, Int. J. Comput. Math. 82 (2005) 1525–1540.

2. Bhatta, D. *Use of modified Bernstein polynomials to solve KdV–Burgers equation numerically,* Appl. Math. Comput. 206 (2008) 457–464.

3. Cabeza, J.M.G., Garcia, J.A.M. and Rodriguez, A.C. *A sequential algorithm of inverse heat conduction problems using singular value decomposition,* Int. J. Therm. Sci. 44 (2005) 235–244.

4. Chen, G.U. and Boyd, J.P. *Analytical and numerical studies of weakly nonlocal solitary waves of the rotation-modified Korteweg–de Vries equation,* Physica D 155 (2001) 201–222.

5. Chen, R. and Wu, Z. *Solving partial differential equation by using multiquadric quasi-interpolation,* Appl. Math. Comput. 186 (2007) 1502–10.

6. de Boor, C. *On the convergence of odd degree spline interpolation,* J. Approx. Theory 1 (1968) 452–463.

7. Esfahani, A. and Levandosky, S. *Solitary waves of the rotation-generalized Benjamin-Ono equation,* Discrete Contin. Dyn. Syst. 33 (2013) 663–700.

8. Foadian, S., Pourgholi, R. and Tabasi, S.H. *Cubic B-spline method for the solution of an inverse parabolic system,* Appl. Anal. 97 (2018) 438–465.

9. Galkin, V.N. and Stepanyants, Y.A. *On the existence of stationary solitary waves in a rotating field,* Appl. Math. Mech. 55 (1991) 939–943.

10. Gilman, O.A., Grimshaw, R. and Stepanyants, Y.A. *Approximate and numerical solutions of the stationary Ostrovsky equation,* Stud. Appl. Math. 95 (1995) 115–126

11. Grimshaw, R. *Internal solitary waves, environmental stratified flows*, Kluwer Academic Publishers, Dordrecht, 2001, Ch. 1, pp. 1–29.

12. Haq, S., Siraj-Ul-Islam, and Uddin, M. *A mesh-free method for the numerical solution of the KdV-Burgers equation,* Appl. Math. Model. 33 (2009) 3442–3449.

13. Hall, C.A. *On error bounds for spline interpolation,* J. Approx. Theory 1 (1968) 209–218.

14. Helal, M.A. and Mehanna, M.S. *A comparison between two different methods for solving KdV–Burgers equation,* Chaos Soliton Fract. 28 (2006) 320–326.

15. Khater, A.H., Temsah, R.S. and Hassan, M.M. *A Chebyshev spectral collocation method for solving Burgers'-type equations,* J. Comput. Appl. Math. 222 (2008) 333–350.

16. Mitsudera, H. and Grimshaw, R. *Effects of friction on a localized structure in a baroclinic current,* J. Physical Oceanography 23 (1993) 2265–2292.

17. Obregon, M.A. and Stepanyants, Y.A. *Oblique magneto-acoustic solitons 460 in rotating plasma,* Phys. Lett. A 249 (1998) 315–323.

18. Ostrovsky, L.A. *Nonlinear internal waves in a rotating ocean*, Oceanologia 18 (1978) 181–191

19. Ott, E. and Sudan, R.N. *Damping of solitary waves,* Phys. Fluids 13 (1970) 1432–1434.

20. Rudin, W. *Principles of mathematical analysis,* Third edition, International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976.

21. Smith, G.D. *Numerical solution of partial differential equation: finite difference method,* Learendom Press, Oxford 1978.

22. Wang, H. and Esfahani, A. *Well-posedness and asymptotic behavior of the dissipative Ostrovsky equation,* Evol. Equ. Control Theory 8 (2019) 709–735.

23. Xu, Y. and Shu, C-W. *Local discontinuous Galerkin methods for the Kuramoto Sivashinsky equations and the Ito-type coupled equations,* Comput. Methods Appl. Mech. 195 (2006) 3430–3447.

24. Zhan, J.M. and Li, Y.S. *Generalized finite spectral method for 1D burgers and KdV equations,* Appl. Math. Mech. (English Ed.) 27 (2006) 1635–1643.

How to cite this article
Ghanadian, F., Pourgholi, R., Tabasi, S. Numerical approximation for inverse problem of the Ostrovsky-Burgers equation. *Iranian Journal of Numerical Analysis and Optimization*, 2022; 12(1): 73-109. doi: 10.22067/ijnao.2021.70478.1036.