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Research Article

The most probable allocation solution for the p-median problem[†]

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Abstract

The most important purpose in location problems is usually to locate some facilities and allocate the demands of nodes so that the total transportation cost of the network is minimized. However, in real networks, there are some other influencing factors, aside from the transportation costs, for determining the allocation mode. In this paper, a minimum information approach is applied to the capacitated p-median problem to estimate the most likely allocation solution based on some prior probabilities. Indeed, the most probable solution is achieved through minimizing a log-based objective function, while the total transportation cost should be less than or equal to a predetermined budget. The problem is solved by using a decomposition method combined with the Karush–Kuhn–Tucker optimality conditions, and some numerical examples are provided to verify the added value of the proposed model and solution approach

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1 Introduction

The p-median problem is one of the known subjects in the field of locating facilities with a minisum objective function, where its purpose is to locate

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p facilities and allocate the demands of nodes so that the total demand-weighted distance is minimized. ReVelle and Swain [26] introduced the first integer linear programming formulation for the p-median, which has played an important role in solving the problem by applying such exact methods as the branch and bound [24] and Lagrangian dual relaxation [5]. Hakimi [18] proved that the p-median problem on the general networks is NP-hard; however, it has at least one nodal solution. Kariv and Hakimi [20] suggested an $O(p^2n^2)$ time algorithm to solve the p-median problem on the tree networks. Then, Tamir [30] improved the time complexity on the tree networks to $O(pn^2)$. For a complete bibliography on the p-median problem; see [25].

In the classical p-median model, which is sometimes referred to as the uncapacitated p-median ($UCpM$) problem, there is no constraint on the capacities of facilities; see [17]. In the case of limited capacities, the problem is usually called the capacitated p-median (CpM) problem. Both the $UCpM$ and CpM problems are NP-hard, although considering capacity limitations makes the latter problem more difficult; see [14]. A column generation approach, along with a Lagrangian/surrogate relaxation technique has been applied by Lorena and Senne [21] for solving the CpM problem. Ceselli and Righini [8] presented a branch and price algorithm that utilizes a column generation, heuristics, and branch-and-bound to determine the optimal solutions. Such Meta-heuristic methods as a Hybrid scatter search and path relinking algorithm [10], variable neighborhood search [12], neural network [27], and genetic algorithm [23] have been recently applied to solve the CpM problem.

When there is a lack of information in the network, the maximum entropy (ME) principle or its extended concept, the minimum information theory, is used to consider the solution with the most unbiased probability distribution; see [22]. In the ME models, there is no assumption about the missing information, and all possible states consistent with the system conditions are considered equally; then the most likely one is selected; see [9, 19]. The application of ME to the location problems was employed by Teye, Bell, and Bliemer [32, 33] to solve a multiuser inter-modal terminal (IMT) location problem wherein there is no priority to use an IMT. Based on the ME principle, the most unbiased probability distribution was achieved through considering all possible states of IMT usage and selecting the most likely one consistent with the problem constraints. Teye, Bell, and Bliemer [31] also proposed an ME approach for locating competitive multiuser freight facilities in general, and inland multiuser IMTs, in particular, when the multiple users have the choice whether to use a facility or not.

The ME approach to the origin-destination ($O-D$) matrix estimation problems was studied by Van Zuylen and Willumsen [34] to determine the most probable $O-D$ demands consistent with the available traffic counts. Abareshi, Zaferanieh, and Keramati [2] introduced an ME path flow estimator for disaggregated flows between $O-D$ pairs with a prespecified level for each disaggregation.

When, in the network, there is some prior information, but insufficient, it is reasonable to choose a solution that adds the least extra information to the available knowledge, which results in the most possible unbiased probability distribution. The minimum information (MI) approach provides an extended measure of the likelihood of a certain macro-state on the existence of some appropriate micro-state space; see [28]. Abareshi and Zaferanieh [1] proposed a new *bi*-level p-median problem in which the total cost of locating facilities and serving demands was minimized through the upper level while the MI approach was applied in the lower level to determine the most unbiased allocation solution based on some prior partial information. The prior information was given in terms of the probabilities of serving the demands of client nodes by different facilities, which might be determined by considering some attributes such as distance, locality, and geographical features, taken into account by multiple attributes decision making procedures; see [1, 29].

In this paper, considering some prior probabilities for serving the demands of nodes, we attempt to locate p capacitated facilities and determine the most probable allocation solution, while the total transportation cost should be less than or equal to a predetermined budget. The model is formulated as a mixed-integer nonlinear programming ($MINLP$) problem, which is known to be NP-hard. Applying such alternative methods as the Lagrangian dual approach or Benders decomposition algorithm to separate the model into some easier subproblems, the solution could be iteratively obtained.

The generalized Benders decomposition (GBD) algorithm has been suggested by Geoffrion [16] as a solution procedure for certain nonlinear programming (NLP) and $MINLP$ problems. The decomposition procedure was first developed by Benders [6] for the solution of mixed-variable programming problems. However, some restrictions regarding the convexity and other properties of the involved functions were considered. Floudas, Aggarwal, and Ciric [13] studied this technique and revealed its potential for applying in a chemical process design. They also proposed a computational implementation to reach the global optimal solution for nonconvex NLP and $MINLP$ problems. This claim was sustained by solving different examples, though it has not been proven mathematically.

Bagajewicz and Manousiouthakis [3] investigated the importance of two properties defined by Geoffrion [16], called *property (P)* and *L-dual-adequacy*, for the proper application of the GBD algorithm. They solved several examples satisfying or violating these properties and analyzed the obtained solutions. Through examples and mathematical analysis, they showed that dual gaps may prevent this procedure from converging to the global solution, and therefore only bounds on the global optimum might be obtained. Here, we apply the GBD algorithm studied in [3] to solve the proposed $MINLP$ model of the p-median problem in the MI approach.

The remainder of this paper is organized as follows: In Section 2, the model of the CpM problem in the MI approach is introduced. Using the scheme proposed by Bagajewicz and Manousiouthakis [3], the GBD algo-

Table 1: The notation used in the model

Notation	Definition
N	$\{v_1, \dots, v_n\}$, the set of nodes
E	The set of links
I	The index set of client nodes
J	The index set of candidate points for establishing facilities
y_j	The binary variable representing whether node $v_j, j \in J$ is used to locate a facility or not
c_j	The capacity of the facility at node $v_j, j \in J$
w_i	The total demand of the client node $v_i,\ i \in I$
p_{ij}	The probability of serving the demand of node $v_i, i \in I$ by the facility at node $v_j, j \in J$
x_{ij}	The decision variable representing the amount of demand of node $v_i, i \in I$ provided by the facility at node $v_j, j \in J$
d_{ij}	Per unit cost to serve the demand of node $v_i, i \in I$ by the facility at node $v_j, j \in J$
B	The total available budget

rithm and the Karush–Kuhn–Tucker (KKT) optimality conditions are applied in Sections 3 and 4 whereby the problem is reduced to easier subproblems. The feasibility and optimality constraints are added iteratively and the upper and lower bounds are updated. The convergence of the GBD algorithm on the proposed problem is also established in Section 4. To investigate the added value of the proposed model, some numerical examples are provided in Section 5, while the efficiency of the implemented algorithm is compared with some existing $MINLP$ solvers by applying the performance profile test proposed by Dolan and More [11]. The summary and conclusions are given in the last section.

2 CpM problem with the most likely allocation solution

In this section, the CpM model applying the MI approach, to determine the most probable allocation solution, is proposed. Consider a network $G = (N, E)$, where the other frequently used notations are listed in Table 1.

To apply the concept of MI to the CpM problem, consider several demand nodes $v_i,\ i \in I$ with w_i clients that should be served by the facilities at candidate nodes $v_j,\ j \in J$. Also, let x_{ij} be the amount of clients of the demand node v_i that are served by the facility at node v_j, where its corresponding probability is given by p_{ij}. Then, using the multinomial distribution, the probability of observing an allocation solution with the components x_{ij} is given as follows (see [1]):

$$E(x) = \prod_i \left(\frac{w_i!}{\prod_j x_{ij}!} \prod_j p_{ij}^{x_{ij}}\right).$$

Following the scheme presented by Abareshi and Zaferanieh [1], by taking the logarithm, using Stirling's approximation, removing constant values, and reducing the problem to the minimum case, the model of the minimum information capacitated p-median ($MICpM$) problem is introduced as follows:

$$MICpM: \min_{x,y} Z_0(x,y) = \sum_{j\in J}\sum_{i\in I} x_{ij}(\ln x_{ij} - 1 - \ln p_{ij}), \tag{1}$$

$$\sum_{j\in J} y_j = p, \tag{2}$$

$$\sum_{j\in J} x_{ij} = w_i \qquad \text{for all } i\in I, \tag{3}$$

$$\sum_{i\in I} x_{ij} \leq y_j c_j \qquad \text{for all } j\in J, \tag{4}$$

$$\sum_{j\in J}\sum_{i\in I} d_{ij}x_{ij} \leq B, \tag{5}$$

$$y_j \in \{0,1\},\ x_{ij} \geq 0 \qquad \text{for all } i\in I, j\in J. \tag{6}$$

Constraint (2) assures that the number of established facilities is equal to p and constraints (3) practically guarantee that the demands of all nodes v_i are supplied. Constraints (4) make all open facilities serve the demands less than or equal to their capacities. In addition, constraint (5) states that the total spent cost does not exceed the available budget B. Due to the existing of nonlinear terms and mixed-integer variables, the $MICpM$ problem (1)–(6) is NP-hard; see [14]. In the following, the GBD algorithm is introduced whereby the problem is reduced to smaller subproblems to which the constraints are added iteratively to update the upper and lower bounds.

3 Generalized Benders decomposition algorithm

Here, the GBD algorithm studied by Bagajewicz and Manousiouthakis [3], for general $MINLP$ is stated, while its application to the $MICpM$ problem (1)–(6) is examined through the next section. Consider the following problem

$$\begin{aligned} &\min_{x,y} F(x,y), \\ &s.t. \ \ G(x,y) \leq 0, \\ &x\in X, \ \ y\in Y. \end{aligned} \tag{7}$$

The projection of the above problem on Y was proposed by Geoffrion [16] as

$$\begin{aligned} &\min_{y} v(y), \\ &s.t. \ \ v(y) = \min_{x\in X}\{F(x,y): \ \ s.t.\ G(x,y) \leq 0\}, \end{aligned} \tag{8}$$

$$y \in Y \cap V,$$

where $V = \{y : \ G(x,y) \leq 0 \text{ for some } x \in X\}$. Introducing the following two problems, Geoffrion [16] proposed a decomposition of (8) when X is a convex set, and the functions $F(x,y)$ and $G(x,y)$ are convex with respect to the variable x:

[1.] Primal problem:

$$\begin{aligned} &\min_x F(x,\bar{y}), \\ &s.t. \ \ G(x,\bar{y}) \leq 0, \\ &x \in X, \end{aligned} \tag{9}$$

where $\bar{y}$ is an arbitrary but fixed point in Y.

[2.] Master problem:

$$\begin{aligned} &\min_{y\in Y}\{\max_{u\geq 0}\{\min_{x\in X} F(x,y) + u^t G(x,y)\}\}, \\ &s.t. \ \ \min_{x\in X}\{\lambda^t G(x,y)\} \leq 0 \quad \text{ for all } \lambda \in \Lambda, \end{aligned} \tag{10}$$

where $\Lambda = \{\lambda \in \mathbb{R}^m : \lambda \geq 0, \ \sum_{i=1}^m \lambda_i = 1\}$ (m is the size of vector G).

Bagajewicz and Manousiouthakis [3] pointed out that the master problem is equivalent to the projection (8), in the case, where X is a convex set and $F(x,y)$ and $G(x,y)$ are convex functions with respect to the variable x. In addition, they showed that the master problem is also equivalent to the following problem:

$$\min_{y,y_0} y_0, \tag{11}$$

$$s.t. \ \ L^*(y,u) = \min_{x\in X}\{F(x,y) + u^t G(x,y)\} \leq y_0 \quad \text{ for all } u \geq 0, \tag{12}$$

$$L_*(y,\lambda) = \min_{x\in X}\{\lambda^t G(x,y)\} \leq 0 \quad \text{ for all } \lambda \in \Lambda. \tag{13}$$

Geoffrion [16] suggested to solve a relaxed version of problem (11)–(13) in which all but a few constraints are ignored. Indeed, in each iteration, the constraints corresponding to a subset of vectors u and λ, instead of all of them, are considered. Due to ignoring some of constraints, the obtained problem is called the relaxed master problem; see [3]. Since constraints are continuously added, the optimal values of this problem form a monotone nondecreasing sequence as the lower bounds to problem (7). Next, the steps of the GBD algorithm in the general form are represented.

3.1 General form of the GBD algorithm

Step 1. Let a point $\bar{y} \in Y \cap V$ be available. Solve the primal problem (9) and obtain the optimal solution x^* and the optimal multiplier vector u^*. Put the counters $K^f = 1$, $K^i = 0$. Set $UB = F(x^*, \bar{y})$. Select a tolerance $\epsilon \geq 0$ and set $u^{(1)} = u^*$. Finally, determine the function $L^*(y, u^{(1)})$.

Step 2. Solve globally the current relaxed master problem:

$$\begin{aligned}&\min_{y,y_0} y_0,\\ &s.t.\ \ L^*(y, u^{(k_1)}) \leq y_0,\ k_1 = 1, \dots, K^f, \qquad \text{(optimality constraint)}\\ &\qquad L_*(y, \lambda^{(k_2)}) \leq 0,\ \ k_2 = 1, \dots, K^i, \quad \text{(feasibility constraint)}.\end{aligned}$$

Let $(\hat{y}, \hat{y}_0)$ be the globally optimal solution. Set $LB = \hat{y}_0$. If $UB \leq LB + \epsilon$, then terminate.

Step 3. Solve globally the primal problem (9) using $\bar{y} = \hat{y}$.

Step 3.a. Feasible primal problem: Solve the primal problem (9) by using $\bar{y}$. If $v(\bar{y}) \leq LB + \epsilon$, then terminate. Otherwise, determine the optimal multiplier vector u^*, set $K^f = K^f + 1$ and $u^{(K^f)} = u^*$. If $v(\bar{y}) < UB$, then set $UB = v(\bar{y})$. Finally, determine the function $L^*(y, u^{(K^f)})$ and return to Step 2.

Step 3.b. Infeasible primal problem: Determine a set of values of $\lambda^* \in \Lambda$ satisfying $\min_{x \in X}\{(\lambda^*)^t G(x, \bar{y})\} > 0$. Set $K^i = K^i + 1$ and $\lambda^{(K^i)} = \lambda^*$. Determine the function $L_*(y, \lambda^{(K^i)})$. Return to Step 2.

Remark 1. To determine the value of λ^* in Step 3.b, the following problem should be solved (see [3]):

$$\begin{aligned}&\min_{x,\alpha} \alpha,\\ &s.t.\ \ G(x, \bar{y}) - \alpha \underline{1} \leq 0,\\ &\qquad x \in X,\ \alpha \in \mathbb{R},\end{aligned}$$

where $\underline{1} = (1, 1, \dots, 1)^t$.

Remark 2. If the functions $F(x, y)$ and $G(x, y)$ are separable with respect to x and y and convex with respect to x, the global optimal solutions for x within constraints (12) and (13) are obtained independently of y; see [3]. In this case, functions $L^*(y, u^{(k^f)})$ and $L_*(y, \lambda^{(k^i)})$ are defined in explicit forms.

Geoffrion stated some conditions ([16, Theorem 2.4] for the finite discrete set Y and [16, Theorem 2.5] for the infinite cardinality set Y) for problem (7), where the lowest solution of the primal and the global solution of the master would approach one another, and consequently, the global optimum of the overall problem would be provided within a prespecified tolerance. However, when the convexity in x does not hold, dual gaps may exist. Nevertheless, the global solution to the master problem will still provide a valid lower bound to problem (7). Next, an application of the GBD algorithm on problem (1)–(6) is investigated.

4 Implementing the GBD algorithm on the $MICpM$ problem

To apply the proposed GBD algorithm, we reformulate problem (1)–(6) as follows:

$$
\begin{aligned}
&\min_{x,y} Z_0(x,y) = \sum_{j\in J}\sum_{i\in I} x_{ij}(\ln x_{ij} - 1 - \ln p_{ij}),\\
&\sum_{i\in I} x_{ij} \le y_j c_j \qquad \text{for all } j\in J, \qquad\qquad (14)\\
&x \in X = \{x_{ij}\ge 0:\ \sum_{j\in J} x_{ij} = w_i \ \text{ for all } i\in I,\ \sum_{j\in J}\sum_{i\in I} d_{ij}x_{ij} \le B\},\\
&y \in Y = \{y:\ y_j \text{ is binary and } \sum_{j\in J} y_j = p\}.
\end{aligned}
$$

Note that the set X as well as functions $Z_0(x,y)$ and $G_j(x,y) = \sum_{i\in I} x_{ij} - y_j c_j$, for all $j \in J$, are convex with respect to x. Following the steps of Algorithm 3.1, the implementation of the GBD algorithm to solve the $MICpM$ problem is summarized as below.

4.1 GBD algorithm on the $MICpM$ problem

Step 1 Select a tolerance $\epsilon \ge 0$ and a feasible vector $\bar{y} \in Y$ for which there exists a feasible solution $x \in X$ satisfying in (14).

Step 1.a. Solve the corresponding primal problem (15)–(16)

$$\min_{x\in X} Z_0(x,\bar{y}) = \sum_{j\in J}\sum_{i\in I} x_{ij}(\ln x_{ij} - 1 - \ln p_{ij}), \qquad (15)$$

$$\sum_{i\in I} x_{ij} \le \bar{y}_j c_j \qquad \text{for all } j\in J, \qquad (16)$$

and obtain the optimal solution x^* and the optimal multipliers $u_j^*,\ j \in J$ corresponding to constraints (16). Set $K^f = 1, K^i = 0$, $UB = Z_0(x^*, \bar{y})$ and vector $u^{(1)} = u^*$.

Step 1.b. Determine the optimal solution $\hat{x}$ corresponding to the following subproblem and set $x^{(K^f)} = \hat{x}$:

$$\min_{x \in X} \sum_{j \in J} \sum_{i \in I} x_{ij}(\ln x_{ij} - 1 - \ln p_{ij}) + \sum_{j \in J} u_j^{(K^f)} \sum_{i \in I} x_{ij}. \tag{17}$$

Step 2 Solve globally the current relaxed master problem:

$$\min_{y_0, y \in Y} y_0, \tag{18}$$

$$\begin{aligned} s.t.\ \ L^*(y, u^{(k_1)}) = &\sum_{j \in J} \sum_{i \in I} x_{ij}^{(k_1)} (\ln x_{ij}^{(k_1)} - 1 - \ln p_{ij}) \\ &+ \sum_{j \in J} u_j^{(k_1)} (\sum_{i \in I} x_{ij}^{(k_1)} - c_j y_j) \le y_0, \quad k_1 = 1, \ldots, K^f, \end{aligned} \tag{19}$$

$$L_*(y, \lambda^{(k_2)}) = \sum_{j \in J} \lambda_{j \in J}^{(k_2)} (\sum_{i \in I} x_{ij}'^{(k_2)} - c_j y_j) \le 0, \quad k_2 = 1, \ldots, K^i,\ \lambda \in \Lambda.$$

Note that in each iteration, the variables $u^{(k_1)}, x^{(k_1)}$, and $x'^{(k_2)}$ are specified fixed values that have been obtained during the previous steps of the algorithm; therefore, problem (18) is a linear mixed-binary model, which could be globally solved by such existing methods as branch and bound. Let $(\hat{y}, \hat{y}_0)$ be the global optimal solution. Set $LB = \hat{y}_0$. If $UB \le LB + \epsilon$, then terminate.

Step 3. Solve globally the following problem to examine the feasibility of the primal problem (15)–(16) corresponding to $\bar{y} = \hat{y}$.

$$\begin{aligned} &\min_{x, \alpha} \alpha, \\ &\sum_i x_{ij} - c_j \bar{y}_j - \alpha \le 0 \quad \text{for all } j \in J, \\ &x \in X,\ \alpha \in \mathbb{R}. \end{aligned} \tag{20}$$

Step 3.a. If $\alpha \le 0$, then the primal problem (15)–(16) is feasible. Solve the primal problem for the given vector $\bar{y}$ to find the optimal solution x^*. If $Z_0(x^*, \bar{y}) \le LB + \epsilon$, then terminate. Otherwise, determine the optimal multiplier vector u^* corresponding to constraints (16), and set $K^f = K^f + 1$ and $u^{(K^f)} = u^*$. If $Z_0(x^*, \bar{y}) < UB$, then put $UB = Z_0(x^*, \bar{y})$. Finally, return to Step 1.b.

Step 3.b. If $\alpha > 0$, then the primal problem (15)–(16) is infeasible. Determine the optimal multipliers λ_j^* corresponding to constraints (20).

Set $K^i = K^i + 1$ and $\lambda^{(K^i)} = \lambda^*$. To find the optimal solution $\tilde{x}$, solve the following model:

$$\min_{x \in X} \sum_{j \in J} \lambda_j^{(K^i)} \sum_{i \in I} x_{ij}.$$

Put $x'^{(K^i)} = \tilde{x}$ and go to Step 2.

The purpose of the algorithm is to minimize the gap between the upper and lower bounds. Next, the convergence conditions stated by Geoffrion [16], are verified for problem (1)–(6).

Theorem 1. The GBD Algorithm 4.1 for the $MICpM$ problem (1)–(6) terminates in a finite number of steps, for any given $\epsilon \geq 0$.

Proof. Since $Y = \{y : y_j$ is binary and $\sum_{j \in J} y_j = p\}$ is a finite discrete set, the convergence of Algorithm 4.1 should be established via [16, Theorem 2.4]. Indeed, the following conditions must hold:

i. X is a nonempty convex set and functions $G_j(x, y) = \sum_{i \in I} x_{ij} - y_j c_j$ are convex on X for each fixed $y \in Y$. Further, the set $Z_y = \{z = (z_1, \ldots, z_{|J|}) : G_j(x, y) \leq z_j$ for some $x \in X\}$ is closed for each fixed $y \in Y$.

ii. Functions $Z_0(x, y)$ and $G_j(x, y)$ are convex on X for each fixed $y \in Y$. In addition, for each fixed $\bar{y} \in Y \cap V$, either the optimal value $Z_0(x, \bar{y})$ is infinite, or the primal problem (15)–(16) possesses an optimal multiplier vector.

Part i. Obviously, the set $X = \{x_{ij} \geq 0 : \sum_{j \in J} x_{ij} = w_i, \ \forall i \in I, \ \sum_{j \in J} \sum_{i \in I} d_{ij} x_{ij} \leq B\}$ is nonempty ($x_{ii} = w_i$, for all $i \in I$ and $x_{ij} = 0$ for $i \neq j$ is a feasible solution), closed and bounded. Noting the continuity of functions $G_j(x, y)$ on X, the closeness of Z_y is implied; see [16].

Part ii. The convexity of functions $Z_0(x, y) = \sum_{j \in J} \sum_{i \in I} x_{ij}(\ln x_{ij} - 1 - \ln p_{ij})$ and $G_j(x, y) = \sum_{i \in I} x_{ij} - y_j c_j$ with respect to x for each fixed y is straightforward by the definition. Following the saddle point optimality conditions stated in [4], the existence of optimal multiplier vector is inferred; see [15]. □

The overall scheme of the algorithm is represented in Figure 1.

4.2 Solving the primal problem

The primal problem (15)–(16) is a convex nonlinear problem, which can be solved by applying the KKT optimality conditions. Let $\bar{y}$ be given. Consider the Lagrangian dual vectors u_j, γ_i, and η corresponding to constraints

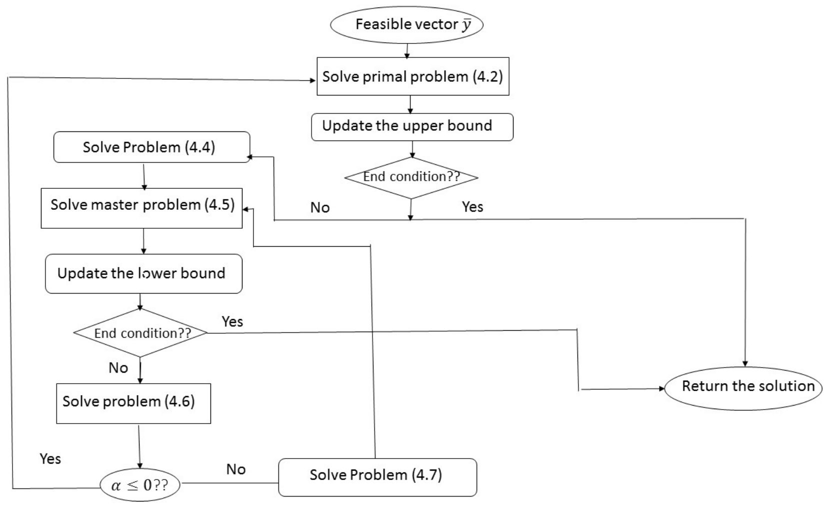

Figure 1: The overall framework of the proposed algorithm

(22), (23), and (24). Then Lemma 1 expresses the relationship between the primal and dual variables:

$$\min_x Z_0(x, \bar{y}) = \sum_{j \in J} \sum_{i \in I} x_{ij}(\ln x_{ij} - 1 - \ln p_{ij}), \tag{21}$$

$$\sum_{i \in I} x_{ij} \leq c_j \bar{y}_j \qquad \text{for all } j \in J, \tag{22}$$

$$\sum_{j \in J} x_{ij} = w_i \qquad \text{for all } i \in I, \tag{23}$$

$$\sum_{j \in J} \sum_{i \in I} d_{ij} x_{ij} \leq B, \tag{24}$$

$$x_{ij} \geq 0 \quad \text{for all } i \in I, j \in J. \tag{25}$$

Lemma 1. The optimal solution of problem (21)–(25) is obtained by $x_{ij} = p_{ij} e^{-(u_j + \gamma_i + \eta d_{ij})}$, for all $j \in J, i \in I$, where u_j, γ_i, and η are the optimal multiplier vectors corresponding to constraints (22), (23), and (24), respectively.

Proof. The KKT optimality conditions of model (21)–(25) with respect to x obtain in the following expressions:

$$\ln x_{ij} - \ln p_{ij} + u_j + \gamma_i + \eta d_{ij} \geq 0, \tag{26}$$

$$x_{ij}(\ln x_{ij} - \ln p_{ij} + u_j + \gamma_i + \eta d_{ij}) = 0. \tag{27}$$

The above conditions state that one of the following cases holds in the optimality:

$$
\begin{aligned}
&x_{ij} = 0 \text{ and } \ln x_{ij} - \ln p_{ij} + u_j + \gamma_i + \eta d_{ij} \geq 0,\\
&\qquad \to x_{ij} \geq p_{ij} e^{-(u_j+\gamma_i+\eta d_{ij})} \geq 0,\\
&\text{or}\\
&x_{ij} \geq 0 \text{ and } \ln x_{ij} - \ln p_{ij} + u_j + \gamma_i + \eta d_{ij} = 0,\\
&\qquad \to x_{ij} = p_{ij} e^{-(u_j+\gamma_i+\eta d_{ij})} \geq 0.
\end{aligned}
$$

The former case holds if $p_{ij} = 0$; therefore, the optimal solution would be written as $x_{ij} = p_{ij} e^{-(u_j+\gamma_i+\eta d_{ij})} \geq 0$ in both cases. □

Note that to obtain the optimal solution of problem (21)–(25), the relation obtained for x_{ij} must be considered together with the other optimality conditions of the problem, resulting in the following system to be solved:

$$x_{ij} = p_{ij} e^{-(u_j+\gamma_i+\eta d_{ij})} \quad \text{for all } i \in I, \quad j \in J, \tag{28}$$

$$\sum_{i\in I} x_{ij} \leq c_j \bar{y}_j \quad \text{for all } j \in J, \tag{29}$$

$$u_j(\sum_{i\in I} x_{ij} - c_j \bar{y}_j) = 0 \quad \text{for all } j \in J, \tag{30}$$

$$\sum_{j\in J} x_{ij} = w_i \quad \text{for all } i \in I, \tag{31}$$

$$\eta(\sum_{j\in J}\sum_{i\in I} d_{ij} x_{ij} - B) = 0, \tag{32}$$

$$\sum_{j\in J}\sum_{i\in I} d_{ij} x_{ij} \leq B, \tag{33}$$

$$\eta, u \geq 0. \tag{34}$$

To transform the nonlinear terms in the system of equations (28)–(34) into linear ones, a linearization method is applied. Setting $v_j = e^{-u_j}$, $s_i = e^{-\gamma_i}$, $k_{ij} = e^{-\eta d_{ij}}$ and noting that $u, \eta \geq 0$, the system of equations (28)–(34) is rewritten as follows:

$$
\begin{aligned}
&x_{ij} = p_{ij} v_j s_i k_{ij} \quad \text{for all } i \in I, \quad j \in J,\\
&\sum_{i\in I} x_{ij} \leq c_j \bar{y}_j \quad \text{for all } j \in J,\\
&(v_j - 1)(\sum_{i\in I} x_{ij} - c_j \bar{y}_j) = 0 \quad \text{for all } j \in J, \qquad (35)\\
&\sum_{j\in J} x_{ij} = w_i \quad \text{for all } i \in I,
\end{aligned}
$$

$$(k_{ij}-1)(B-\sum_{j\in J}\sum_{i\in I} d_{ij}x_{ij}) = 0 \quad \text{for all } i \in I,\ j\in J, \tag{36}$$

$$\sum_{j\in J}\sum_{i\in I} d_{ij}x_{ij} \leq B, \tag{37}$$

$$v_j \leq 1,\quad k_{ij} \leq 1 \qquad \text{for all } i\in I, j\in J, \tag{38}$$

$$v_j, s_i, k_{ij} > 0 \qquad \text{for all } i\in I, \quad j\in J.$$

Since $u_j = 0$ results in $v_j = 1$, for all $j \in J$, and visa versa. Therefore, constraints (30) and (35) are equivalent. The same statement holds for constraints (32) and (36). Using the approach introduced in [7], the product of variables $f_{ij} = h_{ij}k_{ij} = v_j s_i k_{ij}$ can be linearized by adding some constraints; see [1]. The final linear mixed-integer system that should be solved is equivalent to the following:

$$x_{ij} = p_{ij}f_{ij} \text{ for all } i\in I, \quad j\in J, \tag{39}$$

$$\sum_{i\in I} x_{ij} \leq c_j\bar{y}_j \qquad \text{for all } j\in J,$$

$$\sum_{j\in J} x_{ij} = w_i \qquad \text{for all } i\in I,$$

$$s_i - h_{ij} \leq Mr_j,\quad c_j\bar{y}_j - \sum_i x_{ij} \leq M(1-r_j) \qquad \text{for all } j\in J,$$

$$h_{ij} - f_{ij} \leq Mq,\quad B - \sum_i\sum_j d_{ij}x_{ij} \leq M(1-q) \qquad \text{for all } i\in I, j\in J,$$

$$\sum_{j\in J}\sum_{i\in I} d_{ij}x_{ij} \leq B, \tag{40}$$

$$f_{ij} \leq h_{ij},\quad h_{ij}\leq s_i \qquad \text{for all } i\in I,\quad j\in J, \tag{41}$$

$$s_i, h_{ij}, f_{ij} > 0,\ r_j, q \in \{0,1\} \qquad \text{for all } i\in I, \quad j\in J,$$

in which M is a sufficiently large number. Problem (17) in Step 1.b can be similarly solved while the other problems in Steps 2 and 3 are linear or mixed-binary linear that would be solved by existing methods.

5 Numerical examples

In this section, the proposed GBD algorithm is implemented in $GAMS$ 23.7 environment to determine the most probable allocation solution on some small and medium-sized networks. During the steps of the algorithm, the obtained mixed-binary linear problem (18) is solved by such mixed-integer programming solvers as $CPLEX$.

Although by Theorem 1, the algorithm would converge after a finite number of iterations, implementing numerical examples revealed that the optimal solution would be obtained much sooner than the convergence. Indeed, after a number of iterations, the upper bound values might marginally change or even remain unchanged, which results in an optimal or near-optimal solution. Therefore, we have considered some other stopping criteria, by which the algorithm continues until no progress is achieved in the upper or lower bounds during a predetermined number of successive iterations, or the total number of iterations reaches its maximum value. The used stopping criteria are summarized below:

i. To reach a solution with $UB \leq LB + \epsilon$, where ϵ is a predetermined tolerance.

ii. Proceeding the algorithm with no improvement in the values of UB and LB for k successive iterations.

iii. To reach the maximum number of iterations Max_{iter}.

Example 1. Consider the small network shown in Figure 2 wherein $I = J = N$ and the costs of links have been given next to them. The value of d_{ij} for each pair (i, j) is calculated by the shortest distance between i and j. The capacities and demands of nodes, (c, w), are inserted in Table 2. To estimate the probabilities, the approximate distances with 5 additional criteria indicated by random integers, representing such geographical and local features as proximity to transportation services, security, and surrounding facilities, are considered. The compared weights of different nodes j for a certain node i with respect to the first attribute, that is, the approximate distance, are determined as follows:

$$\begin{cases} \frac{0.45}{n_i^1} \text{ if } j \in S_i^1 = \{j : d_{ij} \leq 25\}, \\ \frac{0.25}{n_i^2} \text{ if } j \in S_i^2 = \{j : 25 < d_{ij} \leq 35\}, \\ \frac{0.15}{n_i^3} \text{ if } j \in S_i^3 = \{j : 35 < d_{ij} \leq 45\}, \\ \frac{0.10}{n_i^4} \text{ if } j \in S_i^4 = \{j : 45 < d_{ij} \leq 55\}, \\ \frac{0.05}{n_i^5} \text{ if } j \in S_i^5 = \{j : 55 < d_{ij}\}, \end{cases}$$

where $n_i^q = |S_i^q|, \ q = 1, \ldots, 5$. In addition, the compared weights of nodes with respect to the other 5 attributes $r = 2, \ldots, 6$ are given in Table 2. We use the multiple attributes decision making procedures to calculate the probabilities p_{ij} for different weight vectors $Weight$, which defines the relative importance of 6 criteria; see [1, 29].

To see the effect of the available budget as well as emphasizing on different attributes, the solution of the $MICpM$ problem (1)–(6) for $p = 3, 2$ is estimated for $B = 15000$ and $B = 12000$, by applying weight vectors $Weight_1$ and $Weight_2$; see Table 3. The solutions are estimated via both the GBD algorithm with $Max_{iter} = 15, \ \epsilon = 100$, and $k = 5$ as well as the

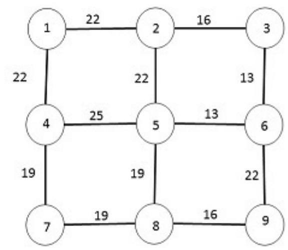

Figure 2: The grid network with 9 nodes

Table 2: The capacities and demands of nodes along with their weights with respect to criteria $r = 2, \ldots, 6$

Nodes	1	2	3	4	5	6	7	8	9
c	300	200	350	320	250	250	350	100	200
w	50	60	80	40	90	100	80	60	50
Criteria	1	2	3	4	5	6	7	8	9
2	0.0541	0.0270	0.1351	0.1622	0.0811	0.1081	0.0541	0.1351	0.2432
3	0.0244	0.0488	0.1220	0.0976	0.1707	0.1951	0.2195	0.0732	0.0488
4	0.0455	0.1136	0.1591	0.0227	0.0682	0.1591	0.2045	0.1818	0.0455
5	0.2162	0.0270	0.0811	0.1351	0.1081	0.0541	0.2432	0.0270	0.1081
6	0.2286	0.0571	0.1429	0.0286	0.1143	0.0857	0.2571	0.0571	0.0286

"Dicopt" solver of $GAMS$ 23.7, where the allocation solutions along with their objective function values ($OBJF$) are inserted in Table 3.

As it is seen, the optimal values of the objective function provided by the GBD algorithm are better than or at least equal to those provided by "*Dicopt*" solver. Note that, the optimal solution might change by changing the $GAMS$ solver. However, examining different cases showed that the solutions obtained by *"Dicopt"* are usually better than or at least near to the other solvers of $GAMS$ and hence have been reported in Table 3.

Referring to the estimated solutions, it can be realized if the capacities are large enough, the shares of the selected facilities in the $MICpM$ allocation solution are proportional to their probabilities. For example, the probabilities $p(1, j), j = 1, \ldots, 9$ for $Weight_1$ are estimated as

$$p(1, :) = [0.1617, 0.1072, 0.1090, 0.1423, 0.0983, 0.0895, 0.1471, 0.0642, 0.0808].$$

Among the selected medians $3, 6, 7$, using the GBD algorithm, the median 7 has the most probability and consequently the most share to provide the demand of node 1. Indeed, the higher is the probability p_{ij}, the more would be the share of the median v_j to provide the demand of node v_i. The same statement holds for medians $1, 3, 7$ found by the $GAMS$ solver. However, decreasing the budget B to 12000 results in a different allocation solution to assign as much demand as possible to the closer facilities; compare the column of $i = 6$ in cases $B = 12000$ and $B = 15000$ for the weight vector $Weight_1$ in the GBD solution.

For the weight vector $Weight_2$, wherein the criterion $r = 6$ is ranked as the most important one, the locations of 2 medians are to be selected. As it is seen, both the GBD and *"Dicopt"* solver determine the nodes $3, 7$ for

Table 3: The allocation solution for the $MICpM$ problem

Facility \ Node	1	2	3	4	5	6	7	8	9	$OBJF$	$Time$
$B = 15000$	$p = 3$, $Weight_1 = [0.3462\ 0.1923\ 0.1154\ 0.0385\ 0.1154\ 0.1923]$										
The GBD algorithm											
3	16.739	27.739	36.479	8.315	26.951	37.234	12.206	13.266	13.765	2533.355	4
6	11.952	17.815	29.058	9.391	34.628	40.470	13.927	15.904	18.429		
7	21.308	14.446	14.463	22.294	28.421	22.296	53.867	30.831	17.806		
GAMS solver *"Dicopt"*											
1	31.095	24.002	15.583	15.006	21.182	16.798	12.873	7.646	5.906	2583.869	8
3	8.491	27.644	54.607	5.016	36.048	60.544	6.251	12.448	19.226		
7	10.414	8.354	9.810	19.978	32.770	22.659	60.876	39.906	24.869		
$B = 12000$	$p = 3$, $Weight_1 = [0.3462\ 0.1923\ 0.1154\ 0.0385\ 0.1154\ 0.1923]$										
The GBD algorithm											
3	18.245	35.331	46.724	5.194	25.673	37.088	4.160	9.138	12.350	2598.169	4
6	9.937	17.310	28.391	7.690	43.242	52.847	6.222	14.362	21.675		
7	21.818	7.359	4.885	27.117	21.086	10.065	69.617	36.499	15.975		
GAMS solver *"Dicopt"*											
1	39.397	34.999	18.405	14.815	14.836	9.744	8.310	4.658	2.983	2655.921	10
6	2.925	17.626	53.375	4.664	50.319	77.113	4.155	13.557	26.266		
7	7.678	7.375	8.219	20.521	24.845	13.144	67.535	41.785	20.751		
$B = 15000$	$p = 2$, $Weight_2 = [0.0714,\ 0.0714,\ 0.0714,\ 0.0714,\ 0.0714,\ 0.6429]$										
The GBD algorithm											
3	20.035	41.620	63.996	7.688	41.899	65.449	6.953	13.305	19.109	2654.990	4
7	29.965	18.380	16.004	32.312	48.101	34.551	73.047	46.695	30.891		
GAMS solver *"Dicopt"*											
1	49.154	32.910	5.434	38.559	27.072	6.799	75.123	21.472	3.478	2867.986	5
3	0.846	27.090	74.566	1.441	62.928	93.201	4.877	38.528	46.522		
$B = 12000$	$p = 2$, $Weight_2 = [0.0714,\ 0.0714,\ 0.0714,\ 0.0714,\ 0.0714,\ 0.6429]$										
The GBD algorithm											
3	21.181	54.045	77.430	3.191	50.387	86.269	0.998	6.688	19.109	2778.247	4
7	28.819	5.955	2.570	36.809	39.613	13.731	79.002	53.312	30.891		
GAMS solver *"Dicopt"*											
3	21.181	54.045	77.430	3.191	50.387	86.269	0.998	6.688	19.109	2778.247	4
7	28.819	5.955	2.570	36.809	39.613	13.731	79.002	53.312	30.891		

$B = 12000$ as the optimal locations, while for $B = 15000$, the GBD provides a better solution. Referring to Table 2, it is realized that the model finds the optimal location of facilities in nodes having more weights with respect to the attribute $r = 6$, while simultaneously the total transportation cost is limited to the given budget. In the case of enough capacities, the shares of the selected facilities to provide the demands of nodes are again based on their corresponding probabilities. In addition, reducing the budget changes the allocation solution so that the closer is the median, the larger will be the assigned share.

To analyze the convergence of the GBD algorithm, the variations of the upper and lower bounds through successive iterations for weight vector $Weight_1$ with $B = 15000$, $p = 3$ and $Weight_2$ with $B = 15000$, $p = 2$, are illustrated in Figures 3 and 4, respectively. Here, no limit has been considered on iterations, that is, the only stopping criterion is satisfying the case i and the other two ones are neglected. In Figure 3, the algorithm has stopped after 76 iterations, where the lower bound becomes greater than the upper bound. However, the upper bound values remain almost unchanged after 20 iterations. Indeed, the optimal value of the objective function (1) has been obtained in the 20th iteration. For the weight vector $Weight_2$ with $B = 15000$, $p = 2$ in Figure 4, the algorithm converges in the 7the iteration while the optimal value of the objective function (1) has been achieved in the first iteration.

Example 2. In the second example, the efficiency of the proposed algorithm on larger problems is investigated. First we consider a 100-node network, where the network topology, the lengths of links, and the demands and capacities of nodes are taken from the URL address $http://people.brunel.ac.uk/mastjjb\ /jeb/orlib/pmedinfo.html$. To calculate the probabilities p_{ij}, ap-

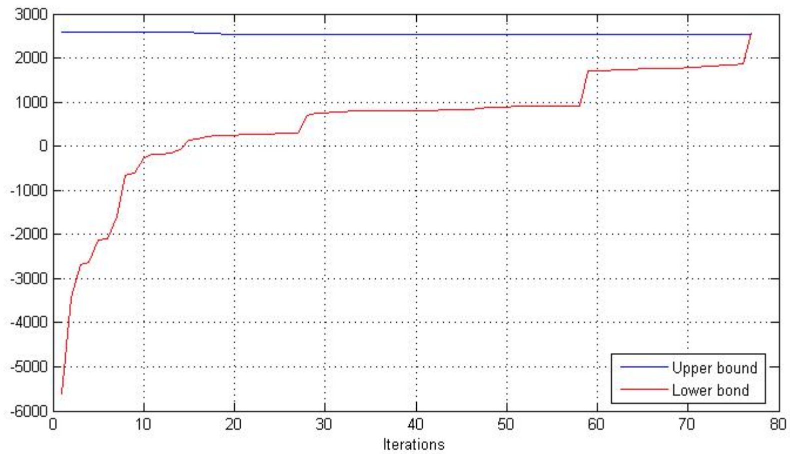

Figure 3: The variations of the upper and lower bounds for $Weight_1$, $B = 15000, p = 3$

Table 4: Comparison between GBD and $GAMS$ solvers

	$B = 800000,\ p = 10$		$B = 900000,\ p = 10$		$B = 500000,\ p = 20$	
Solver	Objective function	Running time	Objective function	Running time	Objective function	Running time
Knitro	–	–	–	–	–	–
Baron	–	–	–	–	–	–
Bonmin	40639.753	2077	41851.221	1004	40396.948	1006
Dicopt	41263.215	90	40557.247	16	40340.818	32
Sbb	–	–	40247.635	1005	–	–
Lindo global	–	–	–	–	–	–
Oqnlp	–	–	–	–	–	–
GBD	41929.25	26	41474.553	29	39942.716	26

proximate distances plus 5 additional criteria, indicated by random integers assigned to network nodes, are considered. The problem is solved for $p = 10$ and $B = 800000,\ 900000$ and $p = 20,\ B = 500000$, by applying different $MINLP$ solvers of $GAMS$ as well as the GBD algorithm. The stopping criteria for the GBD algorithm are the same as those used in Example **??** 1. The optimal value of the objective function and the elapsed running time in each case have been reported in Table 4. The sign $(-)$ indicates that the solver has not been able to find any solution for the problem. As it is seen, the GBD algorithm generally provides an adequate solution in reasonable running time in comparison with the used solvers.

Next, we use the method proposed by Dolan and More [11] to analyze the performance profile of the GBD algorithm as well as three solvers *Bonmin*, *Dicopt*, and *Sbb*. To this purpose, we have considered 30 problems $p \in P$ taken from the URL address $http://people.brunel.ac.uk/\ \ mastjjb/jeb/orlib/pmedinfo.html$ on networks with 100, 250, 300 and 400 nodes, and solved them by the three mentioned solvers $s = 1, 2, 3$ and the GBD algorithm, as well. Let

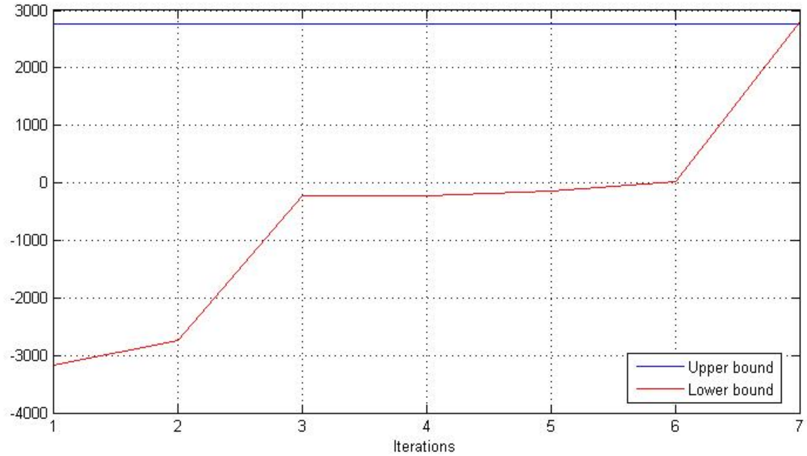

Figure 4: The variations of the upper and lower bounds for $Weight_2$, $B = 15000, p = 2$

$$t_{s,p} = \text{ the running time required to solve problem } p \text{ by solver } s.$$

Referring to [11], the performance on problem $p \in P$ by solver s is compared with the best performance realized among all the used solvers on this problem. Therefore, we use the performance ratio

$$r_{s,p} = \frac{t_{s,p}}{\min\{t_{s,p} : s = 1, 2, 3\}}.$$

A parameter $r_M \geq r_{s,p}$ for all p, s is chosen, where $r_{s,p} = r_M$ if and only if the solver s does not solve problem p. Then, the probability for the solver s that the performance ratio $r_{s,p}$ is less than or equal to a factor $\tau \in \mathbb{R}$, is demonstrated by

$$\rho_s(\tau) = \frac{1}{|P|}\text{size}\{p,\ r_{s,p} \leq \tau\}.$$

The function $\rho_s(\cdot)$ is the (cumulative) distribution function for the performance ratio and is termed as the performance profile; see [11]. Figure 5 illustrates the comparison between the performance profiles of the three solvers and the GBD algorithm for $\tau = 1, \ldots, 200$. As it is realized, the GBD algorithm has much better performance in comparison with the applied $GAMS$ solvers. In addition, even for those problems $p \in P$, where the solvers could not provide any solution, the GBD algorithm reached a reasonable upper bound in less than 200 seconds.

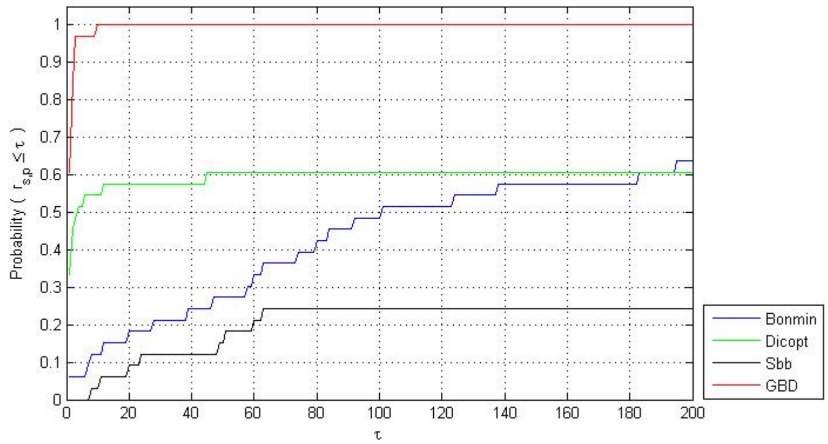

Figure 5: Comparison between the performance profiles of the $GAMS$ solvers and the GBD algorithm

6 Summary and conclusions

In this paper, we proposed a log-based model using the minimum information approach, for the capacitated p-median problem to estimate the most likely allocation solution, while the total transportation cost should be less than or equal to a predetermined budget. Using the generalized benders decomposition method, the proposed problem was reduced to easier subproblems to which some constraints were added iteratively, and the upper and lower bounds have been updated. The convergence of the algorithm was established analytically, while its efficiency in comparison to $MINLP$ solvers was investigated practically by applying the performance profile test on some numerical examples.

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