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Research Article

Discontinuous Galerkin approach for two-parametric convection-diffusion equation with discontinuous source term

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Abstract

In this article, we explore the discontinuous Galerkin finite element method for two-parametric singularly perturbed convection-diffusion problems with a discontinuous source term. Due to the discontinuity in the source term, the problem typically shows a weak interior layer. Also, the presence of multiple perturbation parameters in the problem causes boundary layers on both sides of the boundary. In this work, we develop the nonsymmetric discontinuous Galerkin finite element method with interior penalties to handle the layer phenomenon. With the use of a typical Shishkin mesh, the domain is discretized, and a uniform error estimate is obtained. Numerical experiments are conducted to validate the theoretical conclusions.

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1 Introduction

In this article, we consider a two-parametric singularly perturbed problem with a discontinuous source term

$$\begin{cases} -\varepsilon_1 u'' + \varepsilon_2 b(x)u' + c(x)u = f, \quad x \in \Omega = (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $0 < \varepsilon_1,\ \varepsilon_2 \ll 1$. In addition, the coefficient functions, $b(x)$ and $c(x)$, are assumed to sufficiently smooth such that $b(x) \geq b_0 > 0$, $c(x) \geq c_0 > 0$, and

$$\gamma^2 = c(x) - \frac{1}{2}\varepsilon_2 b'(x) \geq \gamma_0 > 0, \tag{2}$$

where b_0, c_0, and γ_0 are positive constants and the source function, $f(x)$, has discontinuity of type I at some interior point say $x = d$. Moreover, we also assume that $[f](d) \leq C$, where $[f](d) = f(d+) - f(d-)$. Problems of these types have several real-world applications, like transport phenomena in chemistry, biology, financial mathematics, and so on. Several numerical techniques are available in the literature on finite difference methods and finite element methods; for example, see [1, 10, 14] for the finite element method (FEM) and [3] for the finite difference method (FDM). Moreover, few articles [7, 4] are available that deal with two-parametric singularly perturbed boundary value problems, in which the authors have used various difference schemes to establish uniform convergence. In [11], authors have studied the superconvergence of non-symmetric interior penalty Galerkin FEM for two-parametric singularly perturbed boundary value problems.

The FEM is one of the efficient numerical methods among all other numerical methods to compute numerical solutions of differential equations. Kadalbajoo and Yadaw [5] established a second-order convergence of FEM in maximum norm on Shishkin mesh for two-parametric convection-diffusion problem. Brdar and Zarin [2] achieved uniform convergence of FEM in the energy norm on Bakhvalov mesh, while Zarin [12] has proved uniform convergence on exponentially graded mesh with FEM. In spite of these works, no one has considered the two-parametric convection-diffusion problem with discontinuous coefficients or source terms, which also predominately appears in the modeling of real-world problems.

On the other way, the nonsymmetric interior penalty Galerkin (NIPG) method uses penalty parameters to enforce both the continuity of the solution and boundary conditions. Another interesting factor about the NIPG method is to handle the unstructured mesh, discontinuous coefficients, and local mesh refinements. Furthermore, an NIPG method can be made stable and coercive in the sense of convergence by taking some user-chosen penalty parameters. Moreover, in other variants of the discontinuous Galerkin method like symmetric interior penalty Galerkin method and incomplete interior penalty Galerkin method, we have to choose penalty parameters, and it is a herculean task to establish its coercivity.

Therefore, in this present work, we consider a two-parametric convection-diffusion problem with a discontinuous source term and establish a second-order convergence (superconvergence) of the NIPG method in the norm $\|\cdot\|_\varepsilon$ induced by the corresponding bilinear form.

The article is arranged in the following way: In Section 2, we study the continuous problem and the bounds for different components of the exact solution. Section 3 is all about the bilinear form with respect to the NIPG method, its properties, and the existence and uniqueness of the weak solution. Construction of the Shishkin-type mesh, error analysis, and establishment of convergence results are introduced in Section 4. The numerical results and discussions are kept in Section 5 that which supports our theoretical conclusions.

Note: Throughout this paper, we have taken C as a generic constant, which is free from perturbation parameters ε_1, ε_2, and the discretization parameter, N.

2 The continuous problem and bounds for the solutions

This section is designed to discuss the solution of the continuous problem (1), their smooth and layer components, and their respective bounds. In general, the solution to the reduced problem (1) does not satisfy the boundary conditions. Therefore we have boundary and interior layers in the solution. To handle these boundary and interior layers, let p_0 and p_1 be two solutions of the characteristic equation:

$$-\varepsilon_1 p^2(x) + \varepsilon_2 b(x)p(x) + c(x) = 0.$$

Moreover, $p_0(x) < 0$ and $p_1(x) > 0$ describe the boundary layers at $x = 0$ and $x = 1$, respectively. Let

$$\lambda_l = -\max_{x\in[0,1]} p_0(x), \quad \lambda_r = \min_{x\in[0,1]} p_1(x),$$

where λ_l and λ_r are defined as

$$\lambda_l = \min_{x\in[0,1]} \frac{-\varepsilon_2 b(x) - \sqrt{\varepsilon_2^2 b(x)^2 + 4\varepsilon_1 c(x)}}{2\varepsilon_1},$$
$$\lambda_r = \min_{x\in[0,1]} \frac{-\varepsilon_2 b(x) + \sqrt{\varepsilon_2^2 b(x)^2 + 4\varepsilon_1 c(x)}}{2\varepsilon_1}.$$

In that case, the problems are categorized on the basis of the relationship between the perturbation parameters:

Case *I***.** When $\varepsilon_1 \ll \varepsilon_2 = 1$, then $\lambda_l = \mathcal{O}(\varepsilon_1^{-1})$ and $\lambda_r = \mathcal{O}(1)$. It is the case of singular perturbation problem with a only one perturbation parameter.

Case *II***.** When $\varepsilon_1 \ll \varepsilon_2 \ll 1$, then $\lambda_l = \mathcal{O}(\varepsilon_2\varepsilon_1^{-1})$ and $\lambda_r = \mathcal{O}(\varepsilon_1^{-1})$. Clearly one can see that $\lambda_l < \lambda_r$, which results a stronger boundary layer at $x = 1$ than the boundary layer at $x = 0$.

Case *III***.** When $\varepsilon_2^2 \ll \varepsilon_1 \ll 1$, then λ_l and λ_r both are of order $\mathcal{O}(\varepsilon_1^{-1/2})$, and problem behaves like reaction diffusion problems.

Further details about the category of the problems can be found in [11]. In this work, we discuss the a posteriori error estimation for the problem that falls under the category of case II.

Theorem 1. Let $p \in (0,1)$ be arbitrary and let

$$\varepsilon_2 m \left\| b' \right\|_{L^\infty(\Omega)} \leq k(1-p).$$

Then, the following bounds for the derivatives of the exact solution holds:

$$\left| u^k(x) \right| \leq C\big(1 + \lambda_l^k e^{-p\lambda_l x} + \lambda_l^k e^{-p\lambda_l (d-x)} + \lambda_r^k e^{-p\lambda_r (x-d)} + \lambda_r^k e^{-p\lambda_r (1-x)}\big),$$
$$x \in \Omega, \text{ for } 0 \leq k \leq m.$$

Proof. The proof can be found in [6]. ☐

This motivates us to decompose the exact solution u of (1) into smooth and layer components in the following way:

$$u = S + E_{l,1} + E_{l,2} + E_{r,1} + E_{r,2}$$

such that the bounds satisfy the following relations:

$$|S^k(x)| \leq C, \tag{3a}$$
$$|E_{l,1}^k(x)| \leq C\lambda_l^k e^{-p\lambda_l x}, \tag{3b}$$
$$|E_{l,2}^k(x)| \leq C\lambda_l^k e^{-p\lambda_l (d-x)}, \tag{3c}$$
$$|E_{r,1}^k(x)| \leq C\lambda_r^k e^{-p\lambda_r (x-d)}, \tag{3d}$$
$$|E_{r,2}^k(x)| \leq C\lambda_r^k e^{-p\lambda_r (1-x)}, \tag{3e}$$

for $x \in \Omega$ and $0 \leq k \leq m$.

3 The NIPG method

Let $\mathcal{T}_N = I_j = (x_{j-1}, x_j),\ j = 1, 2, \cdots, N$ be a partition of the domain Ω. Define the broken Sobolev space of order k, $H^k(\Omega, \mathcal{T}_N) := \{v : v \in L^2(\Omega), v\big|_{I_j} \in H^k(I_j), \text{ for all } I_j \in \mathcal{T}_N\}$. Definitions of the Sobolev norm and seminorm are as follows:

$$\|v\|_{k,\mathcal{T}_N} := \left(\sum_{j=1}^{N} \|v\|_{k,I_j}^2\right)^{1/2} \quad |v|_{k,\mathcal{T}_N} := \left(\sum_{j=1}^{N} |v|_{k,I_j}^2\right)^{1/2}.$$

We define the finite element space V^N that are related to the partition $\mathcal{T}_N$ as follows:

$$V^N = \left\{ v : v \in L^2(\Omega), v\big|_{I_j} \in P^1(I_j), \text{ for all } I_j \in \mathcal{T}_N \right\}.$$

Here, $P^1(I_j)$ is the space of all polynomials of degree at most one on I_j. It is to note that the elements in V^N are allowed to be discontinuous across the boundaries of each element I_j.

The corresponding weak formulation of (1) can be read as follows: Find $u_N \in V^N$ such that

$$B(u_N, v_N) = l(v_N), \quad \text{for all } v_N \in V^N, \tag{4}$$

where

$$B(u, v) = B_1(u, v) + B_2(u, v).$$

Also, $B_1(u, v)$, $B_2(u, v)$, and $l(v)$ are defined for all $u, v \in H^1(\Omega, \mathcal{T}_N)$ as

$$\begin{aligned} B_1(u, v) &= \sum_{j=1}^{N} \int_{I_j} \varepsilon_1 u' v' dx + \sum_{j=1}^{N} \varepsilon_1 \big([u(x_j)]\{v'(x_j)\} - \{u'(x_j)\}[v(x_j)] \big) \\ &\quad + \sum_{j=1}^{N} \sigma_j [u(x_j)][v(x_j)], \\ B_2(u, v) &= \sum_{j=1}^{N} \int_{I_j} \big(\varepsilon_2 b(x) u' + c(x) u \big) dx + \sum_{j=1}^{N-1} b(x_j)[u(x_j)] v(x_j^-), \\ l(v) &= \sum_{j=1}^{N} \int_{I_j} f v dx. \end{aligned}$$

Here, σ_j's, $j = 0, 1, 2, \cdots, N$, are discontinuity-penalization parameters associated with the grid points x_j. Jumps and averages are defined by

$$\begin{aligned} [v(x_j)] &= v(x_j^+) - v(x_j^-), \qquad j = 1, 2, \cdots, N-1, \\ \{v(x_j)\} &= \frac{1}{2}\big(v(x_j^+) - v(x_j^-) \big), \quad j = 1, 2, \cdots, N-1, \end{aligned}$$

where $v(x_j^+) = \lim_{x \to x_j^+} v(x)$ and $v(x_j^-) = \lim_{x \to x_j^-} v(x)$, and at the boundary nodes these are defined as

$$[v(x_0)] = v(x_0), \quad \{v(x_0)\} = v(x_0), \quad [v(x_N)] = -v(x_N), \quad \{v(x_N)\} = v(x_N).$$

These notations can be seen in [8, 9, 15]. For any $v \in H^2(\Omega, \mathcal{T}_N)$, the norm is defined by

$$\|v\|_\varepsilon^2 = \varepsilon_1 \sum_{i=1}^{N} h_i \sum_{j=1}^{M} w_j v'(x_{i,j})^2 + \sum_{j=1}^{N} \|\gamma v\|_{L^2(I_j)}^2 + \sum_{j=0}^{N} \left(\frac{1}{2} b(x_j) + \sigma_j\right) [v(x_j)]^2.$$

Here, $x_{i,j}$ are the Gaussian points in the elements $I_j = (x_{j-1}, x_j)$, and $w_j > 0$ are the weights for M points Gaussian quadrature rule. The coercivity of bilinear form can be established by a simple calculation as in [8] as

$$B(v, v) = \|v\|_\varepsilon^2 .$$

Using coercivity, one can establish the result,

$$\|u_N\|_\varepsilon \leq C \|f\|_{L^2(\Omega)} , \tag{5}$$

where u_N is the solution of weak formulation (4). More details can be found in [9]. Furthermore, the result (5) implies the uniqueness of the solution of (4). Rank-nullity theorem in a finite-dimensional space and uniqueness give the existence of the solution.

Lemma 1. Let u and u_N be the solutions of (1) and (4), respectively. Then the bilinear form defined in (4) satisfies Galerkin orthogonality property

$$B(u - u_N, v_N) = 0, \quad \text{for all } v_N \in V^N. \tag{6}$$

Proof. Detailed proof can be found in [11]. □

4 Error analysis on Shishkin-type mesh

In this section, we introduce Shishkin-type mesh for the discretizing the domain of definition. Then, we apply the NIPG method on the introduced mesh to avail the convergence result theoretically. Proceeding in this way, the domain, $\overline{\Omega}$ is divided into N subintervals as follows: Let $\Omega_{l,1}$, $\Omega_{l,2}$, $\Omega_{l,3}$, $\Omega_{r,1}$, $\Omega_{r,2}$, and $\Omega_{r,3}$ be subdomains such that

$$\Omega_{l,1} = [0, \delta_0], \quad \Omega_{l,2} = [\delta_0, d - \delta_0], \quad \Omega_{l,3} = [d - \delta_0, d], \quad \Omega_{r,1} = [d, d + \delta_1],$$
$$\Omega_{r,2} = [d + \delta_1, 1 - \delta_1], \quad \Omega_{r,3} = [1 - \delta_1, 1].$$

for some δ_0 and δ_1 defined by

$$\delta_0 = \min\left\{\frac{d}{8}, \frac{\delta}{\lambda_l}\ln N\right\}, \quad \delta_1 = \min\left\{\frac{1-d}{8}, \frac{\delta}{\lambda_r}\ln N\right\}.$$

Here, δ is a user defined parameter. Therefore, the mesh length of each subdomain can be defined as

$$h_j = \begin{cases} \frac{8}{N}\frac{\delta}{\lambda_l}\ln N, & \text{for } \Omega_{l,1} \cup \Omega_{l,3}, \\ \frac{4(d-2\delta_0)}{N}, & \text{for } \Omega_{l,2}, \\ \frac{8}{N}\frac{\delta}{\lambda_r}\ln N, & \text{for } \Omega_{r,1} \cup \Omega_{r,3}, \\ \frac{4(1-d-2\delta_1)}{N}. & \text{for } \Omega_{r,2}. \end{cases} \tag{7}$$

The following bounds can be obtained for the above step sizes:

$$h_j \leq \begin{cases} CN^{-1}\ln N \lambda_l^{-1}, & \text{for } \Omega_{l,1} \cup \Omega_{l,3}, \\ CN^{-1}, & \text{for } \Omega_{l,2} \cup \Omega_{r,2}, \\ CN^{-1}\ln N \lambda_r^{-1}, & \text{for } \Omega_{r,1} \cup \Omega_{r,3}. \end{cases} \tag{8}$$

Lemma 2. Let S^I be the piecewise Lagrange's interpolation of smooth component S of the exact solution u of (1). Then, we have

$$\left\|S - S^I\right\|_\varepsilon \leq CN^{-2}.$$

Proof. Using the definition of jump for the smooth part of the solution, as $S - S^I$ is continuous across interelement boundaries. So, $[(S - S^I)(x_j)] = 0$, it gives

$$\left\|S - S^I\right\|_\varepsilon^2 = \varepsilon_1 \sum_{i=1}^{N} h_i \sum_{j=1}^{M} \left((S - S^I)'(x_{i,j})\right)^2 + \sum_{j=1}^{N} \int_{I_j} \gamma^2 (S - S^I)^2 dx. \tag{9}$$

Now, we have to calculate the bounds for the two factors separately. We make the use of result on interpolation error estimate in [13]

$$\varepsilon_1 \sum_{i=1}^{N} h_i \sum_{j=1}^{M} \big((S-S^I)'(x_{i,j})\big)^2 \le C\varepsilon_1 \sum_{j=1}^{N} h_j^3 |S|^2_{H^2(I_j)} \le C \sum_{j=1}^{N} h_j^5$$
$$\le C\big(N(N^{-1}\ln N\lambda_l^{-1})^5 + N^{-4}\big). \qquad (10)$$

Here, we have used $\lambda_r^{-1} \le \lambda_l^{-1}$ and bounds on the mesh lengths from (8). To establish the result in Lemma 2, we have to find out the essential bound for second term in (9)

$$\sum_{j=1}^{N} \big\|\gamma(S-S^I)\big\|_{L^2(I_j)} \le C \sum_{j=1}^{N} h_j^2 \int_{x_{j-1}}^{x_j} |S''(x)|dx \le C \sum_{j=1}^{N} h_j^3$$
$$\le C\big(N(N^{-1}\ln N\lambda_l^{-1})^3 + N^{-2}\big), \qquad (11)$$

using (10) and (11) in (9). □

Lemma 3. Let $E_{l,i}$ and $E_{r,i}$ for $i = 1, 2$ be the piecewise Lagrange's interpolations of left and right layer components E_l and E_r, respectively. Then, we have the following estimation:

$$\big\|(E_{l,i} - E_{l,i}^I)^s\big\|_{L^\infty(\Omega\setminus\Omega_{l,i})} \le C\lambda_l^s N^{-2}, \qquad i = 1, 2, s = 0, 1, 2.$$
$$\big\|(E_{r,i} - E_{r,i}^I)^s\big\|_{L^\infty(\Omega\setminus\Omega_{r,i})} \le C\lambda_r^s N^{-2}, \qquad i = 1, 2, s = 0, 1, 2.$$

Proof. We will prove only the estimation for $E_{l,1} - E_{l,1}^I$, and other estimations for remaining part will be same.

Making the use of the Cauchy–Schwarz inequality gives us

$$\big\|(E_{l,1} - E_{l,1}^I)^s\big\|_{L^\infty(\Omega\setminus\Omega_{l,1})} \le \|(E_{l,1})^s\|_{L^\infty(\Omega\setminus\Omega_{l,1})} + \big\|(E_{l,1}^I)^s\big\|_{L^\infty(\Omega\setminus\Omega_{l,1})}$$
$$\le 2\,\|(E_{l,1})^s\|_{L^\infty(\Omega\setminus\Omega_{l,1})}.$$

Here, we have used the stability property for special interpolant from Lemma 3.2 in [15]. Now we have to bound only $\|(E_{l,1})^s\|_{L^\infty(\Omega\setminus\Omega_{l,1})}$.

$$\|(E_{l,1})^s\|_{L^\infty(\Omega\setminus\Omega_{l,1})} \le C\lambda_l^s \max_{x\in[0,\delta_0]} e^{-p\lambda_l x} \le C\lambda_l^s N^{-p\delta} \le C\lambda_l^s N^{-2}.$$

Here we have used $p\delta \ge 2$, which is the required estimate, we had to prove. For estimations in the second part of the Lemma 2, we apply the same path of approach as above. So we have done the proof of lemma. □

Theorem 2. Let E^I denote the piecewise Lagrange's interpolations of layer components of solution E. Therefore, the interpolation error of the layer component satisfies

$$\left\|E - E^I\right\|_\varepsilon \le C(N^{-1}\ln N)^2.$$

Proof. To establish the bound in this theorem, we express errors in terms of left boundary layer, right boundary layer, and interior layer components; that is,

$$\left\|E - E^I\right\|_\varepsilon \le \left\|E_{l,1} - E^I_{l,1}\right\|_\varepsilon + \left\|E_{l,2} - E^I_{l,2}\right\|_\varepsilon + \left\|E_{r,1} - E^I_{r,1}\right\|_\varepsilon + \left\|E_{r,2} - E^I_{r,2}\right\|_\varepsilon . \tag{12}$$

We derive the bounds for all the four terms in the right hand side of (12). Taking the first term from (12) and using coercivity of bilinear form from Section 3, we have

$$\left\|E_{l,1} - E^I_{l,1}\right\|^2_\varepsilon = \varepsilon_1 \sum_{i=1}^{N} h_i \sum_{i=1}^{M} \left((E_{l,1} - E^I_{l,1})'(x_{i,j})\right)^2 + \sum_{j=1}^{N} \int_{I_j} \gamma^2 (E_{l,1} - E^I_{l,1})^2 dx. \tag{13}$$

Now, we estimate the first term from (13) over the fine mesh region $\Omega_{l,1}$, using classical result on interpolation error estimates

$$\begin{aligned}
\varepsilon_1 \sum_{i=1}^{N/8} h_i \sum_{j=1}^{M} \left((E_{l,1} - E^I_{l,1})'(x_{i,j})\right)^2 &\le C\varepsilon_1 \sum_{j=1}^{N/8} h_j^3 \left(\int_{x_{j-1}}^{x_j} |E'''_{l,1}| dx\right)^2 \\
&\le C\varepsilon_1 \sum_{j=1}^{N/8} h_j^3 \left(\int_{x_{j-1}}^{x_j} |E'''_{l,1}|^2 dx\right)\left(\int_{x_{j-1}}^{x_j} dx\right) \\
&\le C\varepsilon_1 \sum_{j=1}^{N/8} h_j^4 \int_{x_{j-1}}^{x_j} \lambda_l^4 e^{-2p\lambda_l x} dx \\
&\le C(N^{-1}\ln N)^4.
\end{aligned} \tag{14}$$

$$\begin{aligned}
\sum_{j=1}^{N/8} \int_{I_j} \gamma^2 (E_{l,1} - E^I_{l,1})^2 dx &\le C \sum_{j=1}^{N/8} h_j^4 \int_{x_{j-1}}^{x_j} |E''_{l,1}|^2 dx \\
&\le C \sum_{j=1}^{N/8} h_j^4 \int_{x_{j-1}}^{x_j} \lambda_l^4 e^{-2p\lambda_l x} dx
\end{aligned}$$

$$\leq C(N^{-1}\ln N)^4. \tag{15}$$

Using (14) and (15), we have established the first part of (12) for the finer part of the domain $\Omega_{l,1}$. In the same manner, we can establish the bound for $\left\| E_{l,2} - E^I_{l,2} \right\|_\varepsilon$, $\left\| E_{r,1} - E^I_{r,1} \right\|_\varepsilon$, and $\left\| E_{r,2} - E^I_{r,2} \right\|_\varepsilon$ over $\Omega_{l,2}$, $\Omega_{r,1}$ and $\Omega_{r,2}$, respectively. Therefore, we can obtain

$$\left\| E_{l,2} - E^I_{l,2} \right\|_{\varepsilon,\Omega_{l,2}} \leq C(N^{-1}\ln N)^2, \tag{16a}$$

$$\left\| E_{r,1} - E^I_{r,1} \right\|_{\varepsilon,\Omega_{r,1}} \leq C(N^{-1}\ln N)^2, \tag{16b}$$

$$\left\| E_{r,2} - E^I_{r,2} \right\|_{\varepsilon,\Omega_{r,2}} \leq C(N^{-1}\ln N)^2. \tag{16c}$$

Now, we estimate $\left\| E_{l,1} - E^I_{l,1} \right\|_\varepsilon$ over $\Omega \setminus \Omega_{l,1}$. In this domain, we will derive the bound for the error in maximum norm. Using the result in Lemma 3

$$\varepsilon_1 \sum_{i=N/8+1}^{N} h_i \sum_{j=1}^{M} w_j (E_{l,1} - E^I_{l,1})'(x_{i,j})^2 \leq C\varepsilon_1 \sum_{j=N/8+1}^{N} h_j \lambda_l^2 N^{-4} \leq CN^{-4}.$$

Again, bound on the second term may be calculated using triangular inequality and stability property as

$$\begin{aligned}
\left\| E_{l,1} - E^I_{l,1} \right\|^2_{L^2(\Omega\setminus\Omega_{l,1})} &\leq \left\| E_{l,1} \right\|^2_{L^2(\Omega\setminus\Omega_{l,1})} + \leq \left\| E^I_{l,1} \right\|^2_{L^2(\Omega\setminus\Omega_{l,1})} \\
&\leq 2\left\| E_{l,1} \right\|^2_{L^2(\Omega\setminus\Omega_{l,1})} \\
&\leq C \sum_{j=N/8+1}^{N} \int_{x_{j-1}}^{x_j} e^{-2p\lambda_l x} dx \leq CN^{-4}.
\end{aligned}$$

With the help of above two estimations a conclusion can be made

$$\left\| E_{l,1} - E^I_{l,1} \right\|^2_{\varepsilon,\Omega\setminus\Omega_{l,1}} \leq CN^{-4}. \tag{17}$$

Similar to (17), we can estimate the other remaining terms, and using (14)–(17), we conclude the bound in Theorem 2. □

Theorem 3. Let u^I be the piecewise Lagrange's interpolation of the exact solution u of (1) on Gaussian points. Then interpolation error $\zeta = u - u^I$ satisfies

$$\|\zeta\|_\varepsilon \leq C(N^{-1}\ln N)^2.$$

Proof. Using the solution decomposition and the estimations from Lemma 2 and Theorem 2, we can conclude that

$$\|\zeta\|_{\varepsilon} \leq \left\|S - S^I\right\|_{\varepsilon} + \sum_{k=l,r} \sum_{i=1}^{2} \left\|E_{k,i} - E_{k,i}^I\right\|_{\varepsilon} \leq C(N^{-1}\ln N)^2.$$

□

Now, we conclude the bound for closeness error by means of Galerkin orthogonality and coercivity. Therefore, we need the following theorem.

Theorem 4. Let u and u_N be the exact and discrete solution of (1) and its weak formulation (4), respectively. For u^I, the piecewise Lagrange's interpolation of u, let $\chi = u^I - u_N$ satisfies

$$\|\chi\|_{\varepsilon} \leq C(N^{-1}\ln N)^2.$$

Proof. Coercivity and Galerkin orthogonality give us

$$\|\chi\|_{\varepsilon}^2 = B(\chi,\chi) = -B(\zeta,\chi). \tag{18}$$

From the definition, ζ is continuous on interelement boundaries. That is, $[\zeta(x_j)] = 0$. So, $B_1(\zeta,\chi)$ can be reduced to

$$B_1(\zeta,\chi) = \sum_{j=1}^{N} \varepsilon_1 \int_{x_{j-1}}^{x_j} \zeta'\chi' dx - \sum_{j=1}^{N} \varepsilon_1 \{\zeta'(x_j)\}[\chi(x_j)]. \tag{19}$$

Now, we estimate the first term in (19),

$$\begin{aligned}\sum_{j=1}^{N} \varepsilon_1 \int_{x_{j-1}}^{x_j} \zeta'\chi' dx &\leq \left(\sum_{j=1}^{N} \varepsilon_1 \int_{x_{j-1}}^{x_j} (\zeta')^2 dx\right)^{1/2} \left(\sum_{j=1}^{N} \varepsilon_1 \int_{x_{j-1}}^{x_j} (\chi')^2 dx\right)^{1/2}\\ &\leq C(N^{-1}\ln N)^2 \|\chi\|_{\varepsilon}.\end{aligned}$$

Second term in (19) can be estimated using Cauchy–Schwarz inequality and the exact choice of discontinuity-penalization parameter $\sigma_j = N$,

$$\begin{aligned}\sum_{j=1}^{N} \varepsilon_1 \{\zeta'(x_j)\}[\chi(x_j)] &\leq \left(\sum_{j=0}^{N} \frac{\varepsilon_1}{\sigma_j} \{\zeta'(x_j)\}^2\right)^{1/2} \left(\sum_{j=0}^{N} \sigma_j [\chi(x_j)]^2\right)^{1/2}\\ &\leq C(N^{-1}\ln N)^2 \|\chi\|_{\varepsilon}.\end{aligned}$$

Using both these inequalities in (19), gives us

$$B_1(\zeta,\chi) \le C(N^{-1}\ln N)^2 \left\|\chi\right\|_{\varepsilon}. \tag{20}$$

Last but not the least is to estimate $B_2(\zeta,\chi)$,

$$\begin{aligned}\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\big(\varepsilon_2 b(x)\zeta' + c(x)\zeta\big)\chi dx =& \sum_{j=1}^{N}\varepsilon_2\int_{x_{j-1}}^{x_j} b(x)\zeta\chi' dx \\ &+ \sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\big(c(x)-\varepsilon_2 b'(x)\big)\zeta\chi dx.\end{aligned}$$

We estimate the two terms of right side of above equation separately,

$$\begin{aligned}&\left|\sum_{j=1}^{N}\varepsilon_2\int_{x_{j-1}}^{x_j} b(x)\zeta\chi' dx\right| \\ &\quad\le C\bigg(\sum_{i=1}^{3}\left\|\zeta\right\|_{L^\infty(\Omega_{l,i})}\left\|\chi'\right\|_{L^1(\Omega_{l,i})} + \sum_{i=1}^{3}\left\|\zeta\right\|_{L^\infty(\Omega_{r,i})}\left\|\chi'\right\|_{L^1(\Omega_{r,i})}\bigg) \\ &\quad\le C(N^{-1}\ln N)^2\left\|\chi\right\|_{\varepsilon}.\end{aligned}$$

Here, we have used

$$\left\|\chi'\right\|_{L^1(\Omega_{l,i})} \le C(\lambda_l^{-1}\ln N)^{1/2}\left\|\chi'\right\|_{L^2(\Omega_{l,i})} \le C(\varepsilon_1^{-1}\lambda_l^{-1}\ln N)^{1/2}\left\|\chi\right\|_{\varepsilon}$$

and results of Lemma 3.

Finally, the last term can be bounded as

$$\left|\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\big(c(x)-\varepsilon_2 b'(x)\big)\zeta\chi dx\right| \le C\left\|\zeta\right\|_{L^2(\Omega)}\left\|\chi\right\|_{L^2(\Omega)} \le C(N^{-1}\ln N)^2\left\|\chi\right\|_{\varepsilon}.$$

Keeping in view of above two inequalities, we have

$$B_2(\zeta,\chi) \le C(N^{-1}\ln N)^2 \left\|\chi\right\|_{\varepsilon}. \tag{21}$$

From (18), (20) and (21), we get the result of Theorem 4. □

Theorem 5. Let u and u_N be the solutions of continuous problem (1) and the discrete problem (4), respectively. Then the discretization error satisfies the following estimates:

$$\|u - u_N\|_\varepsilon \leq C(N^{-1}\ln N)^2.$$

Proof. Applying triangular inequality on the estimation from Theorems 3 and 4, we get the desired result. □

5 Numerical results and their assessments

In this section, we consider two test problems to validate the theoretical estimation established in the previous section.

Example 1. In this context, we have the first test problem:

$$\begin{aligned} -\varepsilon_1 u'' + \varepsilon_2(3x^2+2)u' + 7u &= f(x), \\ u(0) = u(1) &= 0, \end{aligned}$$

where $f(x)$ is defined by

$$f(x) = \begin{cases} 3x^2 + 2, & x \leq d, \\ 0.5x, & x > d, \end{cases}$$

and $d = 0.7$.

The above test problem is a singularly perturbed convection-diffusion problem with two parameters and discontinuity of type I in the source term at an interior point $x = d$ of the domain. The solution of the above test problem exhibits boundary layers at both end points $x = 0,\ 1$. In addition, a layer at the point of discontinuity of source term $x = d$ in the interior of the domain is recorded. The curve of the computed solution along with the exact solution is provided in Figure 1. The solutionprofile shows both the boundary layers at $x = 0$ and $x = 1$ and also the interior layer at $x = 0.7$. Here, 0.7 is the point of discontinuity for $f(x)$ in Example 1. The exact choice of discontinuity-penalization parameter used for this is $\sigma_j = N,$ for all j, which are discussed in Section 4.

Error and rate of convergence are examined for various values of discretization parameter N and the singular perturbation parameter ε_1. Two parameters ε_1 and ε_2 are related to each other by the relation $\varepsilon_1 \leq \varepsilon_2^2$. For the sake of convenience in our computation, we take the general case $\varepsilon_1 = \varepsilon_2^2$.

The exact solution to the above test problem is not known. Hence for the computational purpose, we determine errors for DG-norm by $\|u_N - u_{2N}\|_{\mathrm{DG}}$. We apply the double mesh principle to deduce the rate of convergence and errors. Therefore, the following formula is applied

$$R_\varepsilon^N = \log_2\left(\frac{\|u_N - u_{2N}\|_{\mathrm{DG}}}{\|u_{2N} - u_{4N}\|_{\mathrm{DG}}}\right),$$

and the consequence is mentioned in Tables 1 and 2. From these tables, it is clear that the scheme is convergent with an order of convergence up to 2. In the study of [11], we see that if we change our domain discretization from Shishkin mesh to Bakhvalov mesh and Bakhvalov mesh to exponential mesh, then the rate of convergence increases gradually. In this regard, we achieve comparatively better order of convergence, and numerically it will be achieved to 2.

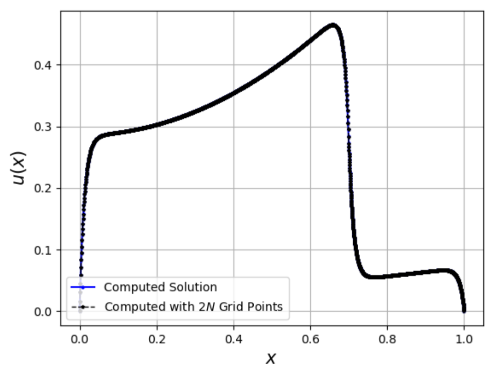

Figure 1: Computed and exact solution for Example 1 for $N = 2048$ and $\varepsilon = 10^{-6}$.

Table 1: $\|u_N - u_{2N}\|_{\mathrm{DG}}$ errors for Example 1

	Number of intervals (N)					
ε_2	64	128	256	512	1024	2048
10^{-2}	2.7829E-03	1.3574E-03	6.7059E-04	3.3331E-04	1.6617E-04	8.2964E-05
10^{-3}	3.7937E-03	1.8284E-03	8.9695E-04	4.4416E-04	2.2100E-04	1.1023E-04
10^{-4}	5.3400E-03	2.5120E-03	1.2140E-03	5.9633E-04	2.9548E-04	1.4707E-04
10^{-5}	7.7506E-03	3.5276E-03	1.6647E-03	8.0656E-04	3.9677E-04	1.9676E-04

Table 2: Convergence rates R_ε^N for Example 1

	Number of intervals (N)				
ε_2	128	256	512	1024	2048
10^{-2}	0.8902	0.9406	0.9705	0.9855	0.9928
10^{-3}	1.1069	0.8868	0.9140	0.9545	0.9774
10^{-4}	1.7920	1.5795	1.0695	0.9100	0.9347
10^{-5}	1.4870	1.8389	1.8514	1.5312	1.0518

Example 2. The second test problem in this context is taken a

$$-\varepsilon_1 u'' + \varepsilon_2(7x+5)u' + 3u = f(x),$$
$$u(0) = u(1) = 0,$$

where $f(x)$ is defined by

$$f(x) = \begin{cases} x, & x \leq d, \\ 2-x, & x > d, \end{cases}$$

and $d = 0.7$.

The test problem given above in Example 2 is also regarded as two-parametric convection-diffusion problem with discontinuous source term. Due to the lack of an exact solution to the above problem, we use the double mesh principle to conclude the error and rate of convergence of the obtained numerical solutions. The rate of convergence is obtained by the following formula:

$$P_\varepsilon^N = \log_2\left(\frac{\|u_N - u_{2N}\|_{\text{DG}}}{\|u_{2N} - u_{4N}\|_{\text{DG}}}\right).$$

The same choice of discontinuity penalization parameter as in the previous example is imposed here. The best possible relation between ε_1 and ε_2 is $\varepsilon_1 = \varepsilon_2^2$. Now, the error $\|u_N - u_{2N}\|_{\text{DG}}$ and the rate of convergence P_ε^N for the computed solution for the test problem in Example 2 with different values of discretization parameter N are given below in Tables 3 and 4, respectively.

The second-order convergence appears to be achieved up to the logarithmic factor from Table 4.

Table 3: $\|u_N - u_{2N}\|_{\mathrm{DG}}$ errors for Example 2

	Number of intervals (N)					
ε_2	64	128	256	512	1024	2048
10^{-2}	9.8446E-03	3.0845E-03	9.9633E-04	3.3445E-04	1.1108E-04	3.9958E-05
10^{-3}	4.0731E-02	2.0132E-02	8.8730E-03	3.5715E-03	1.4107E-03	5.3149E-04
10^{-4}	4.1807E-02	2.0444E-02	8.8440E-03	3.5894E-03	1.3849E-03	5.3839E-04
10^{-5}	4.1841E-02	2.0351E-02	8.8344E-03	3.5877E-03	1.4237E-03	5.2625E-04

Table 4: Convergence rates P_ε^N for Example 2

	Number of intervals (N)				
ε_2	128	256	512	1024	2048
10^{-2}	1.6742	1.6303	1.5748	1.5901	1.4751
10^{-3}	1.8166	1.8819	1.6128	1.3401	1.4083
10^{-4}	1.9320	1.8089	1.3009	1.3739	1.5630
10^{-5}	1.9397	1.8239	1.6000	1.5333	1.5359

5.1 Discussion

Several articles are available in the literature that deal with the convergence and superconvergence of numerical solutions for problems with one and two small parameters. However, in all the articles, problems with continuous coefficients and source terms are discussed. In [11], authors have established the superconvergence result of the discontinuous Galerkin method for a two-parametric singular perturbation problem in which all the coefficients are assumed to be continuous. In that, authors have used three types of meshes usual Shishkin mesh, Bakhvalov mesh, and exponentially graded mesh for domain discretization and shown their rate of convergence of order two by talking linear elements in its finite element space. The convergence results on both the Bakhvalov mesh and exponentially graded mesh have a sharper rate of convergence, that is $\mathcal{O}(N^{-2})$, than the rate of convergence obtained on the usual Shishkin mesh. In the present work, we attempt to obtain the convergence result for two-parametric problem in which the source term is discontinuous. The discontinuous Galerkin method is applied, and the second-order convergence has been obtained up to the logarithmic factor.

6 Conclusion

In this article, we proposed and discussed uniformly convergent discontinuous Galerkin FEM for two-parametric singularly perturbed problems with a discontinuous source term. The problem usually exhibits a weak interior layer because of the discontinuous source term and the presence of the multiple perturbation parameters gives rise to boundary layers on both sides of the boundary. In order to address the layering phenomenon, we developed the non-symmetric discontinuous Galerkin FEM with interior penalties through this work. With the use of a typical Shishkin mesh, the domain is discretized, and a uniform error estimate was obtained. The numerical experiments were conducted to validate the theoretical conclusions.

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