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Uniformly continuous 1-1 functions on ordered fields not mapping interior to interior

Mojtaba Moniri*and Jafar S. Eivazloo

Department of Mathematics,

Tarbiat Modarres University, Tehran

Abstract

In an earlier work we showed that for ordered fields F not isomorphic to the reals R, there are continuous 1-1 functions on $[0,1]_F$ which map some interior point to a boundary point of the image (and so are not open). Here we show that over closed bounded intervals in the rationals Q as well as in all non-Archimedean ordered fields of countable cofinality, there are *uniformly continuous* 1-1 functions not mapping interior to interior. In particular, the minimal non-Archimedean ordered field $Q(x)$, as well as ordered Laurent series fields with coefficients in an ordered field accommodate such pathological functions.

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*E-mail: mojmon@ipm.ir

1 Introduction

A cut C of an ordered field F is a subset satisfying $C < F \setminus C$. A nonempty proper cut is said to be a gap, whenever it fails to have a supremum in the field. A gap G in F is called regular, when for all $\epsilon \in F^{>0}$, $G + \epsilon \nsubseteq G$. An ordered field is Archimedean (has no infinitesimals) if and only if it is (isomorphic to) a subfield of real ordered field R. The latter is, up to isomorphism, the unique ordered field which is Dedekind complete, i.e. does not have any gaps. Any Dedekind incomplete ordered field F has gaps in all its non-degenerated intervals: In the Archimedean case, F is a proper subfield of R which therefore misses some points in any real interval. In the non-Archimedean case, and given any two points $a < b$ of F, downward closure of the set of points x such that $\frac{x-a}{b-a}$ is an infinitesimal, forms a gap in (a, b).[1]

Monotone complete ordered fields were introduced in [2]. They are ordered fields with no bounded strictly increasing divergent functions. From [[4] Corollary 2.7], follows that there are monotone complete ordered fields of any uncountable regular cardinality and so there exist plenty of monotone complete ordered fields not isomorphic to R. On the other hand, it is clear that there are no monotone complete ordered field of countable cofinality, except (those isomorphic to) R. For the notions of cofinality and regular cardinals, we refer to [1]. We use cf for cofinality. If F is a monotone incomplete ordered field, then any non-degenerated interval of F contains the image of a strictly increasing divergent function.

We proved in [[3], Theorem 1.2] that an ordered field F is Dedekind complete, if all continuous 1-1 functions defined on some (equivalently all) non-degenerated closed bounded interval(s) of F map interior points [of the interval(s)] to interior points [of their range(s)]. For proper subfields of R, we show here that the rather strange functions coming from above can not be uniformly continuous provided that their unique continuous extensions to R are 1-1. On the other

[1] Alternatively, one can use linear increasing functions between intervals to map a given gap (somewhere in the field) to a gap in a given interval.

hand, on closed bounded intervals in the rational ordered field, as well as in all non-Archimedean ordered fields with countable cofinality, there are uniformly continuous 1-1 functions which do not map interior to interior. We will finish by presenting some ordered fields over which this phenomenon occurs.

2 The Archimedean case

In this section, a well known property of the real ordered field is treated for its subfields.

Lemma 2.1 *Let F be an Archimedean ordered field. If $f : [0,1]_F \to [0,1]_F$ is a uniformly continuous function, then f can be extended to a unique (uniformly) continuous function $\overline{f}$ on $[0,1]_R$.*

Proof. Given $x \in [0,1]_R$, there exists a sequence $(r_n)_{n\geq 1}$ in $[0,1]_Q$ such that $\lim_R r_n = x$. As f is uniformly continuous on $[0,1]_F$ which contains $[0,1]_Q$ and the sequence $(r_n)_{n\geq 1}$ is Cauchy, $(f(r_n))_{n\geq 1}$ is Cauchy in $[0,1]_F$ and so has a limit in $[0,1]_R$. Let $\overline{f}(x) = \lim_R f(r_n)$. This is well defined, since if $(r_n)_{n\geq 1}$ and $(s_n)_{n\geq 1}$ are two Cauchy sequences in $[0,1]_Q$ such that $\lim_R r_n = \lim_R s_n$, then by uniform continuity of f, we have $\lim_R f(r_n) = \lim_R f(s_n)$. Note that $\overline{f}$ is indeed the unique such extension of f. It is also continuous on $[0,1]_R$: Let $x_0 \in [0,1]_R$ and $\epsilon \in R^{>0}$. By uniform continuity of f, there exists $\delta > 0$ such that for all $x, y \in [0,1]_F, (|x-y| < \delta \to |f(x) - f(y)| < \frac{\epsilon}{2})$ $(*)$. We claim that for this δ, $(\forall x \in [0,1]_R)(|x - x_0| < \delta \to |\overline{f}(x) - \overline{f}(x_0| < \epsilon)$. Let $x \in [0,1]_R$ be such that $|x - x_0| < \delta$. There exist sequences $(r_n)_{n\geq 1}$ and $(s_n)_{n\geq 1}$ in $[0,1]_Q$ such that $\lim_R r_n = x_0$ and $\lim_R s_n = x$ respectively. Now let N be a nonnegative integer such that $(\forall n \geq N)(|r_n - s_n| < \delta)$, so from $(*)$ we have $(\forall n \geq N)(|f(r_n) - f(s_n)| < \frac{\epsilon}{2})$. Thus $|\overline{f}(x) - \overline{f}(x_0)| = \lim_R |f(r_n) - f(s_n)| \leq \frac{\epsilon}{2} < \epsilon$.

Proposition 2.2 *Let F be an Archimedean ordered field. If $f : [0,1]_F \to [0,1]_F$ is a uniformly continuous function whose unique extension (as above) to R is one-to-one, then it maps every open subset of $[0,1]_F$ onto an open subset of $f([0,1]_F)$.*

Proof. Let U be an open subset of $[0,1]_F$. There is an open subset V of R such that $U = V \cap [0,1]_F$. We have $f(U) = \overline{f}(U) = \overline{f}(V \cap [0,1]_F) = \overline{f}(V \cap [0,1]_R) \cap \overline{f}([0,1]_F) = \overline{f}(V \cap [0,1]_R) \cap f([0,1]_F)$. But $\overline{f}$ is open, hence $\overline{f}(V \cap [0,1]_R)$ is a relatively open subset of $\overline{f}([0,1]_R)$ and so there is an open subset W of R, such that $\overline{f}(V \cap [0,1]_R) \cap f([0,1]_F) = (W \cap \overline{f}([0,1]_R)) \cap f([0,1]_F) = W \cap f([0,1]_F)$. Therefore $f(U)$ is a relatively open subset of $f([0,1]_F)$.

Proposition 2.3 *There is a uniformly continuous 1-1 function on $[0,1]_Q$, which its range has empty interior.*

Proof. Consider the function $f(x) = |\frac{1}{4}x^2 + x - \frac{1}{2}|$. By changing the variable $x = 2(u-1)$, the proof of f being 1-1 is based on the fact that the equation $r^2 + s^2 = 3$ is not solvable in Q. A further argument shows that the complement of the range of f with respect to $[0,1]_Q$ is dense in $[0,1]_Q$ and so the range, as a subspace of $[0,1]_Q$, has empty interior.

3 The non-archimedean case of countable cofinality

Theorem 3.1 *Let F be a non-Archimedean ordered field with $cf(F) = \omega$. Then for all $a < b$ in F, there exist 1-1 uniformly continuous functions $f : [a,b]_F \to [a,b]_F$ whose ranges are closed such that f maps some interior point of $[a,b]_F$ to a boundary point of its image.*

Proof. Let $[a,b]$ be a non-degenerated interval in F and $c = \frac{a+b}{2}$. Fix a strictly increasing sequence $(a_k)_{k\in\omega}$ in $[a,c)$ such that $a_0 = a$, $(\forall k \geq 1)|a_{k+1} - a_k| \leq \frac{1}{4}|a_k - a_{k-1}|$ and $\lim_k a_k = c$. Put $b_0 = b$, and for all $k \geq 1$, $b_k = b_0 - (a_k - a_0)$. Downward closure of the monad of 0 is an irregular gap G in $[0,1]_F$. Fix (an infinitesimal) $\gamma \in G^{>0}$. Note that $(\forall x \in G)(\forall y \in F \setminus G)(y - x > \gamma)$. For each $k \in \omega$, let U_k be the image of $G \cap [0,1]$ under the linear increasing function from $[0,1]$ onto $[a_k, a_{k+1}]$ and V_k the image of $G \cap [0,1]$ under the linear increasing function from $[0,1]$ onto $[b_{k+1}, b_k]$. Let $S_0 = U_0$, $T_0 = [a,b] \setminus$ (downward closure of V_0), and for all $k \geq 1$, $S_k = U_k \setminus U_{k-1}$, $T_k = V_{k-1} \setminus V_k$. For all $k \geq 1$, we have

$(\forall x \in S_k)(\forall y \in S_{k+1})(y - x > \gamma(a_{k+1} - a_k))$ $(*)$ and, from a similar observation for the T_k's, $I_k = [b_k - \gamma(b_k - b_{k+1}), b_k + \gamma(b_k - b_{k+1})] \subseteq T_k$. For $k \in \omega \setminus \{0\}$, let h_k be the linear increasing function which maps $[a_{k-1}, a_{k+1}]$ onto I_{2k}. For $k = 0$, first pick out some $d_0 \in T_0 \setminus \{b\}$ such that $b - d_0 < a_1 - a$ and put $I_0 = [d_0, b]$. Then let $h_0 : [a, a_1] \to I_0$ be the onto linear increasing function. Let f_k be the restriction of h_k to S_k; so f_k is a function which maps S_k linearly and increasingly into $I_{2k} \subseteq T_{2k}$. Similarly, for all $k \in \omega$, we can get linear increasing functions $g_k : T_k \to T_{2k+1}$. Let $f = (\bigcup_{k\in\omega} f_k) \cup (\bigcup_{k\in\omega} g_k) \cup \{(c, c)\}$. Then f is a one-to-one function from $[a, b]$ into $[c, b]$ with a closed range which maps $c \in (a, b)$ to the boundary point $c \in f([a, b]) \subseteq [c, b]$.

To prove that f is uniformly continuous, we proceed as follows. Given $\epsilon \in F^{>0}$ which we may assume without loss of generality to be less than $a_2 - a_1$, let $\delta = \frac{1}{4}\gamma\epsilon$. Take $x, y \in [a, b]$ with $|x - y| < \delta$. If either x or y (but to avoid trivialities not both) equals c or $x < c < y$, then one easily checks that $|f(x) - f(y)| < |x - y|$. So assume both x and y are strictly less or greater than c, they will be either both in two S's or both in two T's. The arguments will be similar, we only deal with the S's. Suppose $x \in S_k$ and $y \in S_l$. There are the following exclusive cases of how l is compared to k.

(A) $l = k$. Here $f = f_k$ is the restriction of h_k to S_k. If $k = 0$, then by the condition on d_0, we have $|f(x) - f(y)| = |f_0(x) - f_0(y)| < |x - y|$. Assume $k \geq 1$. Then $h_k : [a_{k-1}, a_{k+1}] \to I_{2k}$ is linear and $|I_{2k}| < (a_{k+1} - a_{k-1})$, so $|f(x) - f(y)| = |f_k(x) - f_k(y)| < |x - y| < \delta$, which is infinitely smaller than ϵ.

(B) $l = k + 1$. By the choice of δ, we must have $k \geq 2$. The reason is that we will have $\delta < \gamma(a_2 - a_1)$ (and in particular $\delta < \gamma(a_1 - a_0)$). We have $|f(x) - f(y)| = |f_k(x) - f_{k+1}(y)| < |b_{2(k+1)+1} - b_{2k-1}| \leq |b_{k+1} - b_k| = |a_{k+1} - a_k| <$ [by $(*)$] $\frac{1}{\gamma}|x - y| < \frac{1}{\gamma}\delta = \frac{1}{4}\epsilon < \epsilon$.

(C) $k \geq 2$ and $l = k + 2$. We will have $|f(x) - f(y)| \leq |f(x) - f(a_{k+1})| + |f(a_{k+1}) - f(y)|$. Both of these terms are less than $\frac{1}{4}\epsilon$, by case (B).

(D) $k \geq 2$ and $l \geq k + 3$. We will have $|f(x) - f(y)| < |b_{2l+1} - b_{2k-1}|$. Using $|a_{i+1} - a_i| \leq \frac{1}{4}|a_i - a_{i-1}|$, one gets $|b_{2l+1} - b_{2k-1}| < |a_{k+1} - a_{l-1}|$. The latter value is less than $|x-y|$.

(E) $k=0$ and $l \geq 2$. We will have $|f(x)-f(y)| \leq |f(x)-f(a_1)|+|f(a_1)-f(a_2)|+|f(a_2)-f(y)|$. The first two terms on the right hand side are, by case (B), less than $\frac{1}{4}\epsilon$. The last one is also less than $\frac{1}{4}\epsilon$: If $l=2$, $l=3$, $l=4$, $l \geq 5$, then one may use the cases (A), (B), (C) and (D) respectively.

(F) $k=1$ and $l \geq 3$. We will have $|f(x)-f(y)| \leq |f(x)-f(a_2)|+|f(a_2)-f(y)|$. The first term on the right is, by case (B), less than $\frac{1}{4}\epsilon$. The second one is also less than $\frac{1}{4}\epsilon$: If $l=3$, $l=4$, $l \geq 5$, then one may use the cases (B), (C) and (D) respectively.

Finally, We present some non-Archimedean ordered fields allowing the mentioned kind of strange functions.

Example 3.2 Consider the ordered field $(Q(x),+,\cdot,<)$ with $x>Q$. It is non-Archimedean and of cofinality ω, hence monotone incomplete. So there are uniformly continuous one-to-one functions $f:[0,1]_{Q(x)} \rightarrow [0,1]_{Q(x)}$, which map some interior point to a boundary point of its image.

Example 3.3 Let F be an ordered field. Then the ordered field of Laurent series with coefficients in F is non-Archimedean of cofinalty ω, and so it is monotone incomplete. Once again, there will exist the mentioned kind of functions over closed bounded intervals of this field.

Note added in the proof. In 2002, Lobachevskii (J. Math.) (it is available online), in addition to the above for non-Archimedean ordered fields of countable cofinality, we constructed uniformly continuous 1-1 functions which although map interior to the interior of the image, but still are not open.

References

[1] Ciesielski, K., Set Theory for the Working Mathematician, Cambridge University Press, Cambridge, 1997.

[2] Keisler, H.J., Monotone Complete Fields, in: Victoria Symposium on Nonstanard Analysis (A. Hurd and P. Loeb, eds.), Lecture Notes in Mathematics 369, pp. 113-115, Springer-Verlag, Berlin, 1974.

[3] Moniri, M. and Eivazloo, J.S., Using nets in dedekind, monotone, or scott incomplete ordered fields and definability issues, Proceedings of the Ninth Prague Topological Symposium, Ed. P. Simon, pp. 195-203, Topology Atlas (available at http://at.yorku.ca), North Bay, ON, 2002.

[4] Schmerl, J.H., Models of Peano arithmetic and a question of sikorski on ordered fields, Israel J. Math. **50**(1985), 145-159.