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**Research Article**

# Approximate solution of a system of singular integral equations of the first kind by using Chebyshev polynomials

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**Abstract**

The aim of the present work is to introduce a method based on the Chebyshev polynomials for numerical solution of a system of Cauchy type singular integral equations of the first kind on a finite segment. Moreover, an estimation error is computed for the approximate solution. Numerical results demonstrate the effectiveness of the proposed method.

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**Keywords:** System of singular linear integral equations; Orthogonal polynomials; Fourier series; Best approximation.

## 1 Introduction

Let us consider a system of singular integral equations of the form

$$A(t)\Phi(t) + B(t)\int_{-1}^{1} \frac{\Phi(\tau)}{\tau - t}\,d\tau + \int_{-1}^{1} K(t,\tau)\Phi(\tau)d\tau = F(t), \quad -1 < t < 1, \; (1)$$

where

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$$K(t,\tau) = [K_{ij}(t,\tau)]\,, \quad i,j = 1,2,\dots,N,$$
$$F(t) = [f_1(t)\,,\, f_2(t),\dots,f_N(t)]^T,$$
$$\Phi(t) = [\phi_1(t),\phi_2(t),\dots,\phi_N(t)]^T,$$
$$A(t) = [a_{ij}(t)]\,, \quad i,j = 1,2,\dots,N,$$
$$B(t) = [b_{ij}(t)]\,, \quad i,j = 1,2,\dots,N.$$

Here, $\{K_{ij}\}_{i,j=1}^{N}$ and $\{f_i\}_{i=1}^{N}$ are given real-valued Hölder functions and $\{\phi_j\}_{j=1}^{N}$ are the unknown functions. The matrices $A$ and $B$ are known such that $S = A + B$ and $D = A - B$ are nonsingular for all $t \in [-1,1]$. In some familiar physical problems, the entries of the matrices $A$ and $B$ are constants.

The singular integral equations play important roles in physics and theoretical mechanics, particularly in the areas of elasticity, aerodynamics, and unsteady aerofoil theory. They are highly effective in solving boundary value problems occurring in the theory of functions of a complex variable, potential theory, the theory of elasticity, and the theory of fluid mechanics. A general theory of the system of equations (1) has given in [12].

We study the system (1) in the case that $A(t) = 0$ and $B(t)$ is a constant matrix. Therefore, the $i$th equation of system (1) takes the form

$$\sum_{j=1}^{N} b_{ij} \int_{-1}^{1} \frac{\phi_j(\tau)}{\tau - t}\, d\tau + \sum_{j=1}^{N} \int_{-1}^{1} K_{ij}(t,\tau)\phi_j(\tau) d\tau = f_i(t)\,, \quad -1 < t < 1. \quad (2)$$

Studies on this singular integral equation can be found in some literatures (see [1,3,6,7]). Chakrabarti and Berghe [3] proposed a method for solving (2), using polynomial approximation and collocation points have chosen to be the zeros of the Chebyshev polynomials of the first kind for all cases. Kashfi and Shahmorad [7] constructed another approximate solution of this equation by using the Chebyshev polynomials of the first and second kinds. Some other methods for solving this equation can be found in [1,6]. A convergence analysis of Galerkin and collocation methods for (2) has been given by Miel [11].

A special type of (2) is the famous Cauchy singular integral equation

$$\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau = f(t)\,, \quad -1 < t < 1, \quad (3)$$

which has the following analytical solutions in four special cases based on boundedness of the unknown function $\phi$ at the endpoints of the interval $[-1,\, 1]$; see [3, 9, 13].

**Case 1.** If the function $\phi$ is unbounded at the endpoints $\tau = \pm 1$, then

$$\phi(\tau) = \frac{a_0}{\sqrt{1-\tau^2}} - \frac{1}{\pi^2\sqrt{1-\tau^2}} \int_{-1}^{1} \frac{\sqrt{1-t^2} f(t)}{t - \tau}\, dt\,, \quad -1 < \tau < 1,$$

where $a_0$ is an arbitrary constant.

**Case 2.** If the function $\phi$ is bounded at the endpoints $\tau = \pm 1$, then

$$\phi(\tau) = -\frac{\sqrt{1-\tau^2}}{\pi^2} \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}\,(t-\tau)}\, dt\,, \quad -1 < \tau < 1,$$

and a necessary and sufficient condition of existing this solution is

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\, dt = 0\,.$$

**Case 3.** If the function $\phi$ is bounded at the endpoint $\tau = -1$ and unbounded at the endpoint $\tau = 1$, then

$$\phi(\tau) = -\frac{1}{\pi^2} \sqrt{\frac{1+\tau}{1-\tau}} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\, \frac{f(t)}{t-\tau}\, dt\,, \quad -1 < \tau < 1.$$

**Case 4.** If the function $\phi$ is bounded at the endpoint $\tau = 1$ and unbounded at the endpoint $\tau = -1$, then

$$\phi(\tau) = -\frac{1}{\pi^2} \sqrt{\frac{1-\tau}{1+\tau}} \int_{-1}^{1} \sqrt{\frac{1+t}{1-t}}\, \frac{f(t)}{t-\tau}\, dt\,, \quad -1 < \tau < 1.$$

An application of (3) were given in [2] by reducing a system of dual integral equations to Cauchy type singular integral equations, and more methods for solving this equation have given in [5,8,13,15].

In the next section, we investigate approximate solutions for system (1) in the above four cases.

## 2 Approximate solution

To find approximate solutions for system (1) in the cases **1,2,3,4**, for $\nu \in \{1,2,3,4\}$, we set

$$\phi_j(\tau) \simeq \varphi_{\nu,j}(\tau) := \frac{\lambda_\nu(\tau)}{\sqrt{1-\tau^2}} \sum_{l=0}^{M} \beta_{jl}\, P_{\nu,l}(\tau)\,, \quad j = 1, 2, \ldots, N, \qquad (4)$$

and

$$K_{ij}(t,\tau) := \sum_{k=0}^{M} \gamma_{ijk}(t)\, P_{\nu,k}(\tau)\,, \quad i,j = 1,2,\ldots,N\,, \tag{5}$$

where

$$P_{\nu,j}(x) = \begin{cases} T_j(x) = \cos(j\theta), & \nu = 1,\\[2mm] U_j(x) = \frac{\sin((j+1)\theta)}{\sin(\theta)}, & \nu = 2,\\[2mm] V_j(x) = \frac{\cos\left((j+\frac{1}{2})\theta\right)}{\cos(\frac{\theta}{2})}, & \nu = 3,\\[2mm] W_j(x) = \frac{\sin\left((j+\frac{1}{2})\theta\right)}{\sin(\frac{\theta}{2})}, & \nu = 4, \end{cases}$$

are the Chebyshev polynomials of the first to fourth kinds and

$$\lambda_\nu(t) = \begin{cases} 1, & \nu = 1,\\ 1-t^2, & \nu = 2,\\ 1+t, & \nu = 3,\\ 1-t, & \nu = 4, \end{cases}$$

in which $x = \cos(\theta)$. The roots of Chebyshev polynomials $P_{\nu,M+1}(x)$ are given by

$$x_{\nu,n} = \begin{cases} \cos\left(\frac{(2n-1)\pi}{2(M+1)}\right), & \nu = 1,\\[2mm] \cos\left(\frac{n\pi}{M+2}\right), & \nu = 2,\\[2mm] \cos\left(\frac{(2n-1)\pi}{2M+3}\right), & \nu = 3,\\[2mm] \cos\left(\frac{2n\pi}{2M+3}\right), & \nu = 4, \end{cases} \tag{6}$$

where $n = 1,2,\ldots,M+1$. These roots are used as the nodes of Gauss–Chebyshev quadrature rules.

**Lemma 1.** *[10] The Chebyshev polynomials satisfy the orthogonality property*

$$\int_{-1}^{1} \frac{\lambda_\nu(t)}{\sqrt{1-t^2}} P_{\nu,i}(t) P_{\nu,j}(t)\, dt \;=\; \begin{cases} 0, & i \neq j, \\ \pi, & i = j = 0\,, \quad \nu = 1, \\ \frac{\pi}{2}, & i = j \neq 0, \quad \nu = 1, \\ \frac{\pi}{2}, & i = j, \qquad\quad \nu = 2, \\ \pi, & i = j, \qquad\quad \nu = 3\,,\, 4. \end{cases} \tag{7}$$

**Theorem 1.** *[10] As the Cauchy principle value of a singular integral, we have*

$$\int_{-1}^{1} \frac{\lambda_\nu(\tau)}{\sqrt{1-\tau^2}}\, \frac{P_{\nu,j}(\tau)}{\tau - t}\, d\tau \;=\pi \begin{cases} U_{j-1}(t), & \nu = 1, \\ -T_{j+1}(t), & \nu = 2, \\ W_j(t), & \nu = 3, \\ -V_j(t), & \nu = 4. \end{cases} \tag{8}$$

Now we describe details of finding approximate solution in cases **1-4**.

**Case 1.** For $\nu = 1$, the relations (4)–(5) take the forms

$$\phi_j(\tau) \simeq \varphi_{1,j}(\tau) := \frac{1}{\sqrt{1-\tau^2}} {\sum_{l=0}^{M}}{}' \beta_{jl}\, T_l(\tau)\,, \qquad j = 1, 2, \ldots, N, \tag{9}$$

and

$$K_{ij}(t,\tau) := {\sum_{k=0}^{M}}{}' \gamma_{ijk}(t)\, T_k(\tau)\,, \qquad i, j = 1, 2, \ldots, N, \tag{10}$$

where $\beta_{jl}$ $\;(j = 1, 2, \ldots, N, \quad l = 0, 1, \ldots, M)$ are unknown coefficients and the symbol $(\sum{}')$ denotes that the first term in the summation is halved. The functions

$$\gamma_{ijk}(t) = \frac{2}{\pi} \int_{-1}^{1} \frac{K_{ij}(t,\tau) T_k(\tau)}{\sqrt{1-\tau^2}}\, d\tau\,, \quad i, j = 1, 2, \ldots, N, \quad k = 0, 1, \ldots, M,$$

can be determined exactly or may be approximated by using the Gauss–Chebyshev quadrature rule, that is,

$$\gamma_{ijk}(t) \simeq \frac{2}{M+1} \sum_{s=1}^{M+1} K_{ij}(t, x_{1,s}) T_k(x_{1,s})\,,$$

where $x_{1,s}$ obtain from (6).

Substituting from (9)–(10) into (2) and using (7)–(8) for $\nu = 1$, gives the system

$$\sum_{j=1}^{N}\sum_{l=1}^{M} b_{ij}\beta_{jl}U_{l-1}(t) + \frac{1}{2}\sum_{j=1}^{N}{\sum_{k=0}^{M}}' \gamma_{ijk}(t)\beta_{jk} = \frac{1}{\pi}f_i(t)\,, \quad i = 1,2,\dots,N. \tag{11}$$

If the given functions $f_i$ and $\gamma_{ijk}$ are square integrable on $[-1\,,\ 1]$ with respect to the weight function $\frac{\lambda_1(t)}{\sqrt{1-t^2}}$, then they can be expanded as

$$\begin{cases} \gamma_{ijk}(t) \simeq \sum_{l=0}^{M-1} G_{ijkl}\, U_l(t),\ i,j = 1,2,\dots,N,\ k = 0\,,1\,,\dots\,,M, \\ \frac{1}{\pi}f_i(t) \simeq \sum_{l=0}^{M-1} c_{il}\, U_l(t), \quad\ i = 1,2,\dots,N, \end{cases} \tag{12}$$

where the coefficients

$$\begin{cases} G_{ijkl} = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-t^2}\gamma_{ijk}(t)U_l(t)dt \\ \qquad\ \ = \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\sqrt{\frac{1-t^2}{1-\tau^2}}K_{ij}(t,\tau)U_l(t)\,T_k(\tau)d\tau dt \\ \qquad\qquad\qquad i,j = 1,2,\dots,N, \quad k = 0,1,\dots,M,\ \ l = 0,1,\dots,M-1, \\ c_{il} \ \ = \frac{1}{\pi^2}\int_{-1}^{1}\sqrt{1-t^2}f_i(t)U_l(t)dt,\ \ i = 1,2,\dots,N,\ l = 0,1,\dots,M-1, \end{cases}$$

can be approximately determined from

$$\begin{cases} G_{ijkl} \simeq \frac{4}{(M+1)^2}\sum_{r=1}^{M}\sum_{s=1}^{M+1}(1-x_{2,r}^2)K_{ij}(x_{2,r},x_{1,s})U_l(x_{2,r})T_k(x_{1,s}), \\ c_{il} \simeq \frac{2}{\pi(M+1)}\sum_{r=1}^{M}(1-x_{2,r}^2)f_i(x_{2,r})U_l(x_{2,r}). \end{cases} \tag{13}$$

Using (12) in (11), we have

$$\sum_{l=0}^{M-1}\sum_{j=1}^{N} b_{ij}\beta_{j\{l+1\}}U_l(t) + \frac{1}{2}\sum_{l=0}^{M-1}\sum_{j=1}^{N}{\sum_{k=0}^{M}}' G_{ijkl}\beta_{jk}U_l(t) = \sum_{l=0}^{M-1} c_{il}\, U_l(t),$$

which leads to the linear system

$$\sum_{j=1}^{N}\left[b_{ij}\beta_{j\{l+1\}} + \frac{1}{2}{\sum_{k=0}^{M}}' G_{ijkl}\beta_{jk}\right] = c_{il},\ i = 1,2,\dots,N,\ l = 0,1,\dots,M-1, \tag{14}$$

for the unknown values $\beta_{jk}$ $(j = 1,2,\dots,N, \quad k = 0,1,\dots,M)$. Taking $\beta_{11},\dots,\beta_{N1}$ arbitrary, the remaining coefficients $\beta_{jk}$ are uniquely found from the linear system (14) which determine the elements of the vector function $\Phi$ via (9).

**Case 2.** We set $\nu = 2$ in (4)–(5) and substitute them in (2) to get

$$-\sum_{j=1}^{N}\sum_{l=0}^{M} b_{ij}\beta_{jl}T_{l+1}(t) + \frac{1}{2}\sum_{j=1}^{N}\sum_{k=0}^{M}\gamma_{ijk}(t)\beta_{jk} = \frac{1}{\pi}f_i(t), \quad i = 1, 2, \ldots, N, \tag{15}$$

where we used the formulas (7)–(8). Then, we expand the functions $f_i$ and $\gamma_{ijk}$ as

$$\begin{cases} \gamma_{ijk}(t) \simeq \sum_{l=0}^{M}{}' G_{ijkl}\, T_l(t)\,, \quad i, j = 1, 2, \ldots, N, \;\; k = 0\,, 1\,, \ldots\,, M \\ \frac{1}{\pi} f_i(t) \simeq \sum_{l=0}^{M}{}' c_{il}\, T_l(t)\,, \qquad i = 1, 2, \ldots, N, \end{cases}$$

where the coefficients are determined by

$$\begin{cases} G_{ijkl} = \frac{2}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\gamma_{ijk}(t)T_l(t)\,dt, \quad i, j = 1, 2, \ldots, N,\, k, l = 0, 1, \ldots, M, \\ \qquad\;\; = \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\sqrt{\frac{1-\tau^2}{1-t^2}}K_{ij}(t,\tau)T_l(t)\,U_k(\tau)\,d\tau dt \\ c_{il} = \frac{2}{\pi^2}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f_i(t)T_l(t)\,dt, \qquad i = 1, 2, \ldots, N, \qquad l = 0, 1, \ldots, M, \end{cases}$$

or approximated by

$$\begin{cases} G_{ijkl} \simeq \frac{4}{(M+1)(M+2)}\sum_{r=1}^{M+1}\sum_{s=1}^{M+1}(1 - x_{2,s}^2)K_{ij}(x_{1,r}, x_{2,s})T_l(x_{1,r})U_k(x_{2,s}), \\ c_{il} \simeq \frac{2}{\pi(M+1)}\sum_{r=1}^{M+1} f_i(x_{1,r})T_l(x_{1,r}). \end{cases}$$

Using the last expansions in (15), returns the following linear system of equations

$$\begin{cases} \frac{1}{2}\sum_{j=1}^{N}\sum_{k=0}^{M} G_{ijkl}\beta_{jk} = c_{il}, \qquad\qquad i = 1, \ldots, N,\, l = 0, \\ \sum_{j=1}^{N}\left\{-b_{ij}\beta_{j\{l-1\}} + \frac{1}{2}\sum_{k=0}^{M} G_{ijkl}\beta_{jk}\right\} = c_{il},\, i = 1, \ldots, N,\, l = 1, \ldots, M, \end{cases}$$

for the unknown values $\beta_{jl}$ $(j = 1, 2, \ldots, N, \quad l = 0, 1, \ldots, M)$. Then elements of the vector function $\Phi(t)$ obtain from (4).

**Cases 3,4.** Similar to cases 1 and 2, we get the linear systems

$$\sum_{j=1}^{N}\left\{b_{ij}\beta_{jl} + \sum_{k=0}^{M} G_{ijkl}\beta_{jk}\right\} = c_{il}, \quad i = 1, 2, \ldots, N, \quad l = 0, 1, \ldots, M, \tag{16}$$

and

$$\sum_{j=1}^{N}\left\{-b_{ij}\beta_{jl}+\sum_{k=0}^{M}G_{ijkl}\beta_{jk}\right\}=c_{il},\quad i=1,2,\ldots,N,\quad l=0,1,\ldots,M,\tag{17}$$

respectively for $\nu = 3$ and $\nu = 4$, and then we determine the elements of corresponding vector $\Phi$ via (4).

## 3 An estimation error and numerical results

In this section, we describe an estimation error for the approximate solution. Let

$$\overline{\Phi}(t) = [\varphi_1(t)\,,\,\varphi_2(t)\,,\,\ldots\,,\,\varphi_N(t)]^T$$

be the vector of approximate solution of the system (1) and let $E(t) = \overline{\Phi}(t) - \Phi(t)$ be the associated vector valued error function. Due to the approximation $\overline{\Phi}(t)$, for $A(t) = 0$, system (1) may be written as

$$B(t)\int_{-1}^{1}\frac{\overline{\Phi}(\tau)}{\tau-t}\,d\tau+\int_{-1}^{1}K(t,\tau)\overline{\Phi}(\tau)d\tau=F(t)+H(t),\quad -1<t<1,\tag{18}$$

where the perturbation term $H(t)$ obtains from

$$H(t)=B(t)\int_{-1}^{1}\frac{\overline{\Phi}(\tau)}{\tau-t}\,d\tau+\int_{-1}^{1}K(t,\tau)\overline{\Phi}(\tau)d\tau-F(t),\quad -1<t<1.$$

Subtracting (18) from (1) yields a system of error equations as

$$B(t)\int_{-1}^{1}\frac{E(\tau)}{\tau-t}\,d\tau+\int_{-1}^{1}K(t,\tau)E(\tau)d\tau=H(t)\,,\quad -1<t<1,$$

which is solvable approximately like system (1).

The following examples illustrate application of the method.

**Example 1.** Let

$$A(t)=0,\quad B(t)=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad K(t,\tau)=\begin{bmatrix}\tau-t&t\\\tau&\tau+t\end{bmatrix},\quad F(t)=\begin{bmatrix}\pi\\2\pi t\end{bmatrix},$$

and find the solution of system (1) in case **1**.

By the above information, the system (1) is reduced to

$$\begin{cases}\int_{-1}^{1}\frac{\phi_1(\tau)}{\tau-t}\,d\tau+\int_{-1}^{1}(\tau-t)\phi_1(\tau)d\tau+\int_{-1}^{1}t\phi_2(\tau)d\tau=\pi, & -1<t<1,\\ \int_{-1}^{1}\frac{\phi_2(\tau)}{\tau-t}\,d\tau+\int_{-1}^{1}\tau\phi_1(\tau)d\tau+\int_{-1}^{1}(\tau+t)\phi_2(\tau)d\tau=2\pi t, & -1<t<1,\end{cases}\tag{19}$$

and since the matrices $S = A+B = I_2$ and $D = A-B = -I_2$ are nonsingular, then system (19) has a unique solution. The kernels $K_{1j}(t,\tau)$, $K_{2j}(t,\tau)$ $(j = 1,2)$, and the functions $f_1$ and $f_2$ are polynomials of degree at most 1, so we set

$$\phi_j(\tau) := \frac{1}{\sqrt{1-\tau^2}} \left\{\beta_{j0}\, T_0(\tau) + \beta_{j1}\, T_1(\tau) + \beta_{j2}\, T_2(\tau)\right\}, \qquad j = 1,2, \tag{20}$$

and

$$\begin{array}{ll} K_{ij}(t,\tau) = \gamma_{ij0}(t)T_0(\tau) + \gamma_{ij1}(t)T_1(\tau)\,, & i,j = 1,2, \\ f_i(t) = c_{i0}U_0(t) + c_{i1}U_1(t), & i = 1,2, \end{array}$$

where

$$\begin{array}{llll} \gamma_{110}(t) = -\frac{1}{2}U_1(t), & \gamma_{111}(t) = U_0(t), & \gamma_{120}(t) = \frac{1}{2}U_1(t), & \gamma_{121}(t) = 0, \\ \gamma_{210}(t) = 0, & \gamma_{211}(t) = U_0(t), & \gamma_{220}(t) = \frac{1}{2}U_1(t), & \gamma_{221}(t) = U_0(t), \\ c_{10}(t) = 1, & c_{11}(t) = 0, & c_{20}(t) = 0, & c_{21}(t) = 1. \end{array}$$

Substituting these expansions into (19) and using (7)–(8), for $\nu = 1$, we obtain

$$\begin{cases} \beta_{11}U_0(t) + \beta_{12}U_1(t) - \frac{1}{2}\beta_{10}U_1(t) + \frac{1}{2}\beta_{11}U_0(t) + \frac{1}{2}\beta_{20}U_1(t) = U_0(t), \\ \\ \beta_{21}U_0(t) + \beta_{22}U_1(t) + \frac{1}{2}\beta_{11}U_0(t) + \frac{1}{2}\beta_{20}U_1(t) + \frac{1}{2}\beta_{21}U_0(t) = U_1(t). \end{cases}$$

Then the linear independency of $\{U_0(t),\ U_1(t)\}$ implies

$$\begin{cases} \frac{3}{2}\beta_{11} = 1, \\ -\frac{1}{2}\beta_{10} + \beta_{12} + \frac{1}{2}\beta_{20} = 0, \\ \frac{1}{2}\beta_{11} + \frac{3}{2}\beta_{21} = 0, \\ \frac{1}{2}\beta_{20} + \beta_{22} = 1. \end{cases}$$

A nonunique solution of this system for the arbitrary values of $\beta_{10}$ and $\beta_{20}$ is given by

$$\beta_{10}, \quad \beta_{11} = \frac{2}{3}, \quad \beta_{12} = \frac{1}{2}\left(\beta_{10} - \beta_{20}\right), \quad \beta_{20}, \quad \beta_{21} = -\frac{2}{9}, \quad \beta_{22} = 1 - \frac{1}{2}\beta_{20}.$$

For example, if $\beta_{10} = \beta_{20} = 2$, then $\beta_{12} = \beta_{22} = 0$ and so we find from (20) that

$$\phi_1(\tau) = \frac{\frac{2}{3}\tau + 2}{\sqrt{1-\tau^2}}\,, \qquad \phi_2(\tau) = \frac{-\frac{2}{9}\tau + 2}{\sqrt{1-\tau^2}}\,,$$

(see Figure 1 for the behavior of these solutions).

**Example 2.** Solve the problem of Example 1 in the case **3**.

In this case, we set

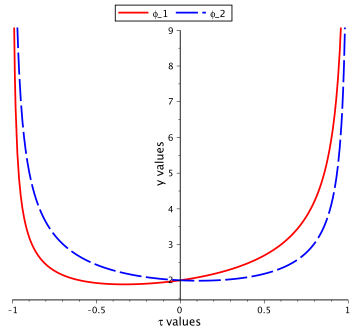


Figure 1: The plots of approximate solutions of Example 1 for M=2.

$$\phi_j(\tau) := \sqrt{\frac{1+\tau}{1-\tau}}\,\{\beta_{j0}\,V_0(\tau) + \beta_{j1}\,V_1(\tau)\}\ , \qquad j = 1, 2, \tag{21}$$

and

$$\begin{array}{ll} K_{ij}(t,\tau) = \gamma_{ij0}(t)V_0(\tau) + \gamma_{ij1}(t)V_1(\tau)\,, & i,j = 1,2, \\ f_i(t) = c_{i0}W_0(t) + c_{i1}W_1(t), & i = 1,2, \end{array}$$

where

$$\begin{array}{lll} \gamma_{110}(t) = W_0(t) - \frac{1}{2}W_1(t), & \gamma_{111}(t) = \gamma_{210}(t) = \gamma_{211}(t) = \gamma_{221}(t) = \frac{1}{2}W_0(t), & \\ \gamma_{120}(t) = -\frac{1}{2}W_0(t) + \frac{1}{2}W_1(t), & \gamma_{121}(t) = 0, \qquad \gamma_{220}(t) = \frac{1}{2}W_1(t), & \\ c_{10}(t) = 1, \qquad c_{11}(t) = 0, & c_{20}(t) = -1, \qquad c_{21}(t) = 1. & \end{array}$$

Substituting these expansions into (19) and using (7)–(8) for $\nu = 3$, result

$$\begin{cases} \beta_{10}W_0(t) + \beta_{11}W_1(t) + \beta_{10}W_0(t) - \frac{1}{2}\beta_{10}W_1(t) \\ \qquad + \frac{1}{2}\beta_{11}W_0(t) - \frac{1}{2}\beta_{20}W_0(t) + \frac{1}{2}\beta_{20}W_1(t) = W_0(t), \\ \\ \beta_{20}W_0(t) + \beta_{21}W_1(t) + \frac{1}{2}\beta_{10}W_0(t) + \frac{1}{2}\beta_{11}W_0(t) \\ \qquad + \frac{1}{2}\beta_{20}W_1(t) + \frac{1}{2}\beta_{21}W_0(t) = -W_0(t) + W_1(t), \end{cases}$$

and from the linear independency of $\{W_0(t)$ and $W_1(t)\}$, we get the algebraic system

$$\begin{cases} 2\beta_{10} + \frac{1}{2}\beta_{11} - \frac{1}{2}\beta_{20} = 1, \\ -\frac{1}{2}\beta_{10} + \beta_{11} + \frac{1}{2}\beta_{20} = 0, \\ \frac{1}{2}\beta_{10} + \frac{1}{2}\beta_{11} + \beta_{20} + \frac{1}{2}\beta_{21} = -1, \\ \frac{1}{2}\beta_{20} + \beta_{21} = 1, \end{cases}$$

which has the unique solution

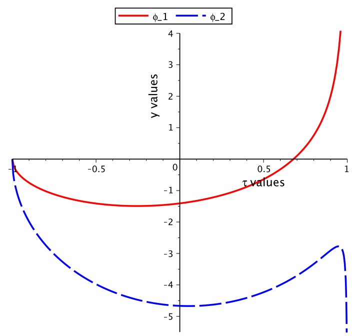


Figure 2: The plots of approximate solutions of Example 2 for M=2.

$$\beta_{10} = -\frac{10}{27}\,, \;\; \beta_{11} = \frac{28}{27}\,, \;\; \beta_{20} = -\frac{22}{9}\,, \;\; \beta_{21} = \frac{20}{9}\,,$$

and the solutions of (19) can be found via (21). The graphs of these solutions plotted in Figure 2.

**Example 3.** Solve the following system in the case **3** [16]:

$$\begin{cases} 3\int_{-1}^{1} \frac{\phi_1(\tau)}{\tau-t}\,d\tau + \int_{-1}^{1} \frac{\phi_2(\tau)}{\tau-t}\,d\tau = (t^2+1)\sin 2t, \; -1 < t < 1, \\ 2\int_{-1}^{1} \frac{\phi_1(\tau)}{\tau-t}\,d\tau + \int_{-1}^{1} \frac{\phi_2(\tau)}{\tau-t}\,d\tau = t\cos 2t, \qquad\quad -1 < t < 1. \end{cases}$$

In this case, we will approximate the unknown functions as

$$\phi_i(\tau) \simeq \varphi_i(\tau) := \sqrt{1-\tau^2}\sum_{j=0}^{M} \beta_{ij}\, U_j(\tau), \qquad i = 1, 2,$$

where the exact solutions are

$$\begin{cases} \phi_1(\tau) = -\frac{\sqrt{1-\tau^2}}{\pi^2} \int_{-1}^{1} \frac{(t^2+1)\sin 2t - t\cos 2t}{\sqrt{1-t^2}(t-\tau)}\,dt, \\ \phi_2(\tau) = -\frac{\sqrt{1-\tau^2}}{\pi^2} \int_{-1}^{1} \frac{3t\cos 2t - 2(t^2+1)\sin 2t}{\sqrt{1-t^2}(t-\tau)}\,dt. \end{cases}$$

We define the error functions as

$$E_i(\tau) = |\phi_i(\tau) - \varphi_i(\tau)|, \qquad i = 1, 2,$$

where the exact solutions $\phi_1$ and $\phi_2$ are calculated by using the Maple code `int(expression,x=a..b, CauchyPrincipalValue)`.

Table 1: Comparison of our results ($E_i$) with the results of [16] ($\varepsilon_i$) for M=8.

| $\tau$ | $\phi_1(\tau)$ | $\phi_2(\tau)$ | $E_1(\tau)$ | $E_2(\tau)$ | $\varepsilon_1(\tau)$ | $\varepsilon_2(\tau)$ |
|---|---|---|---|---|---|---|
| −0.96 | −0.2164694659571311 | 0.5107921932649533 | 1e−8 | 2e−8 | 8e−5 | 2e−4 |
| −0.70 | −0.5394637532569845 | 1.185306580331174 | 3e−7 | 6e−7 | 4e−5 | 1e−4 |
| −0.25 | −0.5831756914063946 | 1.129312022840245 | 4e−7 | 9e−7 | 9e−6 | 2e−5 |
| 0.25 | −0.5831756914063946 | 1.129312022840245 | 4e−7 | 9e−7 | 9e−6 | 2e−5 |
| 0.75 | −0.5394637532569845 | 1.185306580331174 | 3e−7 | 6e−7 | 4e−5 | 1e−4 |
| 0.96 | −0.2164694659571311 | 0.5107921932649533 | 1e−8 | 2e−8 | 8e−5 | 2e−4 |

Table 2: The approximated values $\varphi_i(\tau)$ for $M = 15$ by using 16-digits arithmetic.

| $\tau$ | $\varphi_1(\tau)$ | $\varphi_2(\tau)$ | $E_1(\tau)$ | $E_2(\tau)$ | $\varepsilon_1(\tau)$ | $\varepsilon_2(\tau)$ |
|---|---|---|---|---|---|---|
| −0.96 | −0.2164694659571311 | 0.5107921932649527 | 0 | 6e−16 | - | - |
| −0.70 | −0.5394637532569844 | 1.185306580331173 | 1e−16 | 1e−15 | - | - |
| −0.25 | −0.5831756914063935 | 1.129312022840244 | 1e−15 | 1e−15 | - | - |
| 0.25 | −0.5831756914063935 | 1.129312022840244 | 1e−15 | 1e−15 | - | - |
| 0.75 | −0.5394637532569844 | 1.185306580331173 | 1e−16 | 1e−15 | - | - |
| 0.96 | −0.2164694659571311 | 0.5107921932649527 | 0 | 6e−16 | - | - |

In Table 1, we compared our numerical results (absolute errors $E_1$ and $E_2$) with those reported in [16] (absolute errors $\varepsilon_1$ and $\varepsilon_2$). Table 2 shows our results for $M = 15$, which were not reported in [16].

**Example 4.** Consider the singular integral equation

$$\int_{-1}^{1} \frac{\phi(\tau)}{\tau - t}\, d\tau + \int_{-1}^{1} \mathbf{e}^{\mathbf{i}t}\phi(\tau)\, d\tau = 1 - 2t^2 + \mathbf{i}t + \frac{1}{2}\mathbf{e}^{-\mathbf{i}t}, \quad -1 < t < 1, \quad (22)$$

with the exact solution $\phi(\tau) = \frac{\sqrt{1-\tau^2}}{\pi}(2\tau - \mathbf{i})$ in the complex plane, where $\mathbf{i} = \sqrt{-1}$.

By taking $\phi_1 := Re\{\phi\}$ and $\phi_2 := Im\{\phi\}$, equation (22) is reduced to the system of singular integral equations

$$\begin{cases} \int_{-1}^{1} \frac{\phi_1(\tau)}{\tau - t}\, d\tau + \int_{-1}^{1} \cos(t)\phi_1(\tau)\, d\tau - \int_{-1}^{1} \sin(t)\phi_2(\tau)\, d\tau = 1 - 2t^2 + \frac{1}{2}\sin t, \; -1 < t < 1, \\ \int_{-1}^{1} \frac{\phi_2(\tau)}{\tau - t}\, d\tau + \int_{-1}^{1} \sin(t)\phi_1(\tau)\, d\tau + \int_{-1}^{1} \cos(t)\phi_2(\tau)\, d\tau = t - \frac{1}{2}\cos t, \qquad -1 < t < 1. \end{cases}$$

Similar to Examples 1 and 2, by setting

$$\phi_i(\tau) \simeq \varphi_i(\tau) := \sqrt{1 - \tau^2} \sum_{j=0}^{M} \beta_{ij}\, U_j(\tau), \qquad i = 1, 2,$$

we get the exact solutions for $M = 1$.

**Example 5.** Consider the system of singular integral equations [14]

$$\begin{cases} \frac{1000}{\pi}\int_{-1}^{1}\frac{\phi_1(\tau)}{\tau-t}\,d\tau + \frac{10}{\pi}\int_{-1}^{1}\frac{\phi_2(\tau)}{\tau-t}\,d\tau = f_1(t) + \mathbf{i}g_1(t),\ -1<t<1, \\ \frac{500}{\pi}\int_{-1}^{1}\frac{\phi_1(\tau)}{\tau-t}\,d\tau + \frac{200}{\pi}\int_{-1}^{1}\frac{\phi_2(\tau)}{\tau-t}\,d\tau = f_2(t) + \mathbf{i}g_2(t),\ -1<t<1, \end{cases} \tag{23}$$

where

$$\begin{cases} f_1(t) = -990t^8 + 1089t^7 + 937t^6 - \frac{26704}{25}t^5 - \frac{349161}{1000}t^4 + \frac{792327}{2000}t^3 + \frac{1761}{250}t^2 - \frac{53511}{4000}t - \frac{53929}{2000}, \\ g_1(t) = 990t^8 - 1189t^7 - \frac{8971}{10}t^6 + \frac{119047}{100}t^5 + \frac{163961}{500}t^4 - \frac{279198}{625}t^3 - \frac{30873}{10000}t^2 + \frac{69533}{4000}t + \frac{1130501}{40000}, \\ f_2(t) = -300t^8 + 330t^7 + 215t^6 - \frac{2447}{10}t^5 - \frac{8607}{100}t^4 + \frac{735}{8}t^3 - \frac{17253}{2000}t^2 + \frac{29541}{4000}t - \frac{14701}{1000}, \\ g_2(t) = 300t^8 - 380t^7 - 197t^6 + \frac{1462}{5}t^5 + \frac{9419}{100}t^4 - \frac{27549}{250}t^3 - \frac{183}{400}t^2 - \frac{7957}{2000}t + \frac{57583}{4000}. \end{cases}$$

It is easy to see that system (23) is equivalent to the following disjointed singular integral equations,

$$\begin{cases} \frac{1}{\pi}\int_{-1}^{1}\frac{Re\{\phi_1(\tau)\}}{\tau-t}\,d\tau = \frac{20f_1(t)-f_2(t)}{19500},\ -1<t<1, \\ \frac{1}{\pi}\int_{-1}^{1}\frac{Im\{\phi_1(\tau)\}}{\tau-t}\,d\tau = \frac{20g_1(t)-g_2(t)}{19500},\ -1<t<1, \\ \frac{1}{\pi}\int_{-1}^{1}\frac{Re\{\phi_2(\tau)\}}{\tau-t}\,d\tau = \frac{2f_2(t)-f_1(t)}{390},\ -1<t<1, \\ \frac{1}{\pi}\int_{-1}^{1}\frac{Im\{\phi_2(\tau)\}}{\tau-t}\,d\tau = \frac{2g_2(t)-g_1(t)}{390},\ -1<t<1. \end{cases} \tag{24}$$

The exact solution of system (23) was reported in [14] for the case $\mathbf{r} = 4$. By applying our method for the equation in (24), we get the real parts of unknown functions exactly but, for the imaginary parts, we obtain the error functions

$$E_i(\tau) = |Im\{\phi_i(\tau)\} - Im\{\varphi_i(\tau)\}| = \sqrt{\frac{1-\tau}{1+\tau}}h_i(\tau), \quad i = 1, 2,$$

where

$$\begin{cases} h_1(\tau) = 5.128205128205128 \times 10^{-8}(1+\tau), & -1<\tau<1, \\ h_2(\tau) = 5.128205128205128 \times 10^{-6}(1+\tau), & -1<\tau<1. \end{cases}$$

The plots of $h_1$ and $h_2$ and the regular parts of the error functions, are shown in Figure 3. As shown in these plots, the absolute errors increase when $\tau$ approaches to 1, because in case 4, the singularity for the integral part happens at $\tau = 1$.

**Example 6.** Consider the problem of a half plane containing a crack parallel to the boundary, which is illustrated in Figure 4 and formulated as the system [4]

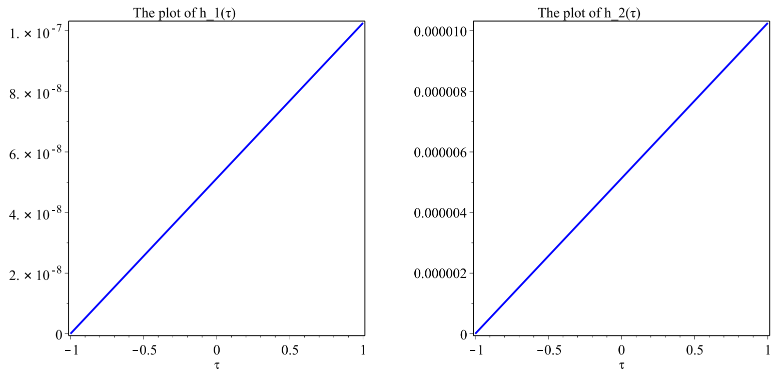


Figure 3: The plots of the regular parts of the error functions in Example 5.

$$\begin{cases} \int_{-1}^{1} \frac{\phi_1(\tau)}{\tau - t}\, d\tau + \int_{-1}^{1} \left[K_{11}(t,\tau)\phi_1(\tau) + K_{12}(t,\tau)\phi_2(\tau) d\tau\right] = 0, \\ \int_{-1}^{1} \frac{\phi_2(\tau)}{\tau - t}\, d\tau + \int_{-1}^{1} \left[K_{21}(t,\tau)\phi_1(\tau) + K_{22}(t,\tau)\phi_2(\tau) d\tau\right] = \pi, \end{cases} \tag{25}$$

with

$$K_{11}(t,\tau) = -\frac{\tau - t}{(\tau-t)^2 + 4h^2} + \frac{8h^2(\tau - t)}{[(\tau-t)^2+4h^2]^2} - \frac{4h^2(\tau-t)\left[12h^2 - (\tau-t)^2\right]}{[(\tau-t)^2+4h^2]^3}\,,$$

$$K_{12}(t,\tau) = K_{21}(t,\tau) = -\frac{8h^3\left[4h^2 - 3(\tau-t)^2\right]}{[(\tau-t)^2+4h^2]^3}\,,$$

$$K_{22}(t,\tau) = -\frac{\tau - t}{(\tau-t)^2 + 4h^2} - \frac{8h^2(\tau - t)}{[(\tau-t)^2+4h^2]^2} - \frac{4h^2(\tau-t)\left[12h^2 - (\tau-t)^2\right]}{[(\tau-t)^2+4h^2]^3},$$

where $h$ is the distance of crack from the boundary. The physical conditions of the problem impose that the relations

$$\int_{-1}^{1} \phi_1(\tau)\, d\tau = 0\,, \qquad \int_{-1}^{1} \phi_2(\tau)\, d\tau = 0, \tag{26}$$

and

$$\phi_1(t) = \phi_1(-t), \quad \phi_2(t) = -\phi_2(-t)$$

are satisfied. Therefore the unknown functions may be expressed as

$$\phi_1(\tau) \simeq \frac{1}{\sqrt{1-\tau^2}} \sum_{j=0}^{M} \beta_{1j}\, T_{2j}(\tau), \quad \phi_2(\tau) \simeq \frac{1}{\sqrt{1-\tau^2}} \sum_{j=1}^{M} \beta_{2j}\, T_{2j-1}(\tau). \tag{27}$$

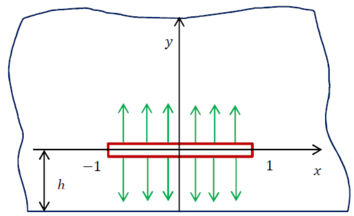


Figure 4: Crack parallel to a free boundary in Example 6.

For $\nu = 1$, it follows from the orthogonality condition (7) that the second condition in (26) satisfies and the first one gives $\beta_{10} = 0$.

Taking $\beta_{11} = \beta_{21} = 0$, the remaining coefficients $\beta_{ij}$ are uniquely determined from the linear algebraic system (14) for each values of $h$ and $M$. This leads to find the functions $\phi_1$ and $\phi_2$ from (27).

The stress intensity factors

$$k_1 = \lim_{\tau \to 1^-} \sqrt{1-\tau^2}\, \phi_2(\tau)$$
$$k_2 = \lim_{\tau \to 1^-} \sqrt{1-\tau^2}\, \phi_1(\tau)$$

and their absolute estimation errors(Est.Err.) are reported in Table 3. For $h = \infty$ and $K_{ij}(t,\tau) = 0$, from (27) and (13)–(14), the exact solutions of (25) are obtained as

$$\phi_1(\tau) = 0, \qquad \phi_2(\tau) = \frac{\tau}{\sqrt{1-\tau^2}},$$

which give $k_1 = 1$ and $k_2 = 0$. This is shown in the last row of Table 3. The table shows the rapid convergence of the results even for relatively small values of $M$.

## 4 Conclusions

We described a new idea of using Chebyshev polynomials for the numerical solution of the system of singular integral equations of the first kind. In Section 3, we illustrated this idea by using system of different kind of singular integral equations (Examples 1–5). In Example 6, we studied a crack problem in solid mechanics and reported the numerical results (see Table 3) to show the efficiency and rapid convergence of the proposed method for all these kinds of problems.

Table 3: Stress intensity factors for the crack parallel to the boundary

| $h$ | $M$ | $k_1$ | Est.Err. $k_1$ | $k_2$ | Est.Err. $k_2$ |
|---|---|---|---|---|---|
| 0.2 | 6 | 4.878800637605022 | 6.3e-14 | 1.750099102171126 | 6.2e-14 |
| | 7 | 4.788277537335018 | 1.1e-14 | 1.727809740547429 | 4.1e-15 |
| | 8 | 4.760729834685963 | 4.8e-14 | 1.719782910590219 | 1.1e-14 |
| 0.4 | 3 | 2.607272141646415 | 4.3e-15 | 0.7745787927510580 | 1.6e-16 |
| | 4 | 2.594500911475041 | 7.3e-15 | 0.7266641783709941 | 5.0e-15 |
| | 6 | 2.594423234973139 | 4.2e-14 | 0.7376171346942053 | 3.3e-16 |
| 0.6 | 2 | 1.834057544899021 | 1.1e-15 | 0.5664257041432605 | 4.1e-16 |
| | 5 | 1.960455689663461 | 6.1e-15 | 0.4297949760368867 | 1.6e-15 |
| 0.8 | 2 | 1.608371955353828 | 1.6e-15 | 0.3323260582700188 | 2.8e-16 |
| | 3 | 1.660617572058080 | 8.7e-16 | 0.2675691556476836 | 4.6e-16 |
| 1.0 | 2 | 1.461157081431933 | 2.0e-15 | 0.2104682299562445 | 1.1e-16 |
| | 4 | 1.485914720666516 | 2.1e-16 | 0.1796691052492212 | 1.1e-16 |
| 1.2 | 4 | 1.372176156193755 | 5.0e-16 | 0.1234414146531335 | 0. |
| 1.5 | 4 | 1.262800608570183 | 1.6e-15 | 0.07465158121522054 | 1.7e-16 |
| 2.0 | 3 | 1.162112249974693 | 1.1e-15 | 0.03662808437088003 | 0. |
| 3.0 | 2 | 1.077621553329114 | 3.3e-16 | 0.01274529646673066 | 0. |
| 10 | 2 | 1.007451045420713 | 2.6e-16 | 0.00037197952964307 | 0. |
| $\infty$ | 1 | 1 | 0 | 0 | 0 |

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