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Statistical inference based on k-records

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Abstract

In this paper, an extension of record models, well known as k-records, is considered. Bayesian estimation as well as prediction based on k-records are presented when the underlying distribution is assumed to have a general form. The proposed procedure is applied to the Exponential, Weibull and Pareto models in one parameter case. Also, the two-parameter Exponential distribution, when both parameters are unknown, is studied in more details. Since the ordinary record values are contained in the k-records, by putting $k = 1$, the results for usual records can be obtained as special case.

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1 Introduction

Let $\{X_i,\ i \geq 1\}$ be a sequence of independent and identically distributed (iid) continuous random variables each distributed according to cumulative distribution

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function (cdf) $F(t)$ and probability density function (pdf) $f(t)$. An observation X_j will be called an upper record value if its value exceeds that of all previous observations. Thus, X_j is an upper record if $X_j > X_i$ for every $i < j$. An analogous definition can be given for lower record values. Today there are over 500 papers and several books published on record-breaking data (see, for instance, Chandler [8], Resnick [19], Shorrock [21], Glick [14], Samaniego and Whitaker [20], Arnold *et al.* [5] and Nevzorov [18]).

There are several situations where the second or third largest values of special interest, insurance claims some non-life insurance can be used as an example, see Kamps [16], so the usual record models is inadequate. Also, in the ordinary record value theory, while inverse sampling considerations have given valuable insights, their practical implementation is greatly hindered by the sparsity of records. These problems caused the researchers to study the theory of k-record models. Upper k-record process is defined in terms of the k-th largest X yet seen. For a formal definition, we consider the definition in Arnold *et al.* [5], p. 43, in the continuous case, let $T_{1(k)} = k, R_{1(k)} = X_{1:k}$ and for $n \geq 2$, let

$$T_{n(k)} = \min\{j : j > T_{n-1(k)}, X_j > X_{T_{n-1(k)}-k+1:T_{n-1(k)}}\},$$

where $X_{i:m}$ denotes the i-th order statistic in a sample of size m. The sequence of *upper k-records* is then defined by $R_{n(k)} = X_{T_{n(k)}-k+1:T_{n(k)}}$ for $n \geq 1$. Arnold *et al.* [5] call this a *Type 2 k-record* sequence. For $k = 1$, note that the usual records are recovered. An analogous definition can be given for *lower k-records* as well. This sequence of k-records was introduced by Dziubdziela and Kopocinski [12] and it has found acceptance in the literature. Some work has been done on the statistical inference, based on k-records. See, for instance, Deheuvels and Nevzorov [11], Berred [7], Ali Mousa *et al.* [4], Malinowska and Szynal [17], Danielak and Raqab [9],[10], Ahmadi *et al.* [2], Fashandi and Ahmadi [13] and references therein.

We assume that this type of k-record data is available and the aim of this paper is to develop inference methods as well as prediction of future k-records based on past observed k-records. The rest of the paper is organized as follows. In Section 2, Bayesian estimation as well as prediction based on k-records are presented when the underlying distribution is assumed to have a general model. In Section 3, a two-parameter exponential distribution is considered; the maximum likelihood and Bayes estimators for the unknown parameters, are obtained. Bayesian prediction of the future k-records, either point or interval, are obtained in Section 4, when the k-records are assumed to come from the two-parameter exponential model.

2 A General Model

In this section, we consider the problems of estimation and prediction based on k-records, when the underlying distribution has a general form. In order to do this, let C be the class of all absolute continuous distribution functions F of the form

$$F_\theta(x) = 1 - e^{-\lambda_\theta(x)}, \quad x > 0, \tag{2.1}$$

where $\lambda'_\theta(x)$ (the derivative of $\lambda_\theta(x)$ w.r.t θ) exists and is a positive function of θ and x. Then

$$f_\theta(x) = \lambda'_\theta(x) e^{-\lambda_\theta(x)}, \quad x > 0. \tag{2.2}$$

This class includes several important life time families such as: Exponential, Weibull, compound Weibull, Pareto, Beta, Gompertz, compound Gompertz and Burr type XII, among others.

2.1 Estimation

Using the joint pdf of usual records, we readily have the joint density of the first m, k-records $R_{1(k)}, R_{2(k)}, ..., R_{m(k)}$ as

$$f(x_1, ..., x_m) = k^m \prod_{i=1}^{m} \frac{f(x_i)}{1 - F(x_i)} (1 - F(x_m))^k, \tag{2.3}$$

(see, Arnold *et al.* [5]). Now, suppose we observe $R_{1(k)} = x_1, \cdots, R_{m(k)} = x_m$ then by substitution of (2.1) and (2.2) in (2.3), the likelihood function $L(\theta)$ is

$$L(\theta) \propto A(\theta; \mathbf{x}) e^{-B(\theta, x_m)}, \tag{2.4}$$

where $\mathbf{x} = (x_1, \cdots, x_m)$,

$$A(\theta; \mathbf{x}) = \prod_{i=1}^{m} \lambda'_\theta(x_i) \quad \text{and} \quad B(\theta, x_m) = k\lambda_\theta(x_m).$$

There is clearly no way in which one can say that one prior is better than any other. Presumably one has own subjective prior and must live with all of its lumps and bumps. It is more frequently the case that we elect to restrict attention to a given flexible family of prior distributions and we choose one from the family which seems to the best of our match and personal believes. With this in mind, let the conjugate prior density function for θ, proposed by AL-Hussaini [3], is given by

$$\pi(\theta; \delta) \propto C(\theta; \delta) e^{-D(\theta; \delta)}, \quad \theta \in \Theta, \ \delta \in \Omega, \tag{2.5}$$

where Ω is the prior parameter(s) space. Then the posterior density function is derived as

$$\pi(\theta|\mathbf{x}) = C_1(M, N) M(\theta; \mathbf{x}, \delta) e^{-N(\theta;\ x_m, \delta)}, \tag{2.6}$$

where

$$\begin{aligned} M(\theta; \mathbf{x}, \delta) &= C(\theta; \delta) A(\theta; \mathbf{x}), \\ N(\theta; x_m, \delta) &= D(\theta; \delta) + B(\theta, x_m), \end{aligned}$$

and $C_1(M, N)$ is the normalizing constant given by

$$C_1(M, N) = \left[\int_\Theta M(\theta; \mathbf{x}, \delta) e^{-N(\theta;\ x_m,\ \delta)} d\theta \right]^{-1}. \tag{2.7}$$

If Θ is one dimensional then the Bayes estimator of θ, under squared error (SE) loss function, is

$$\hat{\theta}_{BS} = \frac{C_1(M, N)}{C_1(M^\star, N)}, \tag{2.8}$$

where $M^{\star}(x) = xM(x)$.

Remark. $\hat{\theta}_{BS}$ in (2.8) is the unique Bayes estimate of θ under SE loss function with respect to the above mentioned proper prior and hence is admissible.

Example 2.1 (One-Parameter Exponential Model)

Let $\lambda_\theta(x) = (x - \mu_0)/\sigma$ where μ_0 is known and $\theta = \sigma$, i.e. we have a one parameter exponential distribution. Then $A(\theta; \mathbf{x}) = 1/\sigma^m$ and $B(\theta; x_m) = k(x_m - \mu_0)/\sigma$. It can be shown that the maximum likelihood estimation of σ is $\hat{\sigma}_M = k(R_{m(k)} - \mu_0)/m$. We use Inverted Gamma with parameters a and b as the conjugate prior, i.e. $\pi(\sigma) = b^a \sigma^{-(a+1)} \exp\{-b/\sigma\}/\Gamma(a)$, where from (2.5), $C(\theta; \delta) = \sigma^{-(m+2)}$, $D(\theta; \delta) = b/\sigma$ and $\delta = (a, b)$. Therefore, $M(\theta; \mathbf{x}, \delta) = 1/\sigma^{m+a+1}$ and $N(\theta; x_m, \delta) = (b + k(x_m - \mu_0))/\sigma$. From (2.8), the Bayes estimate of σ under SE loss function is given by

$$\hat{\sigma}_{BS} = \frac{b + k(R_{m(k)} - \mu_0)}{m + a - 1}.$$

It may be noted that, from (2.6), the posterior distribution of σ^{-1} is $\Gamma(m + a, b + k(x_m - \mu_0))$.

Example 2.2 (Weibull Model)

Suppose $\lambda_\theta(x) = \alpha x^\beta$, where β is known and $\theta = \alpha$. Then $\lambda'_\theta(x) = \alpha\beta x^{\beta-1}$. It can be shown that the maximum likelihood estimate of α is $\hat{\alpha}_M = m/(kR^\beta_{m(k)})$. Assuming a Gamma conjugate prior with parameter a and b, i.e.
$\pi(\alpha) = b^a \alpha^{a-1} \exp\{-b\alpha\}/\Gamma(a)$, the Bayes estimate of α under SE loss function is given by

$$\hat{\alpha}_{BS} = (m + a)/(kR^\beta_{m(k)} + b).$$

Example 2.3 (Pareto Model)

In this model, $\lambda_\theta(x) = \alpha \ln(x/\beta)$ where β is known and $\theta = \alpha$. So, by (2.4), maximum likelihood estimate of α is $\hat{\alpha}_M = m/(k \ln[R_{m(k)}/\beta])$. Assuming a conjugate prior Gamma with parameters a and b, i.e. $\pi(\alpha) = b^a \alpha^{a-1} \exp\{-b\alpha\}/\Gamma(a)$, the Bayes estimate of α under SE loss function is given by

$$\hat{\alpha}_{BS} = \frac{m+a}{k \ln[R_{m(k)}/\beta] + b}.$$

When both of the parameters in the above examples are unknown, In [1] we have obtained similar results based on usual records ($k = 1$).

2.2 Prediction

Assume that we have the first m upper k-records $R_{1(k)} = x_1, R_{2(k)} = x_2, ..., R_{m(k)} = x_m$ from a member of class C in (2.1). Based on such a sample, prediction, either point or interval, is needed for s-th upper k-record, $1 \leq m < s$. Now, let $Y = R_{s(k)}$ be the s-th upper k-record value, $s > m$. The conditional pdf of Y for the given vector parameter θ and that the first m k-record $R_{1(k)}, \cdots, R_{m(k)}$ is given by

$$f(y|\mathbf{x}, \theta) = k^{s-m} \frac{[\lambda_\theta(y) - \lambda_\theta(x_m)]^{s-m-1}}{\Gamma(s-m)} \lambda'_\theta(y) e^{-k(\lambda_\theta(y) - \lambda_\theta(x_m))}. \tag{2.9}$$

Hence, from equations (2.6) and (2.9) we get the Bayes predictive density function of Y

$$\begin{aligned} h^*(y|\mathbf{x}) &= \int_\Theta f(y|\mathbf{x}, \theta)\pi(\theta|\mathbf{x})d\theta \\ &= \frac{k^{s-m} C_1(M, N)}{\Gamma(s-m)} \\ &\quad \int_\Theta M(\theta; \mathbf{x}, \delta)[\lambda_\theta(y) - \lambda_\theta(x_m)]^{s-m-1} \lambda'_\theta(y) e^{-k[\lambda_\theta(y) - \lambda_\theta(x_m)] - N(\theta; x_m, \delta)} d\theta. \end{aligned} \tag{2.10}$$

The Bayes point predictor of the s-th upper k-record based on the first m $(m < s)$ observed k-records is given by

$$\begin{aligned}
\hat{Y}_{BS} &= \int_{x_m}^{+\infty} y h^*(y|\mathbf{x}) dy \\
&= \frac{k^{s-m} C_1(M,N)}{\Gamma(s-m)} \int_{\Theta} M(\theta; \mathbf{x}, \delta) e^{-N(\theta; x_m, \delta)} \\
&\quad \int_{x_m}^{+\infty} y[\lambda_\theta(y) - \lambda_\theta(x_m)]^{s-m-1} \lambda'_\theta(y) e^{-k(\lambda_\theta(y)-\lambda_\theta(x_m))} dy\ d\theta \\
&= \frac{k^{s-m} C_1(M,N)}{\Gamma(s-m)} \int_{\Theta} M(\theta; \mathbf{x}, \delta) e^{-N(\theta; x_m, \delta)} \\
&\quad \left\{ \int_0^{+\infty} \lambda_\theta^{-1}(z + \lambda_\theta(x_m)) z^{s-m-1} e^{-kz} dz \right\} d\theta \\
&= C_1(M,N) \int_{\Theta} M(\theta; \mathbf{x}, \delta) e^{-N(\theta; x_m, \delta)} E\left\{ \lambda_\theta^{-1}(Z + \lambda_\theta(x_m)) \right\} d\theta,
\end{aligned} \tag{2.11}$$

where $Z \sim \Gamma(s-m, k)$ and $\lambda_\theta^{-1}(x)$ is the inverse function of $\lambda_\theta(x)$.

Example 2.4 (Continuation Examples 2.1-2.3)

Using (2.11), we obtain the Bayesian point prediction of $R_{s(k)}$ for the following three models. We have

i. One parameter Exponential model:

$$\hat{Y}_{BS} = \left(\frac{s+a-1}{m+a-1} \right) R_{m(k)} + \left(\frac{s-m}{m+a-1} \right) \left(\frac{b}{k} - \mu_0 \right).$$

ii. Weibull model:

$$\hat{Y}_{BS} = \left(\frac{s+a-1}{m+a-1} \right) R^{\beta}_{m(k)} + \left(\frac{s-m}{m+a-1} \right) \frac{b}{k}.$$

iii. Pareto model:

$$\hat{Y}_{BS} = \frac{[b + k \log(R_{m(k)}/\beta)]^{m+a}}{\Gamma(m+a)} I(R_{m(k)}),$$

where $I(R_{m(k)}) = \int_0^{+\infty} \frac{\alpha^{s+a-1}}{(\alpha-1/k)^{s-m}} e^{-[b+k\log(R_{m(k)}/\beta)]\alpha} d\alpha$.

Remark. It may be noted that one may use (2.10) to obtain Bayesian prediction interval for $R_{s(k)}$.

In the rest of the paper, we consider two-parameter Exponential distribution which does not belong to the class C in (2.1), where the location parameter μ is unknown. Its cdf and pdf are given by

$$F(x;\mu,\sigma) = 1 - e^{-\frac{1}{\sigma}(x-\mu)} \quad x \geq \mu, \quad \sigma > 0, \tag{2.12}$$

and

$$f(x;\mu,\sigma) = \frac{1}{\sigma} e^{-\frac{1}{\sigma}(x-\mu)} \quad x \geq \mu, \quad \sigma > 0, \tag{2.13}$$

respectively, which is denoted by $X \sim Exp(\mu,\sigma)$. Ahmadi *et al.* [2] studied the problem of estimation and prediction in $Exp(\mu,\sigma)$ under LINEX (LINear-EXPonential) loss function based on k-records from Bayesian view point.

3 Estimation in Exponential Model

As mentioned in Section 1, the usual record data are rare in practical situations. In fact, the expected waiting time is infinite for every record after the first; but, this problem will be fixed by considering k-records instead (see Theorem 2.1 of [15]). So, in this section, we shall be concerned with estimation of the two unknown parameters μ and σ of $Exp(\mu,\sigma)$ based on k-record values. Suppose, we observed the first m upper k-records $R_{1(k)} = x_1,\ R_{2(k)} = x_2, ..., R_{m(k)} = x_m$ from an $Exp(\mu,\sigma)$.Then from (2.3), (2.12) and (2.13) the likelihood function is given by

$$L(\mu,\sigma|\mathbf{x}) = (\frac{k}{\sigma})^m e^{-\frac{k}{\sigma}(x_m-\mu)}, \quad \mu \leq x_1 < x_2 < ... < x_m,\ \sigma > 0, \tag{3.1}$$

where $\mathbf{x} = (x_1, x_2, ..., x_m)$.

3.1 Maximum likelihood estimation

In the case $k = 1$, the MLE (maximum likelihood estimation) of the two-parameters of the Exponential distribution can be found in Arnold *et al.* [5], p. 123. We obtained MLE based on k-record values, by (3.1). The natural logarithm of (3.1) is given by

$$l = m \ln k - m \ln \sigma - \frac{k}{\sigma}(x_m - \mu), \quad \mu \leq x_1 < x_2 < ... < x_m. \tag{3.2}$$

Assume that the parameters μ and σ are unknown, from (3.2) we readily obtain the MLE of μ and σ as follows:

$$\hat{\mu}_M = R_{1(k)}, \tag{3.3}$$

and

$$\hat{\sigma}_M = \frac{k}{m}(R_{m(k)} - R_{1(k)}). \tag{3.4}$$

It is easy to verify that

- $R_{1(k)} \sim Exp(\mu, \sigma/k)$,
- $R_{m(k)} - R_{1(k)}$ and $R_{1(k)}$ are independent random variables,
- $R_{m(k)} - R_{1(k)}$ has gamma distribution with parameters $m - 1$ and k/σ.

Then by (3.3) and (3.4) we have

- $E(\hat{\mu}_M) = \mu + \frac{\sigma}{k}$,
- $MSE(\hat{\mu}_M) = 2\frac{\sigma^2}{k^2}$.

Also,

- $E(\hat{\sigma}_M) = \frac{m-1}{m}\sigma$,
- $MSE(\hat{\sigma}_M) = \frac{\sigma^2}{m}$, do not depend on k.
- $Cov(\hat{\mu}_M, \hat{\sigma}_M) = 0$.

Notice that $\hat{\mu}_M$ is a biased estimator μ, while an unbiased estimator for μ is given by

$$\tilde{\mu} = \frac{m + k - 1}{m - 1} R_{1(k)} - \frac{k}{m - 1} R_{m(k)}.$$

3.2 Bayes estimation

Our aim is to obtain Bayes estimate of the unknown parameters based on $x_1, ..., x_m$ under SE loss function. We consider the following two cases for our Bayesian estimation problem.

a) σ *is known.*

Without loss of generality, we may assume $\sigma = 1$ then by (3.1), we have

$$f(\mathbf{x}|\mu) = k^m e^{-k(x_m - \mu)}, \quad \mu < x_1 < x_2 < ... < x_m. \tag{3.5}$$

Assume the Jeffreys non-informative prior distribution (see [6]) of the parameter μ in the form

$$\pi(\mu) \propto 1. \tag{3.6}$$

Hence the posterior distribution of μ is

$$\pi(\mu|\mathbf{x}) \propto f(\mathbf{x}|\mu)\pi(\mu),$$

where $f(\mathbf{x}|\mu)$ is the joint density function given by (3.5) and $\pi(\mu)$ is the prior density given by (3.6). So, we have

$$\pi(\mu|\mathbf{x}) = ke^{k(\mu - x_1)}, \quad \mu < x_1. \tag{3.7}$$

Suppose an SE loss function, the Bayes estimate of a parameter is its posterior mean. Therefore, by (3.7), the Bayes estimate of the parameter μ is given by

$$\hat{\mu}_{1BS} = R_{1(k)} - \frac{1}{k}. \tag{3.8}$$

From Eq. (3.8) we get

- $E(\hat{\mu}_{1BS}) = \mu$,
- $MSE(\hat{\mu}_{1BS}) = \frac{1}{k^2}$.

b) σ *is unknown.*

Under the assumption that both of the parameters μ and σ are unknown, we may consider the joint density as a product of the conditional density of μ for given σ and a two parameter inverted gamma density for σ. So, we have

$$\pi(\mu,\sigma) \propto \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{1}{\sigma^{\alpha+2}} e^{-\frac{\beta}{\sigma}}. \tag{3.9}$$

In fact $\sigma^{-1} \sim \Gamma(\alpha,\beta)$, which is the conjugate prior distribution of the parameter σ for the fixed value of μ, and $\pi_1(\mu|\sigma) \propto \sigma^{-1}$ which is the Jeffreys non-informative prior distribution (see [6]) of the parameter μ for fixed value of the parameter σ. Thus, the joint posterior density is given by

$$\pi(\mu,\sigma|\mathbf{x}) = \frac{k[\beta + k(x_m - x_1)]^{m+\alpha}}{\Gamma(m+\alpha)} \frac{1}{\sigma^{m+\alpha+2}} e^{-\frac{1}{\sigma}[\beta+k(x_m-\mu)]}. \tag{3.10}$$

Therefore, by (3.10) under SE loss function the Bayes estimate of the parameter σ is given by

$$\hat{\sigma}_{2BS} = \frac{\beta + k(R_{m(k)} - R_{1(k)})}{m+\alpha-1}. \tag{3.11}$$

Notice that, as $\beta \to 0$ and $\alpha \to 1$, $\hat{\sigma}_{2BS} \to \hat{\sigma}_{ML}$. By (3.11) we have

- $E(\hat{\sigma}_{2BS}) = \frac{\beta+\sigma(m-1)}{m+\alpha-1}$,
- $MSE(\hat{\sigma}_{2BS}) = \frac{m-1}{(m+\alpha-1)^2}\sigma^2 + \frac{(\beta-\sigma\alpha)^2}{(m+\alpha-1)^2}$.

Also, the Bayes estimate of the parameter μ is given by

$$\hat{\mu}_{2BS} = R_{m(k)} + \beta - \frac{m+\alpha}{k(m+\alpha-1)}[\beta + k(R_{m(k)} - R_{1(k)})]. \tag{3.12}$$

By (3.12), we have

- $E(\hat{\mu}_{2BS}) = \mu + \alpha\frac{\sigma}{k} + [1 - \frac{(m+\alpha)}{k(m+\alpha-1)}]\beta$,
- $Var(\hat{\mu}_{2BS}) = \frac{\sigma^2}{k^2}[1 + \frac{m-1}{(m+\alpha-1)^2}]$,
- $Cov(\hat{\mu}_{2BS}, \hat{\sigma}_{2BS}) = -\frac{(m-1)\sigma^2}{(m+\alpha-1)k}$.

Remark. It may be noted that one may use (3.7) and (3.10) to obtain Bayesian estimation interval for the parameters μ and σ.

4 Prediction in Exponential Model

In this section, we consider the problem of prediction, either point or interval, for future k-record values by Bayesian approach. Assume that we have the first m upper k-records $R_{1(k)} = x_1, R_{2(k)} = x_2, ..., R_{m(k)} = x_m$ from the $Exp(\mu, \sigma)-$distribution. Based on such a sample, prediction, either point or interval, is needed for s-th upper k-record, $1 \leq m < s$. We consider the following two cases:

a) σ *is known.*

Without loss of generality, we may assume $\sigma = 1$, then by (2.12), (2.13) and (2.9), we have

$$f^*(y|x_m, \mu) = \frac{k^{s-m}}{\Gamma(s-m)}(y - x_m)^{s-m-1}e^{-k(y-x_m)}, \quad y > x_m, \tag{4.1}$$

which is independent of μ. So by (2.10) and (4.1) we have

$$\begin{aligned} h^*(y|\mathbf{x}) &= \int_{-\infty}^{x_1} f^*(y|\mathbf{x}, \mu)\pi(\mu|\mathbf{x})d\mu \\ &= \frac{k^{s-m}}{\Gamma(s-m)}(y - x_m)^{s-m-1}e^{-k(y-x_m)}, \quad y > x_m, \end{aligned} \tag{4.2}$$

for any posterior distribution (therefore, for any prior distribution) $\pi(\mu|\mathbf{x})$. By (4.2), we have

$$Y - x_m|\mathbf{x} \sim \Gamma(s-m, k).$$

So,

$$\hat{Y}_1 = R_{m(k)} + \frac{s-m}{k}. \tag{4.3}$$

By (4.3) we have

- $E(\hat{Y}_1) = \mu + \frac{s}{k}$,
- $MSE(\hat{Y}_1) = \frac{s-m}{k^2}$.

Ahmadi *et al.* [2] obtained the $100(1-\gamma)\%$ Bayesian prediction interval for $R_{s(k)}$, with equal tail as $(L_1,\ U_1)$, where L_1 and U_1 are the lower and upper bounds, respectively which are given by

$$L_1 = R_{m(k)} + \frac{\chi^2_{\frac{\gamma}{2}}}{2k},$$

and

$$U_1 = R_{m(k)} + \frac{\chi_{1-\frac{\gamma}{2}}{}^2}{2k}.$$

where χ^2_γ stands for the γ-th percentage of Chi-square distribution with $2(s-m)$ degrees of freedom.

b) σ *is unknown*

Let $Y = R_{s(k)}$ be the s-th upper k-record value, $1 \leq m < s$. So, by (2.12) and (2.13), we have

$$f^*(y|\mathbf{x}, \mu, \sigma) = (\frac{k}{\sigma})^{s-m} \frac{(y - x_m)^{s-m-1}}{\Gamma(s-m)} e^{-\frac{k}{\sigma}(y - x_m)}. \quad (4.4)$$

By (2.13), (3.10) and (4.4) Bayesian predictive density function of $Y = R_{s(k)}$, for the given past m records, is given by

$$\begin{aligned} h(y|\mathbf{x}) &= \int_{-\infty}^{x_1} \int_0^{\infty} f^*(y|\mathbf{x}, \mu, \sigma)\pi(\mu, \sigma|\mathbf{x}) d\sigma d\mu \\ &= \frac{1}{B(m+\alpha, s-m)} (\frac{k(x_m - x_1) + \beta}{k(y - x_1) + \beta})^{m+\alpha} \\ &\quad \times (1 - \frac{k(x_m - x_1) + \beta}{k(y - x_1) + \beta})^{s-m} \frac{1}{y - x_m}, \ y > x_m. \quad (4.5) \end{aligned}$$

Now, by (4.5) the Bayes point predictor of the s-th upper k-record is given by

$$\hat{Y}_2 = \frac{s + \alpha - 1}{m + \alpha - 1} R_{m(k)} + \frac{s - m}{m + \alpha - 1} (\frac{\beta}{k} - R_{1(k)}). \quad (4.6)$$

By (4.6) we have

- $E(\hat{Y}_2) = \frac{k\mu(\alpha+m-1)+s\sigma(m-1)+\beta(s-m)}{k(m+\alpha-1)}$,
- $MSE(\hat{Y}_2) = (s-m)\{\frac{\sigma^2}{k^2}(1 + \frac{(m-1)(s-m)}{(m+\alpha-1)^2}) + (s-m)(\frac{\alpha}{m+\alpha-1} - \frac{\beta}{k})^2\}$.

In this case, Ahmadi *et al.* [2] also derived a Bayesian prediction interval for $R_{s(k)}$ as follow: The $100(1-\gamma)\%$ Bayesian prediction interval for $R_{s(k)}$ is given by

$$(L_2, \ U_2),$$

where

$$L_2 = \frac{R_{m(k)} - R_{1(k)}}{b_{1-\frac{\gamma}{2}}} + \frac{\beta}{k}(\frac{1}{b_{1-\frac{\gamma}{2}}} - 1) + R_{1(k)},$$

and

$$U_2 = \frac{R_{m(k)} - R_{1(k)}}{b_{\frac{\gamma}{2}}} + \frac{\beta}{k}(\frac{1}{b_{\frac{\gamma}{2}}} - 1) + R_{1(k)},$$

where b_γ is the γ-th percentage of $Beta(m + \alpha, s - m)$-distribution.

5 Conclusion

In this paper, we have tackled the problems of estimation and prediction based on k-record data while the underlying distribution is assumed to have a general form. This family contains several life distribution such as Exponential, Weibull and Pareto and so on. A general form of conjugate prior was considered to obtain Bayesian estimation of unknown parameters and prediction of future k-record values. The proposed procedure was applied to the Exponential, Weibull and Pareto models in one parameter case. Moreover, we have developed the proposed procedure for two-parameter Exponential distribution in details.

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