



Discrete collocation method for Volterra type weakly singular integral equations with logarithmic kernels

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Abstract

An efficient discrete collocation method for solving Volterra type weakly singular integral equations with logarithmic kernels is investigated. One of features of these equations is that, in general the first derivative of solution behaves like as a logarithmic function, which is not continuous at the origin. In this paper, to make a compatible approximate solution with the exact ones, we introduce a new collocation approach, which applies the Müntz-logarithmic polynomials (Müntz polynomials with logarithmic terms) as basis functions. Moreover, since implementation of this technique leads to integrals with logarithmic singularities that are often difficult to solve numerically, we apply a suitable quadrature method that allows the exact evaluation of integrals of polynomials with logarithmic weights. To this end, we first remind the well-known Jacobi–Gauss quadrature and then extend it to integrals with logarithmic weights. Convergence analysis of the proposed scheme are presented, and some numerical results are illustrated to demonstrate the efficiency and accuracy of the proposed method.

Keywords: Discrete collocation method; Müntz-logarithmic polynomials; Quadrature method; Volterra type weakly singular integral equations with logarithmic kernels.

1 Introduction

In this paper, we develop an approximate approach to obtain the numerical solution of the following Volterra type weakly singular integral equation with logarithmic kernel

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$$y(x) = g(x) + \int_0^x \ln(x-t)K(x,t)y(t)dt, \quad x \in \Omega = [0, 1], \quad 0 \leq t \leq x \leq 1, \quad (1)$$

where the continuous functions $g(x)$ and $K(x,t)$ are given and $y(x)$ is the unknown solution. Such kinds of equations arise from solution of Dirichlet's problem for the Laplace equation in two dimensions in terms of a single-layer logarithmic potential [24], solution of the reduced wave equation in two dimensions using an integral equation with a kernel, which can be expressed as a Hankel function of order zero that this kernel, has also a logarithmic singularity [25], investigation of electrostatic and low frequency electromagnetic problems [15], methods of computing the conformal mapping of a given domain [28], solution of electromagnetic scattering problems([2], [16]), determination of propagation of acoustical and elastical waves([5], [6]), boundary value problems of plane elasticity theory for regions with a defect [13], problems of diffraction by thin screens [12] and so on.

Weakly singular integral equations with logarithmic kernels are usually difficult to solve analytically; so it is necessary to provide reliable numerical techniques. There are several approximate methods proposed to obtain the numerical solution of these types of equations, which we refer to some of them. In [4], authors designed a computational meshless discrete Galerkin method to solve the second kind Fredholm integral equations with logarithmic kernels. In [17], authors developed a collocation method for the numerical solution of a special integro-differential equation with logarithmic kernel using airfoil polynomials of the first kind. A collocation method based on the periodic splines was introduced in [11] to solve some logarithmic kernel integral equations on open arcs. In [29], authors studied a special integral equation with logarithmic kernel and solved it using product integration method. In [14], a piecewise Chebyshev expansion was considered to solve Volterra integral equations with logarithmic singularities in their kernels. The properties of the integro-differential equations of the convolution on a finite interval with kernel having a logarithmic singularity were studied in [3]. In [10], authors investigated two numerical approaches by means of an analytical integration in the vicinity of the singular point and extraction of the singular part. A Gauss type quadrature method with a logarithmic weight function was extended to evaluate of the Cauchy type integral equations with logarithmic kernels in [30].

In this paper, we design and analyze a reliable discrete collocation technique to obtain a suitable approximate solution of (1). Our strategy is based on the following two stages. From well-known existence and uniqueness theorems [8], we can see that the first derivative of the exact solution has a singularity at the origin and behaves like as a logarithmic function. Then to establish a highly accurate approximate solution, it is necessary to represent the collocation solution as a linear combination of the suitable bases functions with logarithmic asymptotic behavior like as Müntz-logarithmic

polynomials [18]. Since in the implementation procedure, integrals with logarithmic singularities are observed to highly accurate evaluation of them, we use a generalized Gauss type quadrature with logarithmic weight function that calculates exactly integrals of polynomials [7].

The remainder of this paper is organized as follows. In the next section we present the required preliminaries for our subsequent development. Here, we introduce the Müntz-logarithmic polynomials as well as the generalized Gauss quadrature method for the integrals with logarithmic weight functions. In Section 3, we explain the application of the discrete collocation method using Müntz-logarithmic polynomials to approximate the solution of (1). In Section 4, we provide a reliable convergence analysis for the proposed algorithm that justifies the L^2 -norm of the error function tends to zero as the approximation degree tends to infinity. Section 5 devotes to our numerical illustrative and Section 6 contains our conclusions.

2 Preliminaries

In this section, we give some preliminaries that are required in the sequel.

2.1 Müntz-Logarithmic polynomials

This subsection is devoted to a brief introduction on the Müntz-logarithmic polynomials. All of the details presented in this section a long with further details can be found in [18].

The Müntz-logarithmic polynomials are defined as

$$\mathcal{M}_n(x) = \mathcal{R}_n(x) + \mathcal{S}_n(x) \ln x, \quad n \geq 0, \quad x \in \Omega, \quad (2)$$

where $\mathcal{R}_n(x)$ and $\mathcal{S}_n(x)$ are algebraic polynomials of degree $[\frac{n}{2}]$ and $[\frac{n-1}{2}]$, respectively; that is,

$$\mathcal{R}_n(x) = \sum_{i=0}^{[\frac{n}{2}]} r_i x^i, \quad \mathcal{S}_n(x) = \sum_{i=0}^{[\frac{n-1}{2}]} s_i x^i.$$

It is shown that these polynomials are orthogonal with respect to the weight function $w(x) = 1$. Explicit expressions of the coefficients are obtained as follows:

Theorem 1.

- If $n = 2m$, $m \geq 0$, we have

$$\begin{cases} r_i = -\binom{m+i}{m}^2 \binom{m}{i}^2 \left[\frac{2m+1}{2i+1} + 2(m-i) \sum_{j=0, j \neq i}^{m-1} \frac{2j+1}{(j-i)(j+i+1)} \right], \\ s_i = -(m-i) \binom{m+i}{m}^2 \binom{m}{i}^2, \quad 0 \leq i \leq m-1, \end{cases}$$

and

$$r_m = \binom{2m}{m}^2, \quad s_m = 0.$$

- If $n = 2m + 1$, $m \geq 0$, we have

$$\begin{cases} r_i = \binom{m+i}{m}^2 \binom{m}{i}^2 \left[\frac{2m+1}{2i+1} + 2(m+i+1) \sum_{j=0, j \neq i}^m \frac{2j+1}{(j-i)(j+i+1)} \right], \\ s_i = (m+i+1) \binom{m+i}{m}^2 \binom{m}{i}^2, \quad 0 \leq i \leq m. \end{cases}$$

Proof. See [18]. □

The first few Müntz-logarithmic polynomials are given by

$$\begin{aligned} \mathcal{M}_0(x) &= 1, \\ \mathcal{M}_1(x) &= 1 + \ln x, \\ \mathcal{M}_2(x) &= -3 + 4x - \ln x, \\ \mathcal{M}_3(x) &= 9 - 8x + 2(1 + 6x) \ln x, \\ \mathcal{M}_4(x) &= -11 - 24x + 36x^2 - 2(1 + 18x) \ln x, \\ \mathcal{M}_5(x) &= 19 + 276x - 294x^2 + 3(1 + 48x + 60x^2) \ln x, \\ \mathcal{M}_6(x) &= -21 - 768x + 390x^2 + 400x^3 - 3(1 + 96x + 300x^2) \ln x. \end{aligned}$$

2.2 Jacobi–Gauss quadrature

This subsection presents the mechanism of the Jacobi–Gauss quadrature [7, 9, 27], which is to seek the best numerical approximation of an integral by selecting optimal nodes at which the integrand is evaluated. The N th-order Jacobi–Gauss quadrature formula $(\mathcal{JG})^{\alpha, \beta}$ to approximate the integral is given by

$$\int_{-1}^1 f(r)(1-r)^\alpha(1+r)^\beta dr \approx (\mathcal{JG})^{\alpha,\beta}(f) := \sum_{i=1}^N f(x_i^{\alpha,\beta})\mathcal{W}_i^{\alpha,\beta},$$

$$\alpha, \beta > -1, \quad r \in [-1, 1],$$

which the integral is evaluated exactly if $f(r)$ is a polynomial of degree $2N - 1$ or less. The nodes $\{x_i^{\alpha,\beta}\}_{i=1}^N$ and the corresponding weights $\{\mathcal{W}_i^{\alpha,\beta}\}_{i=1}^N$ depend on α, β and are given by the following formulas [7, 27]:

- $\{x_i^{\alpha,\beta}\}_{i=1}^N$ are the zeros of the following orthonormal polynomial

$$\mathcal{P}_N^{\alpha,\beta}(r) = \sqrt{\frac{n!(2N + \alpha + \beta + 1)\Gamma(N + \alpha + \beta + 1)}{2^{\alpha+\beta+1}\Gamma(N + \alpha + 1)\Gamma(N + \beta + 1)}} J_N^{\alpha,\beta}(r),$$

where $J_N^{\alpha,\beta}(r)$ is the classical Jacobi polynomial of degree N .

- The weights $\{\mathcal{W}_i^{\alpha,\beta}\}_{i=1}^N$ are given by

$$(\mathcal{W}_i^{\alpha,\beta})^{-1} = \sum_{n=0}^{N-1} \left(\mathcal{P}_n^{\alpha,\beta}(x_i^{\alpha,\beta}) \right)^2, \quad 1 \leq i \leq N.$$

The error term for the N th-order Jacobi–Gauss quadrature $(\mathcal{JG})^{\alpha,\beta}$ is given by [7]

$$\left(\int_{-1}^1 (1-r)^\alpha(1+r)^\beta f(r) dr \right) - (\mathcal{JG})^{\alpha,\beta}(f) = \mathcal{E}_{N,r}(\alpha, \beta, f(r)) := \delta_N^{\alpha,\beta} \frac{f^{(2N)}(\xi)}{(2N)!},$$

$$(3)$$

where ξ lies somewhere on $[-1, 1]$ and

$$\delta_N^{\alpha,\beta} = \frac{2^{2N+\alpha+\beta+1} N! \Gamma(N + \alpha + 1) \Gamma(N + \beta + 1) \Gamma(N + \alpha + \beta + 1)}{(2N + \alpha + \beta + 1) \Gamma(2N + \alpha + \beta + 1)^2}.$$

2.3 Gauss type quadrature for logarithmic weights integrals

This subsection devotes to introduce a generalized Jacobi–Gauss quadratures [7] to approximate the following integrals

$$\begin{aligned}
(a) \quad & \int_{-1}^1 f(r) \ln(1-r)(1-r)^\alpha dr, \\
(b) \quad & \int_{-1}^1 f(r) \ln(1+r)(1+r)^\beta dr.
\end{aligned} \tag{4}$$

Considering the first integral in (4) we have

$$\begin{aligned}
\frac{\partial}{\partial \alpha} \int_{-1}^1 (1-r)^\alpha f(r) dr &= \frac{\partial}{\partial \alpha} \int_{-1}^1 e^{\alpha \ln(1-r)} f(r) dr \\
&= \int_{-1}^1 f(r) \ln(1-r)(1-r)^\alpha dr,
\end{aligned}$$

and equivalently

$$\begin{aligned}
\int_{-1}^1 f(r) \ln(1-r)(1-r)^\alpha dr &= \frac{\partial}{\partial \alpha} \int_{-1}^1 (1-r)^\alpha f(r) dr \\
&\approx \frac{\partial}{\partial \alpha} ((\mathcal{JG})^{\alpha,0}(f)) = \frac{\partial}{\partial \alpha} \sum_{i=1}^N \left(f(x_i^{\alpha,0}) \mathcal{W}_i^{\alpha,0} \right) \\
&= \sum_{i=1}^N \left(\frac{d\mathcal{W}_i^{\alpha,0}}{d\alpha} f(x_i^{\alpha,0}) + \mathcal{W}_i^{\alpha,0} \frac{dx_i^{\alpha,0}}{d\alpha} f'(x_i^{\alpha,0}) \right) \\
&:= (\mathcal{GJG})^{\alpha,0}(f),
\end{aligned} \tag{5}$$

where “ \mathcal{GJG} ” is an abbreviation for the generalized Jacobi–Gauss. From (3), we can obtain the following formula for the error term of N th order generalized Jacobi–Gauss quadrature $(\mathcal{GJG})^{\alpha,0}$:

$$\begin{aligned}
& \left(\int_{-1}^1 f(r)(1-r)^\alpha \ln(1-r) dr \right) - (\mathcal{GJG})^{\alpha,0}(f) \\
&= \frac{\partial}{\partial \alpha} \mathcal{E}_{N,r}(\alpha, 0, f(r)) := \tilde{\mathcal{E}}_{N,r}(\alpha, 0, f(r)) \\
&:= \left(\ln(2) - \frac{1}{2N + \alpha + 1} \right. \\
&\quad \left. + 2\Psi(N + \alpha + 1) - 2\Psi(2N + \alpha + 1) \right) \mathcal{E}_{N,r}(\alpha, 0, f(r)), \tag{6}
\end{aligned}$$

where $\Psi(z) = \frac{d \ln(\Gamma(z))}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}$ is the psi or digamma function.

Proceeding the same technique with (5)-(6) we can obtain the following approximation for the second integral of (4)

$$\begin{aligned} \int_{-1}^1 f(r) \ln(1+r)(1+r)^\beta dr &= \frac{\partial}{\partial \beta} \int_{-1}^1 (1+r)^\beta f(r) dr \\ &\approx \frac{\partial}{\partial \beta} ((\mathcal{J}G)^{0,\beta}(f)) = \frac{\partial}{\partial \beta} \sum_{i=1}^N \left(f(x_i^{0,\beta}) \mathcal{W}_i^{0,\beta} \right) \\ &= \sum_{i=1}^N \left(\frac{d\mathcal{W}_i^{0,\beta}}{d\beta} f(x_i^{0,\beta}) + \mathcal{W}_i^{0,\beta} \frac{dx_i^{0,\beta}}{d\beta} f'(x_i^{0,\beta}) \right) \\ &:= (G\mathcal{J}G)^{0,\beta}(f), \end{aligned}$$

which has the following error function

$$\begin{aligned} &\left(\int_{-1}^1 f(r)(1+r)^\beta \ln(1+r) dr \right) - (G\mathcal{J}G)^{0,\beta}(f) \\ &= \frac{\partial}{\partial \beta} \mathcal{E}_{N,r}(0, \beta, f(r)) := \tilde{\mathcal{E}}_{N,r}(0, \beta, f(r)) \\ &:= \left(\ln(2) - \frac{1}{2N + \beta + 1} \right. \\ &\quad \left. + 2\Psi(N + \beta + 1) - 2\Psi(2N + \beta + 1) \right) \mathcal{E}_{N,r}(0, \beta, f(r)). \quad (7) \end{aligned}$$

The relations (6) and (7) conclude that the generalized Jacobi–Gauss quadratures $(G\mathcal{J}G)^{\alpha,0}(f)$ and $(G\mathcal{J}G)^{0,\beta}(f)$ calculate the integrals of (4) exactly for polynomials of degree $2N - 1$ or less same as the Jacobi–Gauss quadrature $(\mathcal{J}G)^{\alpha,\beta}(f)$.

3 Numerical approach

The main concern of this section is to obtain the discrete collocation solution of (1) when the Müntz-logarithmic polynomials are applied as the basis functions. To this end we represent the collocation solution of (1) as

$$y_N(x) = \sum_{n=0}^N a_n \mathcal{M}_n(x), \quad (8)$$

such that the unknowns $\{a_n\}_{n=0}^N$ satisfy in the following linear algebraic system:

$$y_N(x_i) = g(x_i) + \int_0^{x_i} \ln(x_i - t)K(x_i, t)y_N(t)dt, \quad 0 \leq i \leq N, \quad (9)$$

where $\{x_i\}_{i=0}^N$ is the shifted Legendre–Gauss quadrature points [27]. Applying the variable transformation

$$t = t_i(\theta) = \frac{x_i}{2}(\theta + 1), \quad \theta \in [-1, 1],$$

we can rewrite (9) as follows:

$$\begin{aligned} y_N(x_i) &= g(x_i) + \frac{x_i}{2} \int_{-1}^1 \ln(x_i - t_i(\theta))K(x_i, t_i(\theta))y_N(t_i(\theta))d\theta \\ &= g(x_i) + \frac{x_i}{2} \left\{ \ln \frac{x_i}{2} \int_{-1}^1 \tilde{K}(x_i, \theta)y_N(t_i(\theta))d\theta \right. \\ &\quad \left. + \int_{-1}^1 \ln(1 - \theta)\tilde{K}(x_i, \theta)y_N(t_i(\theta))d\theta \right\} \end{aligned} \quad (10)$$

for $0 \leq i \leq N$ and $\tilde{K}(x_i, \theta) = K(x_i, t_i(\theta))$. Substituting (8) into (10) yields

$$\sum_{n=0}^N a_n \left\{ \mathcal{M}_n(x_i) - \frac{x_i}{2} \left\{ \ln \frac{x_i}{2} \mathcal{A}_1^{(n, x_i)} + \mathcal{A}_2^{(n, x_i)} \right\} \right\} = g(x_i), \quad 0 \leq i \leq N, \quad (11)$$

where

$$\begin{aligned} \mathcal{A}_1^{(n, x_i)} &= \int_{-1}^1 \tilde{K}(x_i, \theta)\mathcal{M}_n(t_i(\theta))d\theta, \\ \mathcal{A}_2^{(n, x_i)} &= \int_{-1}^1 \ln(1 - \theta)\tilde{K}(x_i, \theta)\mathcal{M}_n(t_i(\theta))d\theta. \end{aligned}$$

Using (2), we have

$$\begin{aligned}
\mathcal{A}_1^{(n,x_i)} &= \int_{-1}^1 \tilde{K}(x_i, \theta) \left(\mathcal{R}_n(t_i(\theta)) + \mathcal{S}_n(t_i(\theta)) \ln(t_i(\theta)) \right) d\theta \\
&= \int_{-1}^1 \tilde{K}(x_i, \theta) \left(\mathcal{R}_n(t_i(\theta)) + \ln \frac{x_i}{2} \mathcal{S}_n(t_i(\theta)) \right) d\theta \\
&\quad + \int_{-1}^1 \ln(1 + \theta) \tilde{K}(x_i, \theta) \mathcal{S}_n(t_i(\theta)) d\theta \\
&= \int_{-1}^1 \mathcal{F}_1^{(n,x_i)}(\theta) d\theta \\
&\quad + \int_{-1}^1 \ln(1 + \theta) \mathcal{F}_2^{(n,x_i)}(\theta) d\theta,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_1^{(n,x_i)}(\theta) &= \tilde{K}(x_i, \theta) \left(\mathcal{R}_n(t_i(\theta)) + \ln \frac{x_i}{2} \mathcal{S}_n(t_i(\theta)) \right), \\
\mathcal{F}_2^{(n,x_i)}(\theta) &= \tilde{K}(x_i, \theta) \mathcal{S}_n(t_i(\theta)).
\end{aligned} \tag{12}$$

Using quadratures $(\mathcal{JG})^{\alpha,\beta}$ and $(G\mathcal{JG})^{0,\beta}$, we obtain

$$\mathcal{A}_1^{(n,x_i)} \approx \mathcal{A}_{1,N}^{(n,x_i)} := (\mathcal{JG})^{0,0}(\mathcal{F}_1^{(n,x_i)}(\theta)) + \left[(G\mathcal{JG})^{0,\beta}(\mathcal{F}_2^{(n,x_i)}(\theta)) \right]_{\beta=0}. \tag{13}$$

On the other hand, from (2) we can write

$$\begin{aligned}
\mathcal{A}_2^{(n,x_i)} &= \int_{-1}^1 \ln(1 - \theta) \tilde{K}(x_i, \theta) \left(\mathcal{R}_n(t_i(\theta)) + \mathcal{S}_n(t_i(\theta)) \ln(t_i(\theta)) \right) d\theta \\
&= \int_{-1}^1 \ln(1 - \theta) \mathcal{F}_1^{(n,x_i)}(\theta) d\theta + \int_{-1}^1 \ln(1 - \theta) \ln(1 + \theta) \mathcal{F}_2^{(n,x_i)}(\theta) d\theta \\
&= \mathcal{A}_2^{(1,n,x_i)} + \mathcal{A}_2^{(2,n,x_i)},
\end{aligned} \tag{14}$$

where

$$\mathcal{A}_2^{(1,n,x_i)} = \int_{-1}^1 \ln(1-\theta) \mathcal{F}_1^{(n,x_i)}(\theta) d\theta,$$

$$\mathcal{A}_2^{(2,n,x_i)} = \int_{-1}^1 \ln(1-\theta) \ln(1+\theta) \mathcal{F}_2^{(n,x_i)}(\theta) d\theta.$$

Trivially we can write

$$\mathcal{A}_2^{(1,n,x_i)} \approx \mathcal{A}_{2,N}^{(1,n,x_i)} = \left[(G\mathcal{J}G)^{\alpha,0}(\mathcal{F}_1^{(n,x_i)}(\theta)) \right]_{\alpha=0}, \quad (15)$$

and

$$\mathcal{A}_2^{(2,n,x_i)} = \int_{-1}^0 \ln(1-\theta) \ln(1+\theta) \mathcal{F}_2^{(n,x_i)}(\theta) d\theta$$

$$+ \int_0^1 \ln(1-\theta) \ln(1+\theta) \mathcal{F}_2^{(n,x_i)}(\theta) d\theta. \quad (16)$$

Using the variable transformation $\theta = \theta_1(s) = \frac{s+1}{2}$ in the second integral and $\theta = -\theta_1(s)$ in the first integral of (16), we obtain

$$\mathcal{A}_2^{(2,n,x_i)} = \frac{1}{2} \int_{-1}^1 \ln \frac{1-s}{2} \ln \frac{3+s}{2} \left(\mathcal{F}_2^{(n,x_i)}(\theta_1(s)) + \mathcal{F}_2^{(n,x_i)}(-\theta_1(s)) \right) ds$$

$$= \frac{1}{2} \left\{ \int_{-1}^1 \ln(1-s) \mathcal{F}_3^{(n,x_i)}(s) ds - \ln 2 \int_{-1}^1 \mathcal{F}_3^{(n,x_i)}(s) ds \right\},$$

where

$$\mathcal{F}_3^{(n,x_i)} = \ln \frac{3+s}{2} \left(\mathcal{F}_2^{(n,x_i)}(\theta_1(s)) + \mathcal{F}_2^{(n,x_i)}(-\theta_1(s)) \right). \quad (17)$$

then we can write

$$\mathcal{A}_2^{(2,n,x_i)} \approx \mathcal{A}_{2,N}^{(2,n,x_i)} = \frac{1}{2} \left(\left[(G\mathcal{J}G)^{\alpha,0}(\mathcal{F}_3^{(n,x_i)}(s)) \right]_{\alpha=0} \right. \\ \left. - \ln 2 (\mathcal{J}G)^{0,0}(\mathcal{F}_3^{(n,x_i)}(s)) \right). \quad (18)$$

Substituting (18) and (15) into (14) yields

$$\mathcal{A}_2^{(n,x_i)} \approx \mathcal{A}_{2,N}^{(n,x_i)} := \mathcal{A}_{2,N}^{(1,n,x_i)} + \mathcal{A}_{2,N}^{(2,n,x_i)}. \quad (19)$$

Inserting (19) and (13) into (11), we can conclude that the discrete collocation solution of (1) is characterized by

$$\bar{y}_N(x) = \sum_{n=0}^N \bar{a}_n \mathcal{M}_n(x),$$

where the unknowns $\{\bar{a}_n\}_{n=0}^N$ satisfy in the following system of linear algebraic equations

$$\sum_{n=0}^N \bar{a}_n \left\{ \mathcal{M}_n(x_i) - \frac{x_i}{2} \left\{ \ln \frac{x_i}{2} \mathcal{A}_{1,N}^{(n,x_i)} + \mathcal{A}_{2,N}^{(n,x_i)} \right\} \right\} = g(x_i), \quad 0 \leq i \leq N, \quad (20)$$

The matrix formulation of (20) is given by

$$\underline{a} \mathcal{M} = \underline{g}, \quad (21)$$

where $\underline{a} = [\bar{a}_0, \bar{a}_1, \dots, \bar{a}_N]$, $\underline{g} = [g(x_0), g(x_1), \dots, g(x_N)]^T$, and

$$\mathcal{M} = \mathcal{M}_1 - (\mathcal{A}_1 \mathcal{D}_1 + \mathcal{A}_2 \mathcal{D}_2),$$

such that

$$\mathcal{M}_1 = \begin{pmatrix} \mathcal{M}_0(x_0) & \mathcal{M}_0(x_1) & \cdots & \mathcal{M}_0(x_N) \\ \mathcal{M}_1(x_0) & \mathcal{M}_1(x_1) & \cdots & \mathcal{M}_1(x_N) \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{M}_N(x_0) & \mathcal{M}_N(x_1) & \cdots & \mathcal{M}_N(x_N) \end{pmatrix},$$

$$\mathcal{A}_1 = \begin{pmatrix} \mathcal{A}_{1,N}^{(0,x_0)} & \mathcal{A}_{1,N}^{(0,x_1)} & \cdots & \mathcal{A}_{1,N}^{(0,x_N)} \\ \mathcal{A}_{1,N}^{(1,x_0)} & \mathcal{A}_{1,N}^{(1,x_1)} & \cdots & \mathcal{A}_{1,N}^{(1,x_N)} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{A}_{1,N}^{(N,x_0)} & \mathcal{A}_{1,N}^{(N,x_1)} & \cdots & \mathcal{A}_{1,N}^{(N,x_N)} \end{pmatrix},$$

$$\mathcal{A}_2 = \begin{pmatrix} \mathcal{A}_{2,N}^{(0,x_0)} & \mathcal{A}_{2,N}^{(0,x_1)} & \cdots & \mathcal{A}_{2,N}^{(0,x_N)} \\ \mathcal{A}_{2,N}^{(1,x_0)} & \mathcal{A}_{2,N}^{(1,x_1)} & \cdots & \mathcal{A}_{2,N}^{(1,x_N)} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{A}_{2,N}^{(N,x_0)} & \mathcal{A}_{2,N}^{(N,x_1)} & \cdots & \mathcal{A}_{2,N}^{(N,x_N)} \end{pmatrix},$$

and \mathcal{D}_1 and \mathcal{D}_2 are the diagonal matrices with diagonal entries

$$(\mathcal{D}_1)_{i,i} = \frac{x_i}{2} \ln \frac{x_i}{2}, \quad (\mathcal{D}_2)_{i,i} = \frac{x_i}{2}, \quad 0 \leq i \leq N.$$

4 Convergence analysis

In this section, we provide a reliable error analysis for the proposed technique to justify convergence of the proposed approach. In our analysis we shall apply the following definitions and lemmas.

Definition 1. [1], [19–23], [27]]

1. $L^2(\Omega) = \{u \mid \|u\|_2^2 := \int_{\Omega} |u(x)|^2 dx < \infty\}$.
2. $C(\Omega)$ is the space of all continuous functions on Ω .
3. $I_N u$ is the Legendre–Gauss interpolation polynomial and defines by

$$I_N u(x) = \sum_{i=0}^N u(x_i) L_i(x),$$

where $L_i(x)$, $0 \leq i \leq N$, are the Lagrange interpolation basis functions associated with the Legendre–Gauss points $\{x_i\}_{i=0}^N$.

4. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed vector spaces, and let $\mathcal{K} : \mathcal{X} \rightarrow \mathcal{Y}$ be a linear operator. Then \mathcal{K} is compact, if the set

$$\{\mathcal{K}u \mid \|u\|_{\mathcal{X}} \leq 1\}$$

has compact closure in \mathcal{Y} . For example, the integral operators with continuous and weakly singular kernels are compact operators on $C(\Omega)$ and $L^2(\Omega)$.

Lemma 1. [see [8]] *Let $K(x, t) \in C(\Omega \times \Omega)$; then for any $g(x) \in C(\Omega)$, the Volterra type weakly singular integral equation with logarithmic kernel (1) possesses a unique solution $y(x) \in C(\Omega)$.*

Lemma 2. [see [1]] *Let \mathcal{X} be a Banach space, and let $\mathcal{K} : \mathcal{X} \rightarrow \mathcal{X}$ be compact. Then the equation $(\lambda - \mathcal{K})u = f$, $\lambda \neq 0$ has a unique solution $u \in \mathcal{X}$ if and only if the homogeneous equations $(\lambda - \mathcal{K})z = 0$ have only the trivial solution $z = 0$. In this case the operator $\lambda - \mathcal{K} : \mathcal{X} \rightarrow \mathcal{X}$ has a bounded inverse $(\lambda - \mathcal{K})^{-1}$.*

Lemma 3. [see [26]] *If $u(x) \in C(\Omega)$, then we have*

$$\|u - I_N u\|_2 \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Lemma 4.[see [19]] *For every bounded function $u(x)$, there exists a constant C independent of $u(x)$ such that*

$$\sup_N \|I_N u\|_2 \leq C \|u\|_\infty.$$

Theorem 2. *Under assumptions of Lemma 1, assume that the following conditions are satisfied:*

1. *The given functions $g(x)$, $K(x, t)$, and the approximate solution $\bar{y}_N(x)$ are continuous on their domains.*
2. *The homogeneous equation (1) with $g(x) = 0$ has the only trivial solution.*
3. *$\bar{y}_N(x) \in C(\Omega)$ and the numerical integral operator*

$$\mathcal{K}_N \bar{y}_N(x) := \sum_{n=0}^N \bar{a}_n \frac{x}{2} \left\{ \ln \frac{x}{2} \mathcal{A}_{1,N}^{n,x} + \mathcal{A}_{2,N}^{n,x} \right\}$$

is a bounded operator on $C(\Omega)$ to $C(\Omega)$ with $\mathcal{A}_{1,N}^{n,x}$ and $\mathcal{A}_{2,N}^{n,x}$ defined in (13) and (19), respectively.

4. *The quadrature errors*

$$\begin{aligned} & \|\mathcal{E}_{N,\theta}(0, 0, \mathcal{F}_1^{n,x}(\theta))\|_\infty, \|\mathcal{E}_{N,s}(0, 0, \mathcal{F}_3^{n,x}(s))\|_\infty, \\ & \left\| \left[\tilde{\mathcal{E}}_{N,\theta}(\alpha, 0, \mathcal{F}_1^{n,x}(\theta)) \right]_{\alpha=0} \right\|_\infty, \left\| \left[\tilde{\mathcal{E}}_{N,\theta}(0, \beta, \mathcal{F}_2^{n,x}(\theta)) \right]_{\beta=0} \right\|_\infty, \\ & \left\| \left[\tilde{\mathcal{E}}_{N,s}(\alpha, 0, \mathcal{F}_3^{n,x}(s)) \right]_{\alpha=0} \right\|_\infty, \quad 0 \leq n \leq N, \end{aligned}$$

converge to zero as $N \rightarrow \infty$. Here the functions $\mathcal{F}_1^{n,x}(\theta)$, $\mathcal{F}_2^{n,x}(\theta)$ and $\mathcal{F}_3^{n,x}(s)$ are given in (12) and (17), respectively, and the error terms $\mathcal{E}_{N,r}(0, 0, f(r))$, $\tilde{\mathcal{E}}_{N,r}(\alpha, 0, f(r))$, $\tilde{\mathcal{E}}_{N,r}(0, \beta, f(r))$ are defined in (3), (6), and (7), respectively.

Then we have

$$\lim_{N \rightarrow \infty} \|y(x) - \bar{y}_N(x)\|_2 = 0.$$

Proof. Using (20), we can conclude that the discrete collocation solution $\bar{y}_N(x)$ for the equation (1) satisfies in the following operator system:

$$\bar{y}_N(x_i) = g(x_i) + \mathcal{K}_N \bar{y}_N(x_i), \quad 0 \leq i \leq N, \quad (22)$$

where the operator \mathcal{K}_N is the numerical integral operator defined in the assumption 3. Multiplying $L_i(x)$ on both sides of (22) and summing up from $i = 0$ to $i = N$ yield

$$I_N(\bar{y}_N) = I_N g + I_N \mathcal{K}_N \bar{y}_N. \quad (23)$$

Subtracting (1) from (23) gives

$$y - I_N(\bar{y}_N) = (g - I_N g) + (\mathcal{K}y - I_N \mathcal{K}_N \bar{y}_N),$$

or equivalently

$$(\text{id} - \mathcal{K})\bar{e}_N = e_{I_N}(g) - e_{I_N}(\bar{y}_N) + e_{I_N}(\mathcal{K}\bar{y}_N) + I_N(\mathcal{K}\bar{y}_N - \mathcal{K}_N \bar{y}_N), \quad (24)$$

where $\bar{e}_N = y(x) - \bar{y}_N(x)$ is the discrete collocation error function, $e_{I_N}u = u - I_N u$ is the interpolation error function, id is the identity operator, and \mathcal{K} is the following continuous and compact integral operator with weakly singular logarithmic kernel

$$\mathcal{K}y(x) = \int_0^x \ln(x-t)K(x,t)y(t)dt.$$

Applying Lemma 2 with $\mathcal{X} = C(\Omega)$ and using the assumption 2, the relation (24) can be rewritten as

$$\|\bar{e}_N\|_2 \leq \|(\text{id} - \mathcal{K})^{-1}\|_\infty \left(\|e_{I_N}(g)\|_2 + \|e_{I_N}(\bar{y}_N)\|_2 + \|e_{I_N}(\mathcal{K}\bar{y}_N)\|_2 + \|I_N(\mathcal{K}\bar{y}_N - \mathcal{K}_N \bar{y}_N)\|_2 \right), \quad (25)$$

with $\|(\text{id} - \mathcal{K})^{-1}\|_\infty < \infty$. Due to Lemma 3 and assumption 1 we have

$$\|e_{I_N}(g)\|_2, \|e_{I_N}(\bar{y}_N)\|_2, \|e_{I_N}(\mathcal{K}\bar{y}_N)\|_2 \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (26)$$

Now, it is sufficient that we show $\|I_N(\mathcal{K}\bar{y}_N - \mathcal{K}_N \bar{y}_N)\|_2 \rightarrow 0$ as $N \rightarrow \infty$. To this end, using Lemma 4 and assumption 3, we can write

$$\|I_N(\mathcal{K}\bar{y}_N - \mathcal{K}_N \bar{y}_N)\|_2 \leq C_1 \|\mathcal{K}\bar{y}_N - \mathcal{K}_N \bar{y}_N\|_\infty, \quad (27)$$

which $\mathcal{K}\bar{y}_N - \mathcal{K}_N \bar{y}_N$ is the numerical integration error function. According to the definition of \mathcal{K}_N and numerical approach proposed in the previous section, we can deduce that the numerical integration error is established by calculating the errors obtained from the applying Jacobi–Gauss and generalized Jacobi–Gauss formulas in (13), (15), (18), and (19). Consequently, we can write (27) as follows:

$$\begin{aligned} \|I_N(\mathcal{K}\bar{y}_N - \mathcal{K}_N\bar{y}_N)\|_2 \leq C_2 \sum_{n=0}^N \left(& \|\mathcal{E}_{N,\theta}(0, 0, \mathcal{F}_1^{n,x}(\theta))\|_\infty \right. \\ & + \left\| \left[\tilde{\mathcal{E}}_{N,\theta}(0, \beta, \mathcal{F}_2^{n,x}(\theta)) \right]_{\beta=0} \right\|_\infty \\ & + \left\| \left[\tilde{\mathcal{E}}_{N,\theta}(\alpha, 0, \mathcal{F}_1^{n,x}(\theta)) \right]_{\alpha=0} \right\|_\infty \\ & + \left\| \left[\tilde{\mathcal{E}}_{N,s}(\alpha, 0, \mathcal{F}_3^{n,x}(s)) \right]_{\alpha=0} \right\|_\infty \\ & \left. + \|\mathcal{E}_{N,s}(0, 0, \mathcal{F}_3^{n,x}(s))\|_\infty \right), \end{aligned}$$

and thereby using assumption 4 in the inequality above, we deduce

$$\|I_N(\mathcal{K}\bar{y}_N - \mathcal{K}_N\bar{y}_N)\|_2 \rightarrow 0, \quad \text{as } N \rightarrow 0. \quad (28)$$

Finally, the required result can be obtained by applying (26) and (28) in (25). \square

5 Numerical Results

In this section, we illustrate some examples using the method proposed in the previous section and confirm its validity. All of calculations performed on a PC running *Mathematica* software. In the obtained results we presented some essential items regarding the L^2 -norms of the error functions and comparison results between our scheme and the well known Chebyshev collocation method [9, 27].

Example 1. Consider the following problem

$$y(x) = g(x) + \int_0^x \ln(x-t)y(t)dt$$

with

$$g(x) = x(\ln x - 1) + \frac{x^2}{12} \left(\pi^2 - 21 + 18 \ln x - 6 \ln^2 x \right)$$

and $y(x) = x(\ln x - 1)$ as the exact solution.

We solve this problem by the proposed method with $N = 3$. Indeed, we seek a discrete collocation solution in the form

$$\bar{y}_3(x) = \sum_{n=0}^3 \bar{a}_n \mathcal{M}_n(x),$$

to approximate this problem and find its unknown coefficients such that they satisfy in the linear system (21) with $N = 3$. Considering the Gauss–Legendre collocation points

$$x_0 = 0.330009, \quad x_1 = 0.669991, \quad x_2 = 0.0694318, \quad x_3 = 0.930568,$$

and the operational matrices

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -0.1086 & 0.5995 & -1.6674 & 0.9280 \\ -0.5713 & 0.0805 & -0.0549 & 0.7942 \\ -0.2477 & -0.3808 & 0.8873 & 0.6080 \end{pmatrix},$$

$$\mathcal{A}_1 = \begin{pmatrix} 2 & 2 & 2 & 2 \\ -2.2173 & -0.8009 & -5.3348 & -0.1439 \\ -0.4627 & -0.5191 & 1.6126 & -0.1338 \\ 0.5550 & -0.2017 & 0.1359 & -0.1194 \end{pmatrix},$$

$$\mathcal{A}_2 = \begin{pmatrix} -0.6137 & -0.6137 & -0.6137 & -0.6137 \\ -0.6095 & -1.0441 & 0.3471 & -1.2457 \\ 0.7718 & 0.1092 & 0.6562 & -0.5302 \\ 0.1912 & 0.6068 & -1.3532 & 0.0376 \end{pmatrix},$$

$$\mathcal{D}_1 = \begin{pmatrix} -0.2973 & 0 & 0 & 0 \\ 0 & -0.3664 & 0 & 0 \\ 0 & 0 & -0.1167 & 0 \\ 0 & 0 & 0 & -0.3560 \end{pmatrix},$$

$$\mathcal{D}_2 = \begin{pmatrix} 0.1650 & 0 & 0 & 0 \\ 0 & 0.3350 & 0 & 0 \\ 0 & 0 & 0.0247 & 0 \\ 0 & 0 & 0 & 0.4653 \end{pmatrix},$$

the system of linear algebraic equations (21) is presented in the following form

$$\begin{cases} 1.0449 + 1.6959\bar{a}_0 - 0.6673\bar{a}_1 - 0.8362\bar{a}_2 - 0.1142\bar{a}_3 = 0, \\ 1.6603 + 1.9383\bar{a}_0 + 0.6558\bar{a}_1 - 0.1463\bar{a}_2 - 0.6580\bar{a}_3 = 0, \\ 0.2955 + 1.2546\bar{a}_0 - 2.3019\bar{a}_1 + 0.1105\bar{a}_2 + 0.9501\bar{a}_3 = 0, \\ 1.8965 + 1.9975\bar{a}_0 + 1.4564\bar{a}_1 + 0.9933\bar{a}_2 + 0.5830\bar{a}_3 = 0, \end{cases}$$

which its solution gives by

$$\bar{a}_0 = -0.75, \quad \bar{a}_1 = -0.25, \quad \bar{a}_2 = -0.0833, \quad \bar{a}_3 = 0.0833.$$

Consequently the discrete collocation solution $\bar{y}_3(x)$ represents by

$$\bar{y}_3(x) = -5.2953 \times 10^{-7} - x - 1.3027 \times 10^{-7} \ln x + 0.9999x \ln x.$$

The L^2 -norm of the error function is 1.3445×10^{-13} that is in a very good agreement with the exact ones whereas the approximation degree is very small ($N = 3$).

Example 2. Consider the following problem;

$$y(x) = g(x) + \int_0^x \ln(x-t)e^{x+t}y(t)dt,$$

with

$$g(x) = e^{-x} \ln x + \frac{xe^x}{6} \left(-12 + \pi^2 - 6 \ln x(-2 + \ln x) \right),$$

and $y(x) = e^{-x} \ln x$, as the exact solution.

We solve the problem and report the obtained results in Table 1 and Figure 1. In Figure 1, we plot the L^2 -norm of the error function in terms of the various values of the degree of approximation N . Figure 1 shows that the proposed algorithm obtains a good accuracy with suitable values of N . Moreover to make a comparison, we also solve this problem by implementing the well-known Chebyshev collocation method [9, 27] and give the obtained results in Table 1. Based on Table 1, we confirm that the Müntz-logarithmic polynomials makes faster rate of convergence for the discrete collocation solution of this problem compared with the classical Chebyshev polynomials.

Example 3. Consider the following problem:

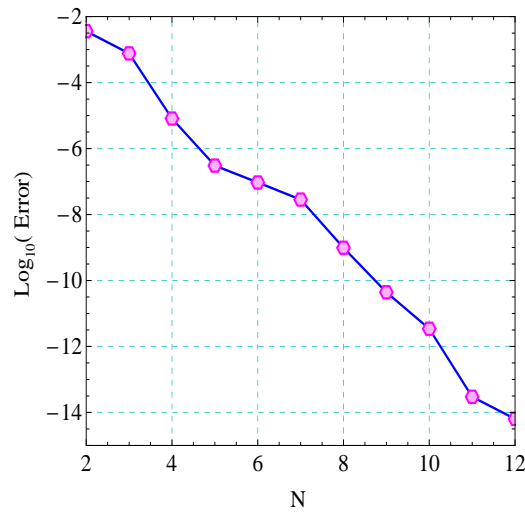
$$y(x) = g(x) + x \int_0^x \ln(x-t)t^2y(t)dt,$$

with

$$g(x) = x^{\frac{5}{2}} - \frac{2x^{\frac{13}{2}}(-13016 + 6930 \ln 2 + 3465 \ln x)}{38115},$$

Table 1: The numerical errors of Example 2

| N | Numerical Errors | |
|-----|------------------------|--------------------------------------|
| | Our Method | Chebyshev collocation Method [9, 27] |
| 2 | 3.61×10^{-3} | 1.3×10^{-1} |
| 4 | 8.13×10^{-6} | 4.22×10^{-2} |
| 6 | 9.38×10^{-8} | 2.14×10^{-2} |
| 8 | 9.84×10^{-10} | 1.3×10^{-2} |
| 10 | 3.44×10^{-12} | 8.73×10^{-3} |
| 12 | 6.37×10^{-15} | 6.28×10^{-3} |

Figure 1: Obtained L^2 - norm of the error function versus N for Example 2

and $y(x) = x^2\sqrt{x}$ as the exact solution.

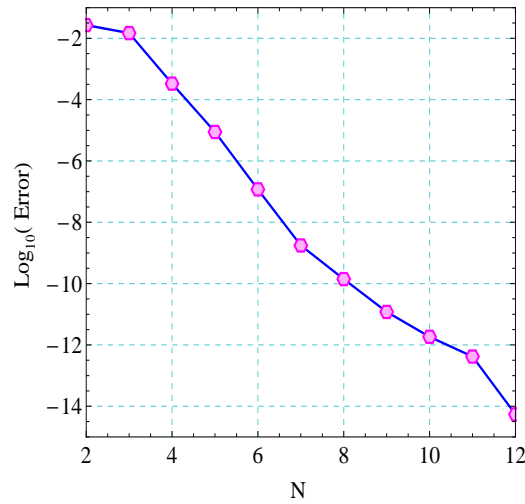
We have calculated the approximate solution with different values of N and displayed the obtained results in Table 2 and Figure 2. Table 2 and Figure 2, present the L^2 -norm of the error functions versus N . As it can be observed although the exact solution has singularity at zero, the numerical errors are decreased with an appropriate rate as the approximation degree N is increased.

Example 4. Consider the following problem:

$$y(x) = g(x) + \int_0^x \ln(x-t)y(t)dt$$

Table 2: The numerical errors of Example 3

| N | Numerical Errors |
|-----|------------------------|
| 2 | 2.69×10^{-2} |
| 4 | 3.32×10^{-4} |
| 6 | 1.18×10^{-7} |
| 8 | 1.4×10^{-10} |
| 10 | 1.84×10^{-12} |
| 12 | 5.43×10^{-15} |

Figure 2: Obtained L^2 - norm of the error function versus N for Example 3

with

$$g(x) = e^x(1 + \gamma + \Gamma(0, x)) + \ln x,$$

where γ is Euler's constant with the numerical value $\simeq 0.577216$ and $\Gamma(0, x)$ is incomplete gamma function satisfies

$$\Gamma(a, z) = \int_z^{\infty} t^{a-1} e^{-t} dt.$$

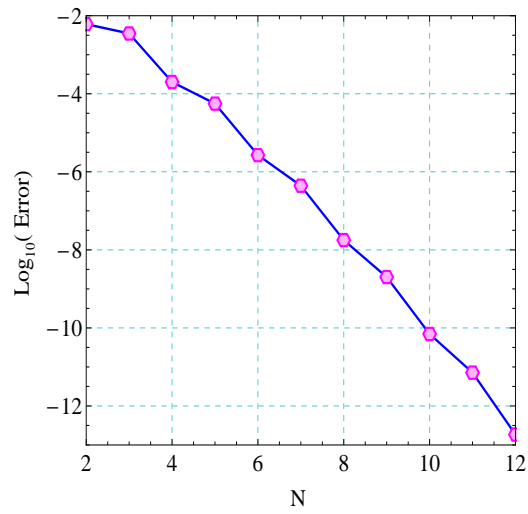
Here the exact solution is given by $y(x) = e^x$.

The obtained numerical results from implementation of the proposed discrete collocation scheme are reported in Table 3 and Figure 3. The presented

Table 3: The numerical errors of Example 4

| N | Numerical Errors |
|-----|------------------------|
| 2 | 6.08×10^{-3} |
| 4 | 2.01×10^{-4} |
| 6 | 2.68×10^{-6} |
| 8 | 1.78×10^{-8} |
| 10 | 6.99×10^{-11} |
| 12 | 1.85×10^{-13} |

results confirm that our scheme provides reliable results for smooth solutions of (1).

Figure 3: Obtained L^2 - norm of the error function versus N for Example 4

6 Conclusion

In this article, we developed a new discrete collocation method based on the Müntz-logarithmic polynomials as basis functions. Moreover, we used highly accurate Jacobi–Gauss and generalized Jacobi–Gauss quadratures to approximate the integrals with Jacobi and logarithmic weights, respectively. Convergence analysis of the proposed method were presented and some numerical examples were illustrated to confirm the applicability of the presented discrete collocation scheme.

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روش هم محلی گسسته برای معادلات انتگرالی ولترا بطور ضعیف تکین با هسته های لگاریتمی

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چکیده : یک روش هم محلی گسسته مناسب به منظور حل معادلات انتگرالی ولترا بطور ضعیف تکین با هسته های لگاریتمی مورد بررسی قرار گرفته است. یکی از ویژگی های این معادلات این است که در حالت کلی مشتق مرتبه اول جواب مانند یک تابع لگاریتمی رفتار می کند که در مبدأ ناپیوسته است. در این مقاله برای ساخت یک جواب تقریبی هم راستا با جواب واقعی یک رویکرد هم محلی جدید را معرفی می کنیم که در آن چند جمله ایهای مونتر-لگاریتمی به عنوان توابع پایه ای بکار برده می شود. بعلاوه، چون پیاده سازی این رویکرد منجر به انتگرال هایی با تکینی های لگاریتمی می شود که اغلب حل آنها به روش عددی مشکل است، یک روش انتگرال گیری عددی مناسب با توابع وزن لگاریتمی را بکار می بریم که در آن انتگرال های چند جمله ای ها با توابع وزن لگاریتمی بطور دقیق محاسبه می شود. بدین منظور روشهای انتگرالگیری گاوس-ژاکوبی را یادآوری نموده و سپس آنرا برای انتگرالهای با توابع وزن لگاریتمی تعمیم می دهیم. آنالیز همگرایی روش پیشنهادی ارائه می شود و برخی نتایج عددی به منظور تایید دقت و مناسب بودن روش پیشنهادی ارائه می شود.

کلمات کلیدی : روش هم محلی گسسته؛ چند جمله ای های مونتر-لگاریتمی؛ روش انتگرالگیری عددی؛ معادلات انتگرالی ولترا بطور ضعیف تکین با هسته های لگاریتمی.