Iranian Journal of Numerical Analysis and Optimization
Vol. 13, No. 1, 2023, pp 59–79
https://doi.org/10.22067/ijnao.2022.75871.1120
https://ijnao.um.ac.ir/

How to cite this article
Research Article

An improvised technique of quintic hermite splines to discretize generalized Burgers–Huxley type equations

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Abstract

A mathematical collocation solution for generalized Burgers–Huxley and generalized Burgers–Fisher equations has been monitored using the weighted residual method with Hermite splines. In the space direction, quintic Hermite splines are introduced, while the time direction is discretized using a finite difference approach. The technique is determined to be unconditionally stable, with order $(h^4 + \triangle t)$ convergence. The technique's efficacy is tested using nonlinear partial differential equations. Two problems of the generalized Burgers–Huxley and Burgers–Fisher equations have been solved using a finite difference scheme as well as the quintic Hermite collocation method (FDQHCM) with varying impacts. The FDQHCM computer codes are written in MATLAB without transforming the nonlinear term to a linear term. The numerical findings are reported in weighted norms and in discrete form. To assess the technique's applicability, numerical and exact values are compared, and a reasonably good agreement is recognized between the two.

AMS subject classifications (2020): 65D07; 65N06; 65N35; 65N12

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Received 16 March 2022; revised 28 April 2022; accepted 6 May 2022

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Keywords: Quintic Hermite splines, Forward finite difference scheme, collocation method, stability analysis.

1 Introduction

Nonlinear partial differential equations nowadays turn out to be basic mathematical methodologies to study multifarious structures like turbulence in fluid dynamics, convection-diffusion, flow through a shock wave roaming in viscid fluid, number theory, continuous stochastic processes, and so on. The diversity of physical phenomena in basic and applied sciences such as physics, chemistry, biology, computer science, electronics, and so on can be preeminently described by these nonlinear equations. Various authors have used these nonlinear partial differential equations in different fields such as hydrodynamics, solid mechanics, and so on; see [7, 9, 10, 11, 18, 25]. The Navier–Stokes equation is a fundamental fluid dynamics equation that may be simplified into a number of mathematical phenomena. By omitting the pressure factor, the generalized Burgers–Huxley equation (GBHE) simplifies this intricate equation. This equation explains how reaction, diffusion, and convection processes interact. Special instances of the GBHE include generalized Burgers–Fisher equation (GBFE) and generalized Burgers equation. These equations are widely studied in fluid dynamics, gas dynamics, traveling wave solutions, and traffic flow. The one-dimensional GBHE, which refers to nerve pulse transmission in nerve fibers and waves in fluid crystals, can be written in the following form:

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - \mu u^\delta \frac{\partial u}{\partial x} + f(u), \qquad (x,t) \in \Omega \times (0,T]. \tag{1}$$

The initial and boundary conditions are presented as

$$u(x,0) = g(x), \tag{2}$$

$$u(0,t) = f_1(t), \quad \text{and} \quad u(1,t) = f_2(t), \tag{3}$$

where $g(x)$, $f_1(t)$, and $f_2(t)$ are continuous functions in x and t, respectively. Moreover, $f(u)$ is a smooth function with a nonlinear nature that is defined on $\Omega \times (0,T)$. In the theory of traveling wave solutions, it is significant. For $f(u) = 0$, equation (1) reduced to modified Burgers equation. On the other hand, in [24], it has studied the generalized Burgers equation with $f(u) \neq 0$, although in a different way than Burgers–Huxley and Burgers–Fisher. For the GBHE, $f(u) = \beta u(1-u^\delta)(u^\delta - \bar{\gamma})$ and for the GBFE, $f(u) = \beta u(1-u^\delta)$, where μ, δ, β, and $\bar{\gamma}$ are real parameters. The viscosity factor ε is another distinction between the Burgers equation and Burgers–Huxley equation. The viscosity factor is typically assumed to be 1 in the Burgers–Huxley equation. However, it plays a vital role in turbulence theory for the Burgers equation. A specific nonlinear evolution equation is represented by the change in parameters.

For example, equation (1) presents the modified Burgers equation to explain wave propagation in nonlinear dissipative systems and various other physical contexts, such as sound waves in a viscous medium, where $\beta = 0$. Therefore, a significant study of GBHE with different case studies helps to analyze the behavior of different nonlinear equations from a wide perspective.
Due to the wide applications of GBHE and GBFE, these equations are studied extensively by many investigators. Hammad and El-Azab [13] have used a collocation method with a $2N$-order compact finite difference scheme, and Kushner and Matviychuk [17] proposed finite-dimensional dynamics to find the exact solution of Burgers–Huxley equation. Celik [8] proposed the Haar wavelet method, whereas Alharbi and Fahmy [1] have proposed an ADM-pade method to study the behavior of Burgers–Huxley equation. Javidi [16] followed a spectral collocation method to solve the GBHE, whereas Saha Ray and Gupta [21] followed a wavelet collocation for the same. The implicit exponential finite difference method has been followed in [14] to find the numerical solution to Burgers–Huxley equation.
To solve the GBHE and GBFE, the present work proposes a quintic Hermite collocation with the forward finite difference technique (FDQHCM). Quintic Hermite collocation is a weighted residual approach with a Hermite basis, whereas finite difference scheme is the variational method. Both approaches are coupled to get numerical outputs that are compatible with numerical codes and are stable.

2 Quintic Hermite collocation

Quintic Hermite collocation method [4, 6, 5] is one of the weighted residual methods. Instead of Jacobi orthogonal polynomials, quintic Hermite interpolating polynomials are used as the foundation function in this approach. Over the area Ω, which is considered to be the union of intervals $[x_{i-1}, x_i]$, the solution function is approximated by an approximation function that uses quintic Hermite interpolating polynomials as its basis.

Hermite interpolating polynomials, which are an extension of Lagrangian interpolating polynomials, are of order $2k + 1$, where k is a positive integer. Hermite interpolating polynomials, on the other hand, are superior to Lagrangian interpolating polynomials in terms of applicability since they interpolate both the function and its kth-order derivative. Furthermore, Lagrangian interpolating polynomials need a requirement of continuity at node locations that are not required by Hermite interpolating polynomials. As a result, quintic Hermite interpolating polynomials with $k = 2$ yield quintic Hermite interpolating polynomials that interpolate the function as well as its first- and second-order derivatives. Arora and Kaur [5] discussed the behavior and structure of quintic Hermite polynomials in depth. Let $u^{\gamma}(x, t)$ be the approximating function to be adjusted to eq. (1). Although the first- and second-order derivatives are interpolated by quintic Hermite interpolating polynomials, the boundary constraints are satisfied at boundary points. The approximating function is

$$u(x,t) = \sum_{i=1}^{2} \Big(P_i(x)a_i(t) + \bar{P}_i(x)b_i(t) + \bar{\bar{P}}_i(x)c_i(t) \Big), \tag{4}$$

where a_i, b_i, and $c_i's$ are continuous functions of $'t'$ and $P_i, \bar{P}_i$, and $\bar{\bar{P}}_i$ are smooth functions of $'x'$ and expressed as

$$P_i(x) = \begin{cases} 6\left(\frac{x_{j+1}-x}{x_{j+1}-x_j}\right)^5 - 15\left(\frac{x_{j+1}-x}{x_{j+1}-x_j}\right)^4 + 10\left(\frac{x_{j+1}-x}{x_{j+1}-x_j}\right)^3, & x_j \le x \le x_{j+1}, \\ 6\left(\frac{x-x_{j-1}}{x_j-x_{j-1}}\right)^5 - 15\left(\frac{x-x_{j-1}}{x_j-x_{j-1}}\right)^4 + 10\left(\frac{x-x_{j-1}}{x_j-x_{j-1}}\right)^3, & x_{j-1} \le x \le x_j, \\ 0 & elsewhere, \end{cases}$$

$$\bar{P}_i(x) = \begin{cases} 3\frac{(x_{j+1}-x)^5}{(x_{j+1}-x_j)^4} - 7\frac{(x_{j+1}-x)^4}{(x_{j+1}-x_j)^3} + 4\frac{(x_{j+1}-x)^3}{(x_{j+1}-x_j)^2}, & x_j \le x \le x_{j+1}, \\ -3\frac{(x-x_{j-1})^5}{(x_j-x_{j-1})^4} + 7\frac{(x-x_{j-1})^4}{(x_j-x_{j-1})^3} - 4\frac{(x-x_{j-1})^3}{(x_j-x_{j-1})^2}, & x_{j-1} \le x \le x_j, \\ 0 & elsewhere, \end{cases}$$

$$\bar{\bar{P}}_i(x) = \begin{cases} 0.5\frac{(x_{j+1}-x)^5}{(x_{j+1}-x_j)^3} - \frac{(x_{j+1}-x)^4}{(x_{j+1}-x_j)^2} + 0.5\frac{(x_{j+1}-x)^3}{(x_{j+1}-x_j)}, & x_j \le x \le x_{j+1}, \\ 0.5\frac{(x-x_{j-1})^5}{(x_j-x_{j-1})^3} - \frac{(x-x_{j-1})^4}{(x_j-x_{j-1})^2} + 0.5\frac{(x-x_{j-1})^3}{(x_j-x_{j-1})}, & x_{j-1} \le x \le x_j, \\ 0 & elsewhere, \end{cases}$$

where

$$P_i(x_j) = \delta_{ji}, \quad P_i^{'}(x_j) = 0, \quad P_i^{''}(x_j) = 0, \quad i,j = 1,2,\ldots,6,$$
$$\bar{P}_i(x_j) = 0, \quad \bar{P}_i^{'}(x_j) = \delta_{ji}, \quad \bar{P}_i^{''}(x_j) = 0, \quad i,j = 1,2,\ldots,6,$$
$$\bar{\bar{P}}_i(x_j) = 0, \quad \bar{\bar{P}}_i^{'}(x_j) = 0, \quad \bar{\bar{P}}_i^{''}(x_j) = \delta_{ji}, \quad i,j = 1,2,\ldots,6.$$

The points $x_j's$ are the node points or mesh points, which are taken equidistant to get uniform mesh grid. The principle of collocation is applied between two consecutive node points, that is, on $[x_j, x_{j+1}]$. The details are given in [2, 4, 5]. Therefore, to apply the principle of collocation, a new variable ξ is introduced within each sub-interval $[x_j \ x_{j+1}]$ in such a way that as x varies from x_j to x_{j+1}, ξ varies from 0 to 1, and h is the length of sub-interval $[x_j \ x_{j+1}]$. To reduce the complexity of system of equations, the interval length h is taken to be uniform. Therefore, after rearranging $P_i, \bar{P}_i$, and $\bar{\bar{P}}_i$, quintic Hermite polynomials take the form as given in Table 1.

Table 1: Presentation of quintic Hermite splines

i	H_i	H_i'	H_i''
1	$1-10\xi^3+15\xi^4-6\xi^5$	$-30\xi^2+60\xi^3-30\xi^4$	$-60\xi+180\xi^2-120\xi^3$
2	$h(\xi-6\xi^3+8\xi^4-3\xi^5)$	$h(1-18\xi^2+32\xi^3-15\xi^4)$	$h(-36\xi+96\xi^2-60\xi^3)$
3	$\frac{h^2}{2}(\xi^2-3\xi^3+3\xi^4-\xi^5)$	$\frac{h^2}{2}(2\xi-9\xi^2+12\xi^3-5\xi^4)$	$\frac{h^2}{2}(2-18\xi+36\xi^2-20\xi^3)$
4	$\frac{h^2}{2}(\xi^3-2\xi^4+\xi^5)$	$\frac{h^2}{2}(3\xi^2-8\xi^3+5\xi^4)$	$\frac{h^2}{2}(6\xi-24\xi^2+20\xi^3)$
5	$(10\xi^3-15\xi^4+6\xi^5)$	$(30\xi^2-60\xi^3+30\xi^4)$	$(60\xi-180\xi^2+120\xi^3)$
6	$h(-4\xi^3+7\xi^4-3\xi^5)$	$h(-12\xi^2+28\xi^3-15\xi^4)$	$h(-24\xi+84\xi^2-60\xi^3)$

The values of quintic Hermite splines at $x = 0$ and $x = 1$ are given in Tables 2 and 3. It is analyzed from these tables that these polynomial are either 0 or 1 at these points, which helps the approximating function to satisfy the Dirichlet or Neumann type boundary conditions. These quintic Hermite splines have the properties $H_1(\xi) = H_5(1-\xi)$, $H_2(\xi) = -H_6(1-\xi)$, and $H_3(\xi) = H_4(1-\xi)$.

Table 2: Presentation of quintic Hermite splines and its corresponding first- and second-order derivatives at ξ=0

H_1	1	H_1'	0	H_1''	0
H_2	0	H_2'	h	H_2''	0
H_3	0	H_3'	0	H_3''	h^2
H_4	0	H_4'	0	H_4''	0
H_5	0	H_5'	0	H_5''	0
H_6	0	H_6'	0	H_6''	0

Table 3: Presentation of quintic Hermite splines and its corresponding first- and second-order derivatives at ξ=1

H_1	0	H_1'	0	H_1''	0
H_2	0	H_2'	0	H_2''	0
H_3	0	H_3'	0	H_3''	0
H_4	0	H_4'	0	H_4''	h^2
H_5	1	H_5'	0	H_5''	0
H_6	0	H_6'	h	H_6''	0

Therefore, equation (4) can be rewritten as

$$u^{\gamma}(\xi, t) = \sum_{m=1}^{6} H_m(\xi) a_m^{\gamma}(t), \tag{5}$$

where $H_m(\xi)$'s are quintic Hermite interpolating polynomials and $a_m^{\gamma}(t)'s$ are continuous functions of $'t'$ with γ being the number of sub-divisions. To apply orthogonal collocation, zeros of fourth-order shifted Legendre polynomials have been taken as collocation points. These polynomials give better results on center as well as on average [2]. Therefore, the zeros of these polynomials have been chosen as collocation points. The diagrammatic representation of these splines is given in Figure 1. In Figure 2, the application of quintic Hermite splines has been shown. Further details of collocation points and the technique of quintic Hermite collocation are given in [5, 2].

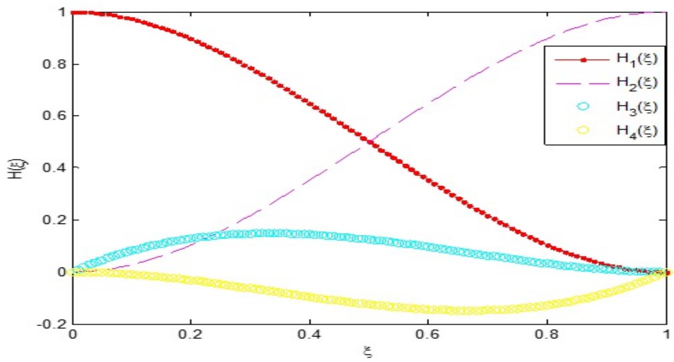

Figure 1: Diagrammatic behavior of quintic Hermite polynomials

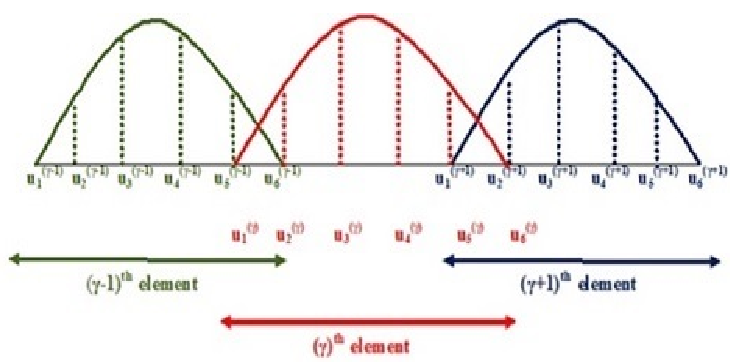

Figure 2: Diagrammatic representation of quintic Hermite collocation scheme

3 Implementation of FDQHCM

To implement the scheme of FDQHCM, equation (1) is discretized in time direction using forward finite difference scheme with step-size $\triangle t$

$$\frac{u_{j+1}-u_j}{\triangle t} = u_{xxj} - \mu u_j^{\delta} u_{xj} + f_j, \tag{6}$$

where f_j is $f(u_j)$. Initial and boundary conditions can be discretized in the following way at $t = t_j$, $u_0(x) = g(x)$; $u_j(0) = f_{1j}$ and $u_j(1) = f_{2j}$, where $f_{1j} = f_1(t_j)$ and $f_{2j} = f_2(t_j)$. By simplifying, equation (6) converts into

$$u_{j+1} = \triangle t u_{xxj} - \triangle t \mu u_j^{\delta} u_{xj} + \triangle t f_j + u_j. \tag{7}$$

After applying quintic Hermite collocation on the variable u_j in the space direction, equation (6) takes the following form:

$$\sum_{m=1}^{6} H_m(\xi)a_m^{\gamma}(t_{j+1}) = \frac{\triangle t}{h^2}\sum_{m=1}^{6} H''_m(\xi)a_m^{\gamma}(t_j) - \mu\frac{\triangle t}{h}\left(\sum_{m=1}^{6} H_m(\xi)a_m^{\gamma}(t_j)\right)^{\delta} \times\left(\sum_{m=1}^{6} H'_m(\xi)a_m^{\gamma}(t_j)\right) + \sum_{m=1}^{6} H_m(\xi)a_m^{\gamma}(t_j) + \triangle t f_j, \quad m = 1, 2, \ldots, 6,\ j = 1, 2, \ldots, n_t. \tag{8}$$

At the kth collocation point, equation (8) can be written as

$$\sum_{m=1}^{6} H_{km}a_{m,j+1}^{\gamma} = \frac{\triangle t}{h^2}\sum_{m=1}^{6} B_{km}a_{m,j}^{\gamma} - \mu\frac{\triangle t}{h}\left(\sum_{m=1}^{6} H_{km}a_{m,j}^{\gamma}\right)^{\delta} \times \sum_{m=1}^{6} A_{km}a_{m,j}^{\gamma} + \sum_{m=1}^{6} H_{km}a_{m,j}^{\gamma} + \triangle t f_{k,j}, \quad m = 1, 2, \ldots, 6,\ k = 2, 3, 4, 5,\ j = 1, 2, \ldots, n_t, \tag{9}$$

where H_{km} is the mth interpolating polynomial at kth collocation point and B_{km} and A_{km} are, respectively, the second-order and the first-order derivatives of mth interpolating polynomial at kth collocation point. Also, $a_{m,j}^{\gamma}$ is the mth collocation function at jth time step in γ^{th} sub-domain and $f_{k,j}$ is the discretized function $f(u)$ at the jth time step and the kth collocation point.

Hence, after implementation of quintic Hermite collocation with forward finite difference scheme, collocation equations reduce to the following matrix form:

$$H\bar{a}_{j+1} = M_1\bar{a}_j + (H\bar{a}_j)^{\delta}(M_2\bar{a}_j) + \bar{f} + K, \tag{10}$$

where $H = [H_m(\xi_k)]$, $M_1 = \left[\frac{\triangle t}{h^2}B_m(\xi_k) + H_m(\xi_k)\right]$, and $M_2 = \left[\frac{\triangle t}{h}A_m(\xi_k)\right]$. Moreover, K is obtained from the boundary conditions and $\bar{f} = [\triangle t f_{k,j}]$. Also, $\bar{a}_{j+1}$ and $\bar{a}_j$ being the unknown collocation vectors at the $(j+1)$th and jth time steps, respectively. In collective form, left-hand side coefficient matrix can be written as

$$J = M_1\bar{a}_j + (H\bar{a}_j)^{\delta}(M_2\bar{a}_j) + \bar{f} + K$$

and $\bar{A} = [\bar{a}_{j+1}]$.

Hence, in the combined form, equation (10) can be written as

$$\bar{A} = H^{-1}J. \tag{11}$$

Details of these matrices are given in [5, 3]. In the case of the first and last elements, the bandwidth is 4×5, whereas for the remaining elements, bandwidth is 4×6. Therefore, after combining all the collocation equations, a set of $4ne \times 4ne$ equations appear, with ne being the total number of subdivisions. The discretized set of equations has been solved in MATLAB. For initial approximation of $\bar{a}_0$, the initial condition of u at $t = 0$ has been taken at different collocation points. Then the loop was introduced to calculate $\bar{a}_1$, $\bar{a}_2$, and so on upto desired accuracy. The details of algorithm are mentioned hereunder.

Algorithm:

The algorithm of the solution technique is

(i) Define the problem.

(ii) Apply the finite difference technique in the time direction.

(iii) Discretize the problem by applying quintic Hermite interpolating polynomials in space direction.

(iv) Apply collocation points on the interpolating equations.

(v) Solve the collocation equations using the following code.

$for \quad ii = 1 : 1 : ti/j$

$P(:, ii) = B - mu * A. * (U^{delta}) + U + beta * j. * U. * (1 - (U^{delta})). * ((U^{delta}) - gamma);$

$Z(:, ii) = inv(H) * P(:, ii);$

$U0 = Z(:, ii);$

end

4 Stability analysis

To study the stability of any partial differential equation, it is necessary that it should be linear. In the case of nonlinear partial differential equations, it is first linearized, and then the stability behavior is analyzed. Therefore, to study the stability of equation (1), it is first linearized by taking $\bar{v} = \max(u^\delta)$ and $f(u) = \beta u \bar{\bar{f}}$, where $\bar{\bar{f}} = max((u^\delta - 1)(\gamma - u^\delta))$, for all $(x, t) \in \Omega \times (0, T)$. As given in [20], a method is said to be A-stable if the region of absolute stability includes the region of $Re(\lambda \triangle t) < 0$.

The region of absolute stability for FDQHCM is the set of all complex numbers $\lambda \triangle t$ such that

$$|u_j| \leq C \quad \text{as} \quad t_j \to \infty. \tag{12}$$

Let $\lambda_j,\ j = 1, 2, \ldots, ne$, be the set of eigenvalues of the coefficient matrix $a^\gamma_{m,j}$. The scheme of FDQHCM is said to be stable if $Re(\lambda_j, \triangle t) < 0$ for all values of μ, β, γ, and δ. Then

$$\sum_{m=1}^{6} H_m(\xi)a^{\gamma}_{m,j+1} = \frac{\triangle t}{h^2}\sum_{m=1}^{6} H''_m(\xi)a^{\gamma}_{m,j} - \mu\frac{\triangle t}{h}\bar{v}\sum_{m=1}^{6} H'_m(\xi)a^{\gamma}_{m,j} + \sum_{m=1}^{6} H_m(\xi)a^{\gamma}_{m,j} + \triangle t\beta\sum_{m=1}^{6} H_m(\xi)a^{\gamma}_{m,j}\bar{\bar{f}}. \tag{13}$$

The stability of the given technique can be checked on a linear equation. Therefore, equation (13) represents the quasi-linearized form of equation (8) to check the stability of the given technique. From Figure 3, it is observed that $Re(\lambda_j, \triangle t) \in [-1, 0]$ for all values of μ, β, γ, and δ, which justifies the stability of the numerical scheme.

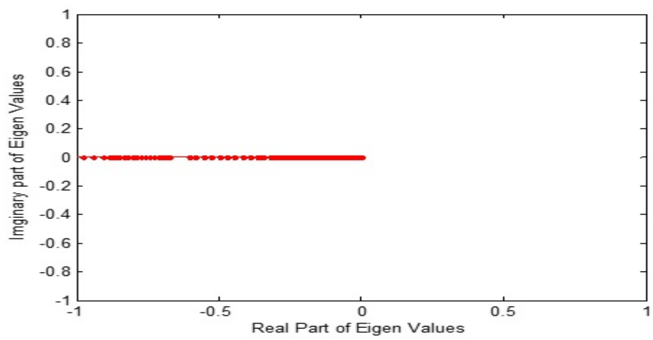

Figure 3: Behavior of eigenvalues for coefficient matrix given in equation (13)

5 Convergence analysis

The operator $\mathfrak{L}$ defined by $\frac{\partial^2}{\partial x^2}$ in spatial and time domains, is positive definite in $L_2(0, 1)$, the space of all real valued Lebesgue measurable functions square integrable $(0, 1)$, for all $t > 0$. The definition given by [19] is quoted here:

Consider a family of mathematical problems parametrized by singular perturbation parameter ε, where ε lies in the semi open interval $0 < \varepsilon \leq 1$. Assume that each problem in the family has the unique solution denoted by u_ε and that each u_ε is approximated by a sequence of numerical solutions $(U_\varepsilon, \bar{\Omega}^N)_{N=1}^{\infty}$, where U_ε is defined on the $\bar{\Omega}^N$, representing the set of points in $\mathbf{R}$ and N is the discretization parameter. Then the numerical solutions U_ε are said to converge to the exact solution u_ε, if their exist a positive integer N_0 and positive numbers C and p, where N_0, C, and p are all independent of N and ε, such that for all $N \geq N_0$, we have

$$sup_{0<\varepsilon\leq 1}||U_\varepsilon - u_\varepsilon||_{\bar{\Omega}^N} \leq CN^{-p}.$$

Here p is the rate of convergence and C is the error constant.

Theorem 1 (Maximum Principle)**.** [19] Let $u(x)$ be a solution of an advection diffusion equation with $u(0) \geq 0$ and $u(1) \geq 0$. Then $\mathbb{L}(u(x)) \geq 0$ for all x on the domain $\Omega, \mathbb{L}$ be the operator. Hence $u(x) \geq 0$ for all x in $\bar{\Omega}$

Lemma 1. [5, 3] Let $\mathbf{H}$ be the space of all Hermite interpolating polynomials of order 5 defined on the interval $0 \leq x \leq 1$. Then

$$\sum_{m=1}^{6} |H_m(x)| \leq 6, \qquad \text{for all } x \in [0,1].$$

Theorem 2. [12] Let $\mathbf{H}$ be the space of all quintic Hermite interpolating polynomials H of function $u(x)$ defined on $[a,b]$. Then the rate of convergence of quintic Hermite interpolation on $[a,b]$ is of order 6. Moreover,

$$||u^{(n)}(x) - H^{(n)}(x)|| \leq C\gamma_n h^{6-n}, \qquad n = 0, 1, \ldots, 5 \tag{14}$$

where, the values of γ_n are given in [12].

Theorem 3. [5, 3] Let $U(x,t_j)$ be the quintic Hermite spline interpolate of $u(x,t)$ from the space $\mathbf{H}$ of all Hermite interpolating polynomials of order 5 defined on the interval $0 \leq x \leq 1$ be the quintic Hermite spline interpolate of $u(x,t)$, such that $P_t(x,t_j) \in C^6([a,b])$. Then the uniform error estimate is given by

$$||u(x,t_j) - U(x,t_j)||_\infty \leq C(\triangle t + h^4). \tag{15}$$

6 Numerical implementation

To check the applicability of FDQHCM, numerical results have been compared to the analytic results. Stability of the proposed technique has been checked by L_2-norm and L_∞-norm as follows:

$$L_2\text{-norm}, \quad ||u||_2 = \sqrt{\sum_{\gamma} h \sum_{i} w_i (u(\xi_i,t) - u^\gamma(\xi_i,t))^2}, \tag{16}$$

where $w_i's$ represents the weight function corresponding to the collocation points and

$$L_\infty\text{-norm}, \quad ||u||_\infty = \max_{0 \leq \xi \leq 1} |u(\xi_i,t) - u^\gamma(\xi_i,t)|, \tag{17}$$

where $u(\xi_i,t)$ is the analytic solution and $u^\gamma(\xi_i,t)$ is the numerical solution obtained from FDQHCM.

Problem 1. Consider the GBHE given in equation (1) with $f(u) = \beta u(1 - u^\delta)(u^\delta - \bar{\gamma})$. The exact solution to the given equation is

$$u(x,t) = \left(\frac{\bar{\gamma}}{2} + \frac{\bar{\gamma}}{2}\tanh(a_1(x - a_2 t))\right)^{1/\delta}, \tag{18}$$

where $a_1 = \frac{\bar{\gamma}}{4(1+\delta)}(-\mu\delta + \delta\sqrt{\mu^2 + 4\beta(1+\delta)})$ and
$a_2 = \frac{2\mu\bar{\gamma} - (1+\delta-\bar{\gamma})(-\mu+\sqrt{\mu^2+4\beta(1+\delta)})}{2(1+\delta)}$.
The initial and boundary conditions can be derived from the exact solution. From Tables 4–8, numerical values obtained from the FDQHCM have been compared to the exact values and of the results obtained from literature [13, 15, 23].

Table 4: Comparison of results for $\mu = 1$, $\beta = 1$, $\delta = 1$, and $\gamma = 0.001$

x	t	Exact	FDQHCM	Absolute Error	[13]	[23]	[15]
0.1	0.05	0.000500019	0.0005000110	7.9750×10^{-9}	7.7006×10^{-9}	7.72768×10^{-9}	1.9372×10^{-7}
	0.1	0.000500025	0.0005000137	1.1300×10^{-8}	1.1268×10^{-8}	1.12968×10^{-8}	3.8743×10^{-7}
	1	0.000500137	0.0005001205	1.6478×10^{-8}	1.6863×10^{-8}	1.68647×10^{-8}	3.8750×10^{-6}
0.5	0.05	0.000500069	0.0005000514	1.7607×10^{-8}	1.7284×10^{-8}	1.73534×10^{-8}	1.9373×10^{-7}
	0.1	0.000500075	0.0005000462	2.8837×10^{-8}	2.8738×10^{-8}	2.88305×10^{-8}	3.8746×10^{-7}
	1	0.000500187	0.0005001405	4.6463×10^{-8}	4.6841×10^{-8}	4.68491×10^{-8}	3.8753×10^{-6}
0.9	0.05	0.000500119	0.0005001110	7.9760×10^{-9}	7.7006×10^{-9}	7.72823×10^{-9}	1.9375×10^{-7}
	0.1	0.000500125	0.0005001137	1.1301×10^{-8}	1.1268×10^{-8}	1.12980×10^{-8}	3.8749×10^{-7}
	1	0.000500237	0.0005002205	1.6481×10^{-8}	1.6863×10^{-8}	1.68669×10^{-8}	3.8756×10^{-6}

Table 5: Comparison of results for $\mu = 1$, $\beta = 1$, and $\bar{\gamma} = 0.001$

		$\delta = 2$			$\delta = 3$		
x	t	Exact	FDQHCM	Absolute Error	Exact	FDQHCM	Absolute Error
0.1	0.05	0.0223614813	0.0223611210	3.60349×10^{-7}	0.0793728109	0.0793714961	1.31479×10^{-6}
	0.1	0.0223617974	0.0223612705	5.26908×10^{-7}	0.0793740199	0.0793720972	1.92270×10^{-6}
	1	0.0223674857	0.0223666991	7.86609×10^{-7}	0.0793957760	0.0793929065	2.86954×10^{-6}
0.5	0.05	0.0223634233	0.0223626133	8.09958×10^{-7}	0.0793790070	0.0793760514	2.95560×10^{-6}
	0.1	0.0223637393	0.0223623937	1.34559×10^{-6}	0.0793802158	0.0793753057	4.91013×10^{-6}
	1	0.0223694271	0.0223672410	2.18610×10^{-6}	0.0794019686	0.0793939936	7.97497×10^{-6}
0.9	0.05	0.0223653650	0.0223650047	3.60299×10^{-7}	0.0793852021	0.0793838875	1.31464×10^{-6}
	0.1	0.0223656810	0.0223651541	5.26903×10^{-7}	0.0793864108	0.0793844882	1.92259×10^{-6}
	1	0.0223713683	0.0223705817	7.86614×10^{-7}	0.0794081601	0.0794052905	2.86962×10^{-6}

It is observed that results obtained from the FDQHCM are at par with [13] and are better than [15, 23]. It is also observed that the FDQHCM gives error varying from 10^{-9} to 10^{-5} for varying values of μ, β, δ, and $\bar{\gamma}$. The results obtained from the FDQHCM appear to be more consistent than of [13]. In Table 9, L_2-norm and L_∞-norm have been calculated for $\mu = \beta = 1$,

Table 6: Comparison of results for $\mu = 1$, $\beta = 1$, and $\bar{\gamma} = 0.001$

		$\delta = 4$			$\delta = 8$		
x	t	Exact	FDQHCM	Absolute Error	Exact	FDQHCM	Absolute Error
0.1	0.05	0.1495399548	0.149537428	2.52679×10^{-6}	0.38670979	0.386702959	6.83447×10^{-6}
	0.1	0.1495423528	0.149538658	3.69481×10^{-6}	0.38671673	0.386706734	9.99399×10^{-6}
	1	0.1495854975	0.149579985	5.51247×10^{-6}	0.38684140	0.386826506	1.48943×10^{-5}
0.5	0.05	0.1495506669	0.149544987	5.67995×10^{-6}	0.38673162	0.386716259	1.53640×10^{-5}
	0.1	0.1495530645	0.149543629	9.43545×10^{-6}	0.38673855	0.386713032	2.55228×10^{-5}
	1	0.1495961998	0.149580879	1.53208×10^{-5}	0.38686318	0.386821783	4.13948×10^{-5}
0.9	0.05	0.1495613768	0.149558851	2.52579×10^{-6}	0.38675344	0.386746612	6.83197×10^{-6}
	0.1	0.1495637738	0.149560080	3.69379×10^{-6}	0.38676037	0.386750381	9.99200×10^{-6}
	1	0.1496068999	0.149601387	5.51287×10^{-6}	0.38688495	0.386870055	1.489155×10^{-5}

Table 7: Comparison of results for $\mu = 1$, $\beta = 1$, $\delta = 16$, and $\bar{\gamma} = 0.001$

x	t	Exact	FDQHCM	Absolute Error
0.1	0.05	0.6218689	0.6218574	1.14208×10^{-5}
	0.1	0.6218811	0.6218644	1.66988×10^{-5}
	1	0.6221000	0.6220752	2.48186×10^{-5}
0.5	0.05	0.6218956	0.6218699	2.56702×10^{-5}
	0.1	0.6219078	0.6218651	4.26434×10^{-5}
	1	0.6221265	0.6220576	6.89814×10^{-5}
0.9	0.05	0.6219223	0.6219108	1.14113×10^{-5}
	0.1	0.6219344	0.6219178	1.66897×10^{-5}
	1	0.6221531	0.6221283	2.48109×10^{-5}

Table 8: Comparison of results for $\mu = 0$, $\beta = 1$, $\delta = 1$, and $\bar{\gamma} = 0.001$

x	t	Exact	FDQHCM	Absolute Error	[13]	[23]	[15]
0.1	0.05	0.000500030	0.00050001989	1.0286×10^{-8}	1.0269×10^{-8}	1.0303×10^{-8}	1.8747×10^{-7}
	0.1	0.000500043	0.00050002762	1.5044×10^{-8}	1.5027×10^{-8}	1.5063×10^{-8}	3.7493×10^{-7}
	1	0.000500268	0.00050024509	2.2466×10^{-8}	2.2488×10^{-8}	2.2488×10^{-8}	3.7500×10^{-6}
0.5	0.05	0.000500101	0.00050007775	2.3132×10^{-8}	2.3049×10^{-8}	2.3138×10^{-8}	1.8749×10^{-7}
	0.1	0.000500113	0.00050007495	3.8429×10^{-8}	3.8324×10^{-8}	3.8441×10^{-8}	3.7498×10^{-7}
	1	0.000500338	0.00050027582	6.2443×10^{-8}	6.2465×10^{-8}	6.2465×10^{-8}	3.7504×10^{-6}
0.9	0.05	0.000500172	0.00050016131	1.0283×10^{-8}	1.0269×10^{-8}	1.0303×10^{-8}	1.8751×10^{-7}
	0.1	0.000500184	0.00050016904	1.5047×10^{-8}	1.5027×10^{-8}	1.5063×10^{-8}	3.7502×10^{-7}
	1	0.000500409	0.00050038651	2.2464×10^{-8}	2.2488×10^{-8}	2.2488×10^{-8}	3.7509×10^{-6}

$\bar{\gamma} = 0.001$, and for $\delta = 1, 2, 3$, respectively. Both the norms are lying within the range of 10^{-8} to 10^{-6}.

Table 9: L_2-norm and L_∞-norm for $\mu = 1$, $\beta = 1$, and $\bar{\gamma} = 0.001$

t	$L_2 - norm$			L_∞-norm		
	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 1$	$\delta = 2$	$\delta = 3$
0.1	6.328299×10^{-9}	2.954021×10^{-7}	1.077846×10^{-6}	2.880650×10^{-8}	1.344790×10^{-6}	4.906733×10^{-6}
0.25	9.224090×10^{-9}	4.305708×10^{-7}	1.571002×10^{-6}	4.271075×10^{-8}	1.993759×10^{-6}	7.274619×10^{-6}
0.5	1.000496×10^{-8}	4.669971×10^{-7}	1.703806×10^{-6}	4.645850×10^{-8}	2.168583×10^{-6}	7.911991×10^{-6}

From Figures 4–7, three-dimensional behavior of $u(x,t)$ has been presented for $\mu = \beta = 1$, $\bar{\gamma} = 0.001$, and $\delta = 1, 2, 3, 4$, respectively, which shows that the values are lying between 0 and 1 for time ranging from 0 to 0.01.

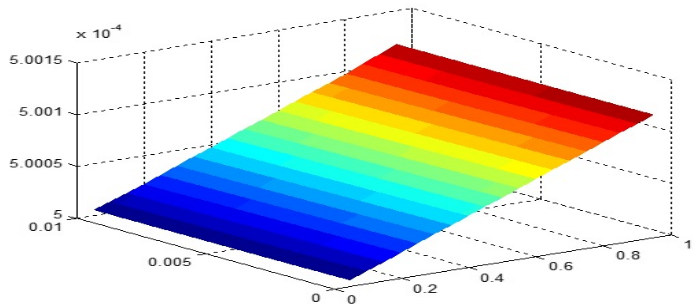

Figure 4: Three-dimensional behavior of $u(x,t)$ for $\mu = 1$, $\beta = 1$, $\gamma = 0.001$, and $\delta = 1$.

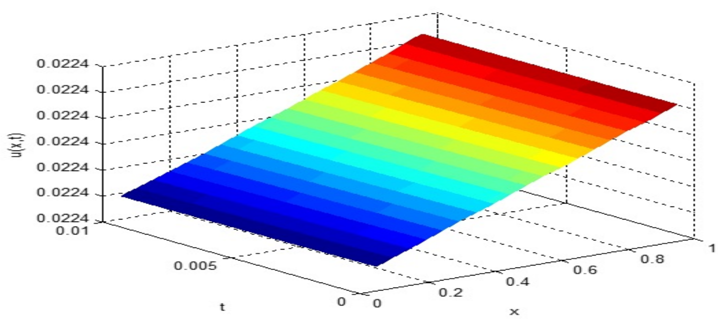

Figure 5: Three-dimensional behavior of $u(x,t)$ for $\mu = 1$, $\beta = 1$, $\gamma = 0.001$, and $\delta = 2$.

Problem 2. Consider the GBFE given in equation (1) for $\varepsilon = 1$ with $f(u) = \beta u(1 - u^{\delta})$. The exact solution for the GBFE is

$$u(x,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh(a_3(x - a_4 t))\right)^{1/\delta}, \tag{19}$$

where $a_3 = \frac{-\mu\delta}{2(1+\delta)}$ and $a_4 = \frac{\mu^2 + \beta(1+\delta)^2}{\mu(1+\delta)}$. The initial condition and boundary conditions can be obtained from the exact solution. From Tables 10–17, numerical values obtained from the FDQHCM have been compared to the exact values and the results obtained from [13, 23]. It is observed that results obtained from the FDQHCM are at par with the results obtained from [13] and are better than [22]. However, in Tables 10 and 17, results by [13] are slightly better than the FDQHCM, but this is overcome by the simplicity of

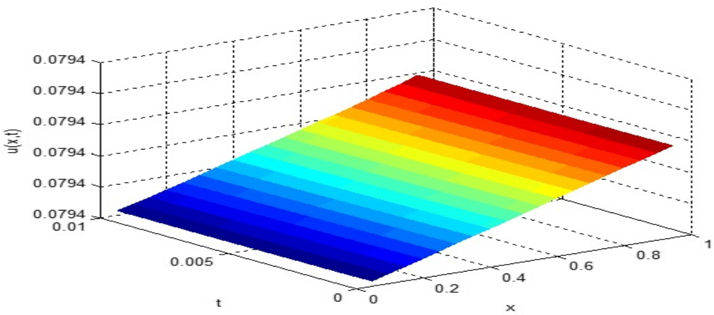

Figure 6: Three-dimensional behavior of $u(x,t)$ for $\mu = 1$, $\beta = 1$, $\gamma = 0.001$, and $\delta = 3$.

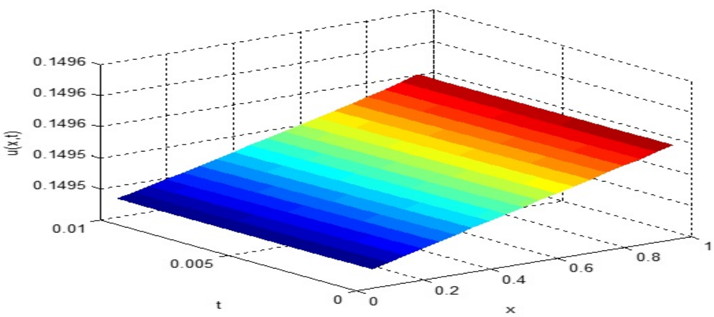

Figure 7: Three-dimensional behavior of $u(x,t)$ for $\mu = 1$, $\beta = 1$, $\gamma = 0.001$, and $\delta = 4$.

the FDQHCM as compared to the technique of former. The absolute error varies from 10^{-11} to 10^{-5} for varying values of μ, β, and δ. In Table 18, L_2-norm and L_∞-norm have been calculated for $\mu = \beta = 0.001$ and $\delta = 1, 2, 3$, respectively.

Table 10: Comparison of results for $\mu = 0.001$, $\beta = 0.001$, and $\delta = 1$

x	t	Exact	FDQHCM	Absolute Error	[13]	[22]	[15]
0.1	0.001	0.499988	0.499988	9.999×10^{-10}	5.8147×10^{-11}	1.01×10^{-7}	9.68763×10^{-6}
	0.005	0.499989	0.499989	6.999×10^{-9}	2.6058×10^{-10}	4.38×10^{-7}	1.93753×10^{-6}
	0.01	0.499990	0.499990	1.100×10^{-8}	4.4599×10^{-10}	7.53×10^{-7}	1.93752×10^{-5}
0.5	0.001	0.499938	0.499938	4.460×10^{-13}	5.6241×10^{-11}	1.04×10^{-7}	9.68691×10^{-6}
	0.005	0.499939	0.499939	9.310×10^{-13}	3.0621×10^{-10}	5.21×10^{-7}	1.93738×10^{-6}
	0.01	0.499940	0.499940	9.985×10^{-10}	6.1867×10^{-10}	1.04×10^{-6}	1.93738×10^{-5}
0.9	0.001	0.499888	0.499888	9.980×10^{-10}	5.8135×10^{-11}	1.01×10^{-7}	9.68619×10^{-6}
	0.005	0.499889	0.499889	6.998×10^{-9}	2.6053×10^{-10}	4.38×10^{-7}	1.93724×10^{-6}
	0.01	0.499890	0.499890	1.100×10^{-8}	4.4591×10^{-10}	7.53×10^{-7}	1.93724×10^{-5}

Table 11: Comparison of results for $\mu = 0.001$, $\beta = 0.001$

		$\delta = 2$			$\delta = 3$		
x	t	Exact	FDQHCM	Absolute Error	Exact	FDQHCM	Absolute Error
0.1	0.001	0.70709535	0.70709535	4.65367×10^{-10}	0.79369100	0.79369100	5.3640×10^{-10}
	0.005	0.70709676	0.70709677	1.02299×10^{-8}	0.79369259	0.79369260	1.1100×10^{-8}
	0.01	0.70709853	0.70709855	1.54395×10^{-8}	0.79369457	0.79369459	1.7813×10^{-8}
0.5	0.001	0.70704821	0.70704821	2.5057×10^{-10}	0.79365131	0.79365131	4.9914×10^{-10}
	0.005	0.70704962	0.70704962	7.9174×10^{-11}	0.79365290	0.79365290	9.5943×10^{-11}
	0.01	0.70705139	0.70705139	1.2627×10^{-11}	0.79365489	0.79365489	4.1912×10^{-10}
0.9	0.001	0.70700106	0.70700106	1.17716×10^{-9}	0.79361162	0.79361162	1.4294×10^{-9}
	0.005	0.70700248	0.70700249	1.07532×10^{-8}	0.79361321	0.79361322	1.1675×10^{-8}
	0.01	0.70700424	0.70700426	1.57272×10^{-8}	0.79361519	0.79361521	1.6992×10^{-8}

Table 12: Comparison of results for $\mu = 0.001$, $\beta = 0.001$, and $\delta = 4$

x	t	Exact	FDQHCM	Absolute Error	[22]
0.1	0.001	0.84088843	0.84088843	1.37598×10^{-9}	1.75×10^{-8}
	0.005	0.84089011	0.84089012	1.25399×10^{-8}	7.37×10^{-7}
	0.01	0.84089221	0.84089223	1.82591×10^{-8}	1.27×10^{-6}
0.5	0.001	0.84085479	0.84085479	2.0918×10^{-10}	1.75×10^{-8}
	0.005	0.84085647	0.84085647	1.7133×10^{-10}	8.77×10^{-7}
	0.01	0.84085857	0.84085857	6.3821×10^{-10}	1.75×10^{-6}
0.9	0.001	0.84082114	0.84082115	1.0779×10^{-9}	1.75×10^{-8}
	0.005	0.84082283	0.84082284	1.1838×10^{-8}	7.38×10^{-7}
	0.01	0.84082493	0.84082495	1.8053×10^{-8}	1.27×10^{-6}

Table 13: Comparison of results for $\mu = 0.001$ and $\beta = 0.001$

		$\delta = 8$			$\delta = 16$		
x	t	Exact	FDQHCM	Absolute Error	Exact	FDQHCM	Absolute Error
0.1	0.001	0.91699941	0.91699941	8.421940×10^{-10}	0.9576009	0.9576009	1.023×10^{-9}
	0.005	0.91700124	0.91697903	2.222702×10^{-5}	0.9576028	0.9576029	4.177×10^{-8}
	0.01	0.91700353	0.91695864	4.491036×10^{-5}	0.9576052	0.9576053	4.778×10^{-8}
0.5	0.001	0.91697903	0.91700124	2.221422×10^{-5}	0.9575897	0.9575897	2.435×10^{-8}
	0.005	0.91698086	0.91698086	3.297860×10^{-10}	0.9575916	0.9575916	2.357×10^{-10}
	0.01	0.91698315	0.91696048	2.267573×10^{-5}	0.9575940	0.9575940	1.435×10^{-8}
0.9	0.001	0.91695864	0.91700353	4.488876×10^{-5}	0.9575784	0.9575784	6.336×10^{-9}
	0.005	0.91696048	0.91698315	2.266323×10^{-5}	0.9578803	0.9575803	2.227×10^{-8}
	0.01	0.91696277	0.91696277	1.947571×10^{-8}	0.9575827	0.9575827	2.092×10^{-8}

Table 14: Comparison of absolute errors for $\mu = 0.1$ and $\beta = -0.0025$

		$\delta = 2$			$\delta = 4$		
x	t	FDQHCM	[13] &	[22]	FDQHCM	[13]	[22]
0.1	0.1	1.9313×10^{-5}	1.76638×10^{-5}	1.121×10^{-5}	5.9667×10^{-6}	1.26230×10^{-5}	1.343×10^{-5}
	0.3	7.0917×10^{-5}	2.51379×10^{-5}	1.600×10^{-5}	2.1779×10^{-5}	1.79797×10^{-5}	1.919×10^{-5}
	0.5	1.2556×10^{-5}	2.61751×10^{-5}	1.667×10^{-5}	3.8540×10^{-5}	1.87212×10^{-5}	2.001×10^{-5}
0.5	0.1	6.4389×10^{-6}	4.49179×10^{-5}	2.904×10^{-5}	2.0158×10^{-6}	3.21358×10^{-5}	3.489×10^{-5}
	0.3	5.0445×10^{-5}	6.91014×10^{-5}	4.468×10^{-5}	1.5604×10^{-5}	4.94694×10^{-5}	5.373×10^{-5}
	0.5	1.0433×10^{-4}	7.24595×10^{-5}	4.687×10^{-5}	3.2214×10^{-5}	5.18702×10^{-5}	5.641×10^{-5}
0.9	0.1	1.9529×10^{-5}	1.75391×10^{-5}	1.154×10^{-5}	6.1067×10^{-6}	1.25646×10^{-5}	1.393×10^{-5}
	0.3	7.1767×10^{-5}	2.50111×10^{-5}	1.643×10^{-5}	2.2305×10^{-5}	1.7920×10^{-5}	1.981×10^{-5}
	0.5	1.2709×10^{-4}	2.60482×10^{-5}	1.711×10^{-5}	3.9456×10^{-5}	1.86610×10^{-5}	2.065×10^{-5}

Table 15: Comparison of absolute errors for $\mu = 0.1$, $\beta = -0.0025$, and $\delta = 8$

x	t	FDQHCM	[13]	[22]
0.1	0.1	4.8295×10^{-7}	7.65875×10^{-6}	1.471×10^{-5}
	0.3	1.5645×10^{-6}	1.09129×10^{-5}	2.107×10^{-5}
	0.5	2.6947×10^{-6}	1.13630×10^{-5}	2.203×10^{-5}
0.5	0.1	1.8448×10^{-7}	1.95143×10^{-5}	3.832×10^{-5}
	0.3	1.1484×10^{-6}	3.00445×10^{-5}	5.911×10^{-5}
	0.5	2.2738×10^{-6}	3.15019×10^{-5}	6.218×10^{-5}
0.9	0.1	4.9302×10^{-7}	7.63553×10^{-5}	1.533×10^{-5}
	0.3	1.6078×10^{-6}	1.08887×10^{-5}	2.183×10^{-5}
	0.5	2.7772×10^{-6}	1.13383×10^{-5}	2.280×10^{-5}

Table 16: Comparison of absolute errors for $\mu = 1$ and $\beta = 0$

		$\delta = 3$		$\delta = 8$	
x	t	FDQHCM	[13]	FDQHCM	[13]
0.1	0.0001	2.3559×10^{-10}	5.7870×10^{-11}	8.59409×10^{-11}	2.7729×10^{-11}
	0.0005	2.6056×10^{-8}	2.0870×10^{-5}	1.35615×10^{-8}	1.0782×10^{-5}
	0.001	6.7400×10^{-7}	4.5521×10^{-5}	3.86425×10^{-7}	2.3520×10^{-5}
0.5	0.0001	4.6311×10^{-10}	5.1671×10^{-11}	9.37×10^{-11}	2.0409×10^{-11}
	0.0005	2.6853×10^{-10}	1.7895×10^{-5}	4.82443×10^{-10}	9.4066×10^{-6}
	0.001	4.8165×10^{-10}	4.0262×10^{-5}	2.49371×10^{-10}	2.1163×10^{-5}
0.9	0.0001	4.5141×10^{-10}	5.3143×10^{-11}	1.40942×10^{-10}	1.8090×10^{-11}
	0.0005	2.9347×10^{-8}	1.6546×10^{-5}	1.7873×10^{-8}	8.7305×10^{-6}
	0.001	7.5936×10^{-7}	3.6150×10^{-5}	4.92825×10^{-7}	1.9075×10^{-5}

Table 17: Comparison of absolute errors for $\mu = 1$ and $\beta = 1$

		$\delta = 1$	$\delta = 2$		
x	t	FDQHCM	FDQHCM	[13]	[22]
0.1	0.0001	1.2969×10^{-6}	3.03026×10^{-10}	4.4788×10^{-10}	1.55×10^{-5}
	0.0005	5.9286×10^{-5}	9.30236×10^{-8}	2.4700×10^{-5}	7.62×10^{-5}
	0.001	1.9592×10^{-4}	1.62366×10^{-6}	5.3872×10^{-5}	1.50×10^{-4}
0.5	0.0001	5.9153×10^{-10}	1.99555×10^{-10}	7.1278×10^{-11}	1.83×10^{-5}
	0.0005	1.5519×10^{-10}	2.98812×10^{-10}	2.0953×10^{-5}	9.14×10^{-5}
	0.001	9.1902×10^{-8}	2.91233×10^{-10}	4.7151×10^{-5}	1.83×10^{-4}
0.9	0.0001	1.1642×10^{-6}	2.34912×10^{-10}	5.5348×10^{-10}	2.07×10^{-5}
	0.0005	5.3457×10^{-5}	9.77826×10^{-8}	1.9314×10^{-5}	1.02×10^{-4}
	0.001	1.7683×10^{-4}	1.70078×10^{-6}	4.2211×10^{-5}	2.00×10^{-4}

Table 18: L_2-norm and L_∞-norm for $\mu = 0.001$ and $\beta = 0.001$

t	$L_2 - norm$			L_∞-norm		
	$\delta = 1$	$\delta = 2$	$\delta = 3$	$\delta = 1$	$\delta = 2$	$\delta = 3$
0.01	6.996123×10^{-9}	8.760110×10^{-9}	9.773331×10^{-9}	4.937500×10^{-8}	6.633549×10^{-8}	7.056646×10^{-8}
0.05	1.243733×10^{-8}	1.776084×10^{-8}	2.003356×10^{-8}	6.687500×10^{-8}	1.007198×10^{-7}	1.096476×10^{-7}
0.1	1.680553×10^{-8}	2.385368×10^{-8}	2.735586×10^{-8}	7.375000×10^{-8}	1.090978×10^{-7}	1.229183×10^{-7}

Both the norms are lying within the range of 10^{-8} to 10^{-7}. From Figures 8–11, three-dimensional behavior of $u(x,t)$ has been presented for $\mu = \beta = 0.001$ and $\delta = 1, 2, 3, 4$, which shows that the values are lying between 0 and 1 for time ranging from 0 to 0.01.

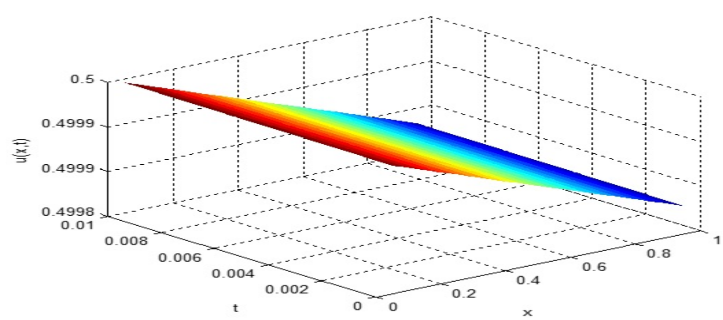

Figure 8: Three-dimensional behavior of $u(x,t)$ for $\mu = 0.001$, $\beta = 0.001$, $\gamma = 0.001$, and $\delta = 1$.

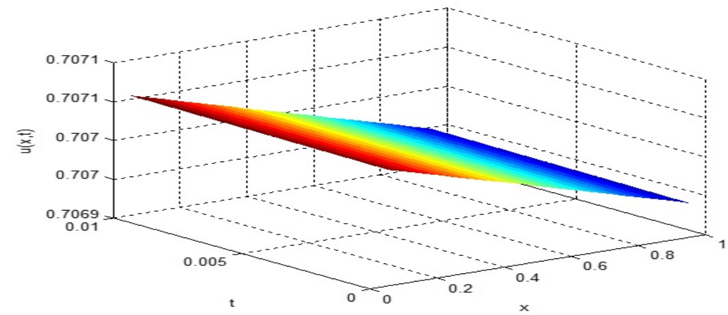

Figure 9: Three-dimensional behavior of $u(x,t)$ for $\mu = 0.001$, $\beta = 0.001$, $\gamma = 0.001$, and $\delta = 2$.

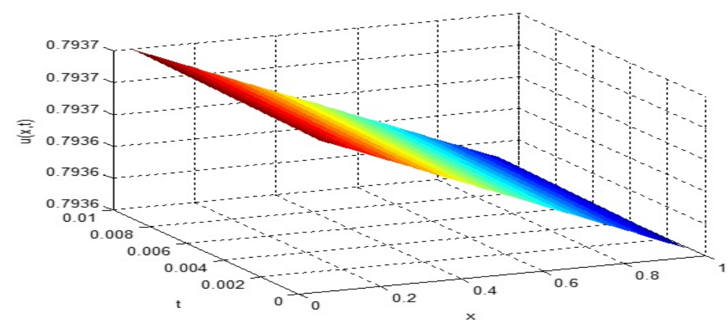

Figure 10: Three-dimensional behavior of $u(x,t)$ for $\mu = 0.001$, $\beta = 0.001$, $\gamma = 0.001$, and $\delta = 3$.

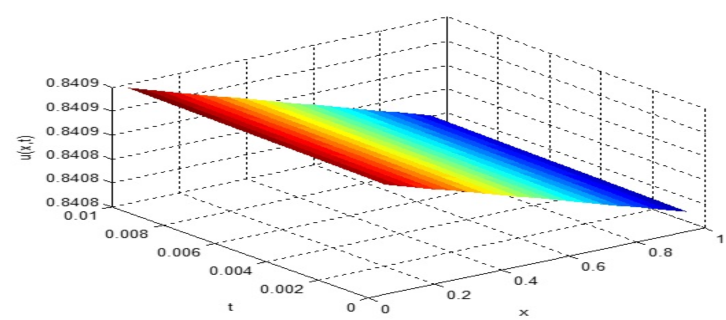

Figure 11: Three-dimensional behavior of $u(x,t)$ for $\mu = 0.001$, $\beta = 0.001$, $\gamma = 0.001$, and $\delta = 4$.

7 Conclusions

The technique of the FDQHCM has been implemented successfully on the GBHE and GBFE. The technique is a combination of the weighted residual method and the finite difference method, which gives stability to the numerical results and is easily adaptable to computer codes. Numerical results have been calculated for a vast range of parameters, even for those parameters not been calculated earlier. The technique is found to be more consistent than [13, 15, 22, 23]. It was also shown that the FDQHCM could be applied to nonlinear partial differential equations of higher order too.

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How to cite this article

Kaur, I., Arora, S. and Bala, I., An improvised technique of quintic hermite splines to discretize generalized Burgers–Huxley type equations. *Iran. j. numer. anal. optim.*, 2023; 13(1): 59-79. https://doi.org/10.22067/ijnao.2022.75871.1120.