Iranian Journal of Numerical Analysis and Optimization
Vol. 9, No. 2, (2019), pp 165–183
DOI:10.22067/ijnao.v9i2.74000

Research Article

Solving linear optimal control problems of the time-delayed systems by Adomian decomposition method

S.M. Mirhosseini-Alizamini*

Abstract

We apply the Adomian decomposition method (ADM) to obtain a suboptimal control for linear time-varying systems with multiple state and control delays and with quadratic cost functional. In fact, the nonlinear two-point boundary value problem, derived from Pontryagin's maximum principle, is solved by ADM. For the first time, we present here a convergence proof for ADM. In order to use the proposed method, a control design algorithm with low computational complexity is presented. Through the finite iterations of algorithm, a suboptimal control law is obtained for the linear time-varying multi-delay systems. Some illustrative examples are employed to demonstrate the accuracy and efficiency of the proposed methods.

AMS(2010): 49N05; 93C05.

Keywords: Multiple time-delay systems; Pontryagin's maximum principle; Adomian decomposition method.

1 Introduction

Optimal control of time-delay systems is one of the most challenging mathematical problems in control theory. Indeed, the presence of delay makes analysis and control design much more complicated. Delays frequently occur in mechanics, physics, population dynamics, biological, chemical, electronic, and transformation systems; see [14]. The theory and the application of optimal control for linear time-delay systems have been developed perfectly. However, as for nonlinear systems, synthesis problems that are solved by classical control theory lead to difficult computations. It is well known

*Corresponding author
Received 8 July 2018; revised 18 June 2019; accepted 24 June 2019
S.M. Mirhosseini-Alizamini
Department of Mathematics, Payame Noor University (PNU), Tehran, Iran. e-mail: m_mirhosseini@pnu.ac.ir

that the nonlinear optimal control time-delay systems can be reduced to a two-point boundary value problem (TPBVP) involving both delay and advance delay terms, implementing Pontryagin's maximum principle (PMP); see [15]. In general, this TPBVP cannot be solved exactly and most researches have been devoted to finding an approximate solution, for nonlinear TPBVP. We briefly review some recent papers that are relevant to the method developed in the current work for time-delay optimal control problem. An averaging approximations for time-delay optimal control problems [5], the B-spline approximation scheme [8,24], the PMP [9], variational iteration method (VIM) [25–28], a novel feedforward-feedback suboptimal control of linear time-delay systems [13], Haar wavelets approach [29], hybrid of block-pulse functions and orthonormal basis [7, 11, 19, 22, 30], composite Chebyshev finite difference method [20], the Hamilton-Jacobi-Bellman equation [6], a delay-dependent stability of neutral systems [4], an interior-point algorithm [34,35], and an embedding process that transfers the problem to a new optimal measure problem [17].

The topic of the Adomian decomposition method (ADM) has been rapidly growing in recent years. It was first proposed by Adomian [1, 2]. In this method, the solution of functional equations is considered as the sum of an infinite series usually converging to the solution. A lot of research works have been conducted recently in applying this method to a class of linear and nonlinear partial differential equations; see [33]. The Adomian's decomposition has many advantages: It does not require any kind of discretization, linearization, or perturbation of the variables and the equation, therefore it does not need any modification of the actual model that could change the solution; it is efficient on providing an approximate or even exact solution in a closed form, to linear and nonlinear problems and provides a fast and accurate convergent series, and therefore it is only necessary to calculate a few terms of the series in order to obtain a reliable approximate solution; the method depends only on the known function $u_0(t)$ and the algorithm is of simple implementation. The method, has been widely applied to solve nonlinear problems, and different modifications are suggested to overcome the demerits arising in the solution procedure [3,32].

This paper concerns with a class of nonlinear quadratic optimal control problem with multi-delay systems. Applying the main ideas of the shooting method to the basic and also an ADM. By applying the necessary optimality conditions, we obtain iterative formulas for the ADM. By using the finite-step iteration of algorithm, we obtain a suboptimal control law. The convergence of the ADM is studied and for illustrating the effectiveness of these methods, some test problems are investigated. Four illustrative examples are given to demonstrate the simplicity and efficiency of the proposed method.

The structure of this paper is arranged as follows: Section 2 is devoted to Pontryagin's maximum principle used for solving linear time-varying multi-delay system. Section 3 is dedicated to the proposed design approach to solve a closed loop optimal control problem based on the ADM and convergence of the method is demonstrated. Section 4 is devoted to the suboptimal control strategy and algorithm for the proposed method. In Section 5, the numerical examples are simulated to show the reasonableness of our theory and demonstrate the performance of our network. Finally, we end this paper with conclusions in Section 6.

2 Problem statement and optimality conditions

Consider the following linear time-varying multi-delay system:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + A_1(t)x(t-\tau_x) + B(t)u(t) + B_1(t)u(t-\tau_u), \\ x(t) = \phi(t), \qquad t_0 - \tau_x \leqslant t \leqslant t_0, \\ u(t) = \psi(t), \qquad t_0 - \tau_u \leqslant t \leqslant t_0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, are the state and control vectors, respectively; $A(t)$, $A_1(t)$, $B(t)$, and $B_1(t)$ are real, piecewise continuous matrices of appropriate dimensions defined on the appropriate intervals; $\phi(t)$ and $\psi(t)$ are specified initial functions; τ_x and τ_u are constant positive scalars. Here, it is assumed that system (1) is controllable and that $\tau_u < \tau_x$. Find the control signal $u(t)$ that minimizes the cost functional:

$$J = \frac{1}{2}x^T(t_f)Q_f x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right) dt, \tag{2}$$

where, the matrix $Q_f \in \mathbb{R}^{n\times n}$ is symmetric positive semidefinite, $Q(t) \in \mathbb{R}^{n\times n}$ and $R(t) \in \mathbb{R}^{m\times m}$ are chosen to be positive semidefinite and positive definite matrices, respectively.

The Hamiltonian function for the problem is

$$\begin{aligned} \mathcal{H}(x,u,\lambda,t) = \frac{1}{2}x^T(t)Q(t)x(t) + \frac{1}{2}u^T(t)R(t)u(t) \\ + \lambda^T(t)[A(t)x(t) + A_1(t)x(t-\tau_x) + B(t)u(t) + B_1(t)u(t-\tau_u)], \end{aligned} \tag{3}$$

where $\lambda(t) \in \mathbb{R}^n$ is the vector of the Lagrange multiplier. According to the necessary conditions for optimality, we can obtain the following nonlinear TPBVP [15,31]:

$$\dot{x}(t) = \begin{cases} A(t)x(t) + A_1(t)x(t-\tau_x) - (S_1(t)+S_2(t))\lambda(t) \\ -S_3(t)\lambda(t+\tau_u) - S_4(t)\lambda(t-\tau_u), \quad t_0 \leqslant t < t_f - \tau_u, \\ A(t)x(t) + A_1(t)x(t-\tau_x) - S_1(t)\lambda(t) \\ -S_4(t)\lambda(t-\tau_u), \quad t_f - \tau_u \leqslant t \leqslant t_f \end{cases} \tag{4}$$

and

$$\dot{\lambda}(t) = \begin{cases} -Q(t)x(t) - A^T(t)\lambda(t) \\ -A_1^T(t+\tau_x)\lambda(t+\tau_x), \quad t_0 \leqslant t < t_f - \tau_x, \\ -Q(t)x(t) - A^T(t)\lambda(t), \quad t_f - \tau_x \leqslant t \leqslant t_f \end{cases} \tag{5}$$

with initial conditions

$$\begin{cases} x(t) = \phi(t), \qquad t_0 - \tau_x \leq t \leqslant t_0, \\ u(t) = \psi(t), \qquad t_0 - \tau_u \leqslant t \leqslant t_0, \\ \lambda(t_f) = Q_f x(t_f), \end{cases} \tag{6}$$

where

$$\begin{aligned} S_1(t) &= B(t)R^{-1}(t)B^T(t), \\ S_2(t) &= B_1(t)R^{-1}(t-\tau_u)B_1^T(t), \\ S_3(t) &= B(t)R^{-1}(t)B_1^T(t+\tau_u), \\ S_4(t) &= B_1(t)R^{-1}(t-\tau_u)B^T(t-\tau_u), \end{aligned}$$

$x(t-\tau)$ is the time-delay term and $\lambda(t+\tau)$ is the time-advance term. Also, the optimal control law is obtained by

$$u^*(t) = \begin{cases} -R^{-1}(t)B^T(t)\lambda(t) \\ -R^{-1}(t)B_1^T(t+\tau_u)\lambda(t+\tau_u), \quad t_0 \leqslant t < t_f - \tau_u, \\ -R^{-1}(t)B^T(t)\lambda(t), \quad t_f - \tau_u \leqslant t \leqslant t_f. \end{cases} \tag{7}$$

The optimal can be implemented as a closed loop optimal if the co-state vector obtained consists of linear function of the states and a nonlinear term, which is the adjoint vector sequence, in the form

$$\lambda(t) = P(t)x(t) + g(t), \qquad \lambda(t_f) = Q_f x(t_f), \tag{8}$$

where $P(t) \in \mathbb{R}^{n\times n}$ is an unknown positive-semidefinite function matrix and $g(t) \in \mathbb{R}^n$ is the adjoint vector.

Substituting (8) into equation (4) yields

$$\begin{aligned} \dot{x}(t) &= [A(t) - S_1(t)P(t)]\,x(t) - S_1(t)g(t) + A_1(t)x(t-\tau_x) + F(t), \\ x(t) &= \phi(t), \qquad t_0 - \tau \leq t \leqslant t_0, \end{aligned} \tag{9}$$

where

$$F(t) = \begin{cases} -S_2(t)\,[P(t)x(t) + g(t)] - S_3(t)\,[P(t+\tau_u)x(t+\tau_u) + g(t+\tau_u)] \\ -S_4(t)\,[P(t-\tau_u)x(t-\tau_u) + g(t-\tau_u)]\,, \ t_0 \leqslant t < t_f - \tau_u, \\ -S_4(t)\,[P(t-\tau_u)x(t-\tau_u) + g(t-\tau_u)]\,, \ \ t_f - \tau_u \leqslant t \leqslant t_f. \end{cases} \tag{10}$$

Computing the derivatives to the both sides with respect to t of equation (8), we have

$$\begin{aligned}\dot{\lambda}(t) &= \dot{P}(t)x(t) + P(t)\dot{x}(t) + \dot{g}(t), \qquad t_0 \leqslant t \leqslant t_f \\ &= \left[\dot{P}(t) + P(t)A(t) - P(t)S_1(t)P(t)\right] x(t) - P(t)S_1(t)g(t) \\ &+ P(t)A_1(t)x(t-\tau_x) + P(t)F(t) + \dot{g}(t). \end{aligned} \tag{11}$$

Putting (8) into equation (5), we get

$$\dot{\lambda}(t) = \begin{cases} -Q(t)x(t) - A^T(t)P(t)x(t) - A^T(t)g(t) \\ -A_1^T(t+\tau_x)[P(t+\tau_x)x(t+\tau_x) + g(t+\tau_x)], t_0 \leqslant t < t_f - \tau_x, \\ -Q(t)x(t) - A^T(t)P(t)x(t) - A^T(t)g(t), t_f - \tau_x \leqslant t \leqslant t_f. \end{cases} \tag{12}$$

Thus, from (11) and (12), we can obtain the following Riccati matrix differential equation:

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) - P(t)S_1(t)P(t) + Q(t), \; P(t_f) = Q_f, \tag{13}$$

and the following adjoint vector differential equation:

$$\dot{g}(t) = -\left[A(t) - S_1(t)P(t)\right]^T g(t) - P(t)A_1(t)x(t-\tau_x) + G(t), \; g(t_f) = 0, \tag{14}$$

where

$$G(t) = \begin{cases} -A_1^T(t+\tau_x)\left[P(t+\tau_x)x(t+\tau_x) + g(t+\tau_x)\right] + P(t)S_2(t)\left[P(t)x(t) + g(t)\right] \\ -P(t)S_3(t)\left[P(t+\tau_u)x(t+\tau_u) + g(t+\tau_u)\right] \\ P(t)S_4(t)\left[P(t-\tau_u)x(t-\tau_u) + g(t-\tau_u)\right], \qquad t_0 \leqslant t < t_f - \tau_x, \\ P(t)S_2(t)\left[P(t)x(t) + g(t)\right] + P(t)S_3(t)\left[P(t+\tau_u)x(t+\tau_u) + g(t+\tau_u)\right] \\ P(t)S_4(t)\left[P(t-\tau_u)x(t-\tau_u) + g(t-\tau_u)\right], \qquad t_f - \tau_x \leqslant t < t_f - \tau_u, \\ P(t)S_4(t)\left[P(t-\tau_u)x(t-\tau_u) + g(t-\tau_u)\right], \qquad t_f - \tau_u \leqslant t \leqslant t_f. \end{cases} \tag{15}$$

Substituting (8) into (7) yields

$$u^*(t) = \begin{cases} -R^{-1}(t)B^T(t)[P(t)x(t) + g(t)] \\ -R^{-1}(t)B_1^T(t+\tau_u)[P(t+\tau_u)x(t+\tau_u) + g(t+\tau_u)], t_0 \leqslant t < t_f - \tau_u, \\ -R^{-1}(t)B^T(t)[P(t)x(t) + g(t)], t_f - \tau_u \leqslant t \leqslant t_f. \end{cases}$$

For the sake of simplicity, let us define the right hand sides of (9) and (14) as follows:

$$f_1(t,x,g) = \left[A(t) - S_1(t)P(t)\right] x(t) - S_1(t)g(t) + A_1(t)x(t-\tau_x) + F(t), \tag{16}$$

$$f_2(t,x,g) == -\left[A(t) - S_1(t)P(t)\right]^T g(t) - P(t)A_1(t)x(t-\tau_x) + G(t), \tag{17}$$

where $F(t)$ and $G(t)$ are relations (10) and (15), respectively.
Thus the TPBVP in (9) and (14) changes to

$$\begin{cases} \dot{x}(t) = f_1(t, x, g), \\ \dot{g}(t) = f_2(t, x, g), \\ x(t_0) = x_0, \quad g(t_f) = 0. \end{cases} \tag{18}$$

Note that, relations (18) form a nonlinear TPBVP with time-varying coefficient involving both delay and advance terms. The exact solution of this problem is, in general, extremely difficult, if not impossible. In the next section, we propose another analytic approximate method based on ADM, for this purpose.

3 Adomian decomposition method

In order to illustrate the basic concepts of the ADM, we consider the following equation:

$$\mathcal{L}(u) + \mathcal{R}(u) + \mathcal{N}(u) = h(t), \tag{19}$$

where $u(t)$ is an unknown function, $\mathcal{L}$ is a linear operator, that is assumed to be invertible, $\mathcal{R}$ is another linear differential operator, $\mathcal{N}(u)$ represents the nonlinear terms, and h is a continuous function. Applying the inverse operator $\mathcal{L}^{-1}$ to both sides of (19) and using the given conditions, we obtain

$$u = f - \mathcal{L}^{-1}(\mathcal{R}(u)) - \mathcal{L}^{-1}(\mathcal{N}(u)), \tag{20}$$

where the function $f(t)$ represents the terms arising from integrating the function $h(t)$ and using the initial condition.

The standard Adomian method defines the solution $u(t)$ of (19) as a series

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{21}$$

where the components $u_n(t)$ are usually determined recurrently. Substituting this infinite series into (20) leads to

$$\sum_{n=0}^{\infty} u_n(t) = f(t) - \mathcal{L}^{-1}\left(\mathcal{R}(\sum_{n=0}^{\infty} u_n(t))\right) - \mathcal{L}^{-1}\left(\mathcal{N}(\sum_{n=0}^{\infty} u_n(t))\right). \tag{22}$$

The nonlinear term in (21) can be computed by substituting

$$\mathcal{N}(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \tag{23}$$

where A_n is the Adomian polynomials, which can be determined by

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial q^n}\mathcal{N}\left[\sum_{n=0}^{\infty} q^k u_k\right]_{q=0}, \quad n = 1, 2, 3, \ldots, . \tag{24}$$

Now, substituting (23) into (22) leads to

$$\sum_{n=0}^{\infty} u_n(t) = f(t) - \mathcal{L}^{-1}\left(\mathcal{R}(\sum_{n=0}^{\infty} u_n(t))\right) - \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} A_n\right).$$

Each term of series (21) is given by the recurrent relation

$$\begin{aligned} u_o &= f(t) \\ u_n &= -\mathcal{L}^{-1}(\mathcal{R}(u_{n-1})) - \mathcal{L}^{-1}(A_{n-1}), \quad n \geq 1. \end{aligned}$$

Now, we briefly describe how to apply the ADM to systems (18). For this purpose, we use a shooting method like procedure combine with the ADM for solving TPBVP in (18).

Remark. It is necessary to transform the boundary value problem into a initial value problem. Therefore, we must find $\alpha \in \mathbb{R}$ such that the condition $g(t_f) = 0$ can be replaced by the condition $g(t_0) = \alpha$. Thus we rewrite the TPBVP (18) as follows:

$$\begin{cases} \dot{x}(t) = f_1(t, x, g), \\ \dot{g}(t) = f_2(t, x, g), \\ x(t_0) = x_0, \ \ g(t_0) = \alpha, \end{cases} \tag{25}$$

where $\alpha \in \mathbb{R}$ is an unknown parameter. This parameter will be identified after sufficient iterations of ADM.

Based on the ADM, we seek the solution $\{x, g\}$ as follows:

$$x = \lim_{N\to\infty}\sum_{n=0}^{N} x_n, \qquad g = \lim_{N\to\infty}\sum_{n=0}^{N} g_n,$$

and hence the recursive relationship is found as

$$\begin{cases} x_{n+1} = \mathcal{L}^{-1}A_{1,n}, \quad n \geq 0, \\ g_{n+1} = \mathcal{L}^{-1}A_{2,n}, \quad n \geq 0, \\ x(t_0) = x_0, \ \ g(t_0) = \alpha, \end{cases}$$

with inverse $\mathcal{L}^{-1}(.) = \int_0^t (.)dt$ and

$$f_k(t, x_n, g_n) = \sum_{n=0}^{\infty} A_{k,n}, \quad k = 1, 2,$$

where $A_{k,n}$ are the Adomian polynomials and are calculated by

$$A_{k,n} = \frac{1}{n!} \frac{\partial^n}{\partial q^n} f_k \left(t, \sum_{n=0}^{\infty} q^k x_k, \sum_{n=0}^{\infty} q^k g_k \right)_{q=0}, \quad n = 0, 1, 2, \ldots. \tag{26}$$

Take the first $n+1$ terms of the nth approximation of x and g as follows:

$$\begin{cases} \Phi_n = x_0 + \sum_{i=1}^{n} \mathcal{L}^{-1}(A_{1,i-1}), \\ \Psi_n = g_0 + \sum_{i=1}^{n} \mathcal{L}^{-1}(A_{2,i-1}). \end{cases} \tag{27}$$

Find the sequences $\Phi_n = x_0 + \cdots + x_n$ and $\Psi_n = g_0 + \cdots + g_n$ such that

$$\begin{cases} \Phi_n = x_0 + \mathcal{L}^{-1}(f_1(t, \Phi_{n-1}, \Psi_{n-1})), & n \geq 1, \\ \Psi_n = g_0 + \mathcal{L}^{-1}(f_2(t, \Phi_{n-1}, \Psi_{n-1})), & n \geq 1, \end{cases} \tag{28}$$

where $x_0(t) = x(t_0) = x_0, \;\; g_0(t) = g(t_0) = \alpha$.

Theorem 1. *Assume that $\{\sum_{k=0}^{n} x_k(t)\}$ and $\{\sum_{k=0}^{n} g_k(t)\}$ are the solution sequences produced by ADM formula (28), which converge, respectively, to $\widehat{x}(t, \alpha)$ and $\widehat{g}(t, \alpha)$, as $n \to \infty$. Then $\widehat{x}(t, \alpha), \widehat{g}(t, \alpha)$ are the exact solutions of (25). Accordingly, $\widehat{x}(t, \widehat{\alpha}), \widehat{g}(t, \widehat{\alpha})$ are the exact solutions of (18) when $\widehat{\alpha}$ is the real root of $\widehat{g}(t_f, \alpha) = 0$.*

Proof. Let consider TPBVP (25) as follows:

$$\begin{cases} x(t) = x(t_0) + \mathcal{L}^{-1}[f_1(t, x, g)], \\ g(t) = g(t_0) + \mathcal{L}^{-1}[f_2(t, x, g)], \\ x(t_0) = x_0, \;\; g(t_0) = \alpha, \end{cases}$$

where $\mathcal{L} = \dfrac{d}{dt}(\cdot)$ and $\mathcal{L}^{-1} = \int_{t_0}^{t}(\cdot)dt$. We have

$$x_n(t) = \int_{t_0}^{t} [A_{1,n-1}(x_0, x_1, \cdots, x_{n-1}, g_0, g_1, \cdots, g_{n-1})] \, ds, \;\; n \geq 1, \tag{29}$$

$$g_n(t) = \int_{t_0}^{t} [A_{2,n-1}(x_0, x_1, \cdots, x_{n-1}, g_0, g_1, \cdots, g_{n-1})] \, ds, \;\; n \geq 1, \tag{30}$$

$$x_0(t) = x(t_0) = x_0, \;\; g_0(t) = g(t_0) = \alpha.$$

Taking limits of both sides of (29) and (30) as $n \to \infty$ implies

$$\lim_{n\to\infty}\sum_{k=1}^{n} x_k(t) = \int_{t_0}^{t}\left[\lim_{n\to\infty}\sum_{k=1}^{n} A_{1,k-1}(x_0, x_1, \cdots, x_{k-1}, g_0, g_1, \cdots, g_{k-1})\right] ds,$$
$$\lim_{n\to\infty}\sum_{k=1}^{n} g_k(t) = \int_{t_0}^{t}\left[\lim_{n\to\infty}\sum_{k=1}^{n} A_{2,k-1}(x_0, x_1, \cdots, x_{k-1}, g_0, g_1, \cdots, g_{k-1})\right] ds.$$

Then

$$\widehat{x}(t,\alpha) = \int_{t_0}^{t}[f_1(s, \widehat{x}(s,\alpha), \widehat{g}(s,\alpha))]ds,$$
$$\widehat{g}(t,\alpha) = \int_{t_0}^{t}[f_2(s, \widehat{x}(s,\alpha), \widehat{g}(s,\alpha))]ds.$$

Differentiating both sides with respect to t yields

$$\dot{\widehat{x}}(t,\alpha) = f_1(t, \widehat{x}(t,\alpha), \widehat{g}(t,\alpha)),$$
$$\dot{\widehat{g}}(t,\alpha) = f_2(t, \widehat{x}(t,\alpha), \widehat{g}(t,\alpha)).$$

Moreover, if $t = t_0$, then from (29) and (30), $x_n(t_0) = 0, g_n(t_0) = 0$ for every $n \geq 1$. Thus

$$\sum_{k=0}^{n} x_k(t_0) = x_0(t_0) = x_0, \quad \sum_{k=0}^{n} g_k(t_0) = g_0(t_0) = \alpha,$$

or equivalently, $\widehat{x}(t_0,\alpha) = x_0$, $\widehat{g}(t_0,\alpha) = \alpha$. Hence, $\widehat{x}(t,\alpha)$ and $\widehat{g}(t,\alpha)$ are the exact solutions of (25). In addition, they are the exact solutions of (18), only if the condition $g(t_f) = 0$ is satisfied. So, it is straightforward to choose the unknown parameter $\alpha \in \mathcal{R}^n$ such that $\widehat{g}(t_f,\alpha) = 0$. Denoting this real root of $\widehat{g}(t_f,\alpha) = 0$ by $\widehat{\alpha}$ completes the proof. □

4 Suboptimal control design strategy

Consider the linear time-varying multi-delay system (1) with cost functional (2). Then, the Nth order suboptimal trajectory-control pair is obtained as follows:

$$\begin{cases} x_N(t) = \sum_{k=0}^{N} x_k(t), \\ g_N(t) = \sum_{k=0}^{N} g_k(t), \end{cases} \tag{31}$$

and

$$u_N(t) = \begin{cases} -R^{-1}(t)B^T(t)[P(t)x_N(t) + g_N(t)] \\ -R^{-1}(t)B_1^T(t+\tau_u)[P(t+\tau_u)x_N(t+\tau_u) \\ +g_N(t+\tau_u)], \quad t_0 \leqslant t < t_f - \tau_u, \\ -R^{-1}(t)B^T(t)[P(t)x_N(t) + g_N(t)], \quad t_f - \tau_u \leqslant t \leqslant t_f. \end{cases} \tag{32}$$

The integer N in (31) and (32) is generally determined according to a concrete control precision. Then, the following cost functional can be calculated:

$$J_N = \frac{1}{2}x_N^T(t_f)Q_f x_N(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left(x_N^T(t)Q(t)x_N(t) + u_N^T(t)R(t)u_N(t)\right) dt, \tag{33}$$

The Nth order in (31) and (32) has the desirable accuracy, if for given positive constant $\epsilon > 0$, the following condition holds jointly:

$$\left| \frac{J_N - J_{N-1}}{J_N} \right| < \epsilon, \tag{34}$$

If the tolerance error bound is chosen small enough, the Nth order suboptimal control law will be very close to $u^*(t)$, and thus, the value of cost functional in (33) and its optimal value J^* will be almost identical.

Algorithm: Suboptimal control law of system (1):

Step 1: Obtain $P(t)$ from (13). Let $x_0(t) = x(t_0) = \phi(t), g_0(t) = g(t_0)$ and $k = 1$.

Step 2: Compute $x_k(t)$ and $g_k(t)$ from (29) and (30).

Step 3: Let $N = k$ and obtain $x_N(t)$ and $u_N(t)$ from (31) and (32).

Step 4: Calculate J_N according to (33). If $\left| \frac{J_N - J_{N-1}}{J_N} \right| < \epsilon$, then stop and output $u_N(t)$, go to step 5; else, replace k by $k+1$ and go to step 2.

Step 5: Stop the algorithm; $x_N(t)$ and $u_N(t)$ are accurate enough.

5 Numerical examples

In this section, the proposed method is illustrated by some test problems. The calculations are performed by using MATLAB software.

Example 1. Consider the following linear time-varying multi-delay systems [20]:

$$\begin{cases} \dot{x}(t) = x(t-\frac{1}{2}) + tx(t-\frac{3}{4}) + u(t), & 0 \leqslant t \leqslant 1, \\ x(t) = t+1, \qquad -\frac{3}{4} \leqslant t \leqslant 0, \end{cases} \tag{35}$$

with the cost functional

$$J = \frac{3}{2}x^2(1) + \frac{1}{2}\int_0^1 u^2(t)dt. \tag{36}$$

The exact solutions of $x(t)$ and $u(t)$ are given by

$$x(t) = \begin{cases} \frac{239075}{420332}t^3 + \frac{3129081}{1681328}t^2 - \frac{1178769}{611392}t + 1, & 0 \leqslant t < \frac{1}{4}, \\ \frac{1}{3}t^3 + \frac{1119201}{840664}t^2 - \frac{680513}{420332}t + \frac{7039811}{7336704}, & \frac{1}{4} \leqslant t < \frac{1}{2}, \\ \frac{239075}{1681328}t^4 + \frac{3375959}{5043984}t^3 - \frac{20932555}{13450624}t^2 + \frac{1156163}{1222784}t + \frac{56216927}{161407488}, & \frac{1}{2} \leqslant t < \frac{3}{4}, \\ \frac{47815}{420332}t^5 + \frac{2306773}{10087968}t^4 - \frac{2461871}{2521992}t^3 + \frac{14865377}{53802496}t^2 + \frac{17419475}{26901248}t & \\ + \frac{2652913}{14673408}, & \frac{3}{4} \leqslant t \leqslant 1, \end{cases}$$

and

$$u(t) = \begin{cases} \frac{296893}{420332}t^2 + \frac{2078251}{840664}t - \frac{1484465}{611392}, & 0 \leqslant t < \frac{1}{4}, \\ \frac{296893}{210166}t - \frac{890679}{420332}, & \frac{1}{4} \leqslant t < \frac{1}{2}, \\ -\frac{296893}{210166}, & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Furthermore, the exact value of the cost functional J is given by $J = 1.70648554$.
In order to obtain an accurate enough suboptimal control law, we applied the proposed algorithm with the tolerance bounds $\epsilon = 2\times 10^{-6}$. In this case, convergence is achieved after 10 iterations, that is, $\left|\frac{J_{10} - J_9}{J_{10}}\right| = 1.757 \times 10^{-6} < 2 \times 10^{-6}$.

To analysis the accuracy and effectiveness of the ADM and VIM, the relative error of objective value is summarized in Table 1, for several iterations. Accordingly, results show that in $N = 10$, ADM converges to the exact solution. The optimal value of the cost functional is obtained as $J_{10} = 1.706485$.

We compared the results obtained from ADM and VIM with exact solution. Figure 1 shows the simulation curves of $u(t)$ at iteration time 10 and the corresponding state trajectory $x(t)$. Results of both methods are very close to exact solution as shown by Figure 1. This confirms that the proposed method yields excellent results.

Example 2. Consider the following linear time-varying multi-delay systems [7, 16, 21, 36]:

Table 1: Simulation results of Example 1 at different iteration times.

method	ADM	ADM	VIM	VIM
N	J_N	$\left\lvert\dfrac{J_N - J_{N-1}}{J_N}\right\rvert$	J_N	$\left\lvert\dfrac{J_N - J_{N-1}}{J_N}\right\rvert$
6	1.682312	–	1.678321	–
7	1.691205	5.255×10^{-2}	1.685342	4.165×10^{-2}
8	1.703156	7.016×10^{-2}	1.692335	4.132×10^{-3}
9	1.706484	1.961×10^{-2}	1.697346	2.952×10^{-3}
10	1.706485	1.757×10^{-6}	1.697349	1.767×10^{-6}

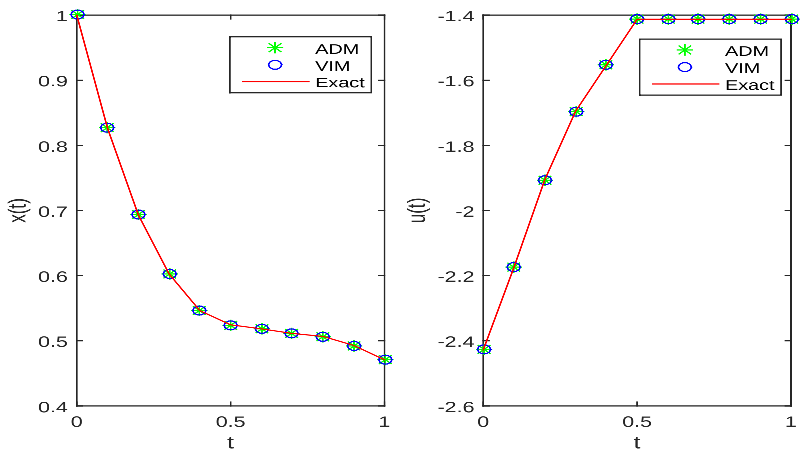

Figure 1: The suboptimal control and state, when $N = 10$ for Example 1.

$$\begin{cases} \dot{x}(t) = -x(t) + x(t - \frac{1}{3}) + u(t) - \frac{1}{2}u(t - \frac{2}{3}), \qquad 0 \leqslant t \leqslant 1, \\ x(t) = 1, \qquad -\frac{1}{3} \leqslant t \leqslant 0, \\ u(t) = 0, \qquad -\frac{2}{3} \leqslant t \leqslant 0, \end{cases} \tag{37}$$

with the cost functional

$$J = \frac{1}{2}\int_0^1 \left(x^2(t) + \frac{1}{2}u^2(t) \right)(t)dt. \tag{38}$$

According to system (1), we have $A = -1, A_1 = B = 1, B_1 = -\frac{1}{2}, Q = 1, R = \frac{1}{2}$.

In order to obtain an accurate enough suboptimal control law, we applied the proposed algorithm with the tolerance bounds $\epsilon = 8.1 \times 10^{-7}$. Simulation results at different iteration times are summarized in Table 2. From Table 2, it is observed that convergence is achieved after 10 iterations, that is, $\left|\frac{J_{10}-J_9}{J_{10}}\right| = 8.040 \times 10^{-7} < 8.1 \times 10^{-7}$.

In Table 3, the minimums of J using the hybrid of block pulse and Legendre polynomials [21], hybrid of general block pulse and Legendre polynomials [36], orthogonal basis [16], hybrid of block pulse and orthogonal Taylor series [7], and present two methods are listed. Also the suboptimal control and state of the proposed ADM and VIM are demonstrated in Figure 2.

Table 2: Simulation results of Example 2 at different iteration times

method	ADM	ADM	VIM	VIM
N	J_N	$\left\|\frac{J_N-J_{N-1}}{J_N}\right\|$	J_N	$\left\|\frac{J_N-J_{N-1}}{J_N}\right\|$
6	0.36430512	–	0.35935108	–
7	0.36512980	2.258×10^{-2}	0.36431299	1.382×10^{-2}
8	0.37311212	2.139×10^{-2}	0.37511283	2.879×10^{-2}
9	0.37311294	2.519×10^{-6}	0.37311210	5.362×10^{-3}
10	0.37311291	8.040×10^{-7}	0.37311310	2.680×10^{-6}

Table 3: The cost functional values for Example 2

Method	Cost functional values
Marzban and Razzaghi [21]	0.37311241
Wang [36]	0.37312682
Kellat [16]	0.3731123
Dadkhah and Farahi [7]	0.373112935
Proposed method ($N = 10$)	
ADM	0.37311291
VIM	0.37311310

Example 3. Consider the following linear time-varying multi-delay systems [12, 18, 23]:

$$\begin{cases} \dot{x}_1(t) = x_2(t) + x_1(t-1), \qquad t \geq 0, \\ \dot{x}_2(t) = t x_1(t) + 2x_1(t-1) + x_2(t-1) + u(t) - u(t-0.5), \\ x_1(t) = x_2(t) = 1, \qquad -1 \leqslant t \leqslant 0, \\ u(t) = 5(t+1), \qquad -0.5 \leqslant t \leqslant 0, \end{cases} \tag{39}$$

with the cost functional

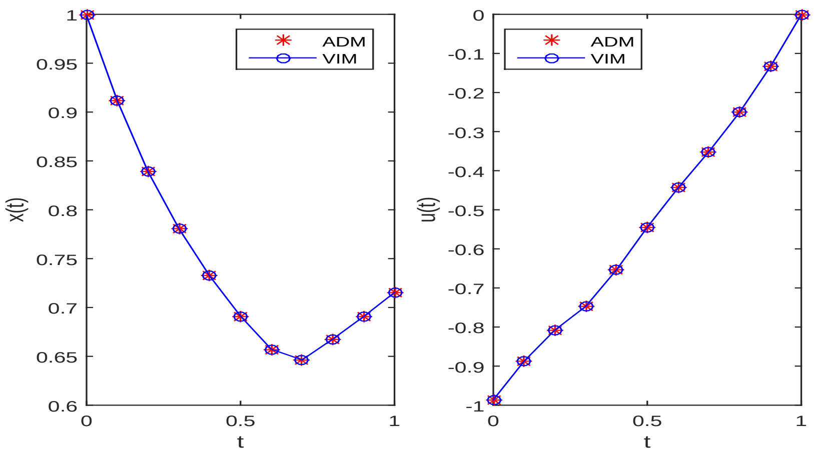

Figure 2: The suboptimal control and state, when $N = 10$ for Example 2.

$$J = \frac{1}{2}x_1^2(3) + x_2^2(3) + \frac{1}{2}\int_0^3 [2x_1^2(t) + 2x_1(t)x_2(t) + x_2^2(t) + \frac{u^2(t)}{(t+2)}]dt. \quad (40)$$

According to system (1), we have

$$A = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

and from (2), we get

$$Q_f = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad R = 1/(t+2).$$

In order to obtain an accurate enough suboptimal control law, we applied the proposed algorithm with the tolerance bounds $\epsilon = 2.3 \times 10^{-6}$. Simulation results at different iteration times are summarized in Table 4. From Table 4, it is observed that convergence is achieved after 12 iterations, that is, $\left|\frac{J_{12} - J_{11}}{J_{12}}\right| = 2.270 \times 10^{-6} < 2.3 \times 10^{-6}$. Table 5, a comparison is made between the value of J obtained by the present method with $N = 12$, together with the value of J reported in the literature: Malek-Zavarei [18], by employing an iterative approach for determining the suboptimal control of the mentioned problem; the method of Hwang and Chen [12], by employing Legendre polynomials of order 20; and the method of Marzban and Pirmoradian [23], by employing a direct approach based on a hybrid of block-pulse functions and Lagrange interpolating polynomials.

Table 4: Simulation results of Example 3 at different iteration times

method	ADM	ADM	VIM	VIM
N	J_N	$\left\lvert \frac{J_N - J_{N-1}}{J_N} \right\rvert$	J_N	$\left\lvert \frac{J_N - J_{N-1}}{J_N} \right\rvert$
8	23.21056	–	22.17650	–
9	23.05198	6.879×10^{-2}	22.15431	1.001×10^{-2}
10	22.02108	1.681×10^{-3}	22.01430	5.903×10^{-3}
11	22.02230	5.539×10^{-5}	22.02143	3.237×10^{-4}
12	22.02235	2.270×10^{-6}	22.02115	1.271×10^{-5}

Table 5: The cost functional values for Example 3

Method	Cost functional values
Malek-Zavarei [18]	24.0200500
Hwang and Chen [12]	22.0212
Marzban and Pirmoradian [23]	22.0230201080
Proposed method ($N = 12$)	
ADM	22.02235
VIM	22.02115

Example 4. We now consider the following nonlinear time-varying multi-delay systems:

$$\begin{cases} \dot{x}(t) = x(t-1)x(t-2)u(t-2), & 0 \leqslant t \leqslant 3, \\ x(t) = 1, \quad -2 \leqslant t \leqslant 0, \\ u(t) = 0, \quad -2 \leqslant t \leqslant 0, \end{cases} \tag{41}$$

with the cost functional

$$J = \int_0^3 (x^2(t) + u^2(t))dt. \tag{42}$$

This optimal control is adopted from [10,34,35]. In order to obtain an accurate enough suboptimal control law, we applied the proposed algorithm with the tolerance bounds $\epsilon = 1.2 \times 10^{-6}$. Table 6, a comparison is made between the value of J obtained by the present method with $N = 13$, together with the value of J reported in the literature. The approximate solution of $x(t)$ and $u(t)$, obtained by the proposed method with $N = 13$ and the results of PMP method generated by Gollmann et al. [10] are plotted in Figure 3. Therefore, in view of the results, the present method is quite effective.

Table 6: The cost functional values for Example 4

Method	Cost functional values
Wachter et al. [35](600 grid points)	2.763044
Vanderbei [34](600 grid points)	2.763044
Gollmann et al. [10](600 grid points)	2.761594156
Proposed method ($N = 13$)	
ADM	2.761591012
VIM	2.761592238

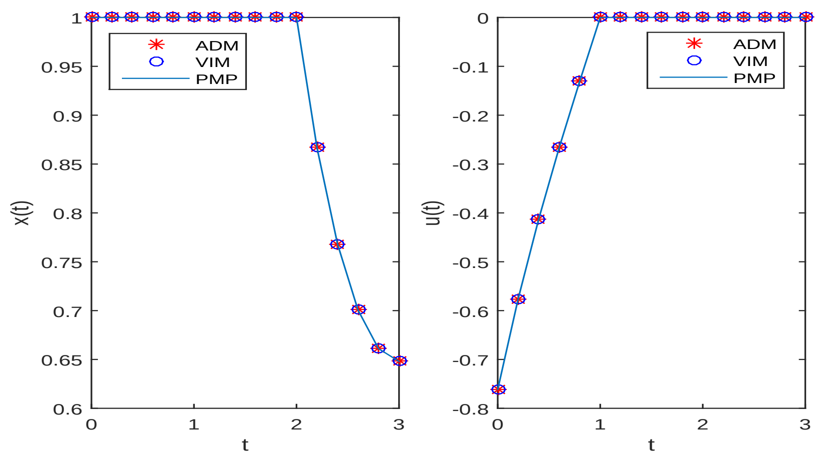

Figure 3: The suboptimal control and state, when $N = 13$ for Example 4.

6 Conclusion

In this work, the ADM has been successfully applied to find the solution of suboptimal control for linear time-varying systems with multiple state and control delays, and quadratic cost functional is presented. By using the ADM and VIM with the finite-step iteration of algorithm, we obtained a suboptimal control law. Some numerical examples have been provided to demonstrate the validity and applicability of the proposed method. The method is general and yields very accurate results.

Acknowledgements

Author is grateful to the anonymous referees and the editors for their constructive comments.

References

1. Adomian, G. *Nonlinear stochastic systems theory and applications to physics*, Kluwer Academic Publishers, 1989, Boston.

2. Adomian, G. *A review of the decomposition method in applied mathematics*, J. Math. Anal. Appl. 135(2) (1989), 501–544.

3. Alizadeh, A. and Effati, S. *Numerical schemes for fractional optimal control problems*, J. Dyn. Sys. Meas. Contr. 4 (2017), 1–14.

4. Balasubramaniam, P., Krishnasamy, R. and Rakkiyappan, R. *Delay-dependent stability of neutral systems with time-varying delays using delay-decomposition approach*, Appl. Math. Model. 36 (2012), 2253–2261.

5. Banks, H.T. and Burns, J.A. *Hereditary control problem: Numerical methods based on averaging approximations*, SIAM J. Contr. Optim. 16(2) (1978), 169–208.

6. Basin, M. and Rodriguez-Gonzalez, J. *Optimal control of linear systems with multiple time delays in control input*, IEEE Trans. Automat. Contr. 51(1) (2006), 91–97.

7. Dadkhah, M. and Farahi, M.H. *Optimal control of time delay systems via hybrid of block-pulse functions and orthogonal Taylor series*, Int. J. Appl. Comput. Math. 2(1) (2016), 137–152.

8. Edrisi-Tabriz, Y., Lakestani, M. and Heydari, A. *Two numerical methods for nonlinear constrained quadratic optimal control problems using linear B-spline functions*, Iranian J. Numer. Anal. Optim. 6(2) (2016), 17–37.

9. Gollmann, L., Kern, D. and Maurer, H. *Optimal control problems with delays in state and control variables subject to mixed control state constraints*, Optim. Contr. Appl. Meth. 30 (2009), 341–365.

10. Gollmann, L. and Maurer, H. *Theory and applications of optimal control problems with mul-tiple time delays*, J. Ind. Manag. Optim. 10(2) (2014), 413–441.

11. Haddadi, N., Ordokhani, Y. and Razzaghi, M.*Optimal control of delay systems by using a hybrid functions approximation*, J. Optim. Theor. Appl. 153 (2012), 338–356.

12. Hwang, G. and Chen, M.Y. *Suboptimal control of linear time-varying multi-delay systems via shifted Legendre polynomials*, Int. J. Syst. Sci. 16(12) (1985), 1517–1537.

13. Jajarmi, A., Dehghan-Nayyeri, M. and Saberi-Nik, H. *A novel feedforward-feedback suboptimal control of linear time-delay systems via shifted Legendre polynomials*, J. Complex. 35 (2016), 46–62.

14. Jamshidi, M. and Wang, C.M. *A computational algorithm for large-scale nonlinear time-delays systems*, IEEE Trans. Syst. Man. Cyber. SMC. 14 (1984), 2–9.

15. Kharatishvili, G.L. *The maximum principle in the theory of optimal process with time-lags*, Doklady Akademii Nauk SSSR. 136 (1961), 39–42.

16. Khellat, F. and Vasegh, N. *Suboptimal control of linear systems with delays in state and input by orthogonal basis*, Int. J. Comput. Math. 88(4) (2011), 781–794.

17. Koshkouei, A.J., Farahi, M.H. and Burnham, K.J. *An almost optimal control design method for nonlinear time-delay systems*, Int. J. Contr. 85(2) (2012), 147–158.

18. Malek-Zavarei, M. *Near-optimum design of nonstationary linear systems with state and control delays*, J. Optim. Theor. Appl. 30 (1980), 73–88.

19. Marzban, H.R. *Optimal control of linear multi-delay systems based on a multi-interval decomposition scheme*, Optim. Contr. Appl. Meth. 37(1) (2016), 190–211.

20. Marzban, H.R. and Hoseini, S.M. *An efficient discretization scheme for solving nonlinear optimal control problems with multiple time delays*, Optim. Contr. Appl. Meth. 37(4) (2016), 682–707.

21. Marzban, H.R. and Razzaghi, M. *Optimal control of linear delay systems via hybrid of block-pulse and Legendre polynomials*, J. Franklin Inst. 341 (2004), 279–293.

22. Marzban, H.R. and Pirmoradian, H. *A direct approach for the solution of nonlinear optimal control problems with multiple delays subject to mixed state-control constraints* , Appl. Math. Model. 53 (2018), 189–213.

23. Marzban, H.R. and Pirmoradian, H. *A novel approach for the numerical investigation of optimal control problems containing multiple delays*, Optim. Contr. Appl. Meth. 39(1) (2018), 302–325.

24. Mehne, H.H., and Farahi, M.H. *Transformation to a fixed domian in LP modelling for a class of optimal shape design problems*, Iranian J. Numer. Anal. Optim. 9(1) (2019), 1–16.

25. Mirhosseini-Alizamini, S.M. *Numerical solution of the controlled harmonic oscillator by homotopy perturbation method*, Contr. Optim. Appl Math. 2(1) (2017), 77–91.

26. Mirhosseini-Alizamini, S.M. and Effati, S. *An iterative method for suboptimal control of a class of nonlinear time-delayed systems*, Int. J. Contr. 92 (12) (2019), 2885–2869.

27. Mirhosseini-Alizamini, S.M., Effati, S. and Heydari, A. *An iterative method for suboptimal control of linear time-delayed systems*, Syst. Contr. Lett. 82 (2015), 40–50.

28. Mirhosseini-Alizamini, S.M., Effati, S. and Heydari, A. *Solution of linear time-varying multi-delay systems via variational iteration method*, J. Math. Comput. Sci. 16 (2016), 282–297.

29. Nazemi, A. and Mansoori, M. *Solving optimal control problems of the time-delayed systems by Haar wavelet*, J. Vib. Contr. 22(11) (2014), 2657–2670.

30. Nazemi, A. and Shabani, M.M. *Numerical solution of the time-delayed optimal control problems with hybrid functions*, IMA J. Math. Contr. Inform. 32(3) (2015), 623–638.

31. Richard, J.P. *Time-delay systems: An overview of some recent advances and open problems*, Automatica, 39 (2003), 1667–1694.

32. Saberi Nik, H., Rebelo, P. and Zahedi, S. *Solution of infinite horizon nonlinear optimal control problems by piecewise Adomian decomposition method*, Math. Model. Anal. 18(4) (2013), 543–560.

33. Shehata, M.M. *A study of some nonlinear partial differential equations by using Adomian decomposition method and variational iteration method*, Am. J. Comput. Math. 5 (2015), 195–203.

34. Vanderbei, R.J. and Shanno, D.F. *An interior-point algorithm for nonconvex nonlinear programming*, Comput. Optim. Appl. 13 (1999), 231–252.

35. Wachter, A. and Biegler, L.T. *On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming*, Math. Program. 106 (2006), 25–57.

36. Wang, X.T. *Numerical solutions of optimal control for linear time-varying systems with delays via hybrid functions*, J. Franklin Inst. 344 (2007), 941–953.