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Research Article

Fuzzy endpoint results for Ćirić-generalized quasicontractive fuzzy mappings

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Abstract

We introduce Ćirić-generalized quasicontractive fuzzy mappings and provide the necessary and sufficient conditions of having a unique endpoint for such mappings. Then we introduce β-ψ-quasicontractive fuzzy mappings, establishing an endpoint result for them. Finally, we provide some results as an application.

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Keywords: Fuzzy endpoint; Ćirić-generalized; Quasicontractive fuzzy mappings; Fuzzy approximate endpoint property.

1 Introduction and preliminaries

The concept of fuzzy set was introduced initially by Zadeh [12] in 1965. In 1981, Heilpern [6] established the fuzzy contraction and proved a fuzzy fixed point theorem, which was a generalization of Nadler's fixed point theorem for multi-valued mappings (see [9]). In 2001, Estruch and Vidal [5] utilized the result of Heilpern to fuzzy fixed point with fixed degree α for some $\alpha \in [0, 1]$, which was later generalized by many authors (see, for instance, [1,3,11]). Recently, Abbas and Turkoglu [11] proved the existence of a fuzzy fixed point for a generalized contractive fuzzy mapping. On the other hand, In 2010, Amini-Harandi [2] proved that some multi-valued mappings $T : X \to CB(X)$ have a unique endpoint if and only if they have the approximate endpoint property. Afterwards, considering the same properties, Moradi and Khojasteh [8] generalized Amini-Harandi's result. In this paper, in the sense of [8], we prove

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that some fuzzy mappings have a unique fuzzy endpoint if and only if they have the fuzzy approximate endpoint property.

Definition 1.(see [6]) Let X be a space of points with generic element x and $I = [0, 1]$. A fuzzy set in X is a function that associates any point of X with a number in interval $[0, 1]$. If A is a fuzzy set in X and $x \in X$, then $A(x)$ is called the grade of membership of x in A.

Definition 2.(see [6]) Let (X, d) be a metric space and let A be a fuzzy set in X. For $\alpha \in [0, 1]$, the α-level set of A denoted by $[A]_\alpha$, is defined as

$$[A]_\alpha = \{x|A(x) \geq \alpha\} \qquad if \qquad \alpha \in (0, 1]$$

and

$$[A]_0 = \overline{\{x|A(x) > 0\}},$$

where $\overline{B}$ denotes the closure of the nonfuzzy set B.

Definition 3.(see [6]) Let X be a nonempty set. For $x \in X$, we write $\boldsymbol{\{}x\boldsymbol{\}}$ the characteristic function of the ordinary subset $\{x\}$ of X. For $\alpha \in (0, 1]$, the fuzzy point x_α of X is the fuzzy set in X given by

$$x_\alpha(y) = \begin{cases} \alpha, & y = x, \\ 0, & y \neq x. \end{cases}$$

Define

$$W_\alpha(X) = \{C \in I^X : [C]_\alpha \ is\ nonempty\ and\ compact\}.$$

Throughout this paper, I^X denotes the collection of all fuzzy sets in X. For $A, B \in I^X$, it is called that A is more accurate than B (denoted by $A \subset B$) whenever $A(x) \leq B(x)$ for all $x \in X$. For $x \in X$, $S \subseteq X$, $A, B \in W_\alpha(X)$, and $\alpha \in (0, 1]$, we define

$$d(x, S) = \inf\{d(x, a) : a \in S\},$$

$$p_\alpha(x, A) = \inf\{d(x, a) : a \in [A]_\alpha\},$$

$$p_\alpha(A, B) = \inf\{d(a, b) : a \in [A]_\alpha, b \in [B]_\alpha\},$$

$$D_\alpha(A, B) = H([A]_\alpha, [B]_\alpha) = \max\{\sup_{x \in A} p_\alpha(x, B), \sup_{y \in B} p_\alpha(y, A)\},$$

where H is the Hausdorff distance. It is easily seen that D_α is the Hausdorff metric on $W_\alpha(X)$ induced by the metric d. Hereafter, we denote by $D_\alpha(x, A)$ the amount $D_\alpha(\boldsymbol{\{}x\boldsymbol{\}}, A) = H(\{x\}, [A]_\alpha)$ for all $x \in X$ and $A \in W_\alpha(X)$.

Definition 4.(see [5]) Let X be a nonempty set, let $T : X \to I^X$, and let $\alpha \in (0, 1]$. A fuzzy point x_α is called a fuzzy fixed point of T if $x_\alpha \subset Tx$ (or equally $x \in [Tx]_\alpha$). This means that the fixed degree of x is at least α. If $\boldsymbol{\{}x\boldsymbol{\}} \subset Tx$, then it is called that x is a fixed point of T.

2 Main results

Now, we are ready to state and prove the main results of this study. Firstly, we give the following definition:

Definition 5. Let X be a nonempty set, let $T : X \to I^X$, and let $\alpha \in (0, 1]$. We say that a point $x \in X$ is a fuzzy endpoint of T if $\{x\} = [Tx]_\alpha$. This means that x is the only point in X that the fixed degree of x is at least α. If $\{x\} = [Tx]_1$, we say that x is an endpoint of T.

Now, we give the following definition of fuzzy approximate endpoint property in the sense of Amini-Harandi [2].

Definition 6. Let (X, d) be a metric space, let $T : X \to I^X$, and let $\alpha \in (0, 1]$. We say that T has the fuzzy approximate endpoint property whenever

$$\inf_{x\in X} \sup_{y\in[Tx]_\alpha} d(x, y) = 0$$

or equally

$$\inf_{x\in X} D_\alpha(x, Tx) = 0.$$

Definition 7. Let (X, d) be a metric space, let $\alpha \in (0, 1]$, and let $T : X \to W_\alpha(X)$. We say that T is a Ćirić-generalized quasicontractive fuzzy mapping whenever there exists an upper semicontinuous (u.s.c) mapping $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(t) < t$, for all $t > 0$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$ satisfying

$$D_\alpha(Tx, Ty) \le \psi(M(x, y)) \quad \text{for all } x, y \in X, \tag{1}$$

where

$$M(x, y) = \max\{d(x, y), D_\alpha(x, Tx), D_\alpha(y, Ty), D_\alpha(x, Ty), D_\alpha(y, Tx)\}.$$

Theorem 1. *Let (X, d) be a complete metric space, let $\alpha \in (0, 1]$, and let $T : X \to W_\alpha(X)$ be a Ćirić-generalized quasicontractive fuzzy mapping. Then, T has a unique fuzzy endpoint if and only if T has the fuzzy approximate endpoint property.*

Proof. If T has a fuzzy endpoint, obviously, it has the fuzzy approximate endpoint property. Conversely, let T has the fuzzy approximate endpoint property. Then, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} D_\alpha(x_n, Tx_n) = 0$. Now for any $n, m \in \mathbb{N}$, we have

$$\begin{aligned} M(x_n, x_m) &= \max\{d(x_n, x_m), D_\alpha(x_n, Tx_n), \\ &\qquad D_\alpha(x_m, Tx_m), D_\alpha(x_n, Tx_m), D_\alpha(x_m, Tx_n)\} \\ &\le D_\alpha(x_n, Tx_n) + D_\alpha(x_m, Tx_m) + D_\alpha(Tx_n, Tx_m) \\ &\le D_\alpha(x_n, Tx_n) + D_\alpha(x_m, Tx_m) + \psi(M(x_n, x_m)). \end{aligned} \tag{2}$$

Therefore, from the above inequality, we have

$$\liminf_{n,m\to\infty} (M(x_n, x_m) - \psi(M(x_n, x_m))) = 0.$$

From the property of ψ, we can conclude that $\limsup_{n,m\to\infty} M(x_n, x_m) < \infty$. Thus from (2) and by upper semicontinuity of ψ, we have

$$\begin{aligned} \limsup_{n,m\to\infty} M(x_n, x_m) &\le \limsup_{n,m\to\infty} \psi(M(x_n, x_m)) \\ &\le \psi(\limsup_{n,m\to\infty} M(x_n, x_m)). \end{aligned}$$

So we have $\limsup_{n,m\to\infty} M(x_n, x_m) = 0$ and so $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$. We shall show that $\{x^*\} = [Tx^*]_\alpha$. To see this, we have

$$\begin{aligned} D_\alpha(x^*, Tx^*) &\le d(x^*, x_n) + D_\alpha(x_n, Tx_n) + D_\alpha(Tx_n, Tx^*) \\ &\le d(x^*, x_n) + D_\alpha(x_n, Tx_n) + \psi(M(x_n, x^*)). \end{aligned} \tag{3}$$

Limiting from both sides of (3), we get

$$D_\alpha(x^*, Tx^*) \le \limsup_{n\to\infty} \psi(M(x_n, x^*)). \tag{4}$$

On the other hand,

$$\begin{aligned} M(x_n, x^*) &= \max\{d(x_n, x^*), D_\alpha(x_n, Tx_n), \\ &\qquad D_\alpha(x^*, Tx^*), D_\alpha(x_n, Tx^*), D_\alpha(x^*, Tx_n)\} \\ &\le d(x_n, x^*) + D_\alpha(x_n, Tx_n) + D_\alpha(x^*, Tx^*), \end{aligned}$$

which implies

$$\limsup_{n\to\infty} M(x_n, x^*) \le D_\alpha(x^*, Tx^*). \tag{5}$$

Consequently, from right upper semicontinuity of ψ, (4) and (5) yield

$$D_\alpha(x^*, Tx^*) \le \psi(D_\alpha(x^*, Tx^*))$$

and so $H(\{x^*\}, [Tx^*]_\alpha) = D_\alpha(x^*, Tx^*) = 0$. This means that $\{x^*\} = [Tx^*]_\alpha$. The uniqueness of endpoint is concluded from (1). □

Definition 8. Let (X, d) be a metric space, $\alpha \in (0, 1]$, and $T : X \to W_\alpha(X)$. We say that T is a Ćirić-generalized β-ψ-quasicontractive fuzzy mapping whenever there exists an upper semicontinuous (u.s.c) mapping ψ :

$[0,+\infty) \to [0,+\infty)$ such that $\psi(t) < t$, for all $t > 0$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$ and a function $\beta : X \times X \to [0,\infty)$ satisfying

$$\beta(x,y)D_\alpha(Tx,Ty) \le \psi(M(x,y)) \quad \text{for all } x,y \in X, \tag{6}$$

where

$$M(x,y) = \max\{d(x,y), D_\alpha(x,Tx), D_\alpha(y,Ty), D_\alpha(x,Ty), D_\alpha(y,Tx)\}.$$

Theorem 2. *Let (X,d) be a complete metric space, let $\alpha \in (0,1]$, and let $T : X \to W_\alpha(X)$ be a Ćirić-generalized β-ψ-quasicontractive fuzzy mapping. Moreover suppose that*

(i) there exists a sequence $\{x_n\}$ in X such that $\beta(x_n,x_m) \ge 1$ for all $n,m \in \mathbb{N}$ with $n < m$ and $\lim_{n\to\infty} D_\alpha(x_n,Tx_n) = 0$,

(ii) for any sequence $\{x_n\}$ in X which $\beta(x_n,x_m) \ge 1$ for all $n,m \in \mathbb{N}$ with $n < m$ and $x_n \to x$, we have $\beta(x_n,x) \ge 1$, for all $n \in \mathbb{N}$.

Then, T has a fuzzy endpoint.

Proof. For any $n,m \in \mathbb{N}$, we have

$$\begin{aligned} M(x_n,x_m) &= \max\{d(x_n,x_m), D_\alpha(x_n,Tx_n), \\ &\qquad D_\alpha(x_m,Tx_m), D_\alpha(x_n,Tx_m), D_\alpha(x_m,Tx_n)\} \\ &\le D_\alpha(x_n,Tx_n) + D_\alpha(x_m,Tx_m) + \beta(x_n,x_m)D_\alpha(Tx_n,Tx_m) \\ &\le D_\alpha(x_n,Tx_n) + D_\alpha(x_m,Tx_m) + \psi(M(x_n,x_m)). \end{aligned} \tag{7}$$

Similar to Theorem 1, we conclude that $\limsup_{n,m\to\infty} M(x_n,x_m) = 0$ and so $\{x_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} d(x_n,x^*) = 0$. We show that $\{x^*\} = [Tx^*]_\alpha$. To see this, we have

$$\begin{aligned} D_\alpha(x^*,Tx^*) &\le d(x^*,x_n) + D_\alpha(x_n,Tx_n) + \beta(x_n,x^*)D_\alpha(Tx_n,Tx^*) \\ &\le d(x^*,x_n) + D_\alpha(x_n,Tx_n) + \psi(M(x_n,x^*)). \end{aligned} \tag{8}$$

Consequently, as in Theorem 1, we obtain

$$D_\alpha(x^*,Tx^*) \le \psi(D_\alpha(x^*,Tx^*)),$$

which implies $H(\{x^*\},[Tx^*]_\alpha) = D_\alpha(x^*,Tx^*) = 0$. This means that $\{x^*\} = [Tx^*]_\alpha$. □

Let $\subset$ be the partial order on $W_\alpha(X)$ defined by $A \subset B$ if and only if $A(x) \le B(x)$ for all $x \in X$. In the following result, we restrict the contraction condition only for $x,y \in X$ with $Tx \subset Ty$.

Corollary 1. *Let (X, d) be a complete metric space, $\alpha \in (0, 1]$, and $T : X \to W_\alpha(X)$ be a fuzzy mapping such that there exists an upper semicontinuous (u.s.c) mapping $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(t) < t$, for all $t > 0$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$ satisfying*

$$D_\alpha(Tx, Ty) \leq \psi(M(x, y)) \quad \text{for all } x, y \in X \text{ with } Tx \subset Ty, \tag{9}$$

where

$$M(x, y) = \max\{d(x, y), D_\alpha(x, Tx), D_\alpha(y, Ty), D_\alpha(x, Ty), D_\alpha(y, Tx)\}.$$

Moreover suppose that

(i) there exists a sequence $\{x_n\}$ in X such that $\{Tx_n\}$ is a nondecreasing sequence in $W_\alpha(X)$ and $\lim_{n\to\infty} D_\alpha(x_n, Tx_n) = 0$,

(ii) for any sequence $\{x_n\}$ in X which $\{Tx_n\}$ is a nondecreasing sequence in $W_\alpha(X)$ and $x_n \to x$, we have $Tx_n \subset Tx$, for all $n \in \mathbb{N}$.

Then, T has a fuzzy endpoint.

Proof. Define the mapping $\beta : X \times X \to [0, \infty)$ by $\beta(x, y) = 1$, whenewer $Tx \subset Ty$ and $\beta(x, y) = 0$ otherwise. Then apply Theorem 2. □

Corollary 2. *Let (X, d) be a complete metric space, let $x^* \in X$ be a fixed element, let $\alpha \in (0, 1]$, and let $T : X \to W_\alpha(X)$ be a fuzzy mapping such that there exists an upper semicontinuous (u.s.c) mapping $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(t) < t$, for all $t > 0$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$ satisfying*

$$D_\alpha(Tx, Ty) \leq \psi(M(x, y)) \quad \text{for all } x, y \in X \text{ with } Tx(x^*) = Ty(x^*), \tag{10}$$

where

$$M(x, y) = \max\{d(x, y), D_\alpha(x, Tx), D_\alpha(y, Ty), D_\alpha(x, Ty), D_\alpha(y, Tx)\}.$$

Moreover suppose that

(i) there are a sequence $\{x_n\}$ in X and $\lambda \in [0, 1]$ such that $Tx_n(x^*) = \lambda$ is fixed for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} D_\alpha(x_n, Tx_n) = 0$,

(ii) for any sequence $\{x_n\}$ in X that $Tx_n(x^*) = \lambda$ is fixed for all $n \in \mathbb{N}$ and $x_n \to x$, we have $Tx(x^*) = \lambda$, for all $n \in \mathbb{N}$.

Then, T has a fuzzy endpoint.

Proof. Define the mapping $\beta : X \times X \to [0, \infty)$ by $\beta(x, y) = 1$, whenever $Tx(x^*) = Ty(x^*)$ and $\beta(x, y) = 0$ otherwise. Then applying Theorem 2 completes the proof. □

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