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Research Article

An approximate method based on Bernstein polynomials for solving fractional PDEs with proportional delays

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Abstract

We apply a new method to solve fractional partial differential equations (FPDEs) with proportional delays. The method is based on expanding the unknown solution of FPDEs with proportional delays by the basis of Bernstein polynomials with unknown control points and uses operational matrices with the least-squares method to convert the FPDEs with proportional delays to an algebraic system in terms of Bernstein coefficients (control points) approximating the solution of FPDEs. We use the Caputo derivatives of degree $0 < \alpha \leq 1$ as the fractional derivatives in our work. The main advantage of using this technique is that the method can easily be employed to a variety of FPDEs with or without proportional delays, and also the method offers a very simple and flexible framework for direct approximating of the solution of FPDEs with proportional delays. The convergence analysis of the present method is discussed. We show the effectiveness and superiority of the method by comparing the results obtained by our method with the results of some available methods in two numerical examples.

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Keywords: Fractional partial differential equation; Bernstein polynomial; Operational matrix; Caputo derivative.

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1 Introduction

Fractional partial differential equations (FPDEs) plays an important role in recent years. Many significant phenomena in mechanics, biology, networks, signal processing, reaction process, and other fields in sciences and engineering are modeled by FPDEs. Although FPDEs are exploited to describe these phenomena carefully, but it is very challenging and irritating to obtain the exact solutions or at last approximate solutions of FPDEs. In recent years, many numerical methods are used to solve FPDEs. For sample, Chebyshev collocation method [1, 6, 19], Legendre collocation method [29], Jacobi collocation method [9, 28], variational iteration method (VIM) [13, 26], homotopy perturbation method (HPM) [15, 23], Laplace transform method [4], Adomian decomposition method [25], and finite difference method [30].

FPDE with proportional delays is a particular type of FPDEs. This case of FPDEs models various phenomena such as shallow-water waves or nonlinear waves, which come from quantum field theory, medicine, biology, and climate models. For more details and various applications, we refer the interested reader to [20, 27]. Recently, some scholars have attempted to solve these types of equations. Sakar, Uludag, and Erdogan [20] applied the homotopy perturbation method to obtain an approximate solution of FPDEs with proportional delays. Singh and Kumar [22] used VIM for solving FPDEs with proportional delays. Shah et al. [21] suggested the natural transform decomposition method for constructing approximate solutions of FPDEs with proportional delays.

Bezier functions are appropriate tools for obtaining the solution to these cases of problems. Easy definition, well performance, and interpolation of end-points are some virtues of these tools. Many scholars have applied control points of the Bernstein–Bezier form in their papers. Zheng, Sederberg, and Johnson [31] introduced the Bernstein–Bezier form for solving ordinary differential equations. Chakrabarti and Martha [3] used Bezier curves for solving the Fredholm integral equations of the second kind. Ghomanjani and Khorram [5] used this approach for solving the quadratic Riccati differential equation. Karimi, Bahadorimehr, and Mansoorzadeh [8] proposed the use of this method to solve PDEs. Also, some researchers applied the Bezier control points method to obtain an approximate solution of fractional problems; one can see [2, 7, 10, 11, 14, 17, 18] and references therein. We suggest a technique similar to the one used in [10], for solving two-dimensional fractional optimal control problems.

In this paper, we consider the following FPDE with proportional delays

$$ {}_0D_t^{\alpha} u(x,t) = h(x,t,u(a_0x,b_0t), \frac{\partial u(a_1x,b_1t)}{\partial x}, \dots, \frac{\partial^{n_1} u(a_{n_1}x,b_{n_1}t)}{\partial x^{n_1}}), \quad (1) $$

with initial conditions

$$u^{(k)}(x,0) = s^{(k)}(x), \qquad k = 0,1,2,\dots,q-1, \tag{2}$$

where $x,t \in [0,1]$, $a_i, b_i \in [0,1]$ for $i \in \{0,1,2,\dots,n_1\}$, also $s^{(k)}(x)$, $k = 0,1,2,\dots,q-1$, where $q \in \mathbb{N}$, are smooth functions and h is the partial differential operator. Also the fractional derivative is defined in Caputo sense. The real numbers a_i's and b_i's are given constant proportional delays. The necessary conditions for the existence of the solution of (1)–(2) and regularity properties of solutions can be found in [20, 21, 22].

Bezier control points with least-squares method are employed for solving FPDEs with proportional delays (1)–(2). The novelty of our strategy is in producing operational matrices —based on Bernstein polynomials— together with the least-squares method converting FPDEs with proportional delays to a simple optimization problem.

This article is formed as follows: Section 2 introduces some properties of fractional derivatives and Bernstein polynomials. In Section 3, we define a class of FPDEs with proportional delays and employ operational matrices based on Bernstein polynomials to convert these equations to a mathematical programming problem in terms of some unknowns control points. In the fourth section, the convergence of the proposed method is studied. The applicability and efficiency of the proposed method are shown through two numerical examples, and comparison with some other methods is investigated in Section 5. Finally, the conclusion is presented in Section 6.

2 Preliminaries

This section contains some basic definitions and properties of fractional calculus, Bernstein polynomials, and function approximation.

2.1 Some preliminaries in fractional derivatives

Definition 1. Let $\mu \in \mathbb{R}$ and let $n \in \mathbb{N}$. A real function $f(x)$, $x > 0$ belongs to the space C_μ if there exist a real number $r > \mu$ and $g(x) \in C[0,\infty]$ such that $f(x) = x^r g(x)$. Moreover, $f \in C_\mu^n$ if and only if $f^{(n)} \in C_\mu$.

Definition 2. The Riemann–Liouville fractional integral of order $\alpha > 0$ is defined as follows:

$${}_0I_x^\alpha\big(f(x)\big) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau)\,d\tau, \qquad x > 0,$$

where $f(x) \in C_\mu$ and $\Gamma(\alpha) = \int_0^\infty \tau^{(\alpha-1)} e^{-\tau} d\tau$ is Euler's gamma function.

Definition 3. The Caputo fractional derivative of order $\alpha > 0$ of the given function $f(x) \in C_\mu$ is defined as

$$\begin{aligned} {}_0D_x^\alpha f(x) &= {}_0I_x^{n-\alpha} D^n f(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \qquad n-1 < \alpha < n, \end{aligned}$$

where n is a nonnegative integer number; in fact $n = [\alpha]+1$ while $[\alpha]$ indicates the integer part of $\alpha > 0$. For $\alpha \in \mathbb{N}$, the Caputo derivative operator is the usual derivative operator of integer order.

We list some properties of fractional integral and fractional derivative for $f \in C_\mu$, $\mu > 1$, and $\alpha, \beta > 0$, as follows:

$$\begin{aligned} &{}_0I_x^0\big(f(x)\big) = f(x), \\ &{}_0I_x^\alpha\big(x^k\big) = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} x^{k+\alpha}, \qquad k \in \mathbb{N} \cup 0, x > 0, \\ &{}_0D_x^\alpha\ {}_0I_x^\alpha\big(f(x)\big) = f(x), \\ &{}_0D_x^\alpha\big(x^k\big) = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha}, \qquad k \in \mathbb{N} \cup 0, x > 0, k > [\alpha], \\ &{}_0D_x^\alpha\big(C\big) = 0, \qquad C \in \mathbb{R}, \\ &{}_0D_x^\alpha\Big(\sum_{i=0}^n c_i f_i(x)\Big) = \sum_{i=0}^n c_i\ {}_0D_x^\alpha f_i(x), \end{aligned}$$

where $\{c_i\}_{i=0}^n$ are constants.

2.2 Some preliminaries in Bernstein polynomials

Definition 4. The Bernstein polynomials of degree n over the interval $[0, 1]$ are defined as follows:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i},$$

for $i = 0, 1, 2, \ldots, n,$, where $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$.
If one uses the binomial expansion for $(1-t)^{n-i}$, then

$$B_i^n(t) = \sum_{k=0}^{n-i} \binom{n-i}{k} \binom{n}{i} (-1)^k t^{i+k}. \tag{3}$$

Definition 5. We define a Bernstein vector $\phi_n(t)$ as

$$\phi_n(t) = \left[B_0^n(t)\ B_1^n(t)\ B_2^n(t)\ \dots\ B_n^n(t)\right]^T;$$

the Bezier polynomial of degree n over the interval $[0,1]$ is defined as follows:

$$P_n(t) = C\phi_n(t), \tag{4}$$

where

$$C = \left[c_0\ c_1\ c_2\ \dots\ c_n\right], \tag{5}$$

is the vector of constant coefficients that we recall its entry as the control points.

Thus by (4),

$$P_n(t) = \sum_{i=0}^{n} c_i B_i^n(t).$$

Lemma 1. By using (3), we define

$$\phi_n(t) = \psi_n T_n(t),$$

where $T_n(t) = [1 \quad t \quad t^2 \quad \cdots \quad t^n]^T$ and $\psi_n = \Psi^{n+1}$ is an $(n+1)\times(n+1)$ upper triangular matrix that can be expressed by

$$\Psi_{i+1,j+1}^{n+1} = \begin{cases} \dfrac{(-1)^{j-i}n!}{(n-j)!(j-i)!i!}, & i \le j, \\ 0, & i > j, \end{cases} \tag{6}$$

where $i,j = 0,1,\dots,n$.

Proof. See [2]. □

Theorem 1. Suppose that $f(x) \in C^m[0,1]$; then for each $k \in \mathbb{N}$, $k \le m$, and $k \le n$, there exists an $(n+1)\times(n+1)$ matrix D such that

$$f^{(k)}(x) \simeq P_n^{(k)}(x) = \mathbf{C}D^k\phi_n(x),$$

where $\mathbf{C}$ is shown in (5) and $D = \psi_n\Lambda\mathcal{V}$, with

$$\Lambda_{i+1,j+1} = \begin{cases} i, & i = j+1, \\ 0 & otherwise, \end{cases}$$

for $i = 0,\dots,n$ and $j = 0,\dots,n-1$, and $\mathcal{V}$ can be expressed by

$$\mathcal{V}_k = \psi_{n,k}^{-1}, \qquad k = 1,2,\dots,n,$$

where $\psi_{n,k}^{-1}$ is the kth row of ψ_n^{-1}.

Proof. For details, see [8]. $\square$

2.3 Function approximation

Definition 6. For the product of two Bernstein polynomials of degree $n, m \in \mathbb{N} \cup \{0\}$, we define

$$ {}_i^n B_j^m(x,t) = B_i^n(x) B_j^m(t), $$

where $t, x \in [0,1]$.

Lemma 2. The set

$$ \mathrm{Y} = Span\{ {}_0^n B_0^m(x,t),\ {}_0^n B_1^m(x,t), \dots,\ {}_0^n B_m^m(x,t),\ {}_1^n B_0^m(x,t), \dots,\ {}_n^n B_m^m(x,t)\} $$

is a complete basis for the Hilbert space $L^2[\bar{\Omega}]$, where $\bar{\Omega} = [0,1] \times [0,1]$.

Proof. Since Y is a finite subset of $L^2[\bar{\Omega}]$, it is complete. For more details, see [12]. $\square$

Lemma 3. If

$$ \mathrm{Y} = Span\{ {}_0^n B_0^m(x,t),\ {}_0^n B_1^m(x,t), \dots,\ {}_0^n B_m^m(x,t),\ {}_1^n B_0^m(x,t), \dots,\ {}_n^n B_m^m(x,t)\} $$

and $u(x,t) \in L^2[\bar{\Omega}]$, then $\phi_n^T(x)\mathbf{U}\phi_m(t)$ is the best approximation of $u(x,t)$ out of Y, that is,

$$ \begin{aligned} u(x,t) \simeq u_{nm}(x,t) &= \sum_{i=0}^{n}\sum_{j=0}^{m} u_{i,j}\ {}_i^n B_j^m(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} u_{i,j} B_i^n(x) B_j^m(t) \\ &= \phi_n^T(x)\mathbf{U}\phi_m(t), \end{aligned} $$

where $\mathbf{U}$ is the $(n+1)\times(m+1)$ matrix such that

$$ \mathbf{U} = Q_n^{-1}\langle \phi_n(x), \langle u(x,t), \phi_m(t)\rangle\rangle Q_m^{-1} $$

and $Q_n = \psi_n G_n \psi_n{}^{\mathsf{T}}$,

$$ G_n = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+3} & \cdots & \frac{1}{2n+1} \end{bmatrix}, $$

where ψ_n is defined in (6). Also the inner product $\langle \cdot, \cdot \rangle$ on the space $C[0,1]$ is defined by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx, \qquad f(x), g(x) \in C[0,1].$$

Proof. See [12]. □

To compute Caputo fractional derivatives for an arbitrary function $f(x)$, we define an operational fractional derivative matrix of order $\alpha > 0$ by using Bernstein polynomials.

Theorem 2. Let $\phi_n(x)$ be a vector of Bernstein polynomials. Then for each $\alpha > 0$, we have

$$ {}_0D_x^\alpha \phi_n(x) = D^\alpha \phi_n(x),$$

while $D^\alpha = \psi_n K_n^\alpha H^\alpha$, where K_n^α is an $(n+1)\times(n+1)$ matrix, defined as follows:

$$K_n^\alpha(i+1, j+1) = \begin{cases} \dfrac{\Gamma(i+1)}{\Gamma(i-\alpha+1)}\left(B(p_i, q_i) - B(\frac{a}{x}; p_i, q_i)\right), & i \in \mathbb{N}\cup 0,\ i \geq \lceil\alpha\rceil \ or \\ & i \notin \mathbb{N}\cup 0,\ i > \lfloor\alpha\rfloor,\ i = j, \\ 0, & otherwise, \end{cases}$$

where the floor function $\lfloor\alpha\rfloor$ and the ceiling function $\lceil\alpha\rceil$ indicate the largest integer less than or equal to α and the smallest integer greater than or equal to α, respectively. Also, $B(p,q)$ and $B(x;p,q)$ are, respectively, the beta function and incomplete beta function, for $i, j = 0, 1, \ldots, n,\ p_i = i - [\alpha]$, and $q_i = [\alpha] - \alpha + 1$. Also H^α is the following $(n+1)\times(n+1)$ matrix:

$$H^\alpha = \left[H^\alpha(1), H^\alpha(2), \ldots, H^\alpha(n+1)\right],$$

where $H^\alpha(i+1) = \langle x^{i+1-\alpha}, \phi_n(x)\rangle A_n^{-1}$, $i = 0, 1, \ldots, n$, and $A_n = \langle \phi_n(x), \phi_n(x)\rangle$. The matrix D^α is called the operational derivative matrix for the fractional Caputo derivative of order α.

Proof. see [10] □

3 Approximate solution of FPDEs with proportional delays

In this section, we consider the FPDE (1)–(2). One may assume that the existence conditions for problem (1)–(2) are satisfied. We develop using Bernstein polynomials for solving FPDE (1)–(2). To achieve this aim, by using the least-squares method, we convert problem (1)–(2) to a mathematical programming problem. For this purpose, define the residual function $R(x,t)$ as follows:

$$R(x,t) = {}_0D_t^{\alpha}u(x,t) - h(x,t,u(a_0x,b_0t),\frac{\partial u(a_1x,b_1t)}{\partial x},\dots,\frac{\partial^{n_1}u(a_{n_1}x,b_{n_1}t)}{\partial x^{n_1}}). \tag{7}$$

It is clear that when we get the exact solution, then the residual function $R(x,t)$ is zero for $(x,t) \in (0,1)\times(0,1)$. To get the approximate solution of problem (1)–(2), we can follow Step 1 up to Step 6.

Step 1: Choose natural numbers n and m and approximate the solution of $u(x,t)$ by Bernstein polynomials.

$$u(x,t) \simeq u_{nm}(x,t) = \omega(x,t)\phi_n^{\intercal}(x)\bar{\mathbf{U}}\phi_m(t) + \xi(x), \tag{8}$$

where $\omega(x,t)$ and $\xi(x)$ are given functions such that $u(x,t)$ satisfies the initial conditions (2). If one chooses $\omega(x,t) = t$ and $\xi(x) = s(x)$, then we have

$$u_{nm}(x,t) = t\phi_n^{\intercal}(x)\bar{\mathbf{U}}\phi_m(t) + \xi(x).$$

But, by Lemma 3.5 in [2], the function $t\phi_n^{\intercal}(x)\bar{\mathbf{U}}\phi_m(t)$ can be rewritten as follows:

$$\begin{aligned} t\phi_n^{\intercal}(x)\bar{\mathbf{U}}\phi_m(t) &= \phi_n^{\intercal}(x)\bar{\mathbf{U}}\phi_m(t)t = \phi_n^{\intercal}(x)\bar{\mathbf{U}}\phi_m(t)\left(\mathbf{C}\phi_m(t)\right)^{\intercal} \\ &= \phi_n^{\intercal}(x)\bar{\mathbf{U}}\phi_m(t)\phi_m^{\intercal}(t)\mathbf{C}^{\intercal} \simeq \phi_n^{\intercal}(x)\bar{\mathbf{U}}\bar{\mathbf{C}}\phi_m(t) \\ &= \phi_n^{\intercal}(x)\mathbf{U}\phi_m(t). \end{aligned}$$

So,

$$u(x,t) \simeq u_{nm}(x,t) = \phi_n^{\intercal}(x)\mathbf{U}\phi_m(t) + \xi(x). \tag{9}$$

Step 2: According to the previous section, one can easily find that

$$\begin{aligned} &{}_0D_t^{\alpha}u(x,t) \simeq \phi_n^{\intercal}(x)\mathbf{U}D^{\alpha}\phi_m(t), \\ &u(a_0x,b_0t) \simeq \phi_n^{\intercal}(a_0x)\mathbf{U}\phi_m(b_0t) + \xi(a_0x), \\ &\frac{\partial u(a_1x,b_1t)}{\partial x} \simeq \phi_n^{\intercal}(a_1x)D^{\intercal}\mathbf{U}\phi_m(b_1t) + \frac{d\xi(a_1x)}{dx}, \\ &\qquad\vdots \\ &\frac{\partial^{n_1}u(a_{n_1}x,b_{n_1}t)}{\partial x^{n_1}} \simeq \phi_n^{\intercal}(a_{n_1}x)D^{n_1^{\intercal}}\mathbf{U}\phi_m(b_{n_1}t) + \frac{d^{n_1}\xi(a_{n_1}x)}{dx^{n_1}}, \end{aligned}$$

and

$$u^{(k)}(x,0) \simeq \phi_n^{\intercal}(x)D^{k^{\intercal}}\mathbf{U}\phi_m(0) + \frac{d^k\xi(x)}{dx^k} = s^{(k)}(x),\ \ k = 0,1,\dots,q-1.$$

Step 3: We express the residual function (7) by the Bernstein polynomials as follows:

$$R(x,t) \simeq \phi_n^{\intercal}(x)\mathbf{R}\phi_m(t) = \sum_{i=0}^{n}\sum_{j=0}^{m} r_{i,j} B_i^n(x) B_j^m(t),$$

where the control points $r_{i,j}$ are functions of coefficients $u_{i,j}, \quad i = 0,1,2,\ldots,n,\ j = 0,1,2,\ldots,m$. In other word,

$$r_{i,j} = r_{i,j}\left(u_{0,0}, u_{0,1}, u_{0,2}, \ldots, u_{0,m}, u_{1,0}, \ldots, u_{n,m}\right).$$

Step 4: Create a cost function $F = \sum\limits_{i=0}^{n}\sum\limits_{j=0}^{m} r_{i,j}^2$. It is obvious that the square function F is a function of $u_{0,0}, u_{0,1}, u_{0,2}, \ldots, u_{0,m}, u_{1,0}, \ldots, u_{n,m}$.

Step 5: Construct a mathematical programming problem as follows:

$$\mathit{minimize}\ F = \sum_{i=0}^{n}\sum_{j=0}^{m} r_{i,j}^2\left(u_{0,0}, u_{0,1}, \ldots, u_{0,m}, u_{1,0}, \ldots, u_{n,m}\right), \tag{10}$$

s.t.

$$\begin{aligned}
&\sum_{i=0}^{n}\sum_{j=0}^{m}\left(\sum_{l=0}^{n} d_{i,l}^k u_{l,j}\right) B_i^n(x) B_j^m(0) + \xi^{(k)}(x) = s^{(k)}(x),\\
&k = 0,1,2,\ldots,q-1,\\
&0 \le x \le 1,\\
&0 \le t \le 1.
\end{aligned} \tag{11}$$

Step 6: Optimization problem (10)–(11) can be solved by some useful computational softwares such as MATLAB. By solving this optimization problem, one can find the values of control points $u_{i,j} \quad i = 0,1,2,\ldots,n,\ j = 0,1,2,\ldots,m$. Then the approximate solution of $u(x,t)$ could be found by using these control points in (8). We need to mention that we have solved optimization problem (10)–(11) by using discrete points $x_p \in (0,1)$ and $t_h \in (0,1)$.

4 Convergence analysis

In this section, we show the convergence of the approximate solution that has been found by Bernstein polynomials for the FPDE (1)–(2) with proportional delays. In this way, without loss of generality, we suppose that $n_1 = 1$, that $q = 1$, and that $0 < \alpha \le 1$ for the FPDE (1)–(2). Consider the following problem:

$$L\left[x,t,u(a_0x,b_0t),\frac{\partial u(a_1x,b_1t)}{\partial x},\ {}_0D_t^{\alpha}u(x,t)\right]$$
$$= {}_0D_t^{\alpha}u(x,t)-h(x,t,u(a_0x,b_0t),\frac{\partial u(a_1x,b_1t)}{\partial x}), \tag{12}$$

with initial conditions

$$u(x,0)=s(x), \tag{13}$$

where $(x,t)\in\bar{\Omega}=[0,1]\times[0,1]$, the constant proportional delays a_0,a_1,b_0, and b_1 are in the interval $[0,1]$, also $s(x)$ is a given function and L is the partial differential operator.

Consider $(C(\bar{\Omega}),\|\cdot\|)$ as the normed space of all differentiable continuous functions, where $\|\cdot\|$ is defined as follows:

$$\|f\|=\|f\|_{\infty}+\left\|\frac{\partial f}{\partial t}\right\|_{\infty}+\left\|\frac{\partial f}{\partial x}\right\|_{\infty},\qquad f\in C(\bar{\Omega}).$$

Lemma 4. If $\bar{\Omega}=[0,1]\times[0,1]$, then the defined function $L:C(\bar{\Omega})\longrightarrow\mathbb{R}$ in (12) is uniformly continuous on $(C(\bar{\Omega}),\|\cdot\|)$.

Proof. See [16]. □

Now, we prove the convergence of the approximate solution for the FPDE (1)–(2) when the degree of the Bernstein polynomials increase. For this intention, we remark the following lemma, that is an obvious result of the Stone–Weirestrass theorem.

Lemma 5. If $\bar{u}(x,t)$ is a continuous real-valued function defined on $\bar{\Omega}$ and $\varepsilon>0$, then there exists a sequence of Bernstein polynomials $\{u_{nm}(x,t)\}_{n,m\in\mathbb{N}}$ and $M\in\mathbb{N}$ such that

$$\|\bar{u}(x,t)-u_{nm}(x,t)\|<\varepsilon,\qquad (x,t)\in\bar{\Omega},\ \ n,m\geq M.$$

Proof. See [24]. □

Theorem 3. Consider the functional L on $C(\bar{\Omega})$. If η_{nm} is the minimum of the functional L on $C(\bar{\Omega})\bigcap Y$, then

$$\eta_{nm}\longrightarrow 0$$

where L and Y are defined in (12) and Lemma 2, respectively.

Proof. Let $u^*(x,t)\in C(\bar{\Omega})$ and $\varepsilon>0$ be such that $L[u^*(x,t)]<\varepsilon$. By the minimum properties, it is possible to find such a function $u^*(x,t)$. Considering Lemma 4, if $u(x,t)\in C(\bar{\Omega})$ and $\|u-u^*\|<\delta$, then

$$|L[u] - L[u^*]| < \varepsilon.$$

We use Lemma 5, for suitable values of n and m, there exist $u_{nm}(x,t) \in C(\bar{\Omega}) \bigcap Y$ such that $\|u_{nm} - u^*\| < \delta$. If we show $L[u_{nm}]$ by η_{nm}, then we have

$$\eta_{nm} = |L[u_{nm}] - L[u^*] + L[u^*]| \leq |L[u_{nm}] - L[u^*]| + |L[u^*]| \leq 2\varepsilon.$$

Because ε is arbitrary, this completes the proof. Note that the minimum value of the functional L in (12) on $C(\bar{\Omega})$ is zero. □

5 Numerical examples

In this section, we give two numerical examples and apply the method presented in Section 3 for solving them. These test problems are solved by using powerful *MATLAB 2018b* software on an *Intel Core i7-7500U*. These test problems demonstrate the validity and efficiency of this technique.

Example 1. Consider the following fractional partial generalized Burgers equation with proportional delay (see [20]):

$${}_0D_t^{\alpha} u(x,t) = \frac{\partial^2 u(x, \frac{t}{2})}{\partial x^2} u(x, \frac{t}{2}) - u(x,t),$$

where $x, t \in [0,1]$, $0 < \alpha < 1$, and the initial condition is

$$u(x,0) = x^2.$$

The exact solution of this example is $u(x,t) = x^2 e^t$ when $\alpha = 1$. If one assume that $\bar{u}(x,t)$ is the approximate solution, then the error between exact and approximate solution can be shown by

$$\mid u(x,t) - \bar{u}(x,t) \mid, \qquad (x,t) \in (0,1) \times (0,1).$$

The comparison of the absolute errors between exact and approximate solution by employing HPM [20], AVIM [22], and the proposed method for $n = 2$ and $m = 5$ is shown in Table 1. Figure 1 shows the graph of the approximate solution of $u(x,t)$ for different values of α when $x = 1$.

Table 1: The absolute value of error for $\alpha = 1$ in Example 1.

x	t	presented method	HPM[20]	AVIM[22]
0.25	0.25	$8.7365E-08$	$5.30E-07$	$2.1974E-08$
0.25	0.5	$5.6597E-08$	$1.7735E-05$	$1.4596E-06$
0.25	0.75	$3.6812E-08$	$1.4087E-04$	$1.7274E-05$
0.25	1	$1.7890E-07$	$6.2178E-04$	$1.0095E-04$
0.5	0.25	$1.0598E-07$	$2.1230E-06$	$8.7896E-08$
0.5	0.5	$1.4489E-07$	$7.0943E-05$	$5.8385E-06$
0.5	0.75	$6.2958E-07$	$5.6348E-04$	$6.9096E-05$
0.5	1	$1.3441E-06$	$2.4871E-03$	$4.0379E-04$
0.75	0.25	$2.4535E-07$	$4.7760E-06$	$1.9777E-07$
0.75	0.5	$9.9618E-07$	$1.5962E-04$	$1.3137E-05$
0.75	0.75	$2.2682E-06$	$1.2678E-03$	$1.5547E-04$
0.75	1	$4.1328E-06$	$5.5960E-03$	$9.0853E-04$

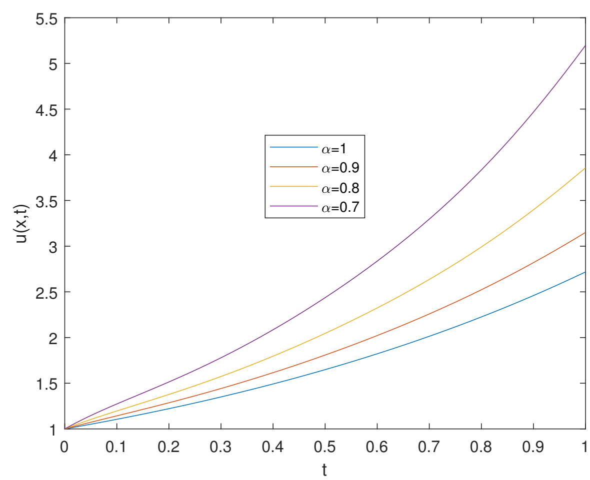


Figure 1: Approximated graphs of $u(x,t)$ in $\alpha = 0.7, 0.8, 0.9, 1$ and $x = 1$ for Example 1.

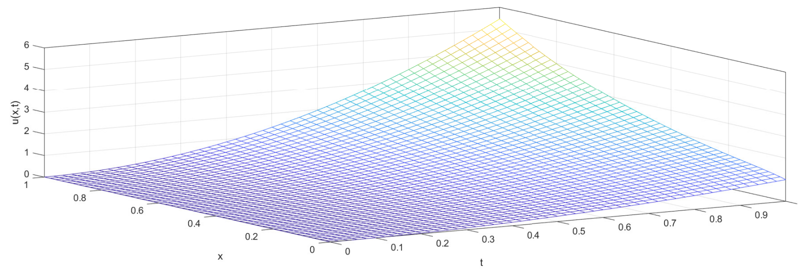
Example 2. In this example, the following fractional generalized Burgers equation with proportional delay is considered (see [20]):

$$ {}_0D_t^{\alpha} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,\frac{t}{2})}{\partial x} u(\frac{x}{2},\frac{t}{2}) + \frac{1}{2} u(x,t), $$

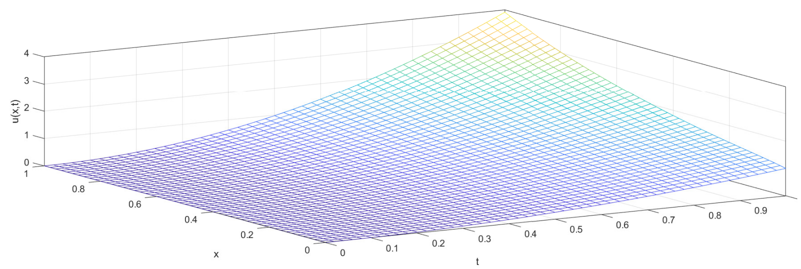
such that

$$ u(x,0) = x, $$

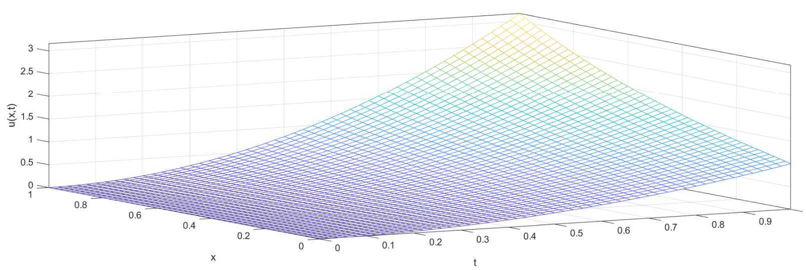
a)



b)



c)



d)

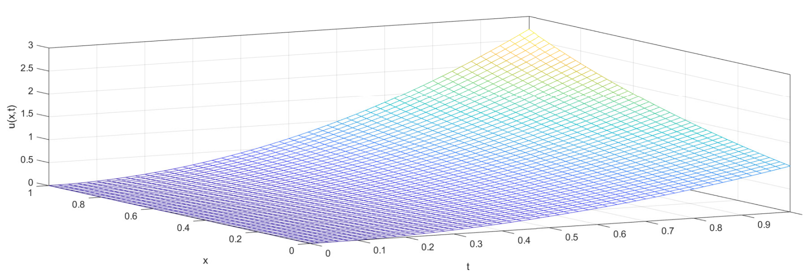


Figure 2: Approximated graphs of $u(x,t)$ in Example 1 for a)$\alpha = 0.7$; b)$\alpha = 0.8$; c)$\alpha = 0.9$; d)$\alpha = 1$;

where $x, t \in [0, 1]$ and $0 < \alpha < 1$.

The exact solution of this example is $u(x,t) = xe^t$ when $\alpha = 1$. The comparison of the absolute errors between exact and approximate solution by employing NTDM [21], HPM [20], AVIM [22], and the proposed method for $n = 2$ and $m = 6$ is shown in Table 2. Figure 3 shows the graph of the approximate solution of $u(x,t)$ for different values of α when $x = 1$.

Table 2: The absolute value of error for $\alpha = 1$ in Example 2.

x	t	presented method	NTDM [21]	HPM [20]	AVIM[22]
0.25	0.25	$1.4976E-06$	$2.1224E-06$	$2.1230E-06$	$8.7896E-08$
0.25	0.5	$7.1515E-06$	$7.0943E-05$	$7.0943E-05$	$5.8386E-06$
0.25	0.75	$1.7332E-05$	$5.6348E-04$	$5.6348E-04$	$6.9096E-05$
0.25	1	$3.4298E-05$	$2.4871E-03$	$2.4871E-03$	$4.0379E-04$
0.5	0.25	$1.0884E-06$	$4.2550E-06$	$4.2450E-06$	$1.7579E-07$
0.5	0.5	$6.5797E-06$	$1.4189E-04$	$1.4189E-04$	$1.1677E-05$
0.5	0.75	$1.6607E-05$	$1.1270E-03$	$1.1270E-03$	$1.3819E-04$
0.5	1	$3.3493E-05$	$4.9742E-03$	$4.9743E-03$	$8.0758E-04$
0.75	0.25	$1.6021E-06$	$6.3675E-06$	$6.3670E-06$	$2.6369E-07$
0.75	0.5	$7.5613E-06$	$2.1283E-04$	$2.1283E-04$	$1.7516E-05$
0.75	0.75	$1.8304E-05$	$1.6904E-03$	$1.6905E-03$	$2.0729E-04$
0.75	1	$3.6085E-05$	$7.4614E-03$	$7.4614E-03$	$1.2114E-03$

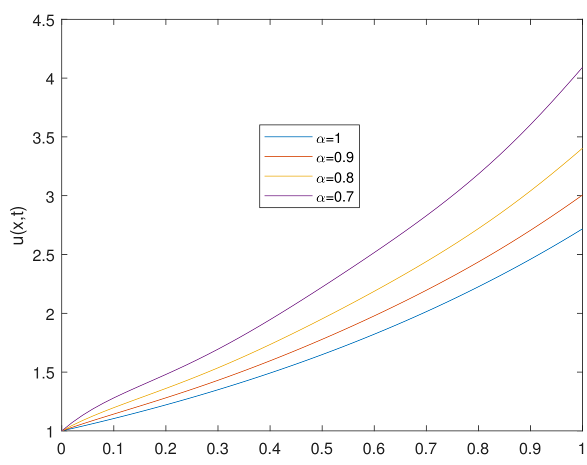


Figure 3: Approximated graphs of $u(x,t)$ in $\alpha = 0.7, 0.8, 0.9, 1$ and $x = 1$ for Example 2.

6 Conclusion

The aim of this paper was to solve fractional partial differential equations with multiple proportional delays by using Bernstein polynomials. To reach this goal, we generated some operational matrices based on Bernstein polynomials for fractional-order derivative in the Caputo sense. Operational matrices, together with the least-squares method, were used in an algorithm for solving these kinds of problems. The steps of the algorithm were easy to implement. The convergence of the method was investigated. Two numerical examples were given to illustrate that this proposed method is accurate and precise.

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