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**Research Article**

# On optimality and duality for multiobjective interval-valued programming problems with vanishing constraints

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**Abstract**

In this study, we explore the theoretical features of a multiobjective interval-valued programming problem with vanishing constraints. In view of this, we have defined a multiobjective interval-valued programming problem with vanishing constraints in which the objective functions are considered to be interval-valued functions, and we define an LU-efficient solution by employing partial ordering relations. Under the assumption of generalized convexity, we investigate the optimality conditions for a (weakly) LU-efficient solution to a multiobjective interval-valued programming problem with vanishing constraints. Furthermore, we establish Wolfe and Mond–Weir duality results under appropriate convexity hypotheses. The study concludes with examples designed to validate our findings.

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**Keywords:** Multiobjective interval-valued optimization problem; vanishing constraints; (weakly) LU-efficient solution; duality.

## 1 Introduction

In modern mathematical research, the concept of mathematical programming with vanishing constraints has emerged as a novel type of constrained optimization problems. Formal analysis was conducted by Achtziger and Kanzow [1]. Dorsch, Shikhman, and Stein [9] presented a topological analysis of mathematical programs with vanishing constraints and introduced the new concept of a $T$-stationary point. By applying the concept of local regularization to mathematical programs with vanishing constraints, Hohesiel, Kanzow, and Schwartz [13] derived a new solution method for solving such a class of optimization problems and proved several convergence results. Later, to compute the mathematical problems involving vanishing constraints numerically, Hoheisel et al. [14] investigated and compared four regularization methods, each impacted by a single regularization parameter. The study of mathematical programming with vanishing constraints has a wide range of real-world applications, including the development of robot motion plans [8, 19], the design of optimal truss topologies for mechanical structures [11], and the design of nonlinear optimal control problems for mixed integers [20].
A multiobjective programming problem involves minimizing multiple objectives over a set of feasible solutions. Multiobjective programming is challenging due to the fact that the objectives for vector optimization problems compete with each other, and an improvement on one objective can reduce goals for other objectives. There is an enormous amount of literature on optimal conditions and numerous kinds of dualities in multiobjective programming problems (see, for example, [7, 22, 23]). A constraint qualification is an element critical to the existence of Lagrange multipliers in multiobjective optimization problems, as it allows Karush–Kuhn–Tucker optimality conditions to hold, thereby assisting with and enhancing optimization algorithms design. There have been several recent articles published on optimality, stationarity, criticality, and constraint qualification; for instance, we refer to [10, 12, 17]. Jayswal and Singh [18] studied about modified objective function approach for an equivalent $\eta$-approximated multiobjective optimization problem with vanishing constraints and also discussed saddle point criteria. The class of differentiable semi-infinite multiobjective programming problems with vanishing constraints was discussed by Antczak [4].
Using separate considerations of minimization and maximization, Ishibuchi and Tanaka [16] investigated multiobjective optimization problems in which the objective functions are interval-valued and developed an ordering relationship between two closed intervals. A general methodology proposed by Urli and Nadeau [26] provides a way of formulating the non-deterministic multiobjective linear programming problem with interval coefficients in a de-

terministic way and then solving it with an interactive approach. Under certain convexity assumptions, The Karush–Kuhn–Tucker necessary optimality conditions for nonlinear differentiable multiobjective programming problems with an interval-valued objective and constraint functions were derived by Hosseinzade and Hassanpour [15]. Studies on optimality conditions and different types of duality for multiobjective programming problems with interval objective function are quite widespread (refer to [6, 27, 28, 15, 21]). In this paper, we aim to investigate the optimality conditions and the duality results for multiobjective interval-valued programming problems with vanishing constraints under the Abadie constraint qualification.
Following is an outline of the rest of this paper: Section 2 consists of some basic definitions, background material, and the necessary optimality conditions. Section 3 represents the sufficient optimality conditions for multiobjective interval-valued optimization problems with vanishing constraints. In Sections 4 and 5, Wolfe type dual and Mond–Weir type dual are presented, and appropriate duality results are also discussed. Section 6 explores special cases. Finally, the paper is concluded in Section 7.

## 2 Preliminaries

This section contains a list of notations and basic definitions which will be used throughout the article. Let $R^n$ be the Euclidean space with n-dimensions and $R^n_+$ be its nonnegative orthant. For a given $a$, $\Theta(a)$ is the system of the neighborhoods of $a$. For $A \subseteq R^n$, $spanA$ and $posA$ stands for its linear hull and convex cone (containing the origin) of $A$, respectively. Let $A \neq \phi$ and let the contingent cone of set $A$ at the point $a$, be denoted by $\mathbb{T}(A, a)$. Let $I(R)$ be the set of all closed and bounded intervals in $R$. For the case where $\Lambda_1 \in I(R)$ is a closed interval, we use the notation $\Lambda_1 = [\alpha_0^L, \alpha_0^U]$, where $\alpha_0^L$ and $\alpha_0^U$ represent the minimum and maximum values of $\Lambda_1$, respectively. Let

$$\Lambda_1 = [\alpha_0^L, \alpha_0^U], \quad \Lambda_2 = [\beta_0^L, \beta_0^U] \in I(R).$$

Then we have

(*i*) $\Lambda_1 + \Lambda_2 = \{\alpha_0 + \beta_0 \mid \alpha_0 \in \Lambda_1 \text{ and } \beta_0 \in \Lambda_2\} = [\alpha_0^L + \beta_0^L, \alpha_0^U + \beta_0^U]$,

(*ii*) $-\Lambda_1 = \{\alpha_0 \mid \alpha_0 \in \Lambda_1\} = [-\alpha_0^U, -\alpha_0^L]$,

(*iii*) $\Lambda_1 - \Lambda_2 = \Lambda_1 + (-\Lambda_2) = [\alpha_0^L - \beta_0^U, \alpha_0^U - \beta_0^L]$,

(*iv*) $k\Lambda_1 = \{k\alpha_0 \mid \alpha_0 \in \Lambda_1\} = \begin{cases} [k\alpha_0^L, k\alpha_0^U], \text{ if } k \geq 0, \\ [k\alpha_0^U, k\alpha_0^L], \text{ if } k < 0, \end{cases}$ where $k$ is a real number.

The real number $k \in R$ is equivalent to the closed interval $\Lambda_{1_k} = [k, k]$. Let $\Lambda_1 = [\alpha_0^L, \alpha_0^U] \in I(R)$ be a closed interval. We write the sum of an interval $\Lambda_1 \in I(R)$ and a real number $k$ as $\Lambda_1 + \Lambda_{1_k}$. Thus, $\Lambda_1 + k = \Lambda_1 + \Lambda_{1_k} = [\alpha_0^L + k, \alpha_0^U + k]$.

For $\Lambda_1 = [\alpha_0^L, \alpha_0^U]$ and $\Lambda_2 = [\beta_0^L, \beta_0^U]$, the order relation $\preceq_{LU}$ is defined as follows:

$(i)$ $\Lambda_1 \preceq_{LU} \Lambda_2$ if and only if $\alpha_0^L \leq \beta_0^L$ and $\alpha_0^U \leq \beta_0^U$.

$(ii)$ $\Lambda_1 \prec_{LU} \Lambda_2$ if and only if $\Lambda_1 \preceq_{LU} \Lambda_2$ and $\Lambda_1 \neq \Lambda_2$.
It is obvious that, $\Lambda_1 \prec_{LU} \Lambda_2$ if and only if

$$\begin{aligned} & \alpha_0^L < \beta_0^L \text{ and } \quad \alpha_0^U < \beta_0^U, \\ \text{or,} \qquad & \alpha_0^L \leq \beta_0^L \text{ and } \quad \alpha_0^U < \beta_0^U, \\ \text{or,} \qquad & \alpha_0^L < \beta_0^L \text{ and } \quad \alpha_0^U \leq \beta_0^U. \end{aligned}$$

Furthermore, for $\dot{u}, \dot{v} \in R^m$, we use the following notations:

$(i)$. $\dot{u} \prec \dot{v} \Leftrightarrow \dot{u}_i < \dot{v}_i$, for all $i \in \{1, 2, \dots, m\}, \dot{u} \not\prec \dot{v}$ is the negation of $\dot{u} \prec \dot{v}$

$(ii)$. $\dot{u} \preceq \dot{v} \Leftrightarrow \begin{cases} \dot{u}_i \leq \dot{v}_i, \text{ for all } i \in \{1, 2, \dots, m\} \\ \dot{u}_{i_0} < \dot{v}_{i_0}, \text{ for at least one } i_0 \in \{1, 2, \dots, m\}, \end{cases}$
$\dot{u} \not\preceq \dot{v}$ is the negation of $\dot{u} \preceq \dot{v}$.

In the present analysis, we consider the following differentiable vector optimization problem with multiple interval-valued objective function with vanishing constraints (MIVVC):

$$\begin{aligned} \text{MIVVC} \qquad & \min \quad \vartheta(\xi) = (\vartheta_1(\xi), \vartheta_2(\xi), \dots, \vartheta_m(\xi)) \\ & \text{subject to} \\ & \tau_i(\xi) \leq 0, \quad \text{for all } i = 1, 2, \dots, p, \\ & \sigma_i(\xi) = 0, \quad \text{for all } i = 1, 2, \dots, q, \\ & \rho_i(\xi) \geq 0, \quad \text{for all } i = 1, 2, \dots, r, \\ & \omega_i(\xi)\rho_i(\xi) \leq 0, \quad \text{for all } i = 1, 2, \dots, r, \end{aligned}$$

where each $\vartheta_i : R^n \to I(R), i \in T = \{1, 2, \dots, m\}$ is an interval-valued function; that is, $\vartheta_i(\xi) = [\vartheta_i^L(\xi), \vartheta_i^U(\xi)], i \in T$ and $\tau_i(i = 1, 2, \dots, p)$, $\sigma_i(i = 1, 2, \dots, q)$, $\rho_i$, $\omega_i(i = 1, 2, \dots, r)$ are assumed to be continuously differentiable functions from $R^n \to R$. Let us denote $T_\tau := \{1, 2, \dots, p\}$, $T_\sigma := \{1, 2, \dots, q\}$, and $T_r := \{1, 2, \dots, r\}$. The feasible solution set of MIVVC is given by

$$\mathbb{F}_{\mathbb{VC}} = \Big\{ \xi \in R^n \mid \tau_i(\xi) \leq 0, \quad \text{for all } i = 1, 2, \dots, p,$$

$$\left.\begin{aligned}&\sigma_i(\xi)=0, \quad \text{for all } i=1,2,\ldots,q,\\&\rho_i(\xi)\geq 0, \quad \text{for all } i=1,2,\ldots,r,\\&\omega_i(\xi)\rho_i(\xi)\leq 0, \quad \text{for all } i=1,2,\ldots,r\end{aligned}\right\}.$$

**Definition 1.** A point $a \in \mathbb{F}_{\mathbb{VC}}$ is said to be a locally LU-efficient solution of MIVVC, if there exists a neighborhood $U \in \Theta(a)$ such that there is no $\xi \in \mathbb{F}_{\mathbb{VC}} \cap \mathbb{U}$ satisfying

$$\vartheta(\xi) \preceq_{LU} \vartheta(a).$$

**Definition 2.** A point $a \in \mathbb{F}_{\mathbb{VC}}$ is said to be a locally weakly LU-efficient solution of MIVVC, if there exists a neighborhood $U \in \Theta(a)$ such that there is no $\xi \in \mathbb{F}_{\mathbb{VC}} \cap \mathbb{U}$ satisfying

$$\vartheta(\xi) \prec_{LU} \vartheta(a).$$

Let $a \in \mathbb{F}_{\mathbb{VC}}$ be any feasible solution of the MIVVC. The following index sets will be used:

$$T_+(a) := \{i \in T_r \mid \rho_i(a) > 0\},$$
$$T_0(a) := \{i \in T_r \mid \rho_i(a) = 0\}.$$

Furthermore, the index set $T_+$ can be divided into the following subsets

$$T_{+0}(a) := \{i \in T_r \mid \rho_i(a) > 0,\ \omega_i(a) = 0\},$$
$$T_{+-}(a) := \{i \in T_r \mid \rho_i(a) > 0,\ \omega_i(a) < 0\}.$$

Similarly, the index set $T_0$ can be partitioned in the following way

$$T_{0+}(a) := \{i \in T_r \mid \rho_i(a) = 0,\ \omega_i(a) > 0\},$$
$$T_{00}(a) := \{i \in T_r \mid \rho_i(a) = 0,\ \omega_i(a) = 0\},$$
$$T_{0-}(a) := \{i \in T_r \mid \rho_i(a) = 0,\ \omega_i(a) < 0\}.$$

**Definition 3.** A point $a \in \mathbb{F}_{\mathbb{VC}}$ is said to be a strong stationary point of MIVVC if and only if there exists $(\alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in R^m_+ \times R^m_+ \times R^p \times R^q \times R^r \times R^r$ with $\sum_{i\in T}\left(\alpha^L_i + \alpha^U_i\right) = 1, \lambda^\rho_{T_+(a)} = 0, \lambda^\rho_{T_{00}(a)\cup T_{0-}(a)} \geq 0$, $\lambda^\omega_{T_{+-}(a)\cup T_{0+}(a)\cup T_{00}(a)\cup T_{0-}(a)} = 0$ and $\lambda^\omega_{T_{+0}(a)} \geq 0$ such that

$$\begin{aligned}&\sum_{i\in T}\alpha^L_i\nabla\vartheta^L_i(a) + \sum_{i\in T}\alpha^U_i\nabla\vartheta^U_i(a) + \sum_{i\in T_\tau}\lambda^\tau_i\nabla\tau_i(a) + \sum_{i\in T_\sigma}\lambda^\sigma_i\nabla\sigma_i(a)\\&+\sum_{i\in T_{+0}}\lambda^\omega_i\nabla\omega_i(a) - \sum_{i\in T_{0+}\cup T_{00}\cup T_{0-}}\lambda^\rho_i\nabla\rho_i(a) = 0.\end{aligned}$$

**Definition 4.** A point $a \in \mathbb{F}_{\mathbb{VC}}$ is said to be a VC-stationary point of MIVVC if and only if there exists $(\alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in R^m_+ \times R^m_+ \times R^p \times R^q \times$

$R^r \times R^r$ with $\sum\limits_{i\in T}\left(\alpha_i^L+\alpha_i^U\right)=1, \lambda^\rho_{T_+(a)}=0, \lambda^\rho_{T_{00}(a)\cup T_{0-}(a)}\geq 0$, $\lambda^\omega_{T_{+-}(a)\cup T_{0+}(a)\cup T_{00}(a)\cup T_{0-}(a)}=0$ and $\lambda^\omega_{T_{+0}(a)\cup T_{00}(a)}\geq 0$ such that

$$\sum_{i\in T}\alpha_i^L\nabla\vartheta_i^L(a)+\sum_{i\in T}\alpha_i^U\nabla\vartheta_i^U(a)+\sum_{i\in T_\tau}\lambda_i^\tau\nabla\tau_i(a)+\sum_{i\in T_\sigma}\lambda_i^\sigma\nabla\sigma_i(a)$$

$$-\sum_{i\in T_{0+}\cup T_{00}\cup T_{0-}}\lambda_i^\rho\nabla\rho_i(a)+\sum_{i\in T_{+0}}\lambda_i^\omega\nabla\omega_i(a)=0.$$

For $a\in\mathbb{F}_{\mathbb{VC}}$ and $(\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in R^p\times R^q\times R^r\times R^r$, let us define

$$T_\tau^+(a):=\{i\in T_\tau(a)\mid \lambda_i^\tau>0\},$$
$$T_\sigma^+(a):=\{i\in T_\sigma(a)\mid \lambda_i^\sigma>0\},\ T_\sigma^-(a):=\{i\in T_\sigma(a)\mid \lambda_i^\sigma<0\},$$
$$\hat{T}_+^+(a):=\{i\in T_+(a)\mid \lambda_i^\rho>0\},$$
$$\hat{T}_0^+(a):=\{i\in T_0(a)\mid \lambda_i^\rho>0\},\ \hat{T}_0^-(a):=\{i\in T_0(a)\mid \lambda_i^\rho<0\},$$
$$\hat{T}_{0+}^+(a):=\{i\in T_{0+}(a)\mid \lambda_i^\rho>0\},\ \hat{T}_{0+}^-(a):=\{i\in T_{0+}(a)\mid \lambda_i^\rho<0\},$$
$$\hat{T}_{00}^+(a):=\{i\in T_{00}(a)\mid \lambda_i^\rho>0\},\ \hat{T}_{00}^-(a):=\{i\in T_{00}(a)\mid \lambda_i^\rho<0\},$$
$$\hat{T}_{0-}^+(a):=\{i\in T_{0-}(a)\mid \lambda_i^\rho>0\},$$
$$T_{+0}^+(a):=\{i\in T_{+0}(a)\mid \lambda_i^\omega>0\},\ T_{+0}^-(a):=\{i\in T_{+0}(a)\mid \lambda_i^\omega<0\},$$
$$T_{+-}^+(a):=\{i\in T_{+-}(a)\mid \lambda_i^\omega>0\},$$
$$T_{0+}^+(a):=\{i\in T_{0+}(a)\mid \lambda_i^\omega>0\},\ T_{0+}^-(a):=\{i\in T_{0+}(a)\mid \lambda_i^\omega<0\},$$
$$T_{00}^+(a):=\{i\in T_{00}(a)\mid \lambda_i^\omega>0\},\ T_{00}^-(a):=\{i\in T_{00}(a)\mid \lambda_i^\omega<0\},$$
$$T_{0-}^+(a):=\{i\in T_{0-}(a)\mid \lambda_i^\omega>0\}.$$

**Definition 5.** Let $a\in\mathbb{F}_{\mathbb{VC}}$.

$(i)$ The linearized cone of MIVVC at $a$ is
$L(a):=\{d\in\mathbb{R}^n\mid \langle\nabla\tau_i(a),\ d\rangle\leq 0\ (i\in T_\tau), \langle\nabla\sigma_i(a),\ d\rangle=0\ (i\in T_\sigma),$
$\langle\nabla\rho_i(a),\ d\rangle=0\ (i\in T_{0+}), \langle\nabla\rho_i(a),\ d\rangle\geq 0\ (i\in T_{00}\cup T_{0-}),$
$\langle\nabla\omega_i(a),\ d\rangle\leq 0\ (i\in T_{+0})\}$.

$(ii)$ The VC-linearized cone of MIVVC at $a$ is
$L_{VC}(a):=\{d\in\mathbb{R}^n\mid \langle\nabla\tau_i(a),\ d\rangle\leq 0\ (i\in T_\tau), \langle\nabla\sigma_i(a),\ d\rangle=0\ (i\in T_\sigma),$
$\langle\nabla\rho_i(a),\ d\rangle=0\ (i\in T_{0+}), \langle\nabla\rho_i(a),\ d\rangle\geq 0\ (i\in T_{00}\cup T_{0-}),$
$\langle\nabla\omega_i(a),\ d\rangle\leq 0\ (i\in T_{+0}\cup T_{00})\}$.

**Definition 6.** The Abadie constraint qualification (MIVVC-ACQ) is said to hold at $a\in\mathbb{F}_{\mathbb{VC}}$ if

$$L(a)\subseteq\mathbb{T}(\mathbb{F}_{\mathbb{VC}},a).$$

**Definition 7.** The vanishing Abadie constraint qualification (MIVVC-VACQ) is said to hold at $a \in \mathbb{F}_{\mathbb{VC}}$ if

$$L_{VC}(a) \subseteq \mathbb{T}(\mathbb{F}_{\mathbb{VC}}, a).$$

The following theorem can be written in a similar way to Proposition 1 of Tung [25].

**Theorem 1** (Necessary optimality conditions)**.** Let $\xi_0$ be a locally weakly LU-efficient solution of primal problem MIVVC and also further assume that if MIVVC-VACQ holds at $\xi_0$ and the set

$$\Delta_1 := pos\left(\bigcup_{i \in T_\tau} \nabla\tau_i(\xi_0) \ \cup \bigcup_{i \in T_{00} \cup T_{0-}} (-\nabla\rho_i(\xi_0)) \ \cup \bigcup_{i \in T_{+0} \cup T_{00}} \nabla\omega_i(\xi_0)\right)$$
$$+ \ span\left(\bigcup_{i \in T_\sigma} \nabla\sigma_i(\xi_0) \ \cup \bigcup_{i \in T_{0+}} \nabla\rho_i(\xi_0)\right)$$

is closed, then there exists $(\alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in R^m_+ \times R^m_+ \times R^p \times R^q \times R^r \times R^r$ with $\sum\limits_{i \in T} (\alpha^L_i + \alpha^U_i) = 1, \lambda^\rho_{T_+(\xi_0)} = 0, \lambda^\rho_{T_{00}(\xi_0) \cup T_{0-}(\xi_0)} \geq 0, \lambda^\omega_{T_{+-}(\xi_0) \cup T_{0+}(\xi_0) \cup T_{0-}(\xi_0)} = 0$ and $\lambda^\omega_{T_{+0}(\xi_0) \cup T_{00}(\xi_0)} \geq 0$ such that

$$\sum_{i \in T} \alpha^L_i \nabla\vartheta^L_i(a) + \sum_{i \in T} \alpha^U_i \nabla\vartheta^U_i(a) + \sum_{i \in T_\tau} \lambda^\tau_i \nabla\tau_i(a) + \sum_{i \in T_\sigma} \lambda^\sigma_i \nabla\sigma_i(a)$$
$$- \sum_{i \in T_r} \lambda^\rho_i \nabla\rho_i(a) + \sum_{i \in T_r} \lambda^\omega_i \nabla\omega_i(a) = 0.$$

## 3 Sufficient optimality conditions

In this section, we establish sufficient optimality conditions for the problem MIVVC using the concept of generalized convexity.

**Theorem 2.** Let $\xi_0$ be a strong stationary point of MIVVC. Suppose that $\hat{T}^-_{0+} \cup T^+_{+0} = \phi$ and $\tau_i$ $(i \in T_\tau)$, $\sigma_i$ $(i \in T^+_\sigma)$, $-\sigma_i$ $(i \in T^-_\sigma)$, $\omega_i$ $(i \in T^+_{+0})$, $-\rho_i$ $(i \in \hat{T}^+_{0+} \cup \hat{T}^+_{00} \cup \hat{T}^+_{0-})$ are quasiconvex functions at $\xi_0$. If $\sum\limits_{i \in T} \alpha^L_i \vartheta^L_i(\cdot) + \sum\limits_{i \in T} \alpha^U_i \vartheta^U_i(\cdot)$ is pseudoconvex function at $\xi_0$, then $\xi_0$ is an LU-efficient solution of MIVVC.

*Proof.* Since $\xi_0$ is a strong stationary point of MIVVC, there exists $(\alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in R^m_+ \times R^m_+ \times R^p \times R^q \times R^r \times R^r$ with $\sum\limits_{i \in T} \left(\alpha^L_i + \alpha^U_i\right) = 1, \lambda^\rho_{T_+} = 0, \lambda^\rho_{T_{00} \cup T_{0-}} \geq 0, \lambda^\omega_{T_{+-} \cup T_{0+} \cup T_{0-}} = 0$ and $\lambda^\omega_{T_{+0}} \geq 0$ such that

$$\sum_{i\in T}\alpha_i^L\nabla\vartheta_i^L(\xi_0)+\sum_{i\in T}\alpha_i^U\nabla\vartheta_i^U(\xi_0)+\sum_{i\in T_\tau}\lambda_i^\tau\nabla\tau_i(\xi_0)+\sum_{i\in T_\sigma}\lambda_i^\sigma\nabla\sigma_i(\xi_0)$$
$$-\sum_{i\in T_r}\lambda_i^\rho\nabla\rho_i(\xi_0)+\sum_{i\in T_r}\lambda_i^\omega\nabla\omega_i(\xi_0)=0. \tag{1}$$

For an arbitrary $\xi\in\mathbb{F}_{\mathbb{VC}}$, we get $\tau_i(\xi)\leq 0=\tau_i(\xi_0)$ for each $i\in T_\tau$. Thus the quasiconvexity at $\xi_0$ of $\tau_i$ $(i\in T_\tau)$ gives that

$$\langle\nabla\tau_i(\xi_0),\xi-\xi_0\rangle\leq 0,\quad \text{for all } i\in T_\tau,$$

consequently, together with $\lambda_i^\tau\in\mathbb{R}^p$ leads that

$$\left\langle\sum_{i\in T_\tau}\lambda_i^\tau\nabla\tau_i(\xi_0),\xi-\xi_0\right\rangle\leq 0. \tag{2}$$

We deduce from $\xi,\xi_0\in\mathbb{F}_{\mathbb{VC}}$ that $\sigma_i(\xi)=\sigma_i(\xi_0)=0,\quad$ for all $i\in T_\sigma$, and hence,

$$\sigma_i(\xi)\leq\sigma_i(\xi_0)=0,\ \text{ for all } i\in T_\sigma^+ \text{ and } -\sigma_i(\xi)\leq-\sigma_i(\xi_0)=0,\ \text{ for all } i\in T_\sigma^-.$$

The above inequalities along with the quasiconvexity at $\xi_0$ of $\sigma_i$ $(i\in T_\sigma^+)$ and $-\sigma_i$ $(i\in T_\sigma^-)$ ensure that

$$\langle\nabla\sigma_i(\xi_0),\xi-\xi_0\rangle\leq 0,\ \text{ for all } i\in T_\sigma^+ \text{ and } \langle-\nabla\sigma_i(\xi_0),\xi-\xi_0\rangle\leq 0,\ \text{ for all } i\in T_\sigma^-.$$

Thus, taking into account the definitions of $T_\sigma^+,T_\sigma^-$ results in

$$\left\langle\sum_{i\in T_\sigma}\lambda_i^\sigma\nabla\sigma_i(\xi_0),\xi-\xi_0\right\rangle\leq 0. \tag{3}$$

Again, we deduce from $\xi\in\mathbb{F}_{\mathbb{VC}}$ that $-\rho_i(\xi)\leq 0,\omega_i(\xi)\geq 0,\quad$ for all $i\in T_r$. Thus,

$$\begin{cases} -\rho_i(\xi)\leq-\rho_i(\xi_0), & i\in\hat{T}_{0+}^+\cup\hat{T}_{00}^+\cup\hat{T}_{0-}^+,\\ \omega_i(\xi)\leq\omega_i(\xi_0), & i\in T_{+0}^+.\end{cases}$$

Therefore, the quasiconvexity of $-\rho_i, i\in\hat{T}_{0+}^+\cup\hat{T}_{00}^+\cup\hat{T}_{0-}^+$ and $\omega_i, i\in T_{+0}^+$ at $\xi_0$ yields that

$$\langle-\nabla\rho_i(\xi_0),\xi-\xi_0\rangle\leq 0,\ \text{ for all } i\in\hat{T}_{0+}^+\cup\hat{T}_{00}^+\cup\hat{T}_{0-}^+, \tag{4}$$
$$\langle\nabla\omega_i(\xi_0),\xi-\xi_0\rangle\leq 0,\ \text{ for all } i\in T_{+0}^+. \tag{5}$$

As $T_{+0}^+\cup\hat{T}_{0+}^-=\phi$, we presume from (1)–(5) that

$$\left\langle \sum_{i\in T} \alpha_i^L \nabla\vartheta_i^L(\xi_0) + \sum_{i\in T} \alpha_i^U \nabla\vartheta_i^U(\xi_0), \xi - \xi_0 \right\rangle$$
$$= \left\langle \sum_{i\in T_\tau} \lambda_i^\tau \nabla\tau_i(\xi_0) + \sum_{i\in T_\sigma} \lambda_i^\sigma \nabla\sigma_i(\xi_0) - \sum_{i\in T_r} \lambda_i^\rho \nabla\rho_i(\xi_0) + \right.$$
$$\left. \sum_{i\in T_r} \lambda_i^\omega \nabla\omega_i(\xi_0), \xi - \xi_0 \right\rangle \geq 0, \tag{6}$$

for all $\xi \in \mathbb{F}_{\mathbb{VC}}$.
On the contrary, suppose $\xi_0$ is not an LU-efficient solution of MIVVC. This leads to the existence of a feasible point $\tilde{\xi} \in \mathbb{F}_{\mathbb{VC}}$ such that

$$\vartheta(\tilde{\xi}) \preceq_{LU} \vartheta(\xi_0);$$

that is, for $i \in T$,

$$\begin{cases} \vartheta_i^L(\tilde{\xi}) < \vartheta_i^L(\xi_0) \\ \vartheta_i^U(\tilde{\xi}) \leq \vartheta_i^U(\xi_0) \end{cases}, \text{or} \begin{cases} \vartheta_i^L(\tilde{\xi}) \leq \vartheta_i^L(\xi_0) \\ \vartheta_i^U(\tilde{\xi}) < \vartheta_i^U(\xi_0) \end{cases}, \text{or} \begin{cases} \vartheta_i^L(\tilde{\xi}) < \vartheta_i^L(\xi_0) \\ \vartheta_i^U(\tilde{\xi}) < \vartheta_i^U(\xi_0) \end{cases}.$$

From the fact $\alpha^L \in R_+^m, \alpha^U \in R_+^m$ with $\sum\limits_{i\in T} \left(\alpha_i^L + \alpha_i^U\right) = 1$, then above inequalities together yield

$$\sum_{i\in T} \alpha_i^L \vartheta_i^L(\tilde{\xi}) + \sum_{i\in T} \alpha_i^U \vartheta_i^U(\tilde{\xi}) < \sum_{i\in T} \alpha_i^L \vartheta_i^L(\xi_0) + \sum_{i\in T} \alpha_i^U \vartheta_i^U(\xi_0),$$

which by the pseudoconvexity of $\sum\limits_{i\in T} \alpha_i^L \vartheta_i^L(\cdot) + \sum\limits_{i\in T} \alpha_i^U \vartheta_i^U(\cdot)$, we obtain

$$\left\langle \sum_{i\in T} \alpha_i^L \nabla\vartheta_i^L(\xi_0) + \sum_{i\in T} \alpha_i^U \nabla\vartheta_i^U(\xi_0), \tilde{\xi} - \xi_0 \right\rangle < 0,$$

contradicting to (6). □

**Theorem 3.** Let $\xi_0$ be a strong stationary point of MIVVC. Suppose that $\hat{T}_{0+}^- \cup T_{+0}^+ = \phi$ and $\tau_i\ (i \in T_\tau), \sigma_i\ (i \in T_\sigma^+), -\sigma_i\ (i \in T_\sigma^-), \omega_i\ (i \in T_{+0}^+), -\rho_i\ (i \in \hat{T}_{0+}^+ \cup \hat{T}_{00}^+ \cup \hat{T}_{0-}^+)$ are quasiconvex functions at $\xi_0$. If $\sum\limits_{i\in T} \alpha_i^L \vartheta_i^L(\cdot) + \sum\limits_{i\in T} \alpha_i^U \vartheta_i^U(\cdot)$ is strictly pseudoconvex function at $\xi_0$, then $\xi_0$ is a weakly LU-efficient solution of MIVVC.

*Proof.* Similar to the proof of Theorem 2, we get

$$\left\langle \sum_{i\in T} \alpha_i^L \nabla\vartheta_i^L(\xi_0) + \sum_{i\in T} \alpha_i^U \nabla\vartheta_i^U(\xi_0), \xi - \xi_0 \right\rangle$$

$$= \left\langle \sum_{i\in T_\tau} \lambda_i^\tau \nabla\tau_i(\xi_0) + \sum_{i\in T_\sigma} \lambda_i^\sigma \nabla\sigma_i(\xi_0) - \sum_{i\in T_r} \lambda_i^\rho \nabla\rho_i(\xi_0) + \sum_{i\in T_r} \lambda_i^\omega \nabla\omega_i(\xi_0), \xi - \xi_0 \right\rangle \geq 0. \tag{7}$$

Reasoning by contraposition, assume that $\xi_0$ is not a weakly LU-efficient solution. Then there exists a feasible point $\tilde{\xi}$ satisfying

$$\vartheta(\tilde{\xi}) \prec_{LU} \vartheta(\xi_0);$$

that is, for $i \in T$,

$$\begin{cases} \vartheta_i^L(\tilde{\xi}) < \vartheta_i^L(\xi_0), \\ \vartheta_i^U(\tilde{\xi}) < \vartheta_i^U(\xi_0). \end{cases}$$

From the fact that $\alpha^L \in R_+^m, \alpha^U \in R_+^m$ with $\sum\limits_{i\in T} \left(\alpha_i^L + \alpha_i^U\right) = 1$, and by the above inequalities, we get

$$\sum_{i\in T} \alpha_i^L \vartheta_i^L(\tilde{\xi}) + \sum_{i\in T} \alpha_i^U \vartheta_i^U(\tilde{\xi}) < \sum_{i\in T} \alpha_i^L \vartheta_i^L(\xi_0) + \sum_{i\in T} \alpha_i^U \vartheta_i^U(\xi_0).$$

By using the strictly pseudoconvexity of $\sum\limits_{i\in T} \alpha_i^L \vartheta_i^L(\cdot) + \sum\limits_{i\in T} \alpha_i^U \vartheta_i^U(\cdot)$ at $\tilde{\xi}$ on $\mathbb{F}_{\mathbb{VC}}$, we get

$$\left\langle \sum_{i\in T} \alpha_i^L \nabla\vartheta_i^L(\xi_0) + \sum_{i\in T} \alpha_i^U \nabla\vartheta_i^U(\xi_0), \tilde{\xi} - \xi_0 \right\rangle < 0,$$

contradicting to (7). □

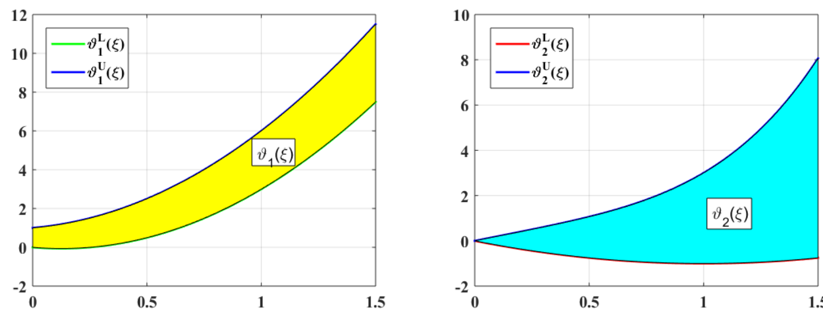
Now, we verify the sufficient optimality conditions by an example.

**Example 1.** Consider the following multiobjective interval-valued programming problem with vanishing constraints (MIVVC-1):

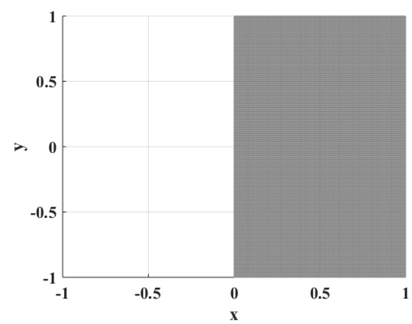
$$\begin{aligned} MIVVC-1\ \mathbb{R}_+ -\min\ \ \vartheta(\xi) &= (\vartheta_1(\xi), \vartheta_2(\xi)) \\ &= \left([4\xi^2 - \xi, 4\xi^2 + \xi + 1],\ [\xi^2 - 2\xi, \xi^4 + 2\xi]\right) \\ &\text{subject to} \\ &\rho_1(\xi) = \xi \geq 0, \\ &\omega_1(\xi)\rho_1(\xi) = (-1 - \xi)\xi \leq 0, \end{aligned}$$

where $\vartheta_1^L(\xi) = 4\xi^2 - \xi, \vartheta_2^L(\xi) = \xi^2 - 2\xi, \vartheta_1^U(\xi) = 4\xi^2 + \xi + 1, \vartheta_2^U(\xi) = \xi^4 + 2\xi$, which is in the form of MIVVC with $m = 2, n = 1, p = q = 0$, and $r = 1$.

The feasible region of MIVVC-1 is $\mathbb{F}_{\mathbb{VC}1} = \{\xi \in R \mid \rho_1(\xi) \geq 0,\ \omega_1(\xi)\rho_1(\xi) \leq 0\}$.




(a) Graphical view of $\vartheta_1(\xi) = [\vartheta_1^L(\xi), \vartheta_1^U(\xi)]$ (b) Graphical view of $\vartheta_2(\xi) = [\vartheta_2^L(\xi), \vartheta_2^U(\xi)]$




Graphical view of the feasible region of MIVVC-1

Note that $\xi_0 = 0$ is a feasible solution of MIVVC-1. By simple calculations, we get $\mathbb{T}(\mathbb{F}_{\mathbb{VC}1}, \xi_0) = \mathbb{F}_{\mathbb{VC}1}$, $\nabla\vartheta_1^L(\xi_0) = \{-1\}$, $\nabla\vartheta_2^L(\xi_0) = \{-2\}$, $\nabla\vartheta_1^U(\xi_0) = \{1\}$, $\nabla\vartheta_2^U(\xi_0) = \{2\}$, $\nabla\rho_1(\xi_0) = \{1\}$, $\nabla\omega_1(\xi_0) = \{-1\}$, $T_+ = T_{0+} = T_{0-} = \phi$, $T_{00} = \{1\}$,

$$\left(\bigcup_{i\in T_{00}} (-\nabla\rho_i(\xi_0))\right)^- = \{\xi \in R \mid \xi \geq 1\},$$

$$\left(\bigcup_{i\in T_{00}} (\nabla\omega_i(\xi_0))\right)^- = \{\xi \in R \mid \xi \geq 1\},$$

$$\left(\bigcup_{i\in T_{00}} (-\nabla\rho_i(\xi_0))\right)^- \cap \left(\bigcup_{i\in T_{00}} \nabla\omega_i(\xi_0)\right)^- = \{\xi \in R \mid \xi \geq 1\}.$$

Hence,

$$\left(\bigcup_{i\in T_{00}}(-\nabla\rho_i(\xi_0))\right)^{-}\cap\left(\bigcup_{i\in T_{00}}(\nabla\omega_i(\xi_0))\right)^{-}\subset T(\mathbb{F}_{\mathbb{VC}1},\xi_0),$$

yields that MIVVC-VACQ satisfies at $\xi_0$. Moreover,

$$\Delta_1 := pos\left(\bigcup_{i\in T_{00}}(-\nabla\rho_i(\xi_0))\cup\bigcup_{i\in T_{00}}\nabla\omega_i(\xi_0)\right)=\{\xi\in R\mid \xi\geq -1\}$$

is closed. Thus, all assumptions in Theorem 1 are satisfied. Then there exist $\alpha_1^L=\alpha_2^L=\frac{1}{2},\alpha_1^U=\alpha_2^U=\frac{1}{2},\lambda_1^\rho=0,\lambda_1^\omega=0$ such that (1) is satisfied at $\xi_0=0$ for the problem MIVVC-1. Furthermore, it can be easily observed that the hypothesis of Theorem 3 hold at $\xi_0=0$, and owing to the fact that for $\xi\neq\xi_0$, $\vartheta(\xi)\not<_{LU}\vartheta(\xi_0)$. Then, we assert that $\xi_0$ is a locally weakly LU-efficient solution of MIVVC-1.

## 4 The Wolfe type duality

In this section, we present the Wolfe type dual problem to MIVVC assuming that all the functions to be convex. For a given $\bar{u}$, $\Theta(\bar{u})$ is the system of the neighborhoods of $\bar{u}$. For $\xi_0\in\mathbb{F}_{\mathbb{VC}}$, $(u,\alpha^L,\ \alpha^U,\ \lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in R^n\times R_+^m\times R_+^m\times R^p\times R^q\times R^r\times R^r$ with $\sum\limits_{i\in T}(\alpha_i^L+\alpha_i^U)=1,\lambda^\rho_{T_+(\xi_0)}\geq 0,\lambda^\omega_{T_{0+}(\xi_0)}\leq 0$, and $\lambda^\omega_{T_{+-}(\xi_0)\cup T_{0-}(\xi_0)}\geq 0$, we define

$$\begin{aligned}\mathcal{L}(u,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho) = \Bigg(&\vartheta_1(u)+\Bigg(\sum_{i\in T_\tau}\lambda_i^\tau\tau_i(u)+\sum_{i\in T_\sigma}\lambda_i^\sigma\sigma_i(u)\\&-\sum_{i\in T_r}\lambda_i^\rho\rho_i(u)+\sum_{i\in T_r}\lambda_i^\omega\omega_i(u)\Bigg)e+\cdots\\&+\vartheta_m(u)+\Bigg(\sum_{i\in T_\tau}\lambda_i^\tau\tau_i(u)+\sum_{i\in T_\sigma}\lambda_i^\sigma\sigma_i(u)-\\&\sum_{i\in T_r}\lambda_i^\rho\rho_i(u)+\sum_{i\in T_r}\lambda_i^\omega\omega_i(u)\Bigg)e\Bigg),\end{aligned}$$

where $e:=(1,\ldots,1)\in R^m$. We consider the Wolfe type dual problem as follows:

$$\begin{aligned}(WD_w(\xi_0))\qquad & R_+^m-\max\mathcal{L}(u,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\\&\text{subject to}\\&\sum_{i\in T}\alpha_i^L\nabla\vartheta_i^L(u)+\sum_{i\in T}\alpha_i^U\nabla\vartheta_i^U(u)+\sum_{i\in T_\tau}\lambda_i^\tau\nabla\tau_i(u)\end{aligned}$$

$$+\sum_{i\in T_\sigma}\lambda_i^\sigma\nabla\sigma_i(u)-\sum_{i\in T_r}\lambda_i^\rho\nabla\rho_i(u)+\sum_{i\in T_r}\lambda_i^\omega\nabla\omega_i(u)=0,$$
$$\sum_{i\in T}\left(\alpha_i^L+\alpha_i^U\right)=1,\lambda^\rho_{T_+(\xi_0)}\geq 0,\lambda^\omega_{T_{0+}(\xi_0)}\leq 0 \text{ and}$$
$$\lambda^\omega_{T_{+-}(\xi_0)\cup T_{0-}(\xi_0)}\geq 0,(u,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in$$
$$R^n\times R^m_+\times R^m_+\times R^p\times R^q\times R^r\times R^r.$$

The feasible set of $(WD_w(\xi_0))$ is defined by

$$\begin{aligned}\mathbb{F}_{\mathbb{VC}w}(\xi_0):=\Bigg\{&(u,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in R^n\times R^m_+\times R^m_+\times R^p\times R^q\times R^r\\ &\times R^r|\sum_{i\in T}\left(\alpha_i^L+\alpha_i^U\right)=1,\lambda^\rho_{T_+(\xi_0)}\geq 0,\lambda^\omega_{T_{0+}(\xi_0)}\leq 0,\text{ and}\\ &\lambda^\omega_{T_{+-}(\xi_0)\cup T_{0-}(\xi_0)}\geq 0,\sum_{i\in T}\alpha_i^L\nabla\vartheta_i^L(u)+\sum_{i\in T}\alpha_i^U\nabla\vartheta_i^U(u)+\\ &\sum_{i\in T_\tau}\lambda_i^\tau\nabla\tau_i(u)+\sum_{i\in T_\sigma}\lambda_i^\sigma\nabla\sigma_i(u)-\sum_{i\in T_r}\lambda_i^\rho\nabla\rho_i(u)+\\ &\sum_{i\in T_r}\lambda_i^\omega\nabla\omega_i(u)=0\Bigg\}.\end{aligned}$$

The Wolfe type duality problem of MIVVC, which is not dependent on $\xi_0$, is

$$\begin{aligned}(WD_w):\qquad & R^m_+-\max\mathcal{L}(\psi,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\\ &\text{subject to}\\ &(\psi,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in\mathbb{F}_{\mathbb{VC}w}:=\bigcap_{\xi_0\in\mathbb{F}_{\mathbb{VC}}}\mathbb{F}_{\mathbb{VC}w}(\xi_0).\end{aligned}$$

**Definition 8.** Let $\xi_0\in\mathbb{F}_{\mathbb{VC}}$. Then $(\bar{u},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda}^\tau,\bar{\lambda}^\sigma,\bar{\lambda}^\omega,\bar{\lambda}^\rho)\in\mathbb{F}_{\mathbb{VC}w}(\xi_0)$ is a locally LU-efficient solution of $(WD_w(\xi_0))$ (locally weakly LU-efficient solution of $(WD_w(\xi_0))$) if there exists $U\in\Theta(\bar{u})$ such that there is no $(u,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in\mathbb{F}_{\mathbb{VC}w}(\xi_0)\cap U$ satisfying

$$\begin{aligned}&\mathcal{L}(\bar{u},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda}^\tau,\bar{\lambda}^\sigma,\bar{\lambda}^\omega,\bar{\lambda}^\rho)\preceq_{LU}\mathcal{L}(u,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho),\\ &\left(\mathcal{L}(\bar{u},\bar{\alpha}^L,\bar{\alpha}^U,\bar{\lambda}^\tau,\bar{\lambda}^\sigma,\bar{\lambda}^\omega,\bar{\lambda}^\rho)\prec_{LU}\mathcal{L}(u,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\right).\end{aligned}$$

**Theorem 4** (Weak Duality)**.** Let $\xi\in\mathbb{F}_{\mathbb{VC}}$ and let $(\psi,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in\mathbb{F}_{\mathbb{VC}w}$. Suppose that $\tau_i(i\in T_\tau^+(\xi)),\sigma_i(i\in T_\sigma^+(\xi)),-\sigma_i(i\in T_\sigma^-(\xi)),\rho_i(i\in\hat{T}_0^-(\xi)),-\rho_i(i\in\hat{T}_+^+(\xi)\cup\hat{T}_0^+(\xi)),\omega_i(i\in T_{+0}^+(\xi)\cup T_{+-}^+(\xi)\cup T_{00}^+(\xi)\cup T_{0-}^+(\xi))$, $-\omega_i(i\in T_{+0}^-(\xi)\cup T_{0+}^-(\xi)\cup T_{00}^-(\xi))$ are convex functions at $\psi$,

$(i)$ If $\vartheta_i^L,\vartheta_i^U(i\in T)$ are convex functions at $\psi$, then

$$\vartheta(\xi) \nless_{LU} \mathcal{L}(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho).$$

$(ii)$ If $\vartheta_i^L, \vartheta_i^U (i \in T)$ are strictly convex functions at $\psi$, then

$$\vartheta(\xi) \nleq_{LU} \mathcal{L}(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho).$$

*Proof.* For $\xi \in \mathbb{F}_{\mathbb{VC}}$ and $(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in \mathbb{F}_{\mathbb{VC}w} = \bigcap_{\xi_0 \in \mathbb{F}_{\mathbb{VC}}} \mathbb{F}_{\mathbb{VC}w}(\xi_0)$, one gets

$$\tau_i(\xi) \leq 0 \ (i \in T_\tau), \sigma_i(\xi) = 0 \ (i \in T_\sigma), \rho_i(\xi) \geq 0 \ (i \in T_r), \omega_i(\xi)\rho_i(\xi) \leq 0 \ (i \in T_r) \tag{8}$$

and

$$\sum_{i \in T} \alpha_i^L \nabla \vartheta_i^L(\psi) + \sum_{i \in T} \alpha_i^U \nabla \vartheta_i^U(\psi) + \sum_{i \in T_\tau} \lambda_i^\tau \nabla \tau_i(\psi) + \sum_{i \in T_\sigma} \lambda_i^\sigma \nabla \sigma_i(\psi) - \sum_{i \in T_r} \lambda_i^\rho \nabla \rho_i(\psi) + \sum_{i \in T_r} \lambda_i^\omega \nabla \omega_i(\psi) = 0 \tag{9}$$

with

$$\sum_{i \in T} \left(\alpha_i^L + \alpha_i^U\right) = 1, \lambda^\rho_{T_+(\xi)} \geq 0, \lambda^\omega_{T_{0+}(\xi)} \leq 0, \lambda^\omega_{T_{+-}(\xi) \cup T_{0-}(\xi)} \geq 0. \tag{10}$$

Therefore we conclude from (8), based on the convexity of $\tau_i \ (i \in T_\tau^+(\xi)), \sigma_i \ (i \in T_\sigma^+(\xi)), -\sigma_i \ (i \in T_\sigma^-(\xi)), \rho_i \ (i \in \hat{T}_0^-(\xi)), -\rho_i \ (i \in \hat{T}_+^+(\xi) \cup \hat{T}_0^+(\xi)), \omega_i \ (i \in T_{+0}^+(\xi) \cup T_{+-}^+(\xi) \cup T_{00}^+(\xi) \cup T_{0-}^+(\xi)), -\omega_i \ (i \in T_{+0}^-(\xi) \cup T_{0+}^-(\xi) \cup T_{00}^-(\xi))$ at $\psi$ and by the definitions of index sets that

$$\tau_i(\psi) + \langle \nabla \tau_i(\psi), \xi - \psi \rangle \leq \tau_i(\xi) \leq 0, \lambda_i^\tau > 0, \text{ for all } i \in T_\tau^+(\xi),$$

$$\sigma_i(\psi) + \langle \nabla \sigma_i(\psi), \xi - \psi \rangle \leq \sigma_i(\xi) = 0, \lambda_i^\sigma > 0, \text{ for all } i \in T_\sigma^+(\xi),$$

$$-\sigma_i(\psi) + \langle -\nabla \sigma_i(\psi), \xi - \psi \rangle \leq -\sigma_i(\xi) = 0, \lambda_i^\sigma < 0, \text{ for all } i \in T_\sigma^-(\xi),$$

$$\rho_i(\psi) + \langle \nabla \rho_i(\psi), \xi - \psi \rangle \leq \rho_i(\xi) = 0, \lambda_i^\rho < 0, \text{ for all } i \in \hat{T}_0^-(\xi),$$

$$-\rho_i(\psi) + \langle -\nabla \rho_i(\psi), \xi - \psi \rangle \leq -\rho_i(\xi) < 0, \lambda_i^\rho > 0, \text{ for all } i \in \hat{T}_+^+(\xi),$$

$$-\rho_i(\psi) + \langle -\nabla \rho_i(\psi), \xi - \psi \rangle \leq -\rho_i(\xi) < 0, \lambda_i^\rho > 0, \text{ for all } i \in \hat{T}_0^+(\xi),$$

$$\omega_i(\psi) + \langle \nabla \omega_i(\psi), \xi - \psi \rangle \leq \omega_i(\xi) = 0, \lambda_i^\omega > 0, \text{ for all } i \in T_{+0}^+(\xi) \cup T_{00}^+(\xi),$$

$$\omega_i(\psi) + \langle \nabla \omega_i(\psi), \xi - \psi \rangle \leq \omega_i(\xi) < 0, \lambda_i^\omega > 0, \text{ for all } i \in T_{+-}^+(\xi) \cup T_{0-}^+(\xi),$$

$$-\omega_i(\psi) + \langle -\nabla \omega_i(\psi), \xi - \psi \rangle \leq -\omega_i(\xi) = 0, \lambda_i^\omega > 0, \text{ for all } i \in T_{+0}^-(\xi) \cup T_{00}^-(\xi),$$

$$-\omega_i(\psi) + \langle -\nabla \omega_i(\psi), \xi - \psi \rangle \leq -\omega_i(\xi) < 0, \lambda_i^\omega < 0, \text{ for all } i \in T_{0+}^-(\xi).$$

The above inequalities imply that

$$\sum_{i\in T_\tau}\lambda_i^\tau\tau_i(\psi)+\sum_{i\in T_\sigma}\lambda_i^\sigma\sigma_i(\psi)-\sum_{i\in T_r}\lambda_i^\rho\rho_i(\psi)+\sum_{i\in T_r}\lambda_i^\omega\omega_i(\psi)$$
$$+\left\langle\sum_{i\in T_\tau}\lambda_i^\tau\nabla\tau_i(\psi)+\sum_{i\in T_\sigma}\lambda_i^\sigma\nabla\sigma_i(\psi)-\sum_{i\in T_r}\lambda_i^\rho\nabla\rho_i(\psi)\right.$$
$$\left.+\sum_{i\in T_r}\lambda_i^\omega\nabla\omega_i(\psi),\xi-\psi\right\rangle\leq 0. \tag{11}$$

By using (9) and (11), we obtain

$$\begin{aligned}&\left\langle\sum_{i\in T}\alpha_i^L\nabla\vartheta_i^L(\psi)+\sum_{i\in T}\alpha_i^U\nabla\vartheta_i^U(\psi),\xi-\psi\right\rangle\\&=-\left\langle\sum_{i\in T_\tau}\lambda_i^\tau\nabla\tau_i(\psi)+\sum_{i\in T_\sigma}\lambda_i^\sigma\nabla\sigma_i(\psi)-\sum_{i\in T_r}\lambda_i^\rho\nabla\rho_i(\psi)+\sum_{i\in T_r}\lambda_i^\omega\nabla\omega_i(\psi),\xi-\psi\right\rangle\\&\geq\sum_{i\in T_\tau}\lambda_i^\tau\tau_i(\psi)+\sum_{i\in T_\sigma}\lambda_i^\sigma\sigma_i(\psi)-\sum_{i\in T_r}\lambda_i^\rho\rho_i(\psi)+\sum_{i\in T_r}\lambda_i^\omega\omega_i(\psi).\end{aligned} \tag{12}$$

$(i)$ Suppose to the contrary that

$$\vartheta(\xi)\prec_{LU}\mathcal{L}(\psi,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho). \tag{13}$$

Then, we deduce from (13) and $\alpha^L\in R_+^m,\alpha^U\in R_+^m$ that

$$\begin{aligned}&\left\langle\alpha^L,\vartheta^L(\xi)-\mathcal{L}(\psi,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\right\rangle<0,\\&\left\langle\alpha^U,\vartheta^U(\xi)-\mathcal{L}(\psi,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\right\rangle<0,\end{aligned}$$

which is equivalent to

$$\sum_{i=1}^m\alpha_i^L\left(\vartheta_i^L(\xi)-\vartheta_i^L(\psi)\right)-\sum_{i=1}^m\alpha_i^L\left(\sum_{i\in T_\tau}\lambda_i^\tau\tau_i(\psi)+\sum_{i\in T_\sigma}\lambda_i^\sigma\sigma_i(\psi)\right.$$
$$\left.-\sum_{i\in T_r}\lambda_i^\rho\rho_i(\psi)+\sum_{i\in T_r}\lambda_i^\omega\omega_i(\psi)\right)<0,$$
$$\sum_{i=1}^m\alpha_i^U\left(\vartheta_i^U(\xi)-\vartheta_i^U(\psi)\right)-\sum_{i=1}^m\alpha_i^U\left(\sum_{i\in T_\tau}\lambda_i^\tau\tau_i(\psi)+\sum_{i\in T_\sigma}\lambda_i^\sigma\sigma_i(\psi)\right.$$
$$\left.-\sum_{i\in T_r}\lambda_i^\rho\rho_i(\psi)+\sum_{i\in T_r}\lambda_i^\omega\omega_i(\psi)\right)<0.$$

On adding, we have

$$\sum_{i=1}^m\alpha_i^L\left(\vartheta_i^L(\xi)-\vartheta_i^L(\psi)\right)+\sum_{i=1}^m\alpha_i^U\left(\vartheta_i^U(\xi)-\vartheta_i^U(\psi)\right)-\sum_{i=1}^m\left(\alpha_i^L+\alpha_i^U\right)$$

$$\left( \sum_{i\in T_\tau} \lambda_i^\tau \tau_i(\psi) + \sum_{i\in T_\sigma} \lambda_i^\sigma \sigma_i(\psi) - \sum_{i\in T_r} \lambda_i^\rho \rho_i(\psi) + \sum_{i\in T_r} \lambda_i^\omega \omega_i(\psi) \right) < 0.$$

It follows from $\sum_{i=1}^{m} \left(\alpha_i^L + \alpha_i^U\right) = 1$ that

$$\sum_{i=1}^{m} \alpha_i^L \left(\vartheta_i^L(\xi) - \vartheta_i^L(\psi)\right) + \sum_{i=1}^{m} \alpha_i^U \left(\vartheta_i^U(\xi) - \vartheta_i^U(\psi)\right)$$
$$< \left( \sum_{i\in T_\tau} \lambda_i^\tau \tau_i(\psi) + \sum_{i\in T_\sigma} \lambda_i^\sigma \sigma_i(\psi) - \sum_{i\in T_r} \lambda_i^\rho \rho_i(\psi) + \sum_{i\in T_r} \lambda_i^\omega \omega_i(\psi) \right). \tag{14}$$

From the convexity of $\vartheta_i^L, \vartheta_i^U (i \in T)$ at $\psi$, we get

$$\begin{aligned} \left\langle \nabla\vartheta_i^L(\psi), \xi - \psi \right\rangle &\le \vartheta_i^L(\xi) - \vartheta_i^L(\psi), \quad \text{for all } i \in T,\\ \left\langle \nabla\vartheta_i^U(\psi), \xi - \psi \right\rangle &\le \vartheta_i^U(\xi) - \vartheta_i^U(\psi), \quad \text{for all } i \in T, \end{aligned}$$

which leads that

$$\left\langle \sum_{i=1}^{m} \alpha_i^L \nabla\vartheta_i^L(\psi), \xi - \psi \right\rangle \le \sum_{i=1}^{m} \alpha_i^L \left(\vartheta_i^L(\xi) - \vartheta_i^L(\psi)\right),$$
$$\left\langle \sum_{i=1}^{m} \alpha_i^U \nabla\vartheta_i^U(\psi), \xi - \psi \right\rangle \le \sum_{i=1}^{m} \alpha_i^U \left(\vartheta_i^U(\xi) - \vartheta_i^U(\psi)\right). \tag{15}$$

We deduce from the above inequalities and (14) that

$$\left\langle \sum_{i=1}^{m} \alpha_i^L \nabla\vartheta_i^L(\psi) + \sum_{i=1}^{m} \alpha_i^U \nabla\vartheta_i^U(\psi), \xi - \psi \right\rangle$$
$$< \left( \sum_{i\in T_\tau} \lambda_i^\tau \tau_i(\psi) + \sum_{i\in T_\sigma} \lambda_i^\sigma \sigma_i(\psi) - \sum_{i\in T_r} \lambda_i^\rho \rho_i(\psi) + \sum_{i\in T_r} \lambda_i^\omega \omega_i(\psi) \right),$$

which contradicts with (12).

$(ii)$ Reasoning by contraposition, suppose that

$$\vartheta(\xi) \preceq_{LU} \mathcal{L}(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho). \tag{16}$$

We deduce from (16) and $\alpha^L \in R_+^m, \alpha^U \in R_+^m$ that

$$\begin{cases} \left\langle \alpha^L, \vartheta^L(\xi) - \mathcal{L}(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \right\rangle < 0, \\ \left\langle \alpha^U, \vartheta^U(\xi) - \mathcal{L}(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \right\rangle \le 0, \end{cases}$$

or

$$\begin{cases} \left\langle \alpha^L, \vartheta^L(\xi) - \mathcal{L}(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \right\rangle \leq 0, \\ \left\langle \alpha^U, \vartheta^U(\xi) - \mathcal{L}(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \right\rangle < 0, \end{cases}$$

or

$$\begin{cases} \left\langle \alpha^L, \vartheta^L(\xi) - \mathcal{L}(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \right\rangle < 0, \\ \left\langle \alpha^U, \vartheta^U(\xi) - \mathcal{L}(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \right\rangle < 0, \end{cases}$$

which is equivalent to

$$\begin{aligned} &\sum_{i=1}^m \alpha_i^L \left( \vartheta_i^L(\xi) - \vartheta_i^L(\psi) \right) + \sum_{i=1}^m \alpha_i^U \left( \vartheta_i^U(\xi) - \vartheta_i^U(\psi) \right) \\ &\leq \sum_{i=1}^m \left( \alpha_i^L + \alpha_i^U \right) \left( \sum_{i \in T_\tau} \lambda_i^\tau \tau_i(\psi) + \sum_{i \in T_\sigma} \lambda_i^\sigma \sigma_i(\psi) - \sum_{i \in T_r} \lambda_i^\rho \rho_i(\psi) + \sum_{i \in T_r} \lambda_i^\omega \omega_i(\psi) \right). \end{aligned}$$

It follows from $\sum_{i=1}^m \left( \alpha_i^L + \alpha_i^U \right) = 1$ that

$$\begin{aligned} &\sum_{i=1}^m \alpha_i^L \left( \vartheta_i^L(\xi) - \vartheta_i^L(\psi) \right) + \sum_{i=1}^m \alpha_i^U \left( \vartheta_i^U(\xi) - \vartheta_i^U(\psi) \right) \\ &\quad \leq \left( \sum_{i \in T_\tau} \lambda_i^\tau \tau_i(\psi) + \sum_{i \in T_\sigma} \lambda_i^\sigma \sigma_i(\psi) - \sum_{i \in T_r} \lambda_i^\rho \rho_i(\psi) + \sum_{i \in T_r} \lambda_i^\omega \omega_i(\psi) \right). \end{aligned} \tag{17}$$

From the strict convexity of $\vartheta_i^L, \vartheta_i^U (i \in T)$ at $\psi$, we get

$$\begin{aligned} \left\langle \nabla \vartheta_i^L(\psi), \xi - \psi \right\rangle < \vartheta_i^L(\xi) - \vartheta_i^L(\psi), \quad &\text{for all } i \in T \\ \left\langle \nabla \vartheta_i^U(\psi), \xi - \psi \right\rangle < \vartheta_i^U(\xi) - \vartheta_i^U(\psi), \quad &\text{for all } i \in T, \end{aligned}$$

which leads that

$$\begin{aligned} \left\langle \sum_{i=1}^m \alpha_i^L \nabla \vartheta_i^L(\psi), \xi - \psi \right\rangle &< \sum_{i=1}^m \alpha_i^L \left( \vartheta_i^L(\xi) - \vartheta_i^L(\psi) \right), \\ \left\langle \sum_{i=1}^m \alpha_i^U \nabla \vartheta_i^U(\psi), \xi - \psi \right\rangle &< \sum_{i=1}^m \alpha_i^U \left( \vartheta_i^U(\xi) - \vartheta_i^U(\psi) \right). \end{aligned} \tag{18}$$

It follows from (17) and (18) that

$$\begin{aligned} &\left\langle \sum_{i=1}^m \alpha_i^L \nabla \vartheta_i^L(\psi) + \sum_{i=1}^m \alpha_i^U \nabla \vartheta_i^U(\psi), \xi - \psi \right\rangle \\ &\quad < \left( \sum_{i \in T_\tau} \lambda_i^\tau \tau_i(\psi) + \sum_{i \in T_\sigma} \lambda_i^\sigma \sigma_i(\psi) - \sum_{i \in T_r} \lambda_i^\rho \rho_i(\psi) + \sum_{i \in T_r} \lambda_i^\omega \omega_i(\psi) \right), \end{aligned}$$

contradicting to (12). □

**Example 2.** Consider the following multiobjective interval-valued programming problem with vanishing constraints (MIVVC-2):

$$
\begin{aligned}
MIVVC-2 \qquad \mathbb{R}_+ - \min\ & \vartheta(\xi) = (\vartheta_1(\xi), \vartheta_2(\xi)) \\
& = \big([2\xi + 4, e^{2\xi}],\ [-2\xi + 1, -\xi - 1]\big) \\
& \text{subject to} \\
& \rho_1(\xi) = \xi \geq 0, \\
& \omega_1(\xi)\rho_1(\xi) = -e^{1-\xi}\xi \leq 0,
\end{aligned}
$$

where $\vartheta_1^L(\xi) = 2\xi + 4, \vartheta_2^L(\xi) = -2\xi + 1, \vartheta_1^U(\xi) = e^{2\xi}, \vartheta_2^U(\xi) = -\xi - 1$, which is in the form of MIVVC with $m = n = 1$, $p = q = 0$ and $r = 1$. The feasible set of MIVVC-2 is $\mathbb{F}_{\mathbb{VC}2} = \{\xi \in R \mid \rho_1(\xi) \geq 0,\ \omega_1(\xi)\rho_1(\xi) \leq 0\}$. For any $\xi_0 \in \mathbb{F}_{\mathbb{VC}2}$, the corresponding Wolfe type dual problem to MIVVC-2 is given by

$$
\begin{aligned}
(WD_w(\xi_0) - 1) R_+^m - \max \mathcal{L}(u, \alpha^L, \alpha^U, \lambda^\omega, \lambda^\rho) \\
= \big([2u + 4, e^{2u}] + (-\lambda_1^\rho(u) + \lambda_1^\omega(-e^{1-u}))(1), \\
[-2u + 1, -u - 1] + (-\lambda_1^\rho(u) + \lambda_1^\omega(-e^{1-u}))(1)\big) \\
\text{subject to} \\
\alpha_1^L(2) + \alpha_1^U(2e^{2u}) + \alpha_2^L(-2) + \alpha_2^U(-1) - \lambda_1^\rho(1) \\
+ \lambda_1^\omega(-1) = 0, \\
\alpha_1^L + \alpha_1^U = 1, \alpha_2^L + \alpha_2^U = 1, \lambda_1^\rho \begin{cases} \geq 0, & if 1 \in T_+(\xi_0), \\ \in R, & if 1 \in T_0(\xi_0), \end{cases} \\
\lambda_1^\omega \begin{cases} \leq 0, & if 1 \in T_{0+}(\xi_0), \\ \geq 0, & if 1 \in T_{+-}(\xi_0) \cup T_{0-}(\xi_0), \\ \in R, & if 1 \in T_{+0}(\xi_0) \cup T_{00}(\xi_0), \end{cases} \\
where\ (u, \alpha_1^L, \alpha_1^U, \alpha_2^L, \alpha_2^U, \lambda^\omega, \lambda^\rho) \in R \times R_+ \times R_+ \times R_+ \times R_+ \times R \times R.
\end{aligned}
$$

Therefore, we get the following feasible set of problem $(WD_w(\xi_0) - 1)$:

$$
\begin{aligned}
(\mathbb{F}_{\mathbb{VC}w}(\xi_0) - 1) := \Big\{ & (u, \alpha_1^L, \alpha_1^U, \alpha_2^L, \alpha_2^U, \lambda^\omega, \lambda^\rho) \in R^n \times R_+^m \times R_+^m \\
& \times R_+^m \times R_+^m \times R^r \times R^r | \alpha_1^L + \alpha_1^U = 1, \alpha_2^L + \alpha_2^U = 1, \\
& \lambda_1^\rho \in R, \lambda_1^\omega \in R, \alpha_1^L \nabla\vartheta_1^L(u) + \alpha_1^U \nabla\vartheta_1^U(u) + \alpha_2^L \nabla\vartheta_2^L(u) \\
& + \alpha_2^U \nabla\vartheta_2^U(u) - \lambda_1^\rho \nabla\rho_1(u) + \lambda_1^\omega \nabla\omega_1(u) = 0 \Big\}.
\end{aligned}
$$

By elementary calculations, we get $\nabla\vartheta_1^L(\xi_0) = \{2\}$, $\nabla\vartheta_1^U(\xi_0) = \{2\}$, $\nabla\vartheta_2^L(\xi_0) = \{-2\}$, $\nabla\vartheta_2^U(\xi_0) = \{-1\}$, $\nabla\rho_1(\xi_0) = \{1\}$, $\nabla\omega_1(\xi_0) = \{e\}$, $T_+ = T_{0+} = T_{0-} = \phi$, $T_{00} = \{1\}$.
Clearly, $(u, \alpha_1^L, \alpha_1^U, \alpha_2^L, \alpha_2^U, \lambda^\omega, \lambda^\rho) = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2})$ is a feasible solution to $(WD_w(\xi_0)-1)$. We also note that $\xi_0 = 0$ is a feasible solution to MIVVC-2. On the other hand, it is easily verified that the hypothesis $(i)$ and $(ii)$ of Theorem 4 are satisfied at $u = 0$.

**Theorem 5** (Strong duality)**.** Let $\xi_0 \in \mathbb{F}_{\mathbb{VC}}$ be a locally weakly efficient solution of MIVVC. If MIVVC-VACQ holds at $\xi_0$ and the set $\Delta_1$ is closed, then there exists $(\bar\alpha^L, \bar\alpha^U, \bar\lambda^\tau, \bar\lambda^\sigma, \bar\lambda^\omega, \bar\lambda^\rho) \in R_+^m \times R_+^m \times R^p \times R^q \times R^r \times R^r$ with $\bar\lambda^\rho_{T_+(\xi_0)} = 0, \bar\lambda^\rho_{T_{00}(\xi_0)\cup T_{0-}(\xi_0)} \geq 0, \bar\lambda^\omega_{T_{+-}(\xi_0)\cup T_{0+}(\xi_0)\cup T_{0-}(\xi_0)} = 0$ and $\bar\lambda^\omega_{T_{+0}(\xi_0)\cup T_{00}(\xi_0)} \geq 0$ such that $(\xi_0, \bar\alpha^L, \bar\alpha^U, \bar\lambda^\tau, \bar\lambda^\sigma, \bar\lambda^\omega, \bar\lambda^\rho) \in \mathbb{F}_{\mathbb{VC}w}(\xi_0)$ and $\vartheta(\xi_0) = \mathcal{L}(\xi_0, \bar\alpha^L, \bar\alpha^U, \bar\lambda^\tau, \bar\lambda^\sigma, \bar\lambda^\omega, \bar\lambda^\rho)$. Furthermore, assume that $\tau_i(i \in T_\tau^+(\xi_0)), \sigma_i\ (i \in T_\sigma^+(\xi_0)), -\sigma_i(i \in T_\sigma^-(\xi_0)), \rho_i(i \in \hat{T}_0^-(\xi_0)), -\rho_i(i \in \hat{T}_+^+(\xi_0) \cup \hat{T}_0^+(\xi_0)), \omega_i(i \in T_{+0}^+(\xi_0) \cup T_{+-}^+(\xi_0) \cup T_{00}^+(\xi_0) \cup T_{0-}^+(\xi_0)), -\omega_i(i \in T_{+0}^-(\xi_0) \cup T_{0+}^-(\xi_0) \cup T_{00}^-(\xi_0))$ are convex functions at $\xi_0$.

(i) If $\vartheta_i^L, \vartheta_i^U (i \in T)$ are convex functions at $\xi_0$, then $(\xi_0, \bar\alpha^L, \bar\alpha^U, \bar\lambda^\tau, \bar\lambda^\sigma, \bar\lambda^\omega, \bar\lambda^\rho)$ is a weakly LU-efficient solution of $WD_w(\xi_0)$.

(ii) If $\vartheta_i^L, \vartheta_i^U (i \in T)$ are strictly convex functions at $\xi_0$, then $(\xi_0, \bar\alpha^L, \bar\alpha^U, \bar\lambda^\tau, \bar\lambda^\sigma, \bar\lambda^\omega, \bar\lambda^\rho)$ is an LU-efficient solution of $WD_w(\xi_0)$.

*Proof.* In view of Theorem 1, there exists $(\bar\alpha^L, \bar\alpha^U, \bar\lambda^\tau, \bar\lambda^\sigma, \bar\lambda^\omega, \bar\lambda^\rho) \in R_+^m \times R_+^m \times R^p \times R^q \times R^r \times R^r$ with $\bar\lambda^\rho_{T_+(\xi_0)} = 0, \bar\lambda^\rho_{T_{00}(\xi_0)\cup T_{0-}(\xi_0)} \geq 0$, $\bar\lambda^\omega_{T_{+-}(\xi_0)\cup T_{0+}(\xi_0)\cup T_{0-}(\xi_0)} = 0$ and $\bar\lambda^\omega_{T_{+0}(\xi_0)\cup T_{00}(\xi_0)} \geq 0$ such that

$$\sum_{i\in T} \bar\alpha_i^L \nabla\vartheta_i^L(\xi_0) + \sum_{i\in T} \bar\alpha_i^U \nabla\vartheta_i^U(\xi_0) + \sum_{i\in T_\tau} \bar\lambda_i^\tau \nabla\tau_i(\xi_0) + \sum_{i\in T_\sigma} \bar\lambda_i^\sigma \nabla\sigma_i(\xi_0)$$
$$- \sum_{i\in T_r} \bar\lambda_i^\rho \nabla\rho_i(\xi_0) + \sum_{i\in T_r} \bar\lambda_i^\omega \nabla\omega_i(\xi_0) = 0.$$

Since $\bar\lambda^\tau \in R^p$, one has $\bar\lambda_i^\tau \tau_i(\xi_0) = 0$ for all $i \in T_\tau$, and thus, $\sum_{i\in T_\tau} \bar\lambda_i^\tau \tau_i(\xi_0) = 0$. The fact $\xi_0 \in \mathbb{F}_{\mathbb{VC}}$ guarantees that $\sum_{i\in T_\sigma} \bar\lambda_i^\sigma \sigma_i(\xi_0) = 0$. Moreover, we observe by $\bar\lambda^\rho_{T_+(\xi_0)} = 0$ and $\rho_i(\xi_0) = 0$ for all $i \in T_0(\xi_0)$ that $\sum_{i\in T_r} \bar\lambda_i^\rho \rho_i(\xi_0) = 0$. Analogously, as $\bar\lambda^\omega_{T_{+-}(\xi_0)\cup T_{0+}(\xi_0)\cup T_{0-}(\xi_0)} = 0$ and $\omega_i(\xi_0) = 0$ for all $i \in T_{+0}(\xi_0) \cup T_{00}(\xi_0)$, we know that $\sum_{i\in T_r} \bar\lambda_i^\omega \omega_i(\xi_0) = 0$. Thus, $(\xi_0, \bar\alpha^L, \bar\alpha^U, \bar\lambda^\tau, \bar\lambda^\sigma, \bar\lambda^\omega, \bar\lambda^\rho) \in \mathbb{F}_{\mathbb{VC}w}(\xi_0)$ and $\sum_{i\in T_\tau} \bar\lambda_i^\tau \tau_i(\xi_0) + \sum_{i\in T_\sigma} \bar\lambda_i^\sigma \sigma_i(\xi_0) - \sum_{i\in T_r} \bar\lambda_i^\rho \rho_i(\xi_0) + \sum_{i\in T_r} \bar\lambda_i^\omega \omega_i(\xi_0) = 0$ which is nothing else but the following equality $\vartheta(\xi_0) = \mathcal{L}(\xi_0, \bar\alpha^L, \bar\alpha^U, \bar\lambda^\tau, \bar\lambda^\sigma, \bar\lambda^\omega, \bar\lambda^\rho)$.

$(i)$. Now, arguing by contradiction, let us suppose that $(\xi_0, \bar\alpha^L, \bar\alpha^U, \bar\lambda^\tau, \bar\lambda^\sigma, \bar\lambda^\omega, \bar\lambda^\rho)$ is not a weakly LU-efficient solution of $WD_w(\xi_0)$. By the definition, there exists $(u, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in \mathbb{F}_{\mathbb{VC}w}(\xi_0)$ such that

$$\mathcal{L}(\xi_0, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^\tau, \bar{\lambda}^\sigma, \bar{\lambda}^\omega, \bar{\lambda}^\rho) \prec_{LU} \mathcal{L}(u, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho).$$

This shows that $\vartheta(\xi_0) \prec_{LU} \mathcal{L}(u, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho)$, which contradicts with Theorem 4(i).

$(ii)$. Reasoning to the contrary, let us assume that $(\xi_0, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^\tau, \bar{\lambda}^\sigma, \bar{\lambda}^\omega, \bar{\lambda}^\rho)$ is not an LU-efficient solution of $WD_w(\xi_0)$. Then it guarantees the existence of $(u, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in \mathbb{F}_{\mathbb{VC}w}(\xi_0)$ such that

$$\mathcal{L}(\xi_0, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^\tau, \bar{\lambda}^\sigma, \bar{\lambda}^\omega, \bar{\lambda}^\rho) \preceq_{LU} \mathcal{L}(u, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho).$$

Consequently, $\vartheta(\xi_0) \preceq_{LU} \mathcal{L}(u, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho)$ which contradicts with Theorem 4(ii). □

**Theorem 6** (Strict converse duality)**.** Let $\tilde{\xi} \in \mathbb{F}_{\mathbb{VC}}$ be a locally weakly efficient solution of MIVVC such that MIVVC-VACQ holds at $\tilde{\xi}$ and the strong duality between the MIVVC and the $(WD_W(\tilde{\xi}))$ as in Theorem 5 holds. Also, let $(\tilde{\psi}, \tilde{\alpha}^L, \tilde{\alpha}^U, \tilde{\lambda}^\tau, \tilde{\lambda}^\sigma, \tilde{\lambda}^\omega, \tilde{\lambda}^\rho) \in \mathbb{F}_{\mathbb{VC}w}$ be an LU-efficient solution of $(WD_W(\tilde{\xi}))$. Moreover, Assume that $\vartheta_i^L, \vartheta_i^U (i \in T)$ are strictly convex functions and that $\tau_i(i \in T_\tau^+(\tilde{\xi})), \sigma_i(i \in T_\sigma^+(\tilde{\xi})), -\sigma_i(i \in T_\sigma^-(\tilde{\xi})), \rho_i(i \in \hat{T}_0^-(\tilde{\xi})), -\rho_i(i \in \hat{T}_+^+(\tilde{\xi}) \cup \hat{T}_0^+(\tilde{\xi})), \omega_i(i \in T_{+0}^+(\tilde{\xi}) \cup T_{+-}^+(\tilde{\xi}) \cup T_{00}^+(\tilde{\xi}) \cup T_{0-}^+(\tilde{\xi})), -\omega_i(i \in T_{+0}^-(\tilde{\xi}) \cup T_{0+}^-(\tilde{\xi}) \cup T_{00}^-(\tilde{\xi}))$ are convex functions at $\tilde{\psi}$, respectively. Then, $\tilde{\xi} = \tilde{\psi}$.

*Proof.* Suppose on the contrary, $\tilde{\xi} \neq \tilde{\psi}$. Then, by Theorem 5, there exist $\tilde{\xi} \in \mathbb{F}_{\mathbb{VC}}$ and $(\tilde{\psi}, \tilde{\alpha}^L, \tilde{\alpha}^U, \tilde{\lambda}^\tau, \tilde{\lambda}^\sigma, \tilde{\lambda}^\omega, \tilde{\lambda}^\rho) \in \mathbb{F}_{\mathbb{VC}w}$, and hence

$$\vartheta(\tilde{\xi}) = \mathcal{L}(\tilde{\psi}, \tilde{\alpha}^L, \tilde{\alpha}^U, \tilde{\lambda}^\tau, \tilde{\lambda}^\sigma, \tilde{\lambda}^\omega, \tilde{\lambda}^\rho). \tag{19}$$

The strict convexity of $\vartheta_i^L, \vartheta_i^U (i \in T)$ at $\tilde{\psi}$ gives that

$$\begin{aligned} &\left\langle \sum_{i=1}^m \alpha_i^L \nabla \vartheta_i^L(\tilde{\psi}) + \sum_{i=1}^m \alpha_i^U \nabla \vartheta_i^U(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \\ &< \left( \sum_{i \in T_\tau} \lambda_i^\tau \tau_i(\tilde{\psi}) + \sum_{i \in T_\sigma} \lambda_i^\sigma \sigma_i(\tilde{\psi}) - \sum_{i \in T_r} \lambda_i^\rho \rho_i(\tilde{\psi}) + \sum_{i \in T_r} \lambda_i^\omega \omega_i(\tilde{\psi}) \right). \end{aligned} \tag{20}$$

The convexity of $\tau_i(i \in T_\tau^+(\tilde{\xi})), \sigma_i(i \in T_\sigma^+(\tilde{\xi})), -\sigma_i(i \in T_\sigma^-(\tilde{\xi}))$, $\rho_i(i \in \hat{T}_0^-(\tilde{\xi})), -\rho_i(i \in \hat{T}_+^+(\tilde{\xi}) \cup \hat{T}_0^+(\tilde{\xi})), \omega_i(i \in T_{+0}^+(\tilde{\xi}) \cup T_{+-}^+(\tilde{\xi}) \cup T_{00}^+(\tilde{\xi}) \cup T_{0-}^+(\tilde{\xi})), -\omega_i(i \in T_{+0}^-(\tilde{\xi}) \cup T_{0+}^-(\tilde{\xi}) \cup T_{00}^-(\tilde{\xi}))$ at $\tilde{\psi}$ and by the definitions of index sets imply that

$$\tau_i(\tilde{\psi}) + \left\langle \nabla \tau_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq \tau_i(\tilde{\xi}) = 0, \lambda_i^\tau > 0, \text{ for all } i \in T_\tau^+(\tilde{\xi}),$$

$$\sigma_i(\tilde{\psi}) + \left\langle \nabla \sigma_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq \sigma_i(\tilde{\xi}) = 0, \lambda_i^\sigma > 0, \text{ for all } i \in T_\sigma^+(\tilde{\xi}),$$

$$-\sigma_i(\tilde{\psi}) + \left\langle -\nabla\sigma_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq -\sigma_i(\tilde{\xi}) = 0, \lambda_i^\sigma < 0, \text{ for all } i \in T_\sigma^-(\tilde{\xi}),$$

$$\rho_i(\tilde{\psi}) + \left\langle \nabla\rho_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq \rho_i(\tilde{\xi}) = 0, \lambda_i^\rho < 0, \text{ for all } i \in \hat{T}_0^-(\tilde{\xi}),$$

$$-\rho_i(\tilde{\psi}) + \left\langle -\nabla\rho_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq -\rho_i(\tilde{\xi}) < 0, \lambda_i^\rho > 0, \text{ for all } i \in \hat{T}_+^+(\tilde{\xi}),$$

$$-\rho_i(\tilde{\psi}) + \left\langle -\nabla\rho_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq -\rho_i(\tilde{\xi}) < 0, \lambda_i^\rho > 0, \text{ for all } i \in \hat{T}_0^+(\tilde{\xi}),$$

$$\omega_i(\tilde{\psi}) + \left\langle \nabla\omega_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq \omega_i(\tilde{\xi}) = 0, \lambda_i^\omega > 0, \text{ for all } i \in T_{+0}^+(\tilde{\xi}) \cup T_{00}^+(\tilde{\xi}),$$

$$\omega_i(\tilde{\psi}) + \left\langle \nabla\omega_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq \omega_i(\tilde{\xi}) < 0, \lambda_i^\omega > 0, \text{ for all } i \in T_{+-}^+(\tilde{\xi}) \cup T_{0-}^+(\tilde{\xi}),$$

which implies that

$$\sum_{i\in T_\tau} \lambda_i^\tau \tau_i(\tilde{\xi}) + \sum_{i\in T_\sigma} \lambda_i^\sigma \sigma_i(\tilde{\xi}) - \sum_{i\in T_r} \lambda_i^\rho \rho_i(\tilde{\xi}) + \sum_{i\in T_r} \lambda_i^\omega \omega_i(\tilde{\xi}) + \Big\langle \sum_{i\in T_\tau} \lambda_i^\tau \nabla\tau_i(\tilde{\psi})$$

$$+ \sum_{i\in T_\sigma} \lambda_i^\sigma \nabla\sigma_i(\tilde{\psi}) - \sum_{i\in T_r} \lambda_i^\rho \nabla\rho_i(\tilde{\psi}) + \sum_{i\in T_r} \lambda_i^\omega \nabla\omega_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \Big\rangle \leq 0. \tag{21}$$

On adding the inequalities (20) and (21) and by using the duality constraint (9) of $(WD_w(\tilde{\xi}))$, we have

$$\mathcal{L}(\tilde{\psi}, \tilde{\alpha}^L, \tilde{\alpha}^U, \tilde{\lambda}^\tau, \tilde{\lambda}^\sigma, \tilde{\lambda}^\omega, \tilde{\lambda}^\rho) \prec_{LU} \vartheta(\tilde{\xi}),$$

which contradicts with (19). □

## 5 The Mond–Weir type duality

The Wolfe dual of the primal problem, which we discussed in the last section, says that all functions must be convex. Wolfe duality does not work for functions, where the objective function is only pseudoconvex and the constraints are only quasiconvex in the primal problem MIVVC (see, Mond [24]). So, in this section, we propose a Mond–Weir type dual to the primal problem MIVVC to weaken the convexity assumptions.

Consider $\xi_0 \in \mathbb{F}_{\mathbb{VC}}, (u, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in R^n \times R_+^m \times R_+^m \times R^p \times R^q \times R^r \times R^r$ with $\sum\limits_{i\in T}(\alpha_i^L + \alpha_i^U) = 1, \lambda^\rho_{T_+(\xi_0)} \geq 0, \lambda^\omega_{T_{0+}(\xi_0)} \leq 0$, and $\lambda^\omega_{T_{+-}(\xi_0)\cup T_{0-}(\xi_0)} \geq 0$. We consider the Mond–Weir type dual problem as follows:

$(MWD_M(\xi_0))$ $R_+^m - \max \ \vartheta(u)$

subject to

$$\sum_{i\in T}\alpha_i^L\nabla\vartheta_i^L(u)+\sum_{i\in T}\alpha_i^U\nabla\vartheta_i^U(u)+\sum_{i\in T_\tau}\lambda_i^\tau\nabla\tau_i(u)+$$
$$\sum_{i\in T_\sigma}\lambda_i^\sigma\nabla\sigma_i(u)-\sum_{i\in T_r}\lambda_i^\rho\nabla\rho_i(u)+\sum_{i\in T_r}\lambda_i^\omega\nabla\omega_i(u)=0,$$
$$\lambda_i^\tau\tau_i(u)\geq 0\ (i\in T_\tau),\lambda_i^\sigma\sigma_i(u)=0\ (i\in T_\sigma),-\lambda_i^\rho\rho_i(u)\geq 0$$
$$(i\in T_r),\lambda_i^\omega\omega_i(u)\geq 0\ (i\in T_r),\sum_{i\in T}\left(\alpha_i^L+\alpha_i^U\right)=1,$$
$$\lambda^\rho_{T_+(\xi_0)}\geq 0,\lambda^\omega_{T_{0+}(\xi_0)}\leq 0 \text{ and } \lambda^\omega_{T_{+-}(\xi_0)\cup T_{0-}(\xi_0)}\geq 0,(u,\alpha^L,\alpha^U,$$
$$\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in R^n\times R^m_+\times R^m_+\times R^p\times R^q\times R^r\times R^r.$$

The feasible set of $(MWD_M(\xi_0))$ is defined by

$$\begin{aligned}\mathbb{F}_{\mathbb{VC}M}(\xi_0):=\Bigg\{&(u,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in R^n\times R^m_+\times R^m_+\times R^p\times R^q\\ &\times R^r\times R^r\mid \lambda_i^\tau\tau_i(u)\geq 0\ (i\in T_\tau),\lambda_i^\sigma\sigma_i(u)=0\ (i\in T_\sigma),\\ &-\lambda_i^\rho\rho_i(u)\geq 0\ (i\in T_r),\lambda_i^\omega\omega_i(u)\geq 0\ (i\in T_r),\\ &\sum_{i\in T}\left(\alpha_i^L+\alpha_i^U\right)=1,\lambda^\rho_{T_+(\xi_0)}\geq 0,\lambda^\omega_{T_{0+}(\xi_0)}\leq 0,\ \text{and}\\ &\lambda^\omega_{T_{+-}(\xi_0)\cup T_{0-}(\xi_0)}\geq 0,\sum_{i\in T}\alpha_i^L\nabla\vartheta_i^L(u)+\sum_{i\in T}\alpha_i^U\nabla\vartheta_i^U(u)\\ &+\sum_{i\in T_\tau}\lambda_i^\tau\nabla\tau_i(u)+\sum_{i\in T_\sigma}\lambda_i^\sigma\nabla\sigma_i(u)-\sum_{i\in T_r}\lambda_i^\rho\nabla\rho_i(u)\\ &+\sum_{i\in T_r}\lambda_i^\omega\nabla\omega_i(u)=0\Bigg\}.\end{aligned}$$

Furthermore, let us denote by $\Gamma_M$ the projection of $\mathbb{F}_{\mathbb{VC}M}$ on $\mathbb{R}^n$; that is,

$$\Gamma_M(\xi_0):=\left\{u\in\mathbb{R}^n|(u,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in\Gamma_M(\xi_0)\right\}.$$

The other Mond–Weir type duality problem of MIVVC, which is not dependent on $\xi_0$, is

$$\begin{aligned}(MWD_M):\qquad & R^m_+-\max\ \vartheta(\psi)\\ & \text{subject to}\\ & (\psi,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in\Gamma_M:=\bigcap_{\xi_0\in\Gamma}\Gamma_M(\xi_0).\end{aligned}$$

**Definition 9.** Let $\xi_0\in\mathbb{F}_{\mathbb{VC}}$. Then $(\bar u,\bar\alpha^L,\bar\alpha^U,\bar\lambda^\tau,\bar\lambda^\sigma,\bar\lambda^\omega,\bar\lambda^\rho)\in\mathbb{F}_{\mathbb{VC}M}(\xi_0)$ is a locally LU-efficient solution of $(MWD_M(\xi_0))$ (locally weakly LU-efficient solution of $(MWD_M(\xi_0))$) if there exists $U\in\Theta(\bar u)$ such that there is no $(u,\alpha^L,\alpha^U,\lambda^\tau,\lambda^\sigma,\lambda^\omega,\lambda^\rho)\in\mathbb{F}_{\mathbb{VC}M}(\xi_0)\cap U$ satisfying

$$\vartheta(\bar{u}) \preceq_{LU} \vartheta(u)$$
$$(\vartheta(\bar{u}) \prec_{LU} \vartheta(u))$$

**Theorem 7** (Weak duality)**.** Let $\xi \in \mathbb{F}_{\mathbb{VC}}$ and $(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in \mathbb{F}_{\mathbb{VC}M}$. Suppose that $\tau_i$ $(i \in T_\tau^+(\xi)), \sigma_i$ $(i \in T_\sigma^+(\xi)), -\sigma_i$ $(i \in T_\sigma^-(\xi)), \rho_i$ $(i \in \hat{T}_0^-(\xi)), -\rho_i$ $(i \in \hat{T}_+^+(\xi) \cup \hat{T}_0^+(\xi)), \omega_i$ $(i \in T_{+0}^+(\xi) \cup T_{+-}^+(\xi) \cup T_{00}^+(\xi) \cup T_{0-}^+(\xi)), -\omega_i$ $(i \in T_{+0}^-(\xi) \cup T_{0+}^-(\xi) \cup T_{00}^-(\xi))$ are quasiconvex functions at $\psi$ on $\mathbb{F}_{\mathbb{VC}M} \cup \Gamma_M$. If $\vartheta_i^L, \vartheta_i^U (i \in T)$ are strictly pseudoconvex functions at $\psi$ on $\mathbb{F}_{\mathbb{VC}M} \cup \Gamma_M$, then $\vartheta(\xi) \not\preceq_{LU} \vartheta(\psi)$.

*Proof.* For $\xi \in \mathbb{F}_{\mathbb{VC}M}$ and

$$(\psi, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in \mathbb{F}_{\mathbb{VC}M} = \bigcap_{\xi_0 \in \mathbb{F}_{\mathbb{VC}M}} \mathbb{F}_{\mathbb{VC}M}(\xi_0),$$

we have

$$\tau_i(\xi) \leq 0 \, (i \in T_\tau), \quad \sigma_i(\xi) = 0 \, (i \in T_\sigma), \quad \rho_i(\xi) \geq 0 \, (i \in T_r), \quad \omega_i(\xi)\rho_i(\xi) \leq 0 \, (i \in T_r), \tag{22}$$

$$\sum_{i\in T} \alpha_i^L \nabla \vartheta_i^L(\psi) + \sum_{i\in T} \alpha_i^U \nabla \vartheta_i^U(\psi) + \sum_{i\in T_\tau} \lambda_i^\tau \nabla \tau_i(\psi) + \sum_{i\in T_\sigma} \lambda_i^\sigma \nabla \sigma_i(\psi)$$
$$- \sum_{i\in T_r} \lambda_i^\rho \nabla \rho_i(\psi) + \sum_{i\in T_r} \lambda_i^\omega \nabla \omega_i(\psi) = 0, \tag{23}$$

and

$$\lambda_i^\tau \tau_i(\psi) \geq 0 \ (i \in T_\tau), \quad \lambda_i^\sigma \sigma_i(\psi) = 0 \ (i \in T_\sigma),$$
$$-\lambda_i^\rho \rho_i(\psi) \geq 0 \ (i \in T_r), \quad \lambda_i^\omega \omega_i(\psi) \geq 0 \ (i \in T_r), \tag{24}$$

with

$$\sum_{i\in T} \left(\alpha_i^L + \alpha_i^U\right) = 1, \quad \lambda^\rho_{T_+(\xi)} \geq 0, \quad \lambda^\omega_{T_{0+}(\xi)} \leq 0, \quad \lambda^\omega_{T_{+-}(\xi)\cup T_{0-}(\xi)} \geq 0. \tag{25}$$

It follows from the above inequalities that

$$\tau_i(\xi) \leq 0 \leq \tau_i(\psi) \leq 0, \qquad \text{for all } i \in T_\tau^+(\xi),$$
$$\sigma_i(\xi) = \sigma_i(\psi) = 0, \qquad \text{for all } i \in T_\sigma^+(\xi) \cup T_\sigma^-(\xi),$$
$$\rho_i(\xi) = 0 \leq \rho_i(\psi), \qquad \text{for all } i \in \hat{T}_0^-(\xi),$$
$$-\rho_i(\xi) \leq 0 \leq -\rho_i(\psi), \qquad \text{for all } i \in \hat{T}_+^+(\xi) \cup \hat{T}_0^+(\xi),$$
$$\omega_i(\xi) \leq 0 \leq \omega_i(\psi), \qquad \text{for all } i \in T_{+0}^+(\xi) \cup T_{+-}^+(\xi) \cup T_{00}^+(\xi) \cup T_{0-}^+(\xi),$$
$$-\omega_i(\xi) \leq 0 \leq -\omega_i(\psi) = 0, \qquad \text{for all } i \in T_{+0}^-(\xi) \cup T_{0+}^-(\xi) \cup T_{00}^-(\xi).$$

Thus, we deduce from the quasiconvexity of $\tau_i$ $(i \in T_\tau^+(\xi)), \sigma_i$ $(i \in T_\sigma^+(\xi)),$ $-\sigma_i$ $(i \in T_\sigma^-(\xi)), \rho_i$ $(i \in \hat{T}_0^-(\xi)), -\rho_i$ $(i \in \hat{T}_+^+(\xi) \cup \hat{T}_0^+(\xi)),$

$\omega_i$ $(i \in T^+_{+0}(\xi) \cup T^+_{+-}(\xi) \cup T^+_{00}(\xi) \cup T^+_{0-}(\xi))$, $-\omega_i$ $(i \in T^-_{+0}(\xi) \cup T^-_{0+}(\xi) \cup T^-_{00}(\xi))$ at $\psi$ and the definitions of index sets that

$$\begin{aligned}
\langle \nabla \tau_i(\psi), \xi - \psi \rangle \leq 0, \lambda_i^\tau > 0, &\quad \text{for all } i \in T^+_\tau(\xi),\\
\langle \nabla \sigma_i(\psi), \xi - \psi \rangle \leq 0, \lambda_i^\sigma > 0, &\quad \text{for all } i \in T^+_\sigma(\xi),\\
\langle -\nabla \sigma_i(\psi), \xi - \psi \rangle \leq 0, \lambda_i^\sigma < 0, &\quad \text{for all } i \in T^-_\sigma(\xi),\\
\langle \nabla \rho_i(\psi), \xi - \psi \rangle \leq 0, \lambda_i^\rho < 0, &\quad \text{for all } i \in \hat{T}^-_0(\xi),\\
\langle -\nabla \rho_i(\psi), \xi - \psi \rangle \leq 0, \lambda_i^\rho > 0, &\quad \text{for all } i \in \hat{T}^+_+(\xi) \cup \hat{T}^+_0(\xi),\\
\langle \nabla \omega_i(\psi), \xi - \psi \rangle \leq 0, \lambda_i^\omega > 0, &\quad \text{for all } i \in T^+_{+0}(\xi) \cup T^+_{+-}(\xi) \cup T^+_{00}(\xi) \cup T^+_{0-}(\xi),\\
\langle -\nabla \omega_i(\psi), \xi - \psi \rangle \leq 0, \lambda_i^\omega < 0, &\quad \text{for all } i \in T^-_{+0}(\xi) \cup T^-_{0+}(\xi) \cup T^-_{00}(\xi),
\end{aligned}$$

Employing this together with (23) gives us the inequality

$$\begin{aligned}
&\left\langle \sum_{i\in T} \alpha_i^L \nabla \vartheta_i^L(\psi) + \sum_{i\in T} \alpha_i^U \nabla \vartheta_i^U(\psi), \xi - \psi \right\rangle\\
&= -\left\langle \sum_{i\in T_\tau} \lambda_i^\tau \nabla \tau_i(\psi) + \sum_{i\in T_\sigma} \lambda_i^\sigma \nabla \sigma_i(\psi) - \sum_{i\in T_r} \lambda_i^\rho \nabla \rho_i(\psi) + \sum_{i\in T_r} \lambda_i^\omega \nabla \omega_i(\psi), \xi - \psi \right\rangle\\
&\geq 0. \qquad (26)
\end{aligned}$$

Assume by contradiction that

$$\vartheta(\xi) \preceq_{LU} \vartheta(\psi).$$

This is equivalent to

$$\begin{cases} \vartheta^L(\xi) < \vartheta^L(\psi) \\ \vartheta^U(\xi) \leq \vartheta^U(\psi) \end{cases}, \quad \text{or} \begin{cases} \vartheta^L(\xi) \leq \vartheta^L(\psi) \\ \vartheta^U(\xi) < \vartheta^U(\psi) \end{cases}, \quad \text{or} \begin{cases} \vartheta^L(\xi) < \vartheta^L(\psi) \\ \vartheta^U(\xi) < \vartheta^U(\psi) \end{cases}.$$

Since $\vartheta_i^L, \vartheta_i^U (i \in T)$ are strictly pseudoconvex functions at $\psi$, we have

$$\begin{aligned}
&\left\langle \nabla \vartheta_i^L(\psi), \xi - \psi \right\rangle < 0, \text{ for all } i \in T,\\
&\left\langle \nabla \vartheta_i^U(\psi), \xi - \psi \right\rangle < 0, \text{ for all } i \in T.
\end{aligned}$$

Taking into account $\alpha^L \in R^m_+, \alpha^U \in R^m_+$ and from $\sum_{i=1}^m \left(\alpha_i^L + \alpha_i^U\right) = 1$, we have

$$\left\langle \sum_{i=1}^m \alpha_i^L \nabla \vartheta_i^L(\psi) + \sum_{i=1}^m \alpha_i^U \nabla \vartheta_i^U(\psi), \xi - \psi \right\rangle < 0,$$

contradicting to (26). □

**Example 3.** Let $m = n = 1$, let $p = q = 0$ and let $r = 1$. Let us investigate the following $(MIVVC-3)$:

$$
\begin{aligned}
MIVVC-3 \qquad \mathbb{R}_+-\min\ \vartheta(\xi) &= (\vartheta_1(\xi), \vartheta_2(\xi))\\
&= \big([4\xi^2-\xi, 4\xi^2+\xi+1],\ [\xi^2-2\xi, \xi^4+2\xi]\big)\\
&\text{subject to}\\
&\rho_1(\xi) = \xi \geq 0,\\
&\omega_1(\xi)\rho_1(\xi) = (-1-\xi)\xi \leq 0.
\end{aligned}
$$

Then, $\mathbb{F}_{\mathbb{VC}3} = \{\xi \in R \mid \rho_1(\xi) \geq 0,\ \omega_1(\xi)\rho_1(\xi) \leq 0\}$. For any $\xi_0 \in \mathbb{F}_{\mathbb{VC}3}$, the corresponding Mond–Weir dual problem to MIVVC-3 is given by

$$
\begin{aligned}
&(MWD_M-1)\ R^m_+ - \max\ \vartheta(u)\\
&\qquad = \big([4u^2-u, 4u^2+u+1], [u^2-2u, u^4+2u]\big)\\
&\qquad \text{subject to}\\
&\qquad \alpha_1^L(8u-1)+\alpha_1^U(8u+1)+\alpha_2^L(2u-2)+\alpha_2^U(4u^3+2)\\
&\qquad -\lambda_1^\rho(1)+\lambda_1^\omega(-1)=0, -\lambda_1^\rho(u)\geq 0, \lambda_1^\omega(-1-u)\geq 0,\\
&\qquad \alpha_1^L+\alpha_1^U=1, \alpha_2^L+\alpha_2^U=1,\\
&\qquad \lambda_1^\rho \begin{cases} \geq 0,\ if 1 \in T_+(\xi_0),\\ \in R,\ if 1 \in T_0(\xi_0), \end{cases} \quad \lambda_1^\omega \begin{cases} \leq 0,\ if 1 \in T_{0+}(\xi_0),\\ \geq 0,\ if 1 \in T_{+-}(\xi_0) \cup T_{0-}(\xi_0),\\ \in R,\ if 1 \in T_{+0}(\xi_0) \cup T_{00}(\xi_0), \end{cases}\\
&\ where\ (u, \alpha_1^L, \alpha_1^U, \alpha_2^L, \alpha_2^U, \lambda_1^\omega, \lambda_1^\rho) \in R \times R_+ \times R_+ \times R_+ \times R_+ \times R \times R.
\end{aligned}
$$

Therefore, we get the following feasible set of problem $(MWD_M(\xi_0)-1)$:

$$
\begin{aligned}
(\mathbb{F}_{\mathbb{VC}M}(\xi_0)-1) := \Big\{ &(u, \alpha_1^L, \alpha_1^U, \alpha_2^L, \alpha_2^U, \lambda^\omega, \lambda^\rho) \in R^n \times R^m_+ \times R^m_+ \times R^m_+\\
&\times R^m_+ \times R^r \times R^r \mid -\lambda_1^\rho(u) \geq 0, \lambda_1^\omega(-1-u) \geq 0,\\
&\alpha_1^L+\alpha_1^U=1, \alpha_2^L+\alpha_2^U=1, \lambda_1^\rho \in R, \lambda_1^\omega \in R,\\
&\alpha_1^L\nabla\vartheta_1^L(u)+\alpha_1^U\nabla\vartheta_1^U(u)+\alpha_2^L\nabla\vartheta_2^L(u)\\
&+\alpha_2^U\nabla\vartheta_2^U(u)-\lambda_1^\rho\nabla\rho_1(u)+\lambda_1^\omega\nabla\omega_1(u)=0\Big\}.
\end{aligned}
$$

By taking $\xi_0 = 0 \in \mathbb{F}_{\mathbb{VC}3}$, we evidence from Examples 1 and 2 that all suppositions of Theorem 1 are fulfilled. Now, by choosing $\alpha_1^L = \alpha_1^U = \frac{1}{2}, \alpha_2^L = \alpha_2^U = \frac{1}{2}, \lambda_1^\omega = 0, \lambda_1^\rho = 0$,
we have

$$
\begin{gathered}
-\lambda_1^\rho(\xi_0) \geq 0, \qquad \lambda_1^\omega(-1-\xi_0) \geq 0,\\
\frac{1}{2}(-1)+\frac{1}{2}(1)+\frac{1}{2}(-2)+\frac{1}{2}(2)-\lambda_1^\rho(1)+\lambda_1^\omega(-1)=0.
\end{gathered}
$$

Finally, by the strict pseudoconvexity of $\vartheta_i^L, \vartheta_i^U (i \in T)$ at $\psi$ on $\mathbb{F}_{\mathbb{VC}M} \cup \Gamma_M$ and by simple calculations, we get $\vartheta(\xi) \not\preceq_{LU} \vartheta(\psi)$.

**Theorem 8** (Strong duality)**.** Let $\xi_0 \in \mathbb{F}_{\mathbb{VC}}$ be a locally weakly efficient solution of MIVVC. If MIVVC-VACQ holds at $\xi_0$ and the set $\Delta_1$ is closed, then there exists $(\bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^\tau, \bar{\lambda}^\sigma, \bar{\lambda}^\omega, \bar{\lambda}^\rho) \in R^m_+ \times R^m_+ \times R^p \times R^q \times R^r \times R^r$ with $\sum_{i=1}^m \left(\bar{\alpha}_i^L + \bar{\alpha}_i^U\right) = 1, \bar{\lambda}^\rho_{T_+(\xi_0)} = 0, \bar{\lambda}^\rho_{T_{00}(\xi_0)\cup T_{0-}(\xi_0)} \geq 0, \bar{\lambda}^\omega_{T_{+-}(\xi_0)\cup T_{0+}(\xi_0)\cup T_{0-}(\xi_0)} = 0$ and $\bar{\lambda}^\omega_{T_{+0}(\xi_0)\cup T_{00}(\xi_0)} \geq 0$ such that $(\xi_0, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^\tau, \bar{\lambda}^\sigma, \bar{\lambda}^\omega, \bar{\lambda}^\rho) \in \mathbb{F}_{\mathbb{VC}M}(\xi_0)$. Furthermore, assume that $\tau_i(i \in T_\tau^+(\xi_0)), \sigma_i(i \in T_\sigma^+(\xi_0)), -\sigma_i(i \in T_\sigma^-(\xi_0)), \rho_i(i \in \hat{T}_0^-(\xi_0)), -\rho_i(i \in \hat{T}_+^+(\xi_0) \cup \hat{T}_0^+(\xi_0)), \omega_i(i \in T_{+0}^+(\xi_0) \cup T_{+-}^+(\xi_0) \cup T_{00}^+(\xi_0) \cup T_{0-}^+(\xi_0)), -\omega_i(i \in T_{+0}^-(\xi_0) \cup T_{0+}^-(\xi_0) \cup T_{00}^-(\xi_0))$ are quasiconvex functions at $\xi_0$. If $\vartheta_i^L, \vartheta_i^U(i \in T)$ are strictly pseudoconvex functions at $\xi_0$, then $(\xi_0, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^\tau, \bar{\lambda}^\sigma, \bar{\lambda}^\omega, \bar{\lambda}^\rho)$ is an LU-efficient solution of $MWD_M(\xi_0)$.

*Proof.* By Theorem (1), there exists $(\bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^\tau, \bar{\lambda}^\sigma, \bar{\lambda}^\omega, \bar{\lambda}^\rho) \in R^m_+ \times R^m_+ \times R^p \times R^q \times R^r \times R^r$ with $\sum_{i=1}^m \left(\alpha_i^L + \alpha_i^U\right) = 1, \bar{\lambda}^\rho_{T_+(\xi_0)} = 0, \bar{\lambda}^\rho_{T_{00}(\xi_0)\cup T_{0-}(\xi_0)} \geq 0, \bar{\lambda}^\omega_{T_{+-}(\xi_0)\cup T_{0+}(\xi_0)\cup T_{0-}(\xi_0)} = 0$ and $\bar{\lambda}^\omega_{T_{+0}(\xi_0)\cup T_{00}(\xi_0)} \geq 0$ such that

$$\sum_{i\in T} \alpha_i^L \nabla \vartheta_i^L(\psi) + \sum_{i\in T} \alpha_i^U \nabla \vartheta_i^U(\psi) + \sum_{i\in T_\tau} \lambda_i^\tau \nabla \tau_i(\psi) + \sum_{i\in T_\sigma} \lambda_i^\sigma \nabla \sigma_i(\psi)$$
$$- \sum_{i\in T_r} \lambda_i^\rho \nabla \rho_i(\psi) + \sum_{i\in T_r} \lambda_i^\omega \nabla \omega_i(\psi) = 0.$$

Since $\bar{\lambda}^\tau \in R^p$, one has $\bar{\lambda}_i^\tau \tau_i(\xi_0) = 0$ for all $i \in T_\tau$. The fact that $\xi_0 \in \mathbb{F}_{\mathbb{VC}}$ guarantees that $\bar{\lambda}_i^\sigma \sigma_i(\xi_0) = 0$. Furthermore, we deduce from $\bar{\lambda}^\rho_{T_+(\xi_0)} = 0$ and $\rho_i(\xi_0) = 0$ for all $i \in T_0(\xi_0)$ that $-\bar{\lambda}_i^\rho \rho_i(\xi_0) = 0$ for all $i \in T_r$. In addition, we get from $\bar{\lambda}^\omega_{T_{+-}(\xi_0)\cup T_{0+}(\xi_0)\cup T_{0-}(\xi_0)} = 0$ and $\omega_i(\xi_0) = 0$ for all $i \in T_{+0}(\xi_0) \cup T_{00}(\xi_0)$, that $\bar{\lambda}_i^\omega \omega_i(\xi_0) = 0$ for all $i \in T_r$. Thus, $(\xi_0, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^\tau, \bar{\lambda}^\sigma, \bar{\lambda}^\omega, \bar{\lambda}^\rho) \in \mathbb{F}_{\mathbb{VC}M}(\xi_0)$.

$(i)$. Now, arguing by contradiction, let us suppose that $(\xi_0, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^\tau, \bar{\lambda}^\sigma, \bar{\lambda}^\omega, \bar{\lambda}^\rho)$ is not a weakly LU-efficient solution of $MWD_M(\xi_0)$. By the definition, there exists $(u, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in \mathbb{F}_{\mathbb{VC}M}(\xi_0)$ such that

$$\vartheta(\xi_0) \prec_{LU} \vartheta(u),$$

which contradicts with Theorem 4(i).

$(ii)$. Reasoning to the contrary, Let us assume that $(\xi_0, \bar{\alpha}^L, \bar{\alpha}^U, \bar{\lambda}^\tau, \bar{\lambda}^\sigma, \bar{\lambda}^\omega, \bar{\lambda}^\rho)$ is not an LU-efficient solution of $MWD_M(\xi_0)$. Then, there exists $(u, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in \mathbb{F}_{\mathbb{VC}M}(\xi_0)$ such that

$$\vartheta(\xi_0) \preceq_{LU} \vartheta(u)$$

which contradicts with Theorem 4(ii), and thus, completes the proof. □

**Theorem 9** (Strict converse duality)**.** Let $\tilde{\xi} \in \mathbb{F}_{\mathbb{VC}}$ be a locally weakly efficient solution of MIVVC such that MIVVC-VACQ holds at $\tilde{\xi}$ and the strong duality between the MIVVC and the $(MWD_M)(\tilde{\xi})$ as in Theorem 8 holds. Also, let $(\tilde{\psi}, \tilde{\alpha}^L, \tilde{\alpha}^U, \tilde{\lambda}^\tau, \tilde{\lambda}^\sigma, \tilde{\lambda}^\omega, \tilde{\lambda}^\rho) \in \mathbb{F}_{\mathbb{VC}M}$ be an LU-efficient solution of $(MWD_M)(\tilde{\xi})$. Moreover, Suppose that $\vartheta_i^L, \vartheta_i^U (i \in T)$ are strictly pseudoconvex functions and that $\tau_i(i \in T_\tau^+(\tilde{\xi})), \sigma_i(i \in T_\sigma^+(\tilde{\xi})), -\sigma_i(i \in T_\sigma^-(\tilde{\xi})), \rho_i(i \in \hat{T}_0^-(\tilde{\xi})), -\rho_i(i \in \hat{T}_+^+(\tilde{\xi}) \cup \hat{T}_0^+(\tilde{\xi})), \omega_i(i \in T_{+0}^+(\tilde{\xi}) \cup T_{+-}^+(\tilde{\xi}) \cup T_{00}^+(\tilde{\xi}) \cup T_{0-}^+(\tilde{\xi}))$, $-\omega_i(i \in T_{+0}^-(\tilde{\xi}) \cup T_{0+}^-(\tilde{\xi}) \cup T_{00}^-(\tilde{\xi}))$ are quasiconvex functions at $\tilde{\psi}$ on $\mathbb{F}_{\mathbb{VC}M} \cup \Gamma_M$, respectively.

*Proof.* Suppose, contrary to the result, that $\tilde{\xi} \neq \tilde{\psi}$. Then, by the strong duality theorem, there exist $(\alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho) \in R_+^m \times R_+^m \times R^p \times R^q \times R^r \times R^r$ such that $(\tilde{\psi}, \alpha^L, \alpha^U, \lambda^\tau, \lambda^\sigma, \lambda^\omega, \lambda^\rho)$ is an LU-efficient solution of $MWD_M(\tilde{\xi})$, and hence

$$\vartheta(\tilde{\xi}) = \vartheta(\tilde{\psi}). \tag{27}$$

By the strict pseudoconvexity of $\vartheta_i^L, \vartheta_i^U (i \in T)$ at $\tilde{\psi}$ on $\mathbb{F}_{\mathbb{VC}M} \cup \Gamma_M$, we have

$$\left\langle \sum_{i=1}^m \alpha_i^L \nabla \vartheta_i^L(\tilde{\psi}) + \sum_{i=1}^m \alpha_i^U \nabla \vartheta_i^U(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle < 0. \tag{28}$$

By the quasiconvexity of $\tau_i(i \in T_\tau^+(\tilde{\xi})), \sigma_i(i \in T_\sigma^+(\tilde{\xi})), -\sigma_i(i \in T_\sigma^-(\tilde{\xi})), \rho_i(i \in \hat{T}_0^-(\tilde{\xi}))$, $-\rho_i(i \in \hat{T}_+^+(\tilde{\xi}) \cup \hat{T}_0^+(\tilde{\xi})), \omega_i(i \in T_{+0}^+(\tilde{\xi}) \cup T_{+-}^+(\tilde{\xi}) \cup T_{00}^+(\tilde{\xi}) \cup T_{0-}^+(\tilde{\xi}))$, $-\omega_i(i \in T_{+0}^-(\tilde{\xi}) \cup T_{0+}^-(\tilde{\xi}) \cup T_{00}^-(\tilde{\xi}))$ at $\tilde{\psi}$ on $\mathbb{F}_{\mathbb{VC}} \cup \Gamma_{MWD}$ and by the definitions of index sets, we have

$$\left\langle \nabla \tau_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq 0, \lambda_i^\tau > 0, \quad \text{for all } i \in T_\tau^+(\tilde{\xi}),$$

$$\left\langle \nabla \sigma_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq 0, \lambda_i^\sigma > 0, \quad \text{for all } i \in T_\sigma^+(\tilde{\xi}),$$

$$\left\langle -\nabla \sigma_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq 0, \lambda_i^\sigma < 0, \quad \text{for all } i \in T_\sigma^-(\tilde{\xi}),$$

$$\left\langle \nabla \rho_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq 0, \lambda_i^\rho < 0, \quad \text{for all } i \in \hat{T}_0^-(\tilde{\xi}),$$

$$\left\langle -\nabla \rho_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq 0, \lambda_i^\rho > 0, \quad \text{for all } i \in \hat{T}_+^+(\tilde{\xi}) \cup \hat{T}_0^+(\tilde{\xi}),$$

$$\left\langle \nabla \omega_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq 0, \lambda_i^\omega > 0, \quad \text{for all } i \in T_{+0}^+(\tilde{\xi}) \cup T_{+-}^+(\tilde{\xi}) \cup T_{00}^+(\tilde{\xi}) \cup T_{0-}^+(\tilde{\xi}),$$

$$\left\langle -\nabla \omega_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq 0, \lambda_i^\omega < 0, \quad \text{for all } i \in T_{+0}^-(\tilde{\xi}) \cup T_{0+}^-(\tilde{\xi}) \cup T_{00}^-(\tilde{\xi}),$$

which implies that

$$\left\langle \sum_{i \in T_\tau} \lambda_i^\tau \nabla \tau_i(\tilde{\psi}) + \sum_{i \in T_\sigma} \lambda_i^\sigma \nabla \sigma_i(\tilde{\psi}) - \sum_{i \in T_r} \lambda_i^\rho \nabla \rho_i(\tilde{\psi}) + \sum_{i \in T_r} \lambda_i^\omega \nabla \omega_i(\tilde{\psi}), \tilde{\xi} - \tilde{\psi} \right\rangle \leq 0. \tag{29}$$

On adding the inequalities (28) and (29) and by using the duality constraint of $(MWD_M(\xi_0))$, we have

$$\vartheta(\tilde{\psi}) \prec_{LU} \vartheta(\tilde{\xi}).$$

which contradicts with (27). □

## 6 Special cases

$(i)$. If $\vartheta_1(\xi) = \vartheta_2(\xi) = \cdots = \vartheta_m(\xi)$ then the MIVVC problem reduces to the following (IVVC) problem of Ahmad et al. [2]:

$$\begin{aligned} \text{(P-1)} \qquad & \min \quad \vartheta(\xi) = (\vartheta_1(\xi)) = [\vartheta_1^L(\xi), \vartheta_1^U(\xi)] \\ & \text{subject to} \\ & \tau_i(\xi) \leq 0, \quad \text{for all } i = 1, 2, \ldots, p, \\ & \sigma_i(\xi) = 0, \quad \text{for all } i = 1, 2, \ldots, q, \\ & \rho_i(\xi) \geq 0, \quad \text{for all } i = 1, 2, \ldots, r, \\ & \omega_i(\xi)\rho_i(\xi) \leq 0, \quad \text{for all } i = 1, 2, \ldots, r. \end{aligned}$$

$(ii)$. If $\vartheta_1(\xi) = \vartheta_2(\xi) = \cdots = \vartheta_m(\xi)$ and $\vartheta_1^L(\xi) = \vartheta_1^U(\xi)$ then the MIVVC problem reduces to the following (MPVC) problem of Hoheisel and Kanzow [12] and the (MPVC) problem of Ahmad, Kummari, and Al-Homidan [3]:

$$\begin{aligned} \text{(P-2)} \qquad & \min \quad \vartheta(\xi) \\ & \text{subject to} \\ & \tau_i(\xi) \leq 0, \quad \text{for all } i = 1, 2, \ldots, p, \\ & \sigma_i(\xi) = 0, \quad \text{for all } i = 1, 2, \ldots, q, \\ & \rho_i(\xi) \geq 0, \quad \text{for all } i = 1, 2, \ldots, r, \\ & \omega_i(\xi)\rho_i(\xi) \leq 0, \quad \text{for all } i = 1, 2, \ldots, r. \end{aligned}$$

$(iii)$. If $\rho_i(\xi) = 0 = \omega_i(\xi)$, for all $i = 1, 2, \ldots, r$, then MIVVC problem reduces to the following IVP problem of Antczak and Michalak [5]:

$$\begin{aligned} \text{(P-3)} \qquad & \min \quad \vartheta(\xi) = (\vartheta_1(\xi), \vartheta_2(\xi), \ldots, \vartheta_m(\xi)) \\ & \text{subject to} \\ & \tau_i(\xi) \leq 0, \quad \text{for all } i = 1, 2, \ldots, p, \\ & \sigma_i(\xi) = 0, \quad \text{for all } i = 1, 2, \ldots, q. \end{aligned}$$

As a result of the above special cases, it is evident that the problem MIVVC presented in this article is more generalized.

## 7 Conclusion

In this paper, we have considered a multiobjective interval-valued programming problem involving vanishing constraints. Based on generalized convexity assumptions, the sufficiency of the Karush–Khun–Tucker necessary optimality conditions has been established. Furthermore, we have anticipated Wolfe and Mond–Weir dual problems for the considered multiobjective programming problem with interval-valued objective function and delved into several duality results under convexity assumptions. The results established in the paper were exemplified by an example.

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