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Research Article

A numerical approximation for the solution of a time-fractional telegraph equation based on the Crank–Nicolson method

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Abstract

In this paper, a two-dimensional time-fractional telegraph equation is considered with derivative in the sense of Caputo and $1 < \beta < 2$. The aim of this work is to extend the Crank–Nicolson method for this time-fractional telegraph equation. The stability and convergence of the numerical method are investigated. Also, the accuracy and efficiency of the proposed method are demonstrated by numerical experiments.

AMS subject classifications (2020): 65M06; 65M12; 35R11.

Keywords: Time-Fractional Telegraph Equation; Crank–Nicolson Method; Stability; Convergence.

1 Introduction

Fractional calculus can be used to model many complex problems. It has been used in many fields of science, engineering, and finance [1, 4, 18, 25, 26]; this fact is the main source of inspiration for most of the recent studies

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conducted on fractional calculus. The classical telegraph equation is used in random walk theory [2]. The time-fractional telegraph equation (TFTE) models the neutron transport process in the core of a nuclear reactor [27, 28].

In recent decades, the fractional telegraph equation has been solved by many researchers. Here, we briefly describe some of the studies that have been conducted in this field of research. Chen, Liu, and Anh [5] proposed the analytic solution of the TFTE using the separating variables method. The homotopy analysis method was used for the TFTE by Das et al. [6]. Yildirim [31] applied the homotopy perturbation method to solve space- and time-fractional telegraph equations. Momani [17] used Adomian decomposition methods to obtain the analytic and approximate solutions of space- and time-fractional telegraph equations. Biazar, Ebrahimi, and Ayati [3] proposed the variational iteration method to solve the fractional telegraph equation. Jiang and Lin [11] presented the exact solution of the TFTE using the reproducing kernel theorem. Nikan, Avazzadeh, and Machado [19] used a mesh-free spectral approach based on LRBF-FD to solve the TFTE with the fractional derivative described in the sense of Caputo. A radial basis function collocation method was used for solving the nonlinear TFTE by Sepehrian and Shamohammadi [22]. Hosseini et al. [9, 10] considered the meshless local radial point interpolation method, and Mohebbi, Abbaszadeh, and Dehghan [16] used the radial basis function technique for the TFTE. Shivanian applied spectral meshless radial point interpolation methods in [23], and the meshless local Petrov–Galerkin scheme was used in [24] to approximate the TFTE.

Many researchers have studied the fractional telegraph equation by using the finite difference method. Liang, Yao, and Wang [15] considered the TFTE by using a fast, high-order difference scheme. The finite difference method was used to solve the linear TFTE by Li and Cao [14]. Wang and Mei [29] considered the TFTE using a Legendre spectral Galerkin method in space and the generalized finite difference scheme in time. For a time-space-fractional telegraph equation, Zhao and Li [32] used a finite difference method in time and a Galerkin finite element method in space. A numerical method for the TFTE was proposed by Wei, Liu, and Sun [30], in which they discretized this equation with a new finite difference scheme in time and a local discontinuous Galerkin (LDG) method in space.

In this work, we find an approximate solution to the following TFTE [13]:

$$\frac{\partial^{\beta} u(x,y,t)}{\partial t^{\beta}} + \frac{\partial^{\beta-1} u(x,y,t)}{\partial t^{\beta-1}} + u(x,y,t) = \Delta u(x,y,t) + f(x,y,t), \quad (x,y) \in \Omega \subset \mathbb{R}^2, 0 \leq t \leq T, \tag{1}$$

with initial and boundary conditions

$$u(x,y,0) = \varphi(x,y), \qquad (x,y) \in \overline{\Omega} = \Omega \cup \partial\Omega, \tag{2}$$

$$\frac{\partial u(x,y,0)}{\partial t} = \psi(x,y), \qquad (x,y) \in \overline{\Omega} = \Omega \cup \partial\Omega, \tag{3}$$

$$u(x,y,t)=h(x,y,t), \qquad (x,y)\in\partial\Omega, t>0, \tag{4}$$

where $1<\beta<2, \Delta$ is the Laplace operator, $\partial\Omega$ is the boundary of Ω, $f(x,y), \varphi(x,y), \psi(x,y)$, and $h(x,y,t)$ are continuous functions, $u(x,y,t)\in C^2(\bar{\Omega}\times[0,T])$ is an unknown function, and the fractional derivatives are defined in the sense of Caputo, as follows:

$$\frac{\partial^{\beta-1}u(x,y,t)}{\partial t^{\beta-1}}=\frac{1}{\Gamma(2-\beta)}\int_0^t\frac{\partial u(x,y,s)}{\partial s}(t-s)^{1-\beta}ds, \qquad 1<\beta<2, \tag{5}$$

$$\frac{\partial^{\beta}u(x,y,t)}{\partial t^{\beta}}=\frac{1}{\Gamma(2-\beta)}\int_0^t\frac{\partial^2 u(x,y,s)}{\partial s^2}(t-s)^{1-\beta}ds, \qquad 1<\beta<2. \tag{6}$$

The Crank–Nicolson difference scheme can be used easily for space-fractional equations, but some manipulations are needed for time-fractional equations [12]. In [19, 13], a semi-discrete scheme based on the Crank–Nicolson method was used to discretize the time-fractional equation. In this work, the discretization of time-fractional derivatives is similar to [12]. The general idea for proving stability and convergence is taken from [19], but our approach differs from that in the details.

The remainder of this paper is organized as follows. In Section 2, the discretization of (1) is described. The stability and the convergence of the proposed method are proved in Sections 3 and 4, respectively. Section 5 is devoted to the numerical tests. Finally, the conclusion is given in Section 6.

2 Discretization of the problem

In this section, we explain the discretization of (1) by using the Crank–Nicolson difference scheme, such that the proposed difference schemes are uniquely solvable.

Consider Δx and Δy as the grid sizes in space for the finite difference scheme, where $\{(x_i,y_i), x_i=i\Delta x,\ \ y_j=j\Delta y, 0\le i\le I, 0\le j\le J; I,J\in\mathbb{R}\}$ covers $\overline{\Omega}$. Also, N is a positive integer, and the grid size in time for the finite difference scheme is $\Delta t=\frac{T}{N}$. Assume that $u_{i,j}^n$ is the value of $u(x_i,y_j,t_n)$.

The following lemma provides suitable tools for the discretization of (1).

Lemma 1. If $g(t)\in C^2[0,T]$ and $1<\beta<2$, then

(a)

$$\begin{aligned}&\int_{t_{k-1}}^{t_k} g'(s)(t_{n-\frac{1}{2}}-s)^{1-\beta}ds\\ &\quad=\frac{(\Delta t)^{1-\beta}}{(2-\beta)}\left[(n-k+\frac{1}{2})^{2-\beta}-(n-k-\frac{1}{2})^{2-\beta}\right][g(t_k)-g(t_{k-1})]\\ &\qquad+O(\Delta t)^{3-\beta}, \qquad k=1,2,\ldots,N-1.\end{aligned}$$

(b)

$$\int_{t_{n-1}}^{t_{n-\frac{1}{2}}} g'(s)(t_{n-\frac{1}{2}}-s)^{1-\beta}ds$$
$$=\frac{(\Delta t)^{1-\beta}}{(2-\beta)2^{2-\beta}}\left[g\left(t_n\right)-g\left(t_{n-1}\right)\right]+O(\quad \Delta t)^{3-\beta}, \qquad n\in\mathbb{N}.$$

Proof. The Taylor expansion allows us to write

$$g'(s)=\frac{g\left(t_k\right)-g\left(t_{k-1}\right)}{\Delta t}-\frac{1}{2\Delta t}[(t_k-s)^2 g''\left(\eta_1\right)-(t_{k-1}-s)^2 g''\left(\eta_2\right)],$$
$$\eta_1\in\left(s,t_k\right),\eta_2\in\left(t_{k-1},s\right).$$

It is easy to show that

$$\int_{t_{k-1}}^{t_k}(t_k-s)^2(t_{n-\frac{1}{2}}-s)^{1-\beta}ds=\omega_1(\Delta t)^{4-\beta}, \qquad \omega_1\in\mathbb{R},$$
$$\int_{t_{k-1}}^{t_k}(t_{k-1}-s)^2(t_{n-\frac{1}{2}}-s)^{1-\beta}ds=\omega_2(\Delta t)^{4-\beta}, \qquad \omega_2\in\mathbb{R}.$$

Thus,

$$\begin{aligned}
&\int_{t_{k-1}}^{t_k} g'(s)(t_{n-\frac{1}{2}}-s)^{1-\beta}ds\\
&=\frac{g\left(t_k\right)-g(t_{k-1})}{\Delta t}\int_{t_{k-1}}^{t_k}(t_{n-\frac{1}{2}}-s)^{1-\beta}ds\\
&\quad-\frac{1}{2\Delta t}g''(\eta_1)\int_{t_{k-1}}^{t_k}\left(t_k-s\right)^2\left(t_{n-\frac{1}{2}}-s\right)^{1-\beta}ds\\
&\quad+\frac{1}{2\Delta t}g''(\eta_2)\int_{t_{k-1}}^{t_k}\left(t_{k-1}-s\right)^2\left(t_{n-\frac{1}{2}}-s\right)^{1-\beta}ds\\
&=\frac{g\left(t_k\right)-g(t_{k-1})}{\Delta t}\times\frac{(\Delta t)^{2-\beta}}{2-\beta}\left[(n-k+\frac{1}{2})^{2-\beta}-(n-k-\frac{1}{2})^{2-\beta}\right]\\
&\quad+\omega(\Delta t)^{3-\beta}, \qquad \omega\in\mathbb{R}.
\end{aligned}$$

This completes the proof of part (a). Part (b) can be proved in the same way. □

By defining $b_s=\left(s+\frac{1}{2}\right)^{2-\beta}-\left(s-\frac{1}{2}\right)^{2-\beta}, \quad s=1,2,\ldots, \quad 1<\beta<2,$ it is easy to show that

$$\sum_{k=1}^{n-1}\left(u_{i,j}^{k}-u_{i,j}^{k-1}\right)\left((n-k+\frac{1}{2})^{2-\beta}-(n-k-\frac{1}{2})^{2-\beta}\right)+\frac{1}{2^{2-\beta}}\left(u_{i,j}^{n}-u_{i,j}^{n-1}\right)$$
$$=-\left[b_{n-1}u_{i,j}^{0}+\sum_{k=1}^{n-2}\left(b_{n-k-1}-b_{n-k}\right)u_{i,j}^{k}+(\frac{1}{2^{2-\beta}}-b_{1})u_{i,j}^{n-1}\right]+\frac{1}{2^{2-\beta}}u_{i,j}^{n}. \tag{7}$$

By using part (b) of Lemma 3, the discretization of (5) at the grid point (x_i, y_j) and the time step $\left(1-\frac{1}{2}\right)$ is as follows:

$$\begin{aligned}\frac{\partial^{\beta-1}u(x,y,t)}{\partial t^{\beta-1}}\Big|_{i,j}^{1-\frac{1}{2}} &= \frac{1}{\Gamma(2-\beta)}\int_{t_0}^{t_{1-\frac{1}{2}}}\frac{\partial u(x_i,y_j,s)}{\partial s}(t_{1-\frac{1}{2}}-s)^{1-\beta}ds\\ &= \frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\times\frac{1}{2^{2-\beta}}\left[u_{i,j}^{1}-u_{i,j}^{0}\right]+O(\Delta t)^{3-\beta}.\end{aligned} \tag{8}$$

By using Lemma 3 and relation (7), the discretization of (5) at the grid point (x_i, y_j) and the time step $\left(n-\frac{1}{2}\right)$ is as follows:

$$\begin{aligned}\frac{\partial^{\beta-1}u(x,y,t)}{\partial t^{\beta-1}}\Big|_{i,j}^{n-\frac{1}{2}} =&\frac{1}{\Gamma(2-\beta)}\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}\frac{\partial u(x_i,y_j,s)}{\partial s}(t_{n-\frac{1}{2}}-s)^{1-\beta}ds\\ &+\frac{1}{\Gamma(2-\beta)}\int_{t_{n-1}}^{t_{n-\frac{1}{2}}}\frac{\partial u\left(x_i,y_j,s\right)}{\partial s}(t_{n-\frac{1}{2}}-s)^{1-\beta}ds\\ =&\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left\{-b_{n-1}u_{i,j}^{0}-\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})u_{i,j}^{k}\right.\\ &\left.-(\frac{1}{2^{2-\beta}}-b_1)u_{i,j}^{n-1}+\frac{1}{2^{2-\beta}}u_{i,j}^{n}\right\}\\ &+O(\Delta t)^{3-\beta},\qquad n\geq 2, 1\leq i\leq I-1, 1\leq j\leq J-1.\end{aligned} \tag{9}$$

In addition, similar to (8) and (9), and by using the relation

$$\left.\frac{\partial u}{\partial t}\right|_{i,j}^{k}=\frac{u_{i,j}^{k}-u_{i,j}^{k-1}}{\Delta t}+\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}\left(x_i,y_j,\eta_1\right),\qquad k\geq 1, \eta_1\in\left(t_{k-1},t_k\right), \tag{10}$$

we obtain

$$\left.\frac{\partial^{\beta}u(x,y,t)}{\partial t^{\beta}}\right|_{i,j}^{1-\frac{1}{2}}=\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\times\frac{1}{2^{2-\beta}}\left[\frac{u_{i,j}^{1}-u_{i,j}^{0}}{\Delta t}-\left.\frac{\partial u}{\partial t}\right|_{i,j}^{0}\right]+O(\Delta t)^{2-\beta}, \tag{11}$$

$$
\begin{aligned}
\frac{\partial^\beta u(x,y,t)}{\partial t^\beta}\Big|_{i,j}^{n-\frac{1}{2}}
&= \frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)} \left\{ -b_{n-1}\frac{\partial u}{\partial t}\Big|_{i,j}^{0} - \sum_{k=1}^{n-2} (b_{n-k-1}-b_{n-k}) \frac{u_{i,j}^{k}-u_{i,j}^{k-1}}{\Delta t} \right. \\
&\quad \left. -(\frac{1}{2^{2-\beta}} - b_1)\frac{u_{i,j}^{n-1}-u_{i,j}^{n-2}}{\Delta t} + \frac{1}{2^{2-\beta}}\frac{u_{i,j}^{n}-u_{i,j}^{n-1}}{\Delta t} \right\} \\
&\quad + O(\Delta t)^{2-\beta}, \qquad n \geq 2, 1 \leq i \leq I-1, 1 \leq j \leq J-1.
\end{aligned} \tag{12}
$$

Having the Taylor expansion in mind, we can write

$$
u(x_i, y_j, t_{n-\frac{1}{2}}) = \frac{u_{i,j}^{n-1} + u_{i,j}^{n}}{2} + O(\Delta t)^2, \qquad n \geq 1, 1 \leq i \leq I-1, 1 \leq j \leq J-1, \tag{13}
$$

$$
\begin{aligned}
\Delta u(x_i, y_j, t_{n-\frac{1}{2}}) =& \frac{\Delta u_{i,j}^{n-1} + \Delta u_{i,j}^{n}}{2} + O(\Delta t)^2 \\
=& \frac{1}{2}\left\{ \frac{u_{i+1,j}^{n-1} - 2u_{i,j}^{n-1} + u_{i-1,j}^{n-1}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n-1} - 2u_{i,j}^{n-1} + u_{i,j-1}^{n-1}}{(\Delta y)^2} \right. \\
& \left. + \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{(\Delta y)^2} \right\} \\
& + O(\Delta x)^2 + O(\Delta y)^2 + O(\Delta t)^2, \\
& \qquad n \geq 1, 1 \leq i \leq I-1, 1 \leq j \leq J-1.
\end{aligned} \tag{14}
$$

Using the finite difference schemes (11), (8), (13), and (14), the discretization of (1) at the grid point (x_i, y_j) and the time step $\left(1-\frac{1}{2}\right)$ is as follows:

$$
\begin{aligned}
& \frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)} \times \frac{1}{2^{2-\beta}} \left[\frac{u_{i,j}^{1} - u_{i,j}^{0}}{\Delta t} - \frac{\partial u}{\partial t}\Big|_{i,j}^{0} \right] + \frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)} \times \frac{1}{2^{2-\beta}} \left(u_{i,j}^{1} - u_{i,j}^{0}\right) \\
& + \frac{1}{2}\left(u_{i,j}^{1} + u_{i,j}^{0}\right) = \frac{1}{2}\left\{ \frac{u_{i+1,j}^{0} - 2u_{i,j}^{0} + u_{i-1,j}^{0}}{(\Delta x)^2} + \frac{u_{i,j+1}^{0} - 2u_{i,j}^{0} + u_{i,j-1}^{0}}{(\Delta y)^2} \right. \\
& \qquad \left. + \frac{u_{i+1,j}^{1} - 2u_{i,j}^{1} + u_{i-1,j}^{1}}{(\Delta x)^2} + \frac{u_{i,j+1}^{1} - 2u_{i,j}^{1} + u_{i,j-1}^{1}}{(\Delta y)^2} \right\} \\
& \qquad + f_{i,j}^{1-\frac{1}{2}} + O(\Delta x)^2 + O(\Delta y)^2 + O(\Delta t)^{2-\beta}, \\
& \qquad\qquad 1 \leq i \leq I-1, 1 \leq j \leq J-1.
\end{aligned} \tag{15}
$$

Using the finite difference schemes (12), (9), (13), and (14), the discretization of (1) at the grid point (x_i, y_j) and the time step $\left(n-\frac{1}{2}\right)$ can be written as follows:

$$
\begin{aligned}
&\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left\{-b_{n-1}\frac{\partial u}{\partial t}\bigg|_{i,j}^{0}-\sum_{k=1}^{n-2}\left(b_{n-k-1}-b_{n-k}\right)\frac{u_{i,j}^{k}-u_{i,j}^{k-1}}{\Delta t}\right.\\
&\left.-\,(\frac{1}{2^{2-\beta}}-b_1)\frac{u_{i,j}^{n-1}-u_{i,j}^{n-2}}{\Delta t}+\frac{1}{2^{2-\beta}}\frac{u_{i,j}^{n}-u_{i,j}^{n-1}}{\Delta t}\right\}\\
&+\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left\{-b_{n-1}u_{i,j}^{0}-\sum_{k=1}^{n-2}\left(b_{n-k-1}-b_{n-k}\right)u_{i,j}^{k}\right.\\
&\left.-\,(\frac{1}{2^{2-\beta}}-b_1)u_{i,j}^{n-1}+\frac{1}{2^{2-\beta}}u_{i,j}^{n}\right\}+\frac{u_{i,j}^{n-1}+u_{i,j}^{n}}{2}\\
&\quad=\frac{1}{2}\left\{\frac{u_{i+1,j}^{n-1}-2u_{i,j}^{n-1}+u_{i-1,j}^{n-1}}{(\Delta x)^2}+\frac{u_{i,j+1}^{n-1}-2u_{i,j}^{n-1}+u_{i,j-1}^{n-1}}{(\Delta y)^2}\right.\\
&\qquad\left.+\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{(\Delta x)^2}+\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{(\Delta y)^2}\right\}\\
&\qquad+f_{i,j}^{n-\frac{1}{2}}+O(\Delta x)^2+O(\Delta y)^2+O(\Delta t)^{2-\beta},\\
&\qquad\qquad\qquad\qquad n\geq 2, 1\leq i\leq I-1, 1\leq j\leq J-1.
\end{aligned}
\tag{16}
$$

Finally, rearranging (15) and (16) and neglecting the truncation errors, it is obvious that the coefficient matrix of the unknowns is strictly diagonally dominant and so, it is nonsingular [8]. Therefore, by neglecting the truncation errors in (15) and (16), the unknowns $\left[u_{i,j}^{n}\right]$ $(1 \leq i \leq I-1, 1 \leq j \leq J-1)$ can be obtained for $n = 1$ and $n \geq 2$, respectively. Hence, the proposed Crank–Nicolson scheme is uniquely solvable.

3 Stability

In this section, we study the stability of the proposed Crank–Nicolson scheme for (1) with initial and boundary conditions (2)–(4). To do so, we introduce the following spaces and recall some theorems and lemmas, which will be used hereafter.

$$
\begin{aligned}
H^1(\Omega) &= \left\{v \in L^2(\Omega) : Dv \in L^2(\Omega)\right\},\\
H_0^1(\Omega) &= \left\{v \in H^1(\Omega) : Dv|_{\partial\Omega} = 0\right\},\\
H^2(\Omega) &= \left\{v \in L^2(\Omega) : D^{\alpha}v \in L^2(\Omega), |\alpha| \leq 2\right\}.
\end{aligned}
$$

Theorem 1 (The Cauchy–Schwarz inequality)**.** [21]
If u and v are members of an inner product space Ω with inner product $\langle\cdot,\cdot\rangle$, then

$$
|\langle u, v\rangle| = \left|\int_{\Omega} uvdx\right| \leq \|u\|_{L^2}\|v\|_{L^2}.
$$

Theorem 2. (Green's theorem) [21]
If Ω is a boundary domain in $\mathbb{R}^n$, then

$$\int_\Omega \nabla u.\nabla v \mathrm{dx} = \int_{\partial\Omega} v\frac{\partial u}{\partial n}ds - \int_\Omega \Delta u\, v\, dx, \qquad for\ u \in H^2(\Omega), v \in H^1(\Omega).$$

Theorem 3 (The Poincare–Friedrich inequality)**.** [21]
Let Ω be a boundary domain in $\mathbb{R}^n$. Then, there exists a constant $c_p > 0$ such that

$$\|u\|_{L^2}^2 \leq c_p\|\nabla u\|_{L^2}^2, \qquad for\ all\ u \in H_0^1(\Omega).$$

Theorem 4 (The discrete Gronwall theorem)**.** [20]
Assume that k_n is a nonnegative sequence and that the sequence ϕ_n satisfies the following relations:

$$\phi_0 \leq g_0,$$
$$\phi_n \leq g_0 + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s\phi_s, \qquad n \geqslant 1.$$

If $g_0 \geq 0$ and $p_n \geq 0$ (for $n \geq 0$), then

$$\phi_n \leq \left(g_0 + \sum_{s=0}^{n-1} p_s\right) \exp\left(\sum_{s=0}^{n-1} k_s\right), \qquad n \geqslant 1.$$

We state some useful relations in Lemmas 2 and 3. These are easy to prove.

Lemma 2. It holds that $\|u\|\|v\| \leq \frac{\gamma^2}{2}\|u\|^2 + \frac{1}{2\gamma^2}\|v\|^2,$ for all $u, v \in \Omega,$ for all $\gamma \in \mathbb{R}$.

Lemma 3. If $b_s = \left(s+\frac{1}{2}\right)^{2-\beta} - \left(s-\frac{1}{2}\right)^{2-\beta}$ $(s = 1, 2, \ldots, 1 < \beta < 2)$, then $b_n < b_{n-1} < \cdots < b_2 < b_1 < 1$.

Neglecting the truncation errors, equations (15) and (16) can be written as

$$\begin{aligned} &\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)} \times \frac{1}{2^{2-\beta}}\left[\frac{u_{i,j}^1 - u_{i,j}^0}{\Delta t} - \frac{\partial u}{\partial t}\bigg|_{i,j}^0\right] \\ &+ \frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)} \times \frac{1}{2^{2-\beta}}(u_{i,j}^1 - u_{i,j}^0) + \frac{1}{2}(u_{i,j}^1 + u_{i,j}^0) \\ &\qquad = \frac{1}{2}\left(\Delta u_{i,j}^1 + \Delta u_{i,j}^0\right) + f_{i,j}^{1-\frac{1}{2}}, \quad 1 \leqslant i \leqslant I-1, 1 \leqslant j \leqslant J-1, \end{aligned} \tag{17}$$

and

$$
\begin{aligned}
&\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left\{-\,b_{n-1}\frac{\partial u}{\partial t}\Big|_{i,j}^{0}-\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\frac{u_{i,j}^{k}-u_{i,j}^{k-1}}{\Delta t}\right.\\
&\left.-\,(\frac{1}{2^{2-\beta}}-b_1)\frac{u_{i,j}^{n-1}-u_{i,j}^{n-2}}{\Delta t}+\frac{1}{2^{2-\beta}}\frac{u_{i,j}^{n}-u_{i,j}^{n-1}}{\Delta t}\right\}\\
&+\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left\{-b_{n-1}u_{i,j}^{0}-\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\,u_{i,j}^{k}\right.\\
&\left.-(\frac{1}{2^{2-\beta}}-b_1)u_{i,j}^{n-1}+\frac{1}{2^{2-\beta}}u_{i,j}^{n}\right\}+\frac{u_{i,j}^{n-1}+u_{i,j}^{n}}{2}\\
&\qquad=\frac{\Delta u_{i,j}^{n-1}+\Delta u_{i,j}^{n}}{2}+f_{i,j}^{n-\frac{1}{2}},\qquad n\geqslant 2, 1\leqslant i\leqslant I-1, 1\leqslant j\leqslant J-1,
\end{aligned}
\tag{18}
$$

respectively. Let $\widetilde{u}_{i,j}^{n}$ $(1\leq i\leq I-1, 1\leq j\leq J-1, n=1,2,\ldots)$ be the approximate solution of (17) and (18) with respect to the round-off error, and let $u_{i,j}^{n}$ $(1\leq i\leq I-1, 1\leq j\leq J-1, n=1,2,\ldots)$ be the exact solution of (17) and (18). Define

$$
e_{i,j}^{n}=u_{i,j}^{n}-\widetilde{u}_{i,j}^{n}\quad (0\leq i\leq I,\quad 0\leq j\leq J,\quad and\quad n=0,1,\ldots).
$$

By considering e^n instead of $e_{i,j}^{n}$, we obtain the following round-off error equations:

$$
\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\times\frac{1}{2^{2-\beta}}\left\{\left[\frac{e^1-e^0}{\Delta t}-\delta e^0\right]+(e^1-e^0)\right\}+\frac{1}{2}(e^1+e^0)=\frac{1}{2}(\Delta e^1+\Delta e^0),
\tag{19}
$$

$$
\begin{aligned}
&\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left\{-b_{n-1}\delta e^0-\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\,\frac{e^k-e^{k-1}}{\Delta t}\right.\\
&\left.-(\frac{1}{2^{2-\beta}}-b_1)\frac{e^{n-1}-e^{n-2}}{\Delta t}+\frac{1}{2^{2-\beta}}\frac{e^n-e^{n-1}}{\Delta t}\right\}\\
&+\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left\{-b_{n-1}e^0-\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\,e^k-(\frac{1}{2^{2-\beta}}-b_1)e^{n-1}\right.\\
&\left.+\frac{1}{2^{2-\beta}}e^n\right\}+\frac{e^{n-1}+e^n}{2}=\frac{\Delta e^{n-1}+\Delta e^n}{2},\qquad n\geqslant 2,
\end{aligned}
\tag{20}
$$

where $\delta e^0=\frac{\partial u}{\partial t}\Big|_{i,j}^{0}-\frac{\partial\widetilde{u}}{\partial t}\Big|_{i,j}^{0}$. Now, we are ready to present the following theorem.

Theorem 5. If $e^k\in H_0^1(\Omega)$, then the solutions of the finite difference approaches (17) and (18) are unconditionally stable.

Proof. Let $\alpha=\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}$. If we multiply (19) by $\left(e^1-e^0\right)$, then we obtain

$$\frac{\alpha}{2^{2-\beta}(\Delta t)}\left\langle e^1-e^0,e^1-e^0\right\rangle+\frac{\alpha}{2^{2-\beta}}\left\langle e^1-e^0,e^1-e^0\right\rangle+\frac{1}{2}\left\langle e^1+e^0,e^1-e^0\right\rangle$$
$$-\frac{1}{2}\left\langle \Delta e^1+\Delta e^0,e^1-e^0\right\rangle=\frac{\alpha}{2^{2-\beta}}\left\langle \delta e^0,e^1-e^0\right\rangle. \tag{21}$$

Applying Theorem 2 (Green's theorem) to $\left\langle \Delta e^1+\Delta e^0,e^1-e^0\right\rangle$ in the left side of (51) and applying Theorem 1 (the Cauchy–Schwarz inequality) and Lemma 2 to the right side of (51), we can write

$$\frac{\alpha}{2^{2-\beta}(\Delta t)}\left\|e^1-e^0\right\|^2+\frac{\alpha}{2^{2-\beta}}\left\|e^1-e^0\right\|^2+\frac{1}{2}(\left\|e^1\right\|^2-\left\|e^0\right\|^2)$$
$$+\frac{1}{2}(\left\|\nabla e^1\right\|^2-\left\|\nabla e^0\right\|^2)\leq\frac{\alpha}{2^{2-\beta}}\left\{\frac{\left\|\delta e^0\right\|^2}{2}+\frac{\left\|e^1-e^0\right\|^2}{2}\right\}.$$

Therefore,

$$\left\|\nabla e^1\right\|^2\leq\left\|e^0\right\|^2+\left\|\nabla e^0\right\|^2+\frac{\alpha}{2^{2-\beta}}\left\|\delta e^0\right\|^2,$$

and by applying Theorem 3 (the Poincaré–Friedrich inequality), we find a constant $c_p>0$ such that

$$\left\|\nabla e^1\right\|^2\leq(c_p+1)\left\|\nabla e^0\right\|^2+\frac{\alpha c_p}{2^{2-\beta}}\left\|\nabla\delta e^0\right\|^2. \tag{22}$$

If we multiply (20) by $\left(e^n-e^{n-1}\right)$, then we find

$$\frac{\alpha}{2^{2-\beta}(\Delta t)}\left\langle e^n-e^{n-1},e^n-e^{n-1}\right\rangle+\frac{\alpha}{2^{2-\beta}}\left\langle e^n,e^n-e^{n-1}\right\rangle$$
$$+\frac{1}{2}\left\langle e^n+e^{n-1},e^n-e^{n-1}\right\rangle-\frac{1}{2}\left\langle \Delta e^n+\Delta e^{n-1},e^n-e^{n-1}\right\rangle$$
$$=\alpha b_{n-1}\left\langle \delta e^0,e^n-e^{n-1}\right\rangle$$
$$+\alpha\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\langle\frac{e^k-e^{k-1}}{\Delta t},e^n-e^{n-1}\rangle$$
$$+\alpha(\frac{1}{2^{2-\beta}}-b_1)\langle\frac{e^{n-1}-e^{n-2}}{\Delta t},e^n-e^{n-1}\rangle+\alpha b_{n-1}\left\langle e^0,e^n-e^{n-1}\right\rangle$$
$$+\alpha\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\left\langle e^k,e^n-e^{n-1}\right\rangle$$
$$+\alpha(\frac{1}{2^{2-\beta}}-b_1)\left\langle e^{n-1},e^n-e^{n-1}\right\rangle. \tag{23}$$

Applying Theorem 2 (Green's theorem) to $\left\langle \Delta e^n+\Delta e^{n-1},e^n-e^{n-1}\right\rangle$ in the left side of (23) and applying Theorem 1 (the Cauchy–Schwarz inequality) and Lemma 2 to the right side of (23), we obtain

$$
\begin{aligned}
&\frac{\alpha}{2^{2-\beta}(\Delta t)}\|e^n - e^{n-1}\|^2 + (\frac{\alpha}{2^{2-\beta}}\|e^n\|^2 - \frac{\alpha}{2^{2-\beta}}\left\langle e^n, e^{n-1}\right\rangle) \\
&+ \frac{1}{2}(\|e^n\|^2 - \left\|e^{n-1}\right\|^2) + \frac{1}{2}(\|\nabla e^n\|^2 - \left\|\nabla e^{n-1}\right\|^2) \\
&\quad \leq \alpha b_{n-1}(\frac{\gamma^2}{2}\left\|\delta e^0\right\|^2 + \frac{\left\|e^n - e^{n-1}\right\|^2}{2\gamma^2}) \\
&\quad\quad + \alpha \sum_{k=1}^{n-2}(b_{n-k-1} - b_{n-k})\,(\frac{\gamma^2}{2}\|\frac{e^k - e^{k-1}}{\Delta t}\|^2 + \frac{1}{2\gamma^2}\left\|e^n - e^{n-1}\right\|^2) \\
&\quad\quad + \alpha(\frac{\gamma^2}{2}\|\frac{e^{n-1} - e^{n-2}}{\Delta t}\|^2 + \frac{1}{2\gamma^2}\|e^n - e^{n-1}\|^2) \\
&\quad\quad + \alpha b_{n-1}(\frac{\gamma^2}{2}\left\|e^0\right\|^2 + \frac{\left\|e^n - e^{n-1}\right\|^2}{2\gamma^2}) \\
&\quad\quad + \alpha \sum_{k=1}^{n-2}(b_{n-k-1} - b_{n-k})\,(\frac{\gamma^2}{2}\left\|e^k\right\|^2 + \frac{1}{2\gamma^2}\left\|e^n - e^{n-1}\right\|^2) \\
&\quad\quad + \alpha(\frac{\gamma^2}{2}\|e^{n-1}\|^2 + \frac{1}{2\gamma^2}\|e^n - e^{n-1}\|^2), \qquad\qquad \gamma \in \mathbb{R}.
\end{aligned}
\tag{24}
$$

Furthermore, from Lemma 3, we deduce that

$$
\frac{\alpha b_{n-1}}{\gamma^2} + \frac{\alpha}{\gamma^2}\sum_{k=1}^{n-2}(b_{n-k-1} - b_{n-k}) + \frac{\alpha}{\gamma^2} \leq \frac{2\alpha}{\gamma^2}, \quad \gamma \in \mathbb{R}. \tag{25}
$$

Having (25) in mind, equation (23) gives

$$
\begin{aligned}
&\frac{\alpha}{2^{2-\beta}(\Delta t)}\|e^n - e^{n-1}\|^2 + \frac{\alpha}{2^{2-\beta}}\|e^n\|^2 + \frac{1}{2}\|e^n\|^2 + \frac{1}{2}\|\nabla e^n\|^2 \\
&\quad \leq \alpha b_{n-1}\frac{\gamma^2}{2}\left\{\left\|\delta e^0\right\|^2 + \left\|e^0\right\|^2\right\} + \frac{\alpha\gamma^2}{2}\sum_{k=1}^{n-2}(b_{n-k-1} - b_{n-k})\left\|e^k\right\|^2 \\
&\quad\quad + \frac{\alpha\gamma^2}{2(\Delta t)^2}\sum_{k=1}^{n-2}(b_{n-k-1} - b_{n-k})\left\|e^k - e^{k-1}\right\|^2 + \frac{\alpha\gamma^2}{2(\Delta t)^2}\left\|e^{n-1} - e^{n-2}\right\|^2 \\
&\quad\quad + (\frac{\alpha\gamma^2}{2} + \frac{1}{2})\left\|e^{n-1}\right\|^2 + \frac{2\alpha}{\gamma^2}\left\|e^n - e^{n-1}\right\|^2 + \frac{\alpha}{2^{2-\beta}}\left\langle e^n, e^{n-1}\right\rangle \\
&\quad\quad + \frac{1}{2}\left\|\nabla e^{n-1}\right\|^2, \quad \gamma \in \mathbb{R}.
\end{aligned}
\tag{26}
$$

By using Theorem 1 (the Cauchy–Schwarz inequality) and Lemma 2, we obtain

$$
\frac{\alpha}{2^{2-\beta}}\left\langle e^n, e^{n-1}\right\rangle \leq \frac{\alpha}{2^{2-\beta}}\left\{\frac{\|e^n\|^2}{2} + \frac{\left\|e^{n-1}\right\|^2}{2}\right\}. \tag{27}
$$

Consider the following relations:

$$\begin{aligned}\frac{\alpha\gamma^2}{2(\Delta t)^2}\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\left\|e^k-e^{k-1}\right\|^2+\frac{\alpha\gamma^2}{2(\Delta t)^2}\left\|e^{n-1}-e^{n-2}\right\|^2\\ \leq\frac{\alpha\gamma^2}{(\Delta t)^2}\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\left(\left\|e^k\right\|^2+\left\|e^{k-1}\right\|^2\right)\\ +\frac{\alpha\gamma^2}{(\Delta t)^2}\left(\left\|e^{n-1}\right\|^2+\left\|e^{n-2}\right\|^2\right).\end{aligned}\tag{28}$$

If we use (27)–(28) and assume that $\gamma^2 = 2^{3-\beta}(\Delta t)$, then relation (26) allows us to write

$$\begin{aligned}&\frac{1}{2}\times\frac{\alpha}{2^{2-\beta}}\left\|e^n\right\|^2+\frac{1}{2}\left\|e^n\right\|^2+\frac{1}{2}\left\|\nabla e^n\right\|^2\\ &\leq\alpha b_{n-1}\frac{\gamma^2}{2}\left\{\left\|\delta e^0\right\|^2+\left\|e^0\right\|^2\right\}+\frac{\alpha\gamma^2}{2}\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\left\|e^k\right\|^2\\ &+\frac{\alpha\gamma^2}{(\Delta t)^2}\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\left(\left\|e^k\right\|^2+\left\|e^{k-1}\right\|^2\right)+\frac{\alpha\gamma^2}{(\Delta t)^2}\left\|e^{n-2}\right\|^2\\ &+(\frac{\alpha\gamma^2}{2}+\frac{\alpha\gamma^2}{(\Delta t)^2}+\frac{\alpha}{2\times 2^{2-\beta}}+\frac{1}{2})\left\|e^{n-1}\right\|^2+\frac{1}{2}\left\|\nabla e^{n-1}\right\|^2,\quad n\geq 2.\end{aligned}\tag{29}$$

By using Theorem 3 (the Poincaré–Friedrich inequality), we find a constant $c_p > 0$ such that relation (29) implies

$$\begin{aligned}\frac{1}{2}\left\|\nabla e^n\right\|^2\leq&\frac{\alpha b_{n-1}\gamma^2}{2}c_p\left\|\nabla e^0\right\|^2+\frac{\alpha b_{n-1}\gamma^2}{2}c_p\left\|\nabla\delta e^0\right\|^2\\ &+\frac{\alpha\gamma^2}{2}c_p\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\left\|\nabla e^k\right\|^2\\ &+\frac{\alpha\gamma^2}{(\Delta t)^2}c_p\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})(\left\|\nabla e^k\right\|^2+\left\|\nabla e^{k-1}\right\|^2)\\ &+\frac{\alpha\gamma^2}{(\Delta t)^2}c_p\left\|\nabla e^{n-2}\right\|^2+(\frac{\alpha\gamma^2}{2}+\frac{\alpha\gamma^2}{(\Delta t)^2}\\ &+\frac{\alpha}{2\times 2^{2-\beta}}+\frac{1}{2})c_p\left\|\nabla e^{n-1}\right\|^2+\frac{1}{2}\left\|\nabla e^{n-1}\right\|^2,\quad n\geq 2.\end{aligned}\tag{30}$$

We may assume without loss of generality that there exist constants $\theta_1, \theta_2 \geq 0$ such that relations (22) and (30) can be written as

$$\begin{aligned}
&\left\|\nabla e^1\right\|^2 \leq \theta_1 \left\|\nabla e^0\right\|^2 + \theta_2 \left\|\nabla \delta e^0\right\|^2 , \\
&\|\nabla e^n\|^2 \leq \left(\theta_1 \left\|\nabla e^0\right\|^2 + \theta_2 \left\|\nabla \delta e^0\right\|^2\right) + \sum_{k=1}^{n-1} \left(c_k \left\|\nabla e^k\right\|^2\right), \\
&\qquad n \geq 2, \quad \theta_1, \theta_2 \geq 0, \quad c_k > 0 \ for \quad k = 1, \dots, n-1.
\end{aligned} \tag{31}$$

By Theorem 4 (the discrete Gronwall theorem), equation (31) yields

$$\|\nabla e^n\|^2 \leq \left(\theta_1 \left\|\nabla e^0\right\|^2 + \theta_2 \left\|\nabla \delta e^0\right\|^2\right) \exp\left(\sum_{k=1}^{n-1} c_k\right), \quad n \geq 1, \theta_1, \theta_2 \geq 0,$$

and according to Theorem 3 (the Poincare–Friedrich inequality), there exists a constant $\widehat{c}_p > 0$ such that

$$\|e^n\|^2 \leq \widehat{c}_p \left(\theta_1 \left\|\nabla e^0\right\|^2 + \theta_2 \left\|\nabla \delta e^0\right\|^2\right) \exp\left(\sum_{k=1}^{n-1} c_k\right), \quad n \geq 1, \theta_1, \theta_2 \geq 0. \tag{32}$$

By using Lemma 3, it is easy to show that

$$\sum_{k=1}^{n-1} c_k \leq \left(2\alpha\gamma^2 + \frac{8\alpha\gamma^2}{(\Delta t)^2} + \frac{\alpha}{2^{2-\beta}} + 1\right) c_p + 1. \tag{33}$$

Set $\theta = \left(2\alpha\gamma^2 + \frac{8\alpha\gamma^2}{(\Delta t)^2} + \frac{\alpha}{2^{2-\beta}} + 1\right) c_p + 1$. Then, it follows from relations (32) and (33) that

$$\|e^n\| \leq \sqrt{\widehat{c}_p \left(\theta_1 \|\nabla e^0\|^2 + \theta_2 \|\nabla \delta e^0\|^2\right) \exp(\theta)}, \quad n \geq 1, \theta_1, \theta_2, \theta \geq 0, \widehat{c}_p > 0,$$

where θ_1, θ_2, θ, $\widehat{c}_p$ are independent of n. □

4 Convergence

In this section, we study the convergence of the proposed Crank–Nicolson scheme for (1) with initial and boundary conditions (2)–(4).

Let $u_{i,j}^n$ $(1 \leq i \leq I-1, 1 \leq j \leq J-1, n = 1, 2, \dots)$ be the exact solution of (17) and (18), and let $U_{i,j}^n$ $(1 \leq i \leq I-1, 1 \leq j \leq J-1, n = 1, 2, \dots)$ be the exact solution of (15) and (16). Define $\xi_{i,j}^n = U_{i,j}^n - u_{i,j}^n$ $(1 \leq i \leq I-1, 1 \leq j \leq J-1, n = 1, 2, \dots)$. By considering ξ^n instead of $\xi_{i,j}^n$ we obtain

$$\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\times\frac{1}{2^{2-\beta}}\frac{\xi^1}{\Delta t}+\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\times\frac{1}{2^{2-\beta}}\xi^1+\frac{1}{2}\xi^1$$
$$=\frac{\Delta\xi^1}{2}+O(\Delta x)^2+(O(\Delta y)^2+O(\Delta t)^{2-\beta}), \tag{34}$$

and

$$\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left\{-\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\frac{\xi^k-\xi^{k-1}}{\Delta t}-(\frac{1}{2^{2-\beta}}-b_1)\frac{\xi^{n-1}-\xi^{n-2}}{\Delta t}+\frac{1}{2^{2-\beta}}\frac{\xi^n-\xi^{n-1}}{\Delta t}\right\}$$
$$+\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}\left\{-\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\,\xi^k-(\frac{1}{2^{2-\beta}}-b_1)\xi^{n-1}+\frac{1}{2^{2-\beta}}\xi^n\right\} \tag{35}$$
$$+\frac{\xi^{n-1}+\xi^n}{2}$$
$$=\frac{\Delta\xi^{n-1}+\Delta\xi^n}{2}+\left(O(\Delta x)^2+O(\Delta y)^2+O(\Delta t)^{2-\beta}\right),\quad n\geq 2.$$

Now, we are ready to present our next theorem.

Theorem 6. If $\xi^k\in H_0^1(\Omega)$, then the solutions of the finite difference approaches (17) and (18) are unconditionally convergent.

Proof. Let $\alpha=\frac{(\Delta t)^{1-\beta}}{\Gamma(3-\beta)}$. If we multiply (34) by $\left(\xi^1\right)$, then we obtain

$$\frac{\alpha}{2^{2-\beta}(\Delta t)}\left\langle\xi^1,\xi^1\right\rangle+\frac{\alpha}{2^{2-\beta}}\left\langle\xi^1,\xi^1\right\rangle+\frac{1}{2}\left\langle\xi^1,\xi^1\right\rangle-\frac{1}{2}\left\langle\Delta\xi^1,\xi^1\right\rangle$$
$$=<\left(O(\Delta x)^2+O(\Delta y)^2+O(\Delta t)^{2-\beta}\right),\xi^1>0. \tag{36}$$

Applying Theorem 2 (Green's theorem) to $\left\langle\Delta\xi^1,\xi^1\right\rangle$ in the left side of (36) and applying Theorem 1 (the Cauchy–Schwarz inequality) and Lemma 2 to the right side of (36), we find that

$$\frac{\alpha}{2^{2-\beta}(\Delta t)}\left\|\xi^1\right\|^2+\frac{\alpha}{2^{2-\beta}}\left\|\xi^1\right\|^2+\frac{\left\|\xi^1\right\|^2}{2}+\frac{1}{2}\left\|\nabla\xi^1\right\|^2$$
$$\leq\frac{\left\|O(\Delta x)^2+O(\Delta y)^2+O(\Delta t)^{2-\beta}\right\|^2}{2}+\frac{\left\|\xi^1\right\|^2}{2}.$$

Therefore,

$$\|\nabla\xi^1\|^2\leq\|O(\Delta x)^2+O(\Delta y)^2+O(\Delta t)^{2-\beta}\|^2. \tag{37}$$

If we multiply (35) by $\left(\xi^n-\xi^{n-1}\right)$, then we obtain

$$
\begin{aligned}
&\frac{\alpha}{2^{2-\beta}(\Delta t)}\left\langle \xi^n-\xi^{n-1},\xi^n-\xi^{n-1}\right\rangle+\frac{\alpha}{2^{2-\beta}}\left\langle \xi^n,\xi^n-\xi^{n-1}\right\rangle\\
&+\frac{1}{2}\left\langle \xi^n+\xi^{n-1},\xi^n-\xi^{n-1}\right\rangle-\frac{1}{2}\left\langle \Delta\xi^n+\Delta\xi^{n-1},\xi^n-\xi^{n-1}\right\rangle\\
&\quad=\alpha\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\langle\frac{\xi^k-\xi^{k-1}}{\Delta t},\xi^n-\xi^{n-1}\rangle\\
&\qquad+\alpha(\frac{1}{2^{2-\beta}}-b_1)\langle\frac{\xi^{n-1}-\xi^{n-2}}{\Delta t},\xi^n-\xi^{n-1}\rangle\\
&\qquad+\alpha\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\left\langle \xi^k,\xi^n-\xi^{n-1}\right\rangle\\
&\qquad+\alpha(\frac{1}{2^{2-\beta}}-b_1)\left\langle \xi^{n-1},\xi^n-\xi^{n-1}\right\rangle\\
&\qquad+<\left(O(\Delta x)^2+O(\Delta y)^2+O(\Delta t)^{2-\beta}\right),\xi^n-\xi^{n-1}\right\rangle.
\end{aligned}
\tag{38}
$$

Again, using Theorem 1 (the Cauchy–Schwarz inequality) and Lemma 2, we can write

$$
\begin{aligned}
&<(O(\Delta x)^2+O(\Delta y)^2+O(\Delta t)^{2-\beta}),\xi^n-\xi^{n-1}>\\
&\qquad\le\|O(\Delta x)^2+O(\Delta y)^2+O(\Delta t)^{2-\beta}\|^2+\frac{\|\xi^n\|^2}{2}+\frac{\left\|\xi^{n-1}\right\|^2}{2}.
\end{aligned}
$$

Simplifying relation (38) (similar to Theorem 5, in which the simplification of (23) resulted in (29)) and using the recent relation, we obtain

$$
\begin{aligned}
&\frac{1}{2}\times\frac{\alpha}{2^{2-\beta}}\left\|\xi^n\right\|^2+\frac{1}{2}\left\|\nabla\xi^n\right\|^2\\
&\quad\le\left\|O(\Delta x)^2+O(\Delta y)^2+O(\Delta t)^{2-\beta}\right\|^2+\frac{\alpha\gamma^2}{2}\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})\left\|\xi^k\right\|^2\\
&\qquad+\frac{\alpha\gamma^2}{(\Delta t)^2}\sum_{k=1}^{n-2}(b_{n-k-1}-b_{n-k})(\left\|\xi^k\right\|^2+\left\|\xi^{k-1}\right\|^2)+\frac{\alpha\gamma^2}{(\Delta t)^2}\left\|\xi^{n-2}\right\|^2\\
&\qquad+(\frac{\alpha\gamma^2}{2}+\frac{\alpha\gamma^2}{(\Delta t)^2}+\frac{\alpha}{2\times 2^{2-\beta}}+1)\left\|\xi^{n-1}\right\|^2+\frac{1}{2}\left\|\nabla\xi^{n-1}\right\|^2,\quad n\ge 2.
\end{aligned}
$$

By Theorem 3 (the Poincaré–Friedrich inequality), there exists a constant $c_p>0$ such that

$$\begin{aligned}\frac{1}{2}\left\|\nabla\xi^{n}\right\|^{2}\leq &\left\|O(\Delta x)^{2}+O(\Delta y)^{2}+O(\Delta t)^{2-\beta}\right\|^{2}\\
&+\frac{\alpha\gamma^{2}}{2}c_{p}\sum_{k=1}^{n-2}\left(b_{n-k-1}-b_{n-k}\right)\left\|\nabla\xi^{k}\right\|^{2}\\
&+\frac{\alpha\gamma^{2}}{(\Delta t)^{2}}c_{p}\sum_{k=1}^{n-2}\left(b_{n-k-1}-b_{n-k}\right)\left(\left\|\nabla\xi^{k}\right\|^{2}+\left\|\nabla\xi^{k-1}\right\|^{2}\right)\\
&+\frac{\alpha\gamma^{2}}{(\Delta t)^{2}}c_{p}\left\|\nabla\xi^{n-2}\right\|^{2}\\
&+\left(\frac{\alpha\gamma^{2}}{2}+\frac{\alpha\gamma^{2}}{(\Delta t)^{2}}+\frac{\alpha}{2\times 2^{2-\beta}}+1\right)c_{p}\left\|\nabla\xi^{n-1}\right\|^{2}\\
&+\frac{1}{2}\left\|\nabla\xi^{n-1}\right\|^{2},\quad n\geq 2.\end{aligned}\tag{39}$$

As we know, $\xi^0 = 0$. Without loss of generality, relations (37) and (39) can be written as

$$\begin{aligned}&\left\|\nabla\xi^{1}\right\|^{2}\leq\left\|O(\Delta x)^{2}+O(\Delta y)^{2}+O(\Delta t)^{2-\beta}\right\|^{2},\\
&\left\|\nabla\xi^{n}\right\|^{2}\leq 2\left\|O(\Delta x)^{2}+O(\Delta y)^{2}+O(\Delta t)^{2-\beta}\right\|^{2}+\sum_{k=1}^{n-1}C_{k}\left\|\nabla\xi^{k}\right\|^{2},\\
&\qquad n\geq 2,\quad C_{k}>0\quad for\quad k=1,\ldots,n-1.\end{aligned}\tag{40}$$

Thus, by using Theorem 4 (the discrete Gronwall theorem), the set of equations (40) yields

$$\left\|\nabla\xi^{n}\right\|^{2}\leq 2\left\|O(\Delta x)^{2}+O(\Delta y)^{2}+O(\Delta t)^{2-\beta}\right\|^{2}\exp\left(\sum_{k=1}^{n-1}C_{k}\right),\quad n\geq 1,$$

and according to Theorem 3 (the Poincaré–Friedrich inequality), there exists a constant $\overline{c}_p > 0$ such that

$$\left\|\xi^{n}\right\|^{2}\leq 2\overline{c}_{p}\left\|O(\Delta x)^{2}+O(\Delta y)^{2}+O(\Delta t)^{2-\beta}\right\|^{2}\exp\left(\sum_{k=1}^{n-1}C_{k}\right),\quad n\geq 1.\tag{41}$$

By using Lemma 3, it is easy to show that

$$\sum_{k=1}^{n-1}C_{k}\leq\left(2\alpha\gamma^{2}+\frac{8\alpha\gamma^{2}}{(\Delta t)^{2}}+\frac{\alpha}{2^{2-\beta}}+2\right)c_{p}+1.\tag{42}$$

Set $\zeta = \left(2\alpha\gamma^2 + \frac{8\alpha\gamma^2}{(\Delta t)^2} + \frac{\alpha}{2^{2-\beta}} + 2\right) c_p + 1$. Then, relations (41) and (42) allow us to write

$$\|\xi^n\| \leq \sqrt{2\bar{c}_p \exp(\zeta)} \left\|O(\Delta x)^2 + O(\Delta y)^2 + O(\Delta t)^{2-\beta}\right\|, \quad n \geq 1, \zeta \geq 0, \bar{c}_p > 0. \tag{43}$$

□

5 Numerical experiments

In this section, we present some numerical tests that confirm the validity of the proposed numerical method. To measure the accuracy of the proposed method, we use the maximum absolute error given by

$$L_\infty = \max_{1\leq i\leq I, 1\leq j\leq J} \left|\tilde{U}_{i,j}(T) - U_{i,j}(T)\right|,$$

where $\tilde{U}_{i,j}(T)$ and $U_{i,j}(T)$ denote the numerical solution and the exact solution of (1) with initial and boundary conditions (2)–(4) at (x_i, y_j) and time T, respectively.

Example 1. Consider a two-dimensional test problem of the form (1), with $\Omega = [0,1] \times [0,1]$, $f(x,t) = \left(\frac{24t^{4-\beta}}{\Gamma(5-\beta)} + \frac{24t^{5-\beta}}{\Gamma(6-\beta)} + 2t^4\pi^2\right) \sin(\pi x + \pi y) + t^4 \sin(\pi x + \pi y)$, and suppose that the initial and boundary conditions are assumed using the exact solution $u(x,y,t) = t^4 \sin(\pi x + \pi y)$; see [13]. Now, we provide some tests.

Test 1 Kumar, Bhardwaj, and Dubey [13] considered this example using a local meshless method with 2025 points on Ω. They reported the maximum absolute errors and CPU time with $\beta = 1.7, 1.9$ and different values for Δt at the time $T = 1.0$. Using the proposed method, we repeated this test. We considered this example by assuming $I = J = 45$ (2025 points on $\overline{\Omega}$). To solve the linear system of equations, we used the GMRES-m method with $m = 20$.

Table 1 presents the maximum absolute errors and CPU time obtained by Kumar, Bhardwaj, and Dubey [13] and the results of the proposed method with $\beta = 1.7$, different values for Δt, and 2025 points on $[0,1] \times [0,1]$ at $T = 1.0$. Table 2 presents the maximum absolute errors and CPU time obtained in [13] and the results of the proposed method with $\beta = 1.9$, different values for Δt, and 2025 points on $[0,1] \times [0,1]$ at $T = 1.0$.

As Tables 1 and 2 show, the maximum absolute errors and the CPU time of [13] and those of the proposed method are close, but the CPU time of the proposed method is smaller than that of [13].

The following tests show that the proposed method provides acceptable accuracy with a smaller number of points on Ω.

Test 2 We considered this example by the proposed method with $\Delta x = \Delta y = 0.1$, $\beta = 1.5, 1.9$, and different values for Δt. According to Table 3, with different values for Δt, the maximum absolute errors were small enough at $T = 1.0$. Also, decreasing the size of the time step increased the CPU time very slowly and improved the accuracy. The value $\Delta t = \frac{1}{80}$ was selected for the next test.

Test 3 We considered this example by the proposed method with $\Delta t = \frac{1}{80}$, $\beta = 1.5, 1.9$, and different values for $\Delta x, \Delta y$. According to Table 4, the accuracy was acceptable. Also, the CPU time was reasonable with $\Delta x = \Delta y = \frac{1}{10}, \frac{1}{20}$. Moreover, by decreasing Δx and Δy to $\frac{1}{40}, \frac{1}{80}$, the CPU time increased rapidly, and the accuracy did not improve significantly. According to relation (43), the convergence rate of our method depends on Δx, Δy, and Δt. In this case, the space steps decrease, but the time step is constant. Therefore the accuracy does not improve.

As shown in Tests 2 and 3, a very small size the of space step is not recommended, but small size of a time step is recommended. According to Diethelm, Garrappa, and Stynes [7], a high-order space discretization for a time-fractional partial differential equation is not advisable. They believe that to reach a high convergence, we must choose very small size of the time step in comparison with the size of the space step. Our experiments confirmed this idea.

Table 1: Comparison of the maximum absolute errors and CPU time with $\beta = 1.7$, different values for Δt, and 2025 points on $[0, 1] \times [0, 1]$ at $T = 1.0$

Δt	L_∞ [13]	L_∞	CPU (s) [13]	CPU (s)
$\frac{1}{10}$	$1.2917e-02$	$1.2575e-02$	1.751	1.414
$\frac{1}{20}$	$5.4532e-03$	$8.7100e-03$	2.210	1.996
$\frac{1}{40}$	$2.3351e-03$	$5.0648e-03$	3.062	2.746

Table 2: Comparison of the maximum absolute errors and CPU time with $\beta = 1.9$, different values for Δt, and 2025 points on $[0, 1] \times [0, 1]$ at $T = 1.0$

Δt	L_∞ [13]	L_∞	CPU (s) [13]	CPU (s)
$\frac{1}{10}$	$2.7619e-02$	$1.6456e-02$	1.751	1.298
$\frac{1}{20}$	$1.3079e-02$	$1.0294e-02$	2.210	1.613
$\frac{1}{40}$	$6.1953e-03$	$5.7166e-03$	3.062	2.119

6 Conclusion

The Crank–Nicolson difference scheme can be used easily for space-fractional equations, but some manipulations are needed for time-fractional equations. In this paper, the Crank–Nicolson method was extended for the discretization of a TFTE. The solvability, stability, and convergence of this proposed

Table 3: Maximum absolute errors and CPU time for different values of Δt and β, with $\Delta x = \Delta y = 0.1$ at $T = 1.0$

	$\beta = 1.5$		$\beta = 1.9$	
Δt	L_∞	$CPU(s)$	L_∞	CPU (s)
$\frac{1}{10}$	$1.3159e-02$	0.1069	$1.9190e-03$	0.1041
$\frac{1}{20}$	$1.0567e-02$	0.1249	$1.2971e-03$	0.1269
$\frac{1}{40}$	$7.5001e-03$	0.1673	$8.3675e-03$	0.1713
$\frac{1}{80}$	$5.5027e-03$	0.2944	$5.7025e-03$	0.2908
$\frac{1}{160}$	$4.3813e-03$	0.6513	$1.9956e-03$	0.6423
$\frac{1}{320}$	$3.7898e-03$	1.7490	$3.3609e-03$	1.7547

Table 4: Maximum absolute errors and CPU time for different values of Δx, Δy, and β, with $\Delta t = \frac{1}{80}$ at $T = 1.0$

	$\beta = 1.5$		$\beta = 1.9$	
$\Delta x = \Delta y$	L_∞	$CPU(s)$	L_∞	CPU (s)
$\frac{1}{10}$	$5.5027e-03$	0.2944	$5.7025e-03$	0.2908
$\frac{1}{20}$	$3.1200e-03$	0.4722	$3.6015e-03$	0.4273
$\frac{1}{40}$	$2.5315e-03$	3.6787	$3.0738e-03$	2.2135
$\frac{1}{80}$	$2.3744e-03$	74.7221	$2.9432e-03$	35.9883

method were proved. The numerical results were accurate enough. According to the numerical tests, to reach a high convergence, a very small size of the space step is not recommended, but a small size of the time step is recommended.

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