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Solving nonlinear Volterra integro-differential equation by using Legendre polynomial approximations

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Abstract

In this paper, we construct a new iterative method for solving nonlinear Volterra Integral Equation of the second kind, by approximating the Legendre polynomial basis. Error analysis is worked using property of interpolation. Finally, some examples are given to compare the results with some of the existing methods.

Keywords: Nonlinear Volterra integro-differential equation; Legendre polynomial; Error analysis.

1 Introduction

The area of orthogonal polynomials is an active research area in mathematics as well as with applications in mathematical physics, engineering, and computer science [6, 16]. Several numerical methods were used to solve integro-differential equations such as successive approximation method, Adomian decomposition method, Chebyshev and Taylor collocation methods, Haar Wavelet method, Wavelet Galerkin method, monotone iterative technique, Tau method, Walsh series method and Bezier curves method [2, 3, 4, 6, 13, 22]. One of the most common set of orthogonal polynomials is the set of the Legendre polynomials $L_0(x), L_1(x), ..., L_M(x)$, which are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = 1$. The

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Legendre polynomials $L_n(x)$, for $-1 \leq x \leq 1$ and $n \geq 0$, are given by the forms [6, 7, 11, 12, 21]

$$L_n(x) = \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}, n = 0, 1, \ldots, \quad (1)$$

where $[n/2] = n/2$ if n is even, otherwise $\frac{n-1}{2}$. To use the Legendre polynomials for our purposes, it is preferable to map this to $[0, 1]$. Then we can also define them by the following recursive formula [11, 12]: $L_0(x) = 1$; and $L_1(x) = 2x - 1$ and for $n = 1, 2, ...$

$$(n+1)L_{n+1}(x) = (2n+1)(2x-1)L_n(x) - nL_{n-1}(x). \quad (2)$$

On the other hand, the methods based on Legendre polynomials may be more appropriate for solving linear and nonlinear differential and Fredholm-Volterra integral and integro-differential-difference equations [5, 6, 7, 15, 18, 21]. Legendre polynomials are examples of eigen functions of singular Strum Liouville problems and have been used extensively in the solution of the boundary value problems and in computational fluid dynamics [5, 20] . Several ways for solving nonlinear integro differential equations are exist, for example Ghasemi et al. [8] with homotopy perturbation method and in [9] with wavelet Galerkin method and in [10] with sine−cosine wavelet method, Zhao and Corless in [23] adopted finite difference method, Lepik and Tamme in [19] with Haar wavelet method. In this paper, by means of the matrix relations between the Legendre polynomials and their derivatives, the mentioned methods above are modified and developed for solving the following nonlinear Volterra Integro-differential equation with variable coefficients

$$f_1(x)u(x) + f_2(x)u'(x) = g(x) + \int_0^x K(x, t, u(t))dt, \quad (3)$$

where $u \in X := C([0, 1], \mathbb{R})$, $f : [0, 1] \to \mathbb{R}$, $K : [0, 1]^2 \times \mathbb{R} \to \mathbb{R}$, also is assumed K is a continuous function, and $u : [0, 1] \to \mathbb{R}$ is an unknown function. We have obtained a solution expressed in the form

$$u(x) \approx \sum_{n=0}^{M} a_n L_n(x). \quad (4)$$

Next sections of this paper are organized as follows. In Section 2, expansion of Legendre basis properties and matrix relations, and its discretization of a integro-differential equation are given. In Section 3, the convergence of the method is described. In Section 4, the efficiency of the method by solving some examples and comparison of the numerical solutions with some other existing methods, is shown. A short conclusion is given in Section 5.

2 Expansion of Legendre basis and method of solution

If we define $\mathbf{L(x)} = [L_0(x)\ L_1(x)\ ...\ L_M(x)]$ and $\mathbf{A} = [a_0\ a_1\ ...\ a_M]^T$ then

$$u(x) \approx \sum_{n=0}^{M} a_n L_n(x) = \mathbf{L(x)A}. \tag{5}$$

Simillary if we define $\mathbf{L'(x)} = [L'_0(x)\ L'_1(x)\ ...\ L'_M(x)]$ we have

$$u'(x) \approx \sum_{n=0}^{M} a_n L'_n(x) = \mathbf{L'(x)A}, \tag{6}$$

where $'$ denotes the derivative with respect to x. By using Legendre recursive formula (1) for $n = 0, 1, 2, ..., M$, we can also obtain the matrix form of the equation as follows

$$\mathbf{L'(x)} = \mathbf{L(x)\Omega^T}, \tag{7}$$

where $\mathbf{\Omega}$ has two forms different for odd and even values of M, that is, for odd values of M we have

$$\mathbf{\Omega} = \begin{pmatrix} 0 & 0 & 0 & 0 \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 3 & 0 & 7 \cdots & 2M-3 & 0 & 0 \\ 1 & 0 & 5 & 0 \cdots & 0 & 2M-1 & 0 \end{pmatrix},$$

and for even values of M

$$\mathbf{\Omega} = \begin{pmatrix} 0 & 0 & 0 & 0 \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0 \cdots & 2M-3 & 0 & 0 \\ 0 & 3 & 0 & 7 \cdots & 0 & 2M-1 & 0 \end{pmatrix}.$$

From (6) and (7) we get

$$u'(x) \approx \mathbf{L(x)\Omega^T A}.$$

We use this method to approximate left hand side of Volterra integro-differential equation (3) as follows

$$f_1(x)u(x) + f_2(x)u'(x) \approx f_1(x)\mathbf{L(x)A} + f_2(x)\mathbf{L(x)\Omega^T A} = (f_1(x)\mathbf{L(x)} + f_2(x)\mathbf{L(x)\Omega^T})\mathbf{A}. \tag{8}$$

Thus

$$f_1(x)u(x) + f_2(x)u'(x) \approx \mathbf{S(x)A}, \tag{9}$$

where $\mathbf{S(x)} = [s_0(x)\ s_1(x)\ ...\ s_M(x)]$, and for $i = 0, 1, ..., M$, we define

$$s_i(x) = f_1(x)L_i(x) + f_2(x)\mathbf{L(x)}(\mathbf{\Omega^T})^i.$$

To obtain a solution of the problem (3), for each $x, t \in [0, 1]$ we define

$$K(x, t, u(t)) \approx \mathbf{L(x)K^*L^T(t)}$$

where $\mathbf{K}^* = [k_{nm}]$, and

$$k_{nm} = \frac{< L_n(x), < K(x, t, u(t)), L_m(t) >>}{\|L_n\|^2 \|L_m\|^2}.$$

We use this method to approximate the right hand side of Volterra integro-differential equation (3) as follows

$$g(x) + \int_0^x \mathbf{L(x)K^*L^T(t)}dt = g(x) + \mathbf{L(x)K^*} \int_0^x \mathbf{L^T(t)}dt.$$

By using Legendre formulas

$$\int_0^x \mathbf{L(t)}dt = (\mathbf{L(x)} - \mathbf{L(0)})(\mathbf{\Omega^T})^{-1},$$

we have

$$g(x) + \int_0^x K(x, t, u(t))dt = g(x) + \mathbf{L(x)K^*(\Omega)}^{-1}(\mathbf{L(x)^T} - \mathbf{L(0)^T}). \tag{10}$$

Let

$$h(x) = g(x) + \mathbf{L(x)K^*(\Omega)}^{-1}(\mathbf{L(x)^T} - \mathbf{L(0)^T}), \tag{11}$$

then from (9) and (11) we have

$$\mathbf{S(x)A} = h(x). \tag{12}$$

We can use a matrix method based on Legendre collocation points defined by

$$x_i = \frac{i}{M} \quad i = 0, 1, \ldots, M. \tag{13}$$

Now, by substituting the collocation points into Eq. (12) we have the following system

$$\mathbf{S(x_i)A} = h(x_i) \quad i = 0, 1, \ldots, M. \tag{14}$$

Thus, we use this numerical method to approximate the solutions of nonlinear Volterra integro-differential equation, which correspond to a system of $(M+1)$ algebraic equations for $(M+1)$ unknown Legendre coefficients $a_0, a_1, ..., a_M$. Briefly, Eq. (14) in the matrix form is as follows

$$\mathbf{SA} = \mathbf{H}, \tag{15}$$

where for $i = 0, 1, ..., M$

$$\mathbf{S} = [\mathbf{S}(x_0)\ \mathbf{S}(x_1)\ ...\ \mathbf{S}(x_M)]^T,$$

and

$$\mathbf{H} = [h(x_1)\ h(x_2)\ ...\ h(x_M)].$$

3 Error analysis

We assume that $u(x)$ is a sufficiently smooth function and $P_M(x)$ is the polynomial that interpolates u at points x_i, $i = 0, 1, ..., M$ that are the roots of $M+1$ degree shifted Chebyshev polynomial in $[0, 1]$. Then we have

$$u(x) - P_M(x) = \frac{d^{M+1}u}{dx^{M+1}}(\xi)\frac{\prod_{i=0}^{M}(x - x_i)}{(M+1)!}, \tag{16}$$

where $\xi \in [0, 1]$, therefore

$$|u(x) - P_M(x)| \leq max|\frac{d^{M+1}u(x)}{dx^{M+1}}|\frac{\prod_{i=0}^{M}(x - x_i)}{(M+1)!}. \tag{17}$$

If we assume that c is an upper bound for $max\frac{d^{M+1}u(x)}{dx^{M+1}}$, then

$$|u(x) - P_M(x)| \leq c\frac{1}{(M+1)!2^{2M+1}}. \tag{18}$$

Theorem 3.1. *Let* $u_M(x) = U_M^T\mathbf{L}(\mathbf{x})$ *where* $U_M = [u_0\ u_1\ ...\ u_M]^T$ *and*

$$u_m = (2m+1)\int_0^1 u(x)L_m(x)dx,$$

then, there exists a real number c' *such that*

$$\|u(x) - u_M(x)\|_2 \leq c'\frac{1}{(M+1)!2^{2M+1}}. \tag{19}$$

Proof. Suppose $f : [0, 1] \to \mathbf{R}$ be an arbitrary continuous function. We define

$$||f||_2 = \int_0^1 |f(x)|^2 dx. \quad (20)$$

Let X_M be the space of all polynomials that their degrees are equal or less than M and $f(x)$ be an arbitary function. Since X_M is a finite dimensional vector space, f has an uniqe best approximation u_M, such that

$$\|u(x) - u_M\|_2 \leq \|u - g\|_2 \qquad \forall g \in X_M. \quad (21)$$

In particular, we have

$$\|u(x) - u_M(x)\|_2^2 = \int_0^1 |u(x) - u_M(x)|^2 dx \leq \int_0^1 |u(x) - P_M(x)|^2 dx, \quad (22)$$

where P_M interpolates f. Thus

$$\|u(x) - u_M(x)\|_2^2 = \int_0^1 (c\frac{1}{(M+1)!2^{2M+1}})^2 dx, \quad (23)$$

so

$$\|u(x) - u_M(x)\|_2 \leq c\frac{1}{(M+1)!2^{2M+1}}. \quad (24)$$

4 Numerical examples

In this section, several numerical examples are given to show the efficiency of our proposed method for approximating the solution of Volterra integro-differential equation by comparing with other methods. In all examples N denotes the number of itereations

Example 4.1. *Consider the following nonlinear Volterra integro-differential equation of the second kind with the exact solution* $u(x) = x^3$

$$(x-1)u'(x)+xu(x) = 3(x-1)x^2-\frac{1}{3}x+\frac{1}{3}xcos(x^3)+\int_0^x xt^2 sin(u(t))dt. \quad (25)$$

Comparison of the absolute errors between Block-Pulse functions method [1] and the proposed method for $N = 7$ is shown in Table 2 . Also, Figure 2 shows the comparison between exact and approximate solutions for $N = 2$ and $N = 7$.

Example 4.2. *Consider the following nonlinear Volterra integro-differential equation of the second kind with the exact solution* $u(x) = x - x^2$

$$3(x-1)u(x) + x^2u'(x) = f(x) + \int_0^x (x-t)u(t)dt, \quad (26)$$

Table 1: Absolute errors for Example 4.1

t	BPFs method [1] for N=7	proposed method for N=7
0.0100	4.654×10^{-4}	3.585×10^{-6}
0.3537	8.098×10^{-5}	1.726×10^{-7}
0.6101	6.675×10^{-5}	6.052×10^{-7}
0.9500	3.581×10^{-5}	2.138×10^{-7}

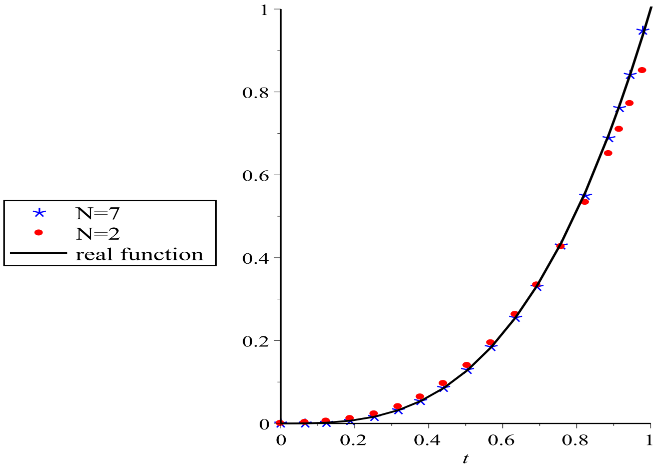

Figure 1: Comparison between exact and approximate solutions for Example 4.1

where
$f(x) = 3(x-1)(x-x^2) + x^2(1-2x) - \frac{1}{4}x^4 + \frac{1}{3}(x+1))x^3 - \frac{1}{2}(1/2)x^3.$

Comparison of absolute errors between CAS wavelet method [3] and the proposed method for $N = 7$ is shown in Table 2 . Also, Figure 2 shows the comparison between exact and approximate solutions for $N = 2$ and $N = 7$.

Example 4.3. *Consider the following nonlinear Volterra integral equation of the second kind with the exact solution* $u(x) = ln(x+1)$

$$u'(x) = f(x) + \int_0^x xt^2(u(t))^2 dt, \tag{27}$$

where

$$f(x) = \frac{1}{x+1} + (\frac{11}{9} + \frac{2}{3}x - \frac{1}{3}x^2 + \frac{2}{9}x^3)x\ln(x+1) - \frac{1}{3}(1+x^3)x(\ln(x+1))^2 - \frac{1}{9}x^2(11 - \frac{5}{2}x + \frac{2}{3}x^2). \tag{28}$$

Table 2: Absolute errors for Example 4.2

t	CAS wavelet method [3] for N=7	proposed method for N=7
0.0100	3.37×10^{-3}	4.8×10^{-4}
0.3446	4.72×10^{-3}	5.7×10^{-5}
0.7075	5.87×10^{-3}	3.4×10^{-4}
0.9178	3.42×10^{-2}	2.13×10^{-6}
1.0000	6.20×10^{-2}	5.8×10^{-5}

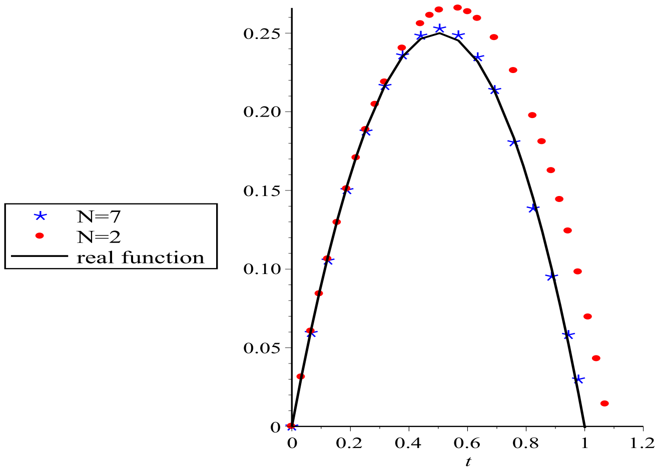

Figure 2: Comparison between exact and approximate solutions for Example 4.2

Comparison of absolute errors between DT wavelet method [4] and proposed method for $N = 7$ is shown in Table 3 . Also, Figure 3 shows the comparison between exact and approximate solutions for $N = 2$ and $N = 7$.

5 Conclusion

In this paper, we have solved nonlinear Volterra integro-differential equation of the second kind by using Legendre polynomial. A considerable advantage of this method is to find the approximation of analytical solution that is a polynomial of degree up to N. An other advantage of the method is that Legendre coefficients of the solution can be found very easily by using computer programs. The convergence of this method has been presented by Theorem 3.1.

Table 3: Absolute errors for Example 4.3

t	DT wavelet method [4] for N=7	proposed method for N=7
0.054	3.29×10^{-2}	3.90×10^{-6}
0.600	1.49×10^{-2}	1.83×10^{-6}
0.851	1.82×10^{-1}	2.88×10^{-5}
1.000	4.71×10^{-1}	7.52×10^{-6}

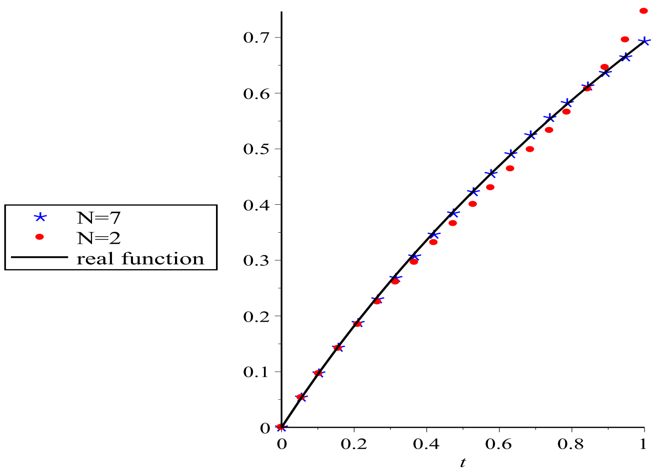

Figure 3: Comparison between exact and approximate solutions for Example 4.3

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حل تقریبی معادلات دیفرانسیل انتگرال غیر خطی با استفاده از چندجمله ای های لژاندر

مرتضی گچ پزان، مجید عرفانیان، و حسین بیگلو

دانشگاه فردوسی مشهد، دانشکده علوم ریاضی، گروه ریاضی کاربردی

چکیده : در این مقاله، یک روش تکراری برای بدست آوردن جوابهای معادلات دیفرانسیل انتگرالی ولترای نوع دوم بر اساس چندجمله ای های لژاندر ساخته شده است. تحلیل خطا با استفاده از درونیابی انجام گرفته است. سرانجام چند مثال با استفاده از روش پیشنهادی حل و با برخی روش های موجود مقایسه شده است.

کلمات کلیدی : معادله دیفرانسیل انتگرال ولترای غیر خطی؛ چندجمله ای های لژاندر؛ تحلیل خطا.