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Research Article

Goursat problem in Hyperbolic partial differential equations with variable coefficients solved by Taylor collocation method

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Abstract

The hyperbolic partial differential equation (PDE) has important practical uses in science and engineering. This article provides an estimate for solving the Goursat problem in hyperbolic linear PDEs with variable coefficients. The Goursat PDE is transformed into a second kind of linear Volterra integral equation. A convergent algorithm that employs Taylor polynomials is created to generate a collocation solution, and the error using the maximum norm is estimated. The paper includes numerical examples to prove the method's effectiveness and precision.

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1 Introduction

Partial differential equations (PDEs) are commonly used to address problems in various fields, such as engineering, physics, and finance. These equations are crucial in the study of various phenomena such as electric currents, gravity, heat transfer, water wave movement, fluid mechanics, electromagnetism, elasticity, quantum mechanics, population dynamics, stock and option pricing, chemical reaction-diffusion, as well as in the modeling of Schrödinger's equation.

A second-order linear hyperbolic PDE with two variables for $\omega(\epsilon, \eta)$ has the form

$$A\frac{\partial^2\omega}{\partial\epsilon^2} + B\frac{\partial^2\omega}{\partial\epsilon\partial\eta} + C\frac{\partial^2\omega}{\partial\eta^2} + D\frac{\partial\omega}{\partial\epsilon} + E\frac{\partial\omega}{\partial\eta} + F\omega + G = 0, \tag{1}$$

where $B^2 - 4AC > 0$, and A, B, C, D, E, F, and G are functions of the variables ϵ and η. By a suitable change of the independent variables, we shall show that any equation of the form (1) can be reduced to the canonical form or normal form of the hyperbolic equation called the Goursat problem

$$\frac{\partial^2\omega(t,s)}{\partial t\partial s} = \phi\left(t, s, \omega, \frac{\partial\omega(t,s)}{\partial t}, \frac{\partial\omega(t,s)}{\partial s}\right). \tag{2}$$

Equations (1) and (2) are hyperbolic PDEs and appear frequently in the study of ecological and cosmological phenomena [26]. To accurately simulate these events, it is important to develop efficient and reliable methods for solving these equations. Several numerical techniques have been proposed for this purpose, including finite difference methods [14, 4], finite element methods [2], Taylor matrix method [7, 6], Legendre multi-wavelet Galerkin method [28, 27], rational Chebyshev method [10], modified variational iteration method [1], and Chebyshev wavelet scheme [13].

The numerical solution of Goursat problem (2) has been investigated by many authors. For instance, Scott [25] considered the homogeneous Goursat problem (3) with the coefficients depending on the same variable. Evans and Sanugi [12] proposed a nonlinear trapezoidal formula based on geometric means. Day [9] used the Runge–Kutta method to approximate the solution of (2). Pandey [24] presented a novel exponential finite difference to obtain a numerical solution of (2). Drignei [11] developed an algorithm to find the quadruple solution to a Goursat problem in a triangular domain.

In this paper, we consider a collocation solution of the linear Goursat problem of the second-order with variable coefficients

$$\begin{aligned}&\frac{\partial^2\omega(t,s)}{\partial t\partial s}+a(t,s)\frac{\partial\omega(t,s)}{\partial t}+b(t,s)\frac{\partial\omega(t,s)}{\partial s}+c(t,s)\omega(t,s)=f(t,s), \qquad (3)\\ &\omega(0,s)=\alpha(s),\omega(t,0)=\beta(t),\alpha(0)=\beta(0),\quad (t,s)\in[0,X]\times[0,Y],\end{aligned}$$

where a, b, c, and f are smooth functions through the domain of discussion. For the existence and uniqueness of the solution, see [15].

The Goursat PDE (3) is converted to the linear Volterra integral equation (VIE) of the second kind

$$\begin{aligned}\omega(\tau,z)=&g(\tau,z)+\int_0^\tau\kappa_1(t,z)\omega(t,z)dt+\int_0^z\kappa_2(\tau,s)\omega(\tau,s)ds\\ &+\int_0^\tau\int_0^z\kappa_3(t,s)\omega(t,s)dsdt, \qquad (4)\end{aligned}$$

where g,κ_1,κ_2, and κ_3 are defined in (7). Moreover, the special case of (4) for $\kappa_1=\kappa_2=0$ is considered in [19]. We develop the collocation method introduced in [3, 16, 17, 18] to solve (4). This method is based on approximating the exact solution of a given integral equation with a suitable function belonging to a chosen finite-dimensional space (8) such that the approximated solution satisfies the integral equation on collocation points (9). An important aspect of this method is that the number of subintervals and the degree of Taylor polynomials can be changed to get the best possible result. It is also easy to implement, and the approximate solution is based on iterative formulas without needing to solve any algebraic equations.

The paper is organized as follows: In section 2, the converting of PDE to the two-dimensional VIE. In Section 3, the approximating solution of (6) in each collocation point by a Taylor polynomial. The convergence analysis is investigated in section 4. Numerical examples are provided in section 5 to illustrate the theoretical results. Finally, a conclusion is given in section 6.

2 Converting PDE to two-dimensional VIE

In this section, we study the technique that will convert PDE (3) to an equivalent VIE (4).

Integrating both sides of (3) with respect to s yields

$$\begin{aligned}\frac{\partial\omega(t,z)}{\partial t}-\frac{\partial\omega(t,0)}{\partial t}+\int_0^z a(t,s)\frac{\partial\omega(t,s)}{\partial t}ds+\int_0^z b(t,s)\frac{\partial\omega(t,s)}{\partial s}ds\\+\int_0^z c(t,s)\omega(t,s)ds=\int_0^z f(t,s)ds,\end{aligned}$$

which implies

$$\begin{aligned}\int_0^z f(t,s)ds=&\frac{\partial\omega(t,z)}{\partial t}-\frac{\partial\omega(t,0)}{\partial t}+\int_0^z a(t,s)\frac{\partial\omega(t,s)}{\partial t}ds+b(t,z)\omega(t,z)\\&-b(t,0)\omega(t,0)-\int_0^z\frac{\partial b(t,s)}{\partial s}\omega(t,s)ds+\int_0^z c(t,s)\omega(t,s)ds.\end{aligned}\tag{5}$$

Integrating again both sides of (5) with respect to t yields

$$\begin{aligned}\int_0^\tau\int_0^z f(t,s)dsdt=&\omega(\tau,z)-\omega(0,z)-\omega(\tau,0)+\omega(0,0)\\&+\int_0^\tau\int_0^z a(t,s)\frac{\partial\omega(t,s)}{\partial t}dsdt\\&+\int_0^\tau b(t,z)\omega(t,z)dt-\int_0^\tau b(t,0)\omega(t,0)dt\end{aligned}$$

$$-\int_0^{\tau}\int_0^{z}\frac{\partial b(t,s)}{\partial s}\omega(t,s)dsdt+\int_0^{\tau}\int_0^{z}c(t,s)\omega(t,s)dsdt.$$

Hence,

$$\begin{aligned}\int_0^{\tau}\int_0^{z}f(t,s)dsdt =&\omega(\tau,z)-\alpha(z)-\beta(\tau)+\omega(0,0)\\&+\int_0^{z}a(\tau,s)\omega(\tau,s)ds+\int_0^{\tau}b(t,z)\omega(t,z)dt\\&-\int_0^{z}a(0,s)\alpha(s)ds-\int_0^{\tau}b(t,0)\beta(t)dt\\&-\int_0^{\tau}\int_0^{z}\frac{\partial a(t,s)}{\partial t}\omega(t,s)dsdt\\&-\int_0^{\tau}\int_0^{z}\frac{\partial b(t,s)}{\partial s}\omega(t,s)dsdt+\int_0^{\tau}\int_0^{z}c(t,s)\omega(t,s)dsdt,\end{aligned}$$

which is equivalent to the following 2D-VIE:

$$\begin{aligned}\omega(\tau,z)=&g(\tau,z)+\int_0^{\tau}\kappa_1(t,z)\omega(t,z)dt+\int_0^{z}\kappa_2(\tau,s)\omega(\tau,s)ds\\&+\int_0^{\tau}\int_0^{z}\kappa_3(t,s)\omega(t,s)dsdt,\end{aligned}\tag{6}$$

where

$$\begin{aligned}\kappa_1(t,z):=&-b(t,z),\\\kappa_2(\tau,s):=&-a(\tau,s),\\\kappa_3(t,s):=&\frac{\partial a(t,s)}{\partial t}+\frac{\partial b(t,s)}{\partial s}-c(t,s),\\g(\tau,z):=&\alpha(z)+\beta(\tau)-\omega(0,0)+\int_0^{z}a(0,s)\alpha(s)ds\\&+\int_0^{\tau}b(t,0)\beta(t)dt+\int_0^{\tau}\int_0^{z}f(t,s)dsdt.\end{aligned}\tag{7}$$

3 Taylor collocation method

In this section, we approximate solutions of 2D-VIE (6) in the space

$$S_{p-1}^{(-1)}(\Pi_{N,M})=\{v:v_{n,m}=v|_{D_{n,m}}\in\pi_{p-1},n=0,1,\dots,N-1;m=0,1,\dots,M-1\}\tag{8}$$

of the real bivariate polynomial spline functions of degree (at most) $p-1$ in τ and z. Its dimension is NMp^2. Here, $\Pi_N=\{\tau_i=ih,i=0,1,\dots,N\}$ and

$\Pi_M = \{z_j = jk, j = 0, 1, \dots, M\}$ denote, respectively, uniform partitions of the intervals $[0, T]$ and $[0, Z]$ with the step-sizes given by $h = \frac{T}{N}$ and $k = \frac{Z}{M}$. These partitions are define a grid for D

$$\Pi_{N,M} = \Pi_N \times \Pi_M = \{(\tau_n, z_m), 0 \le n \le N, 0 \le m \le M\}. \tag{9}$$

Set the subintervals

$$\sigma_n = [\tau_n; \tau_{n+1}), n = 0, 1, \dots, N-2; \quad \sigma_{N-1} = [\tau_{N-1}, \tau_N],$$

$$\delta_m = [z_m; z_{m+1}), m = 0, 1, \dots, M-2; \quad \delta_{M-1} = [z_{M-1}, z_M],$$

and $D_{n,m} := \sigma_n \times \delta_m$ for all $n = 0, 1, \dots, N-1; m = 0, 1, \dots, M-1$.

To define the collocation solution, we use the Taylor polynomial on each rectangle $D_{n,m}; n = 0, 1, \dots, N-1; m = 0, 1, \dots, M-1$. Note that the solution ω of (6) is known at point $(0, 0)$: $\omega(0, 0) = g(0, 0)$.

First, we approximate ω in the rectangle $D_{0,0}$ by the polynomial

$$v_{0,0}(\tau, z) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j}\omega(0,0)}{\partial\tau^i \partial z^j} \tau^i z^j \; ; \quad (\tau, z) \in D_{0,0}, \tag{10}$$

where $\dfrac{\partial^{i+j}\omega(0,0)}{\partial\tau^i \partial z^j}$ is the exact value of $\dfrac{\partial^{i+j}\omega}{\partial\tau^i \partial z^j}$ at point $(0, 0)$.

Differentiating equation (6) j-times with respect to z and i-times with respect to τ, we obtain

$$\begin{aligned}
\frac{\partial^{i+j}\omega(\tau,z)}{\partial\tau^i \partial z^j} =& \partial_1^{(i)} \partial_2^{(j)} g(\tau, z) \\
&+ \sum_{l=0}^{j} \sum_{\eta=0}^{i-1} \binom{j}{l} \binom{i-1}{\eta} \frac{\partial^{i-1-\eta}}{\partial\tau^{i-1-\eta}} \left[\partial_2^{(j-l)} \kappa_1(\tau, z)\right] \frac{\partial^{\eta+l}\omega(\tau,z)}{\partial\tau^\eta \partial z^l} \\
&+ \sum_{l=0}^{j-1} \sum_{\eta=0}^{i} \binom{r}{l} \binom{i}{\eta} \frac{\partial^{i-\eta}}{\partial\tau^{i-\eta}} \left[\frac{\partial^{j-1-l}}{\partial z^{j-1-l}} \kappa_2(\tau, z)\right] \frac{\partial^{\eta+l}\omega(\tau,z)}{\partial\tau^\eta \partial z^l} \\
&+ \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} \binom{j-1}{l} \binom{i-1}{\eta} \frac{\partial^{i-1-\eta}}{\partial\tau^{i-1-\eta}} \left[\frac{\partial^{j-1-l}}{\partial z^{j-1-l}} \kappa_3(\tau, z)\right] \frac{\partial^{\eta+l}\omega(\tau,z)}{\partial\tau^\eta \partial z^l}.
\end{aligned} \tag{11}$$

Second, we approximate ω in the rectangles $D_{n,0}, n = 1, \dots, N-1$ by the polynomials

$$v_{n,0}(\tau, z) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j}\hat{v}_{n,0}(\tau_n, 0)}{\partial\tau^i \partial z^j} (\tau - \tau_n)^i z^j \; ; \quad (\tau, z) \in D_{n,0}, \tag{12}$$

where $\hat{v}_{n,0}$ is the exact solution of the integral equation:

$$\begin{aligned}\hat{v}_{n,0}(\tau,z) =& g(\tau,z)+\int_0^z \kappa_2(\tau,s)\hat{v}_{n,0}(\tau,s)ds\\&+\sum_{\xi=0}^{n-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\kappa_1(t,z)v_{\xi,0}(t,z)dt+\int_{\tau_n}^{\tau}\kappa_1(t,z)\hat{v}_{n,0}(t,z)dt\\&+\sum_{\xi=0}^{n-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\int_0^z \kappa_3(t,s)v_{\xi,0}(t,s)dsdt+\int_{\tau_n}^{\tau}\int_0^z\kappa_3(t,s)\hat{v}_{n,0}(t,s)dsdt,\end{aligned} \tag{13}$$

and $\frac{\partial^{i+j}\hat{v}_{n,0}(\tau_n,0)}{\partial\tau^i\partial z^j}$ is the exact value of $\frac{\partial^{i+j}\hat{v}_{n,0}}{\partial\tau^i\partial z^j}$ at point $(\tau_n,0)$.

Differentiating equation (13) j-times with respect to z and i-times with respect to τ, we obtain

$$\begin{aligned}\frac{\partial^{i+j}\hat{v}_{n,0}(\tau,z)}{\partial\tau^i\partial z^j} =& \partial_1^{(i)}\partial_2^{(j)}g(\tau,z)\\&+\sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial\tau^{i-1-\eta}}\left[\partial_2^{(j-l)}\kappa_1(\tau,z)\right]\frac{\partial^{\eta+l}\hat{v}_{n,0}(\tau,z)}{\partial\tau^\eta\partial z^l}\\&+\sum_{l=0}^{j-1}\sum_{\eta=0}^{i}\binom{j-1}{l}\binom{i}{\eta}\frac{\partial^{i-\eta}}{\partial\tau^{i-\eta}}\left[\frac{\partial^{j-1-l}}{\partial z^{j-1-l}}\kappa_2(\tau,z)\right]\frac{\partial^{\eta+l}\hat{v}_{n,0}(\tau,z)}{\partial\tau^\eta\partial z^l}\\&+\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1}\binom{j-1}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial\tau^{i-1-\eta}}\left[\frac{\partial^{j-1-l}}{\partial z^{j-1-l}}\kappa_3(\tau,z)\right]\frac{\partial^{\eta+l}\hat{v}_{n,0}(\tau,z)}{\partial\tau^\eta\partial z^l}.\end{aligned} \tag{14}$$

Third, we approximate ω by $v_{n,m}$ in the rectangles $D_{n,m}, n=0,1,\ldots,N-1$ and $m=1,\ldots,M-1$ such that

$$v_{n,m}(\tau,z)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{v}_{n,m}(\tau_n,z_m)}{\partial\tau^i\partial z^j}(\tau-\tau_n)^i(z-z_m)^j\ ;\quad (\tau,z)\in D_{n,m}, \tag{15}$$

where $\hat{v}_{n,m}$ is the exact solution of the integral equation:

$$\begin{aligned}\hat{v}_{n,m}(\tau,z) =& g(\tau,z)+\sum_{\xi=0}^{n-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\kappa_1(t,z)v_{\xi,m}(t,z)dt+\int_{\tau_n}^{\tau}\kappa_1(t,z)\hat{v}_{n,m}(t,z)dt\\&+\sum_{\rho=0}^{m-1}\int_{z_\rho}^{z_{\rho+1}}\kappa_2(\tau,s)v_{n,\rho}(\tau,s)ds+\int_{z_m}^{z}\kappa_2(\tau,s)\hat{v}_{n,m}(\tau,s)ds\\&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\int_{z_\rho}^{z_{\rho+1}}\kappa_3(t,s)v_{\xi,\rho}(t,s)dsdt\end{aligned}$$

$$+\sum_{\xi=0}^{n-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\int_{z_m}^{z}\kappa_3(t,s)v_{\xi,m}(t,s)dsdt$$
$$+\sum_{\rho=0}^{m-1}\int_{\tau_n}^{\tau}\int_{z_\rho}^{z_{\rho+1}}\kappa_3(t,s)v_{n,\rho}(t,s)dsdt$$
$$+\int_{\tau_n}^{\tau}\int_{z_m}^{z}\kappa_3(t,s)\hat{v}_{n,m}(t,s)dsdt, \tag{16}$$

and $\frac{\partial^{i+j}\hat{v}_{n,0}(\tau_n,z_m)}{\partial\tau^i\partial z^j}$ is the exact value of $\frac{\partial^{i+j}\hat{v}_{n,0}}{\partial\tau^i\partial z^j}$ at point (τ_n, z_m).

Differentiating equation (16) j-times with respect to z and i-times with respect to τ, we obtain

$$\frac{\partial^{i+j}\hat{v}_{n,m}(\tau,z)}{\partial\tau^i\partial z^j} \tag{17}$$
$$=\partial_1^{(i)}\partial_2^{(j)}g(\tau,z)$$
$$+\sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial\tau^{i-1-\eta}}\left[\partial_2^{(j-l)}\kappa_1(\tau,z)\right]\frac{\partial^{\eta+l}\hat{v}_{n,m}(\tau,z)}{\partial\tau^\eta\partial z^l}$$
$$+\sum_{l=0}^{j-1}\sum_{\eta=0}^{i}\binom{j-1}{l}\binom{i}{\eta}\frac{\partial^{i-\eta}}{\partial\tau^{i-\eta}}\left[\frac{\partial^{j-1-l}}{\partial z^{j-1-l}}\kappa_2(\tau,z)\right]\frac{\partial^{\eta+l}\hat{v}_{n,m}(\tau,z)}{\partial\tau^\eta\partial z^l}$$
$$+\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1}\binom{j-1}{l}\binom{i-1}{\eta}\frac{\partial^{i-1-\eta}}{\partial\tau^{i-1-\eta}}\left[\frac{\partial^{j-1-l}}{\partial z^{j-1-l}}\kappa_3(\tau,z)\right]\frac{\partial^{\eta+l}\hat{v}_{n,m}(\tau,z)}{\partial\tau^\eta\partial z^l}.$$

4 Convergence analysis

We consider the space $L^\infty(D)$ with the norm

$$\|\varphi\|_{L^\infty(D)}=\inf\{C\in\mathbb{R}:|\varphi(\tau,z)|\le C \text{ for a.e. } (\tau,z)\in D\}<\infty.$$

The following lemmas will be used in proving the convergence of the presented method.

Lemma 1 (Taylor's theorem for functions of two independent variables)**.** Let f be p times continuously differentiable on $D=[a,b]\times[c,d]$ and let $(\tau_0,z_0)\in D$. Then for all $(\tau,z)\in D$, we have

$$f(\tau,z)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}f(\tau_0,z_0)}{\partial\tau^i\partial z^j}(\tau-\tau_0)^i(z-z_0)^j$$
$$+\sum_{i+j=p}\frac{1}{i!j!}\frac{\partial^{i+j}f(\tau_1,z_1)}{\partial\tau^i\partial z^j}(\tau-\tau_0)^i(z-z_0)^j,$$

where

$$\begin{cases}\tau_1=\theta\tau+(1-\theta)\tau_0\in[a,b],\\ z_1=\theta z+(1-\theta)z_0\in[c,d],\end{cases}\quad \theta\in(0,1).$$

Lemma 2 (Gronwall-type inequality [21])**.** Let $\omega(\tau,z)$ and $p(\tau,z)$ be non-negative continuous functions in $\Omega=[a,b]\times[c,d]$, and let $p(\tau,z)$ be nondecreasing in each of the variables in Ω and satisfy the following inequality:

$$\omega(\tau,z)\le p(\tau,z)+\kappa\int_a^\tau\omega(t,z)dt+\kappa\int_c^z\omega(\tau,s)ds$$
$$+\kappa\int_a^\tau\int_c^z\omega(t,s)dsdt,\qquad(\tau,z)\in\Omega,$$

where κ is positive constant. Then there exists a positive constant ν such that

$$\omega(\tau,z)\le\nu p(\tau,z).$$

Lemma 3 (Discrete Gronwall-type inequality [5])**.** Let $\{\kappa_j\}_{j=0}^n$ be a given nonnegative sequence and let the sequence $\{\varepsilon_n\}$ satisfy$\varepsilon_0\le p_0$ and

$$\varepsilon_n\le p_0+\sum_{j=0}^{n-1}\kappa_j\varepsilon_j,\quad n\ge1,$$

with $p_0\ge0$. Then

$$\varepsilon_n\le p_0\exp\left(\sum_{j=0}^{n-1}\kappa_j\right),\quad n\ge1.$$

Lemma 4 (Discrete Gronwall-type inequality of two variables [23])**.** Let $\omega_{n,m}$ be a given nonnegative sequence, and let b_1,b_2,b_3 and β be independent of h and k and strictly positive. If the sequence $\omega_{n,m}$ satisfies

$$\omega_{n,m} \leq hb_1 \sum_{\xi=0}^{n-1} \omega_{\xi,m} + kb_2 \sum_{\rho=0}^{m-1} \omega_{n,\rho} + hkb_3 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \omega_{\xi,\rho} + \beta, \tag{18}$$

for all $n = 0, 1, \ldots, N, m = 0, 1, \ldots, M$, then

$$\omega_{n,m} \leq \beta \exp(\gamma(Nh + Mk)), \tag{19}$$

where $\gamma = \frac{1}{2}\left(b_1 + b_2 + \sqrt{(b_1 + b_2)^2 + 4b_3}\right)$.

Lemma 5. Let g, κ_1,κ_2, and κ_3 be p times continuously differentiable on their respective domains. Then, there exists a positive number $\alpha(p)$ such that for all $n = 0, 1, \ldots, N - 1$, $m = 0, \ldots, M - 1$, and $i + j = 0, 1, \ldots, p$, we have

$$\left\| \frac{\partial^{i+j} \hat{v}_{n,m}}{\partial \tau^i \partial z^j} \right\|_{L^\infty(D_{n,m})} \leq \alpha(p),$$

where $\hat{v}_{0,0}(\tau, z) = \omega(\tau, z)$ for $(\tau, z) \in D_{0,0}$.

Proof. Let $a_{n,m}^{i+j} = \| \frac{\partial^{i+j} \hat{v}_{n,m}}{\partial \tau^i \partial z^j} \|_{L^\infty(D_{n,m})}$.

First, we have for all $i + j = 0, 1, \ldots, p$,

$$a_{0,0}^{i+j} \leq \max \left\{ \left\| \frac{\partial^{i+j} \omega}{\partial \tau^i \partial z^j} \right\|_{L^\infty(D_{0,0})}, i + j = 0, 1, \ldots, p \right\} = \alpha_1(p). \tag{20}$$

Second, for $i + j = 0$, from (13), we have for all $n = 1, \ldots, N - 1$

$$\begin{aligned} a_{n,0}^{0+0} \leq & c + c_{1,1} \sum_{\xi=0}^{n-1} \int_{\tau_\xi}^{\tau_{\xi+1}} \sum_{\alpha+\beta=0}^{p-1} a_{\xi,0}^{\alpha+\beta} dt + c_{1,1} \int_{\tau_n}^{\tau} a_{n,0}^{0+0} dt + c_{2,2} \int_0^z a_{n,0}^{0+0} ds \\ & + c_{3,2} \sum_{\xi=0}^{n-1} \int_{\tau_\xi}^{\tau_{\xi+1}} \int_0^z \sum_{\alpha+\beta=0}^{p-1} a_{\xi,0}^{\alpha+\beta} ds dt + c_{3,2} \int_{\tau_n}^{\tau} \int_0^z a_{n,0}^{0+0} ds dt, \end{aligned}$$

and for $i + j = 1, \ldots, p$, from (14), we have for all $n = 1, \ldots, N - 1$,

$$a_{n,0}^{i+j} \leq c + c_{1,2} \sum_{l=0}^{j} \sum_{\eta=0}^{i-1} a_{n,0}^{\eta+l} + c_{2,1} \sum_{l=0}^{j-1} \sum_{\eta=0}^{i} a_{n,0}^{\eta+l} + c_{3,3} \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} a_{n,0}^{\eta+l},$$

where, the constants $c, c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}, c_{3,2}$, and $c_{3,3}$ are positive and independent of N and M.

Hence, for all $i + j = 0, 1, \ldots, p$,

$$\begin{aligned} a_{n,0}^{i+j} \leq & c + c_{1,1} h \sum_{\xi=0}^{n-1} \sum_{\alpha+\beta=0}^{p-1} a_{\xi,0}^{\alpha+\beta} + c_{1,2} \sum_{l=0}^{j} \sum_{\eta=0}^{i-1} a_{n,0}^{\eta+l} + c_{1,1} h \sum_{l=0}^{j} a_{n,0}^{0+l} \\ & + c_{2,1} \sum_{l=0}^{j-1} \sum_{\eta=0}^{i} a_{n,0}^{\eta+l} + c_{2,2} k \sum_{\eta=0}^{i} a_{n,0}^{\eta+0} \\ & + c_{3,2} h k \sum_{\xi=0}^{n-1} \sum_{\alpha+\beta=0}^{p-1} a_{\xi,0}^{\alpha+\beta} + c_{3,3} \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} a_{n,0}^{\eta+l} + c_{3,2} h k a_{n,0}^{0+0}, \end{aligned}$$

which implies that

$$a_{n,0}^{i+j} \leq c + c_1 h \sum_{\xi=0}^{n-1} \sum_{\eta+l=0}^{p-1} a_{\xi,0}^{\eta+l} + c_2 \sum_{\eta+l=0}^{i+j-1} a_{n,0}^{\eta+l}, \tag{21}$$

where $c_1 = c_{1,1} + c_{3,2} k$, $c_2 = c_{1,2} + c_{1,1} h + c_{2,1} + c_{2,2} k + c_{3,3} + c_{3,2} h k$.

Now, we consider the sequence $\Gamma_n = \max\{a_{n,0}^{i+j}, i+j = 0, \ldots, p\}$, $n = 0, 1, \ldots, N-1$, from (21), we obtain

$$a_{n,0}^{i+j} \leq c + c_1 p^2 h \sum_{\xi=0}^{n-1} \Gamma_\xi + c_2 \sum_{\eta+l=0}^{i+j-1} a_{n,0}^{\eta+l}.$$

Using Lemma 3 with the following notations

$$\varepsilon_{i+j} = a_{n,0}^{i+j}, \quad p_0 = c + c_1 p^2 h \sum_{\xi=0}^{n-1} \Gamma_\xi, \quad k_{\eta+l} = c_2,$$

we obtain

$$\begin{aligned} a_{n,0}^{i+j} & \leq \left(c + c_1 p^2 h \sum_{\xi=0}^{n-1} \Gamma_\xi \right) \exp\left(\sum_{\eta+l=0}^{i+j-1} c_2 \right) \\ & \leq \underbrace{c \exp(p^2 c_2)}_{c_3} + \underbrace{c_1 p^2 \exp(p^2 c_2)}_{c_4} h \sum_{\xi=0}^{n-1} \Gamma_\xi, \end{aligned}$$

which implies that

$$\Gamma_n \leq c_3 + c_4 h \sum_{\xi=0}^{n-1} \Gamma_\xi.$$

Again, using Lemma 3, for all $n = 0, 1, \ldots, N-1$ and $i+j = 0, \ldots, p$

$$a_{n,0}^{i+j} \leq \Gamma_n \leq \underbrace{c_3 \exp(a c_4)}_{\alpha_2(p)}. \tag{22}$$

Third, for $i+j=0$, from (16), we have for all $n=0,1,\ldots,N-1$, $m=1,\ldots,M-1$,

$$\begin{aligned}a^{0+0}_{n,m} \leq & c + c_{1,1}h\sum_{\xi=0}^{n-1}\sum_{\alpha+\beta=0}^{p-1} a^{0+0}_{\xi,m} + c_{1,1}ha^{0+0}_{n,m} + c_{2,2}k\sum_{\rho=0}^{m-1}\sum_{\alpha+\beta=0}^{p-1} a^{0+0}_{n,\rho} + c_{2,2}ka^{0+0}_{n,m}\\ & + c_{3,2}hk\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\alpha+\beta=0}^{p-1} a^{0+0}_{\xi,\rho} + c_{3,2}hk\sum_{\xi=0}^{n-1}\sum_{\alpha+\beta=0}^{p-1} a^{0+0}_{\xi,m}\\ & + c_{3,2}hk\sum_{\rho=0}^{m-1}\sum_{\alpha+\beta=0}^{p-1} a^{0+0}_{n,\rho} + c_{3,2}hka^{0+0}_{n,m},\end{aligned}$$

and for $i+j=1,\ldots,p$, we have from (17) that

$$a^{i+j}_{n,m} \leq c + c_{1,2}\sum_{l=0}^{j}\sum_{\eta=0}^{i-1} a^{\eta+l}_{n,m} + c_{2,1}\sum_{l=0}^{j-1}\sum_{\eta=0}^{i} a^{\eta+l}_{n,m} + c_{3,3}\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} a^{\eta+l}_{n,m}.$$

Hence, for all $i+j=0,1,\ldots,p$,

$$\begin{aligned}a^{i+j}_{n,m} \leq & c + c_{1,1}h\sum_{\xi=0}^{n-1}\sum_{\eta+l=0}^{p-1} a^{\eta+l}_{\xi,m} + c_{1,2}\sum_{\eta+l=0}^{i+j-1} a^{\eta+l}_{n,m} + c_{1,1}h\sum_{\eta+l=0}^{i+j-1} a^{\eta+l}_{n,m}\\ & + c_{2,2}k\sum_{\rho=0}^{m-1}\sum_{\eta+l=0}^{p-1} a^{\eta+l}_{n,\rho} + c_{2,1}\sum_{\eta+l=0}^{i+j-1} a^{\eta+l}_{n,m} + c_{2,2}k\sum_{\eta+l=0}^{i+j-1} a^{\eta+l}_{n,m}\\ & + c_{3,2}hk\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\eta+l=0}^{p-1} a^{\eta+l}_{\xi,\rho} + c_{3,2}hk\sum_{\xi=0}^{n-1}\sum_{\eta+l=0}^{p-1} a^{\eta+l}_{\xi,m}\\ & + c_{3,2}hk\sum_{\rho=0}^{m-1}\sum_{\eta+l=0}^{p-1} a^{\eta+l}_{n,\rho}\\ & + c_{3,3}\sum_{\eta+l=0}^{i+j-1} a^{\eta+l}_{n,m} + c_{3,2}hk\sum_{\eta+l=0}^{i+j-1} a^{\eta+l}_{n,m}. \qquad (23)\end{aligned}$$

Consider, the sequence $\Gamma_{n,m}=\max\{a^{i+j}_{n,m}, i+j=0,\ldots,p\}$, $n=0,1,\ldots,N-1; m=0,1,\ldots,M-1$. Then by (23), we have

$$a^{i+j}_{n,m} \leq c + hb_{1,1}\sum_{\xi=0}^{n-1}\Gamma_{\xi,m} + kb_{1,2}\sum_{\rho=0}^{m-1}\Gamma_{n,\rho}$$

$$+ hkb_{1,3} \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + b_{1,4} \sum_{\eta+l=0}^{i+j-1} a_{n,m}^{\eta+l}, \tag{24}$$

where $b_{1,1} = (c_{1,1} + c_{3,2}k)p^2$, $b_{1,2} = (c_{2,2} + c_{3,2}h)p^2$, $b_{1,3} = c_{3,2}p^2$ and $b_{1,4} = c_{1,2} + c_{1,1}h + c_{2,1} + c_{2,2}k + c_{3,3} + c_{3,2}hk$.

Using Lemma 3 with the following notations

$$\varepsilon_{i+j} = a_{n,m}^{i+j},$$

$$p_0 = c + hb_{1,1} \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + kb_{1,2} \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + hkb_{1,3} \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho},$$

$$k_{\eta+l} = b_{1,4},$$

we obtain from (24) that

$$a_{n,m}^{i+j} \leq \underbrace{c \exp(pb_{1,4})}_{b_4} + h \underbrace{b_{1,1} \exp(pb_{1,4})}_{b_1} \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + k \underbrace{b_{1,2} \exp(pb_{1,4})}_{b_2} \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}$$

$$+ hk \underbrace{b_{1,3} \exp(pb_{1,4})}_{b_3} \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho}.$$

It follows that, for all $n = 0, 1, \ldots, N-1; m = 0, 1, \ldots, M-1$,

$$\Gamma_{n,m} \leq b_4 + hb_1 \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + kb_2 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + hkb_3 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho}.$$

Using Lemma 4, we obtain, for all $n = 0, 1, \ldots, N-1, m = 0, 1, \ldots, N-1$, that

$$\Gamma_{n,m} \leq b_4 \exp(\gamma_1(Nh + Mk)) \leq \underbrace{b_4 \exp(\gamma_1(a+b))}_{\alpha_3(p)}, \tag{25}$$

where $\gamma_1 = \frac{1}{2}\left(b_1 + b_2 + \sqrt{(b_1+b_2)^2 + 4b_3}\right)$.

Hence, from (20), (22), and (25), by setting

$$\alpha(p) = \max\left\{\alpha_1(p), \alpha_2(p), \alpha_3(p)\right\},$$

the proof of Lemma 5 is completed. □

Theorem 1. Let g, κ_1, κ_2, and κ_3 be p times continuously differentiable on their respective domains. Then equations (10), (12), (15) define a unique approximation $v \in S_{p-1}^{(-1)}(\Pi_{N,M})$, and the resulting error function $e(\tau, z) = \omega(\tau, z) - v(\tau, z)$ satisfies

$$\|e\|_{L^\infty(D)} \leq C(h+k)^p,$$

where C is a finite constant independent of h and k.

Proof. Define the error $e(\tau, z)$ on $D_{n,m}$ by $e_{n,m}(\tau, z) = \omega(\tau, z) - v_{n,m}(\tau, z)$ for all $n = 0, 1, \ldots, N-1$ and $m = 0, 1, \ldots, M-1$.

The proof is split into three steps.

Claim 1. There exists a constant C_1 independent of h and k such that

$$\|e_{0,0}\|_{L^\infty(D_{0,0})} \leq C_1(h+k)^p.$$

Let $(\tau, z) \in D_{0,0}$. By using Lemma 1, we obtain from (10) that

$$|e_{0,0}(\tau, z)| \leq \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j}\omega}{\partial \tau^i \partial z^j} \right\| h^i k^j.$$

Hence, by Lemma 5, we have

$$|e_{0,0}(\tau, z)| \leq \alpha(p) \sum_{i+j=p} \frac{1}{i!j!} h^i k^j = \underbrace{\frac{\alpha(p)}{p!}}_{C_1}(h+k)^p. \tag{26}$$

Claim 2. There exists a constant C_2 independent of h and k such that

$$\|e_{n,0}\|_{L^\infty(D_{n,0})} \leq C_2(h+k)^p,$$

for all $n = 1, \ldots, N-1$. Let $(\tau, z) \in D_{n,0}$, we have from (13) that

$$
\begin{aligned}
\omega(\tau,z)-\hat{v}_{n,0}(\tau,z) &= \int_0^z \kappa_2(\tau,s)(\omega(\tau,s)-\hat{v}_{n,0}(\tau,s))ds \\
&+ \sum_{\xi=0}^{n-1}\int_{\tau_\xi}^{\tau_{\xi+1}} \kappa_1(z,t)e_{\xi,0}(t,z)dt \\
&+ \int_{\tau_n}^{\tau} \kappa_1(z,t)(\omega(t,z)-\hat{v}_{n,0}(t,z))dt \\
&+ \sum_{\xi=0}^{n-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\int_0^z \kappa_3(t,s)e_{\xi,0}(t,s)dsdt \\
&+ \int_{\tau_n}^{\tau}\int_0^z \kappa_3(t,s)(\omega(t,s)-\hat{v}_{n,0}(t,s))dsdt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\omega(\tau,z)-\hat{v}_{n,0}(\tau,z)| &\le \sum_{\xi=0}^{n-1} h\kappa\|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} + \sum_{\xi=0}^{n-1} hk\kappa\|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} \\
&+ \kappa\int_{\tau_n}^{\tau}|\omega(t,z)-\hat{v}_{n,0}(t,z)|dt \\
&+ \kappa\int_0^z|\omega(\tau,s)-\hat{v}_{n,0}(\tau,s)|ds \\
&+ \kappa\int_{\tau_n}^{\tau}\int_0^z|\omega(t,s)-\hat{v}_{n,0}(t,s)|dsdt,
\end{aligned}
$$

where $\kappa=\max\{\|\kappa_i\|_{L^\infty(D)}, i=1,2,3\}$. Then by Lemma 2, we have

$$
\begin{aligned}
|\omega(\tau,z)-\hat{v}_{n,0}(\tau,z)| &\le \left(\sum_{\xi=0}^{n-1} h\kappa\|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} + \sum_{\xi=0}^{n-1} hk\kappa\|e_{\xi,0}\|_{L^\infty(D_{\xi,0})}\right)\nu \\
&\le \sum_{\xi=0}^{n-1} h\underbrace{\kappa(1+b)\nu}_{\lambda_1}\|e_{\xi,0}\|_{L^\infty(D_{\xi,0})},
\end{aligned}
$$

which implies, by using Lemma 1, that

$$
\begin{aligned}
\|e_{n,0}\|_{L^\infty(D_{n,0})} &\le \|\omega-\hat{v}_{n,0}\|+\|\hat{v}_{n,0}-v_{n,0}\| \\
&\le \sum_{\xi=0}^{n-1} h\lambda_1\|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} + \sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{v}_{n,0}}{\partial\tau^i\partial z^j}\right\| h^i k^j.
\end{aligned}
$$

Hence, by Lemma 5, we obtain

$$\|e_{n,0}\|_{L^\infty(D_{n,0})} \leq \sum_{\xi=0}^{n-1} h\lambda_1 \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} + \frac{\alpha(p)}{p!}(h+k)^p.$$

Then, by Lemma 3, we have

$$\|e_{n,0}\|_{L^\infty(D_{n,0})} \leq \underbrace{\frac{\alpha(p)}{p!}\exp(T\lambda_1)}_{C_2}(h+k)^p.$$

Claim 3. There exists a constant C_3 independent of h and k such that

$$\|e_{n,m}\|_{L^\infty(D_{n,m})} \leq C_3(h+k)^p,$$

for all $n = 0, 1, \ldots, N-1$ and $m = 1, \ldots, M-1$. Let $(\tau, z) \in D_{n,m}$. Then from (16) we have

$$\begin{aligned}
|\omega(\tau,z) - \hat{v}_{n,m}(\tau,z)| \leq & \sum_{\xi=0}^{n-1} h\kappa\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} k\kappa\|e_{n,\rho}\| \\
& + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\kappa\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hk\kappa\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hk\kappa\|e_{n,\rho}\| \\
& + \kappa\int_{\tau_n}^{\tau} |\omega(t,z) - \hat{v}_{n,m}(t,z)|dt \\
& + \kappa\int_{z_m}^{z} |\omega(\tau,s) - \hat{v}_{n,m}(\tau,s)|ds \\
& + \kappa\int_{\tau_n}^{\tau}\int_{z_m}^{z} |\omega(t,s) - \hat{v}_{n,m}(t,s)|dsdt.
\end{aligned}$$

Then by Lemma 2,

$$\begin{aligned}
|\omega(\tau,z) - \hat{v}_{n,m}(\tau,z)| \leq & \sum_{\xi=0}^{n-1} h\underbrace{\kappa(1+k)\nu}_{\lambda_2}\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} k\underbrace{\kappa(1+h)\nu}_{\lambda_3}\|e_{n,\rho}\| \\
& + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\underbrace{\kappa\nu}_{\lambda_4}\|e_{\xi,\rho}\|,
\end{aligned}$$

which implies, by using Lemma 1, that

$$\begin{aligned}\|e_{n,m}\|_{L^\infty(D_{n,0})} \leq & \|\omega - \hat{v}_{n,m}\| + \|\hat{v}_{n,m} - v_{n,m}\| \\ & \leq \sum_{\xi=0}^{n-1} h\lambda_2 \|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} k\lambda_3 \|e_{n,\rho}\| + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\lambda_4 \|e_{\xi,\rho}\| \\ & + \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j}\hat{v}_{n,m}}{\partial \tau^i \partial z^j} \right\| h^i k^j.\end{aligned}$$

Hence, by Lemma 5, we obtain

$$\begin{aligned}\|e_{n,m}\| \leq & \sum_{\xi=0}^{n-1} h\lambda_2 \|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} k\lambda_3 \|e_{n,\rho}\| + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\lambda_4 \|e_{\xi,\rho}\| \\ & + \frac{\alpha(p)}{p!}(h+k)^p.\end{aligned} \tag{27}$$

Using Lemma 4, we obtain

$$\|e_{n,m}\| \leq \underbrace{\frac{\alpha(p)}{p!}\exp(\gamma_3(T+Z))}_{C_3}(h+k)^p, \tag{28}$$

where $\gamma_3 = \frac{1}{2}\left(\lambda_2 + \lambda_3 + \sqrt{(\lambda_2+\lambda_3)^2 + 4\lambda_3}\right)$.

Thus, the proof is completed by taking $C = \max\{C_1, C_2, C_3\}$. □

5 Numerical examples

The method presented in this paper is used to find numerical solutions to some illustrative examples. Our results are compared with the exact solutions by calculating the absolute error function $|e_{n,m}| = |\omega - v_{n,m}|$ for all $n = 0, 1, \ldots, N-1$ and $m = 0, 1, \ldots, M-1$, where ω and v are the exact and approximate solution, respectively. The values of errors are computed for different values of p, N, M and collected in Tables 1, 3, and 4, which are displayed in Figure 1 for Example 1. In Table 5, the presented method is compared with the numerical results obtained by using the Chelyshkov polynomials method (2D-CPs) [20] and the two-dimensional block-pulse functions method (2D-BPFs) [22]. Moreover, the exact and approximate solution over the region $([0,1]\times[0,1])$ are displayed in Figure 2 for Example 2. The results in these examples confirm the theoretical estimates and suggest that the experimental order of convergence (EOC) is p (see Table 2).

Example 1. Considering the Goursat problem, which is linear and homogeneous in hyperbolic PDE [9],

$$3\frac{\partial^2\omega(t,s)}{\partial s\partial t} = \frac{\partial\omega(t,s)}{\partial t} + \frac{\partial\omega(t,s)}{\partial s} + \omega(t,s), \quad (t,s) \in [0,1] \times [0,1],$$

with initial conditions $\omega(t,0) = e^t$ and $\omega(0,s) = e^s$. This equation is equivalent to the linear two-dimensional VIE defined as follows:

$$\begin{aligned}\omega(\tau,z) =& \frac{2}{3}(e^\tau + e^z) - \frac{1}{3} \\ &+ \frac{1}{3}\left(\int_0^\tau \omega(t,z)dt + \int_0^z \omega(\tau,s)ds + \int_0^\tau \int_0^z \omega(t,s)dsdt\right). \quad (29)\end{aligned}$$

The exact solution is $\omega(\tau,z) = e^{\tau+z}$.

The numerical results for $p = 3,4$ and $h = k = 0.05, 0.025$ of the Taylor collocation method are presented in Table 1 and Figure 1.

Table 1: Comparison between the approximate and the exact solution for Example 1

(τ, z)	$N = M = 20$,$p = 3$	$N = M = 20$,$p = 4$	$N = M = 40$, $p = 3$
$(0.1, 0.1)$	$1.62e-05$	$1.57e-05$	$4.05e-06$
$(0.2, 0.2)$	$3.82e-05$	$3.65e-05$	$9.47e-06$
$(0.3, 0.3)$	$6.67e-05$	$6.37e-05$	$1.66e-05$
$(0.4, 0.4)$	$1.05e-04$	$9.89e-05$	$2.59e-05$
$(0.5, 0.5)$	$1.56e-04$	$1.44e-04$	$3.80e-05$
$(0.6, 0.6)$	$2.21e-04$	$2.01e-04$	$5.35e-05$
$(0.7, 0.7)$	$3.04e-04$	$2.74e-04$	$7.34e-05$
$(0.8, 0.8)$	$4.11e-04$	$3.66e-04$	$9.86e-05$
$(0.9, 0.9)$	$5.48e-04$	$4.82e-04$	$1.30e-04$
$(1.0, 1.0)$	$1.99e-03$	$6.39e-04$	$3.38e-04$
CPU time/sec	30.81	47.25	742.68

Example 2. Consider the linear nonhomogeneous Goursat problem [8]

$$\frac{\partial^2\omega(t,s)}{\partial s\partial t} = 4ts - t^2s^2 + \omega(t,s), \quad (t,s) \in [0,1] \times [0,1],$$

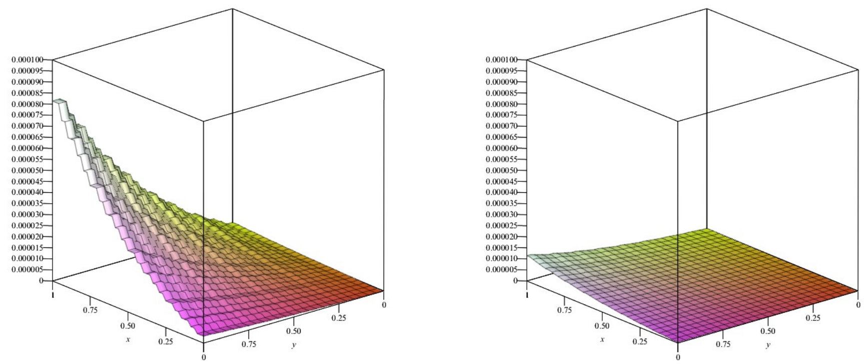


Figure 1: Plot of the absolute error function for Example 1; left: $h = k = 0.05$, right:$h = k = 0.025$.

Table 2: EOC for Example 1

(N, M)	$(2, 2)$	$(4, 4)$	$(8, 8)$	$(16, 16)$	$(32, 32)$	$(64, 64)$
$p = 2$	/	1.45	1.71	1.85	1.94	1.96
$p = 3$	/	2.40	2.69	2.84	2.92	2.96

with initial conditions $\omega(t, 0) = e^t$ and $\omega(0, s) = e^s$. This equation is equivalent to the linear two-dimensional VIE defined as follows:

$$\omega(\tau, z) = e^{\tau} + e^{z} + \tau^2 z^2 - \frac{\tau^3 z^3}{9} + \int_0^{\tau} \int_0^{z} \omega(t, s) ds dt.$$

The exact solution is $\omega(\tau, z) = \tau^2 z^2 + e^{\tau+z}$.

The numerical results for $p = 3$ and $h = k = 0.05, 0.025$ of the Taylor collocation method are presented in Table 3 and Figure 2.

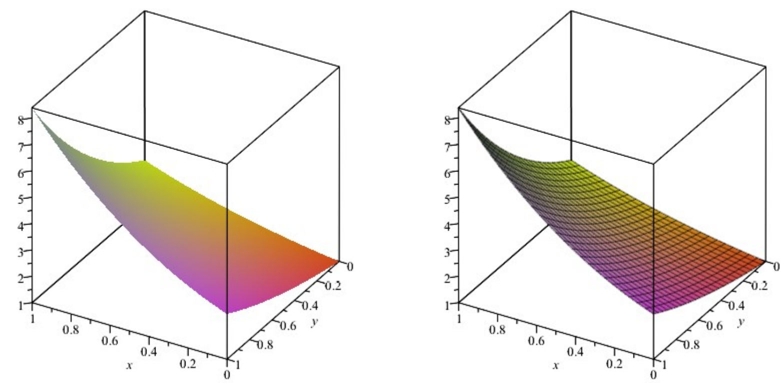
Example 3. Consider the PDE with variable coefficients

$$\begin{aligned}\frac{\partial^2 \omega(t, s)}{\partial s \partial t} =& (t + s^2)\frac{\partial \omega(t, s)}{\partial t} + (t^2 + s)\frac{\partial \omega(t, s)}{\partial s} \\ &+ ts\omega(t, s) + f(t, s), \qquad (t, s) \in [0, 1] \times [0, 1],\end{aligned}$$

with initial conditions $\omega(t, 0) = \cos(t) + e^t$ and $\omega(0, s) = \cos(s) + 1 + s^2$. This equation is equivalent to the linear two-dimensional VIE:

Table 3: Comparison between the approximate and the exact solution for Example 2

(τ, z)	$N = M = 20, p = 3$	$N = M = 40,\ p = 3$
$(0.1, 0.1)$	$3.64e-07$	$4.80e-08$
$(0.2, 0.2)$	$1.77e-06$	$2.29e-07$
$(0.3, 0.3)$	$4.77e-06$	$6.13e-07$
$(0.4, 0.4)$	$1.00e-05$	$1.28e-06$
$(0.5, 0.5)$	$1.83e-05$	$2.32e-06$
$(0.6, 0.6)$	$3.07e-05$	$3.93e-06$
$(0.7, 0.7)$	$4.85e-05$	$6.21e-06$
$(0.8, 0.8)$	$7.32e-05$	$9.36e-06$
$(0.9, 0.9)$	$1.06e-04$	$1.36e-05$
$(1.0, 1.0)$	$1.76e-03$	$2.28e-04$
CPU time/sec	27.71	701.59



(a) Exact solution (b) Approximate solution

Figure 2: Numerical results of Example 2.

$$\omega(\tau, z) = g(\tau, z) + \int_0^\tau (t^2 + z)\omega(t, z)dt + \int_0^z (\tau + s^2)\omega(\tau, s)ds + \int_0^\tau \int_0^z (-2 - ts)\omega(t, s)dsdt,$$

where $g(\tau, z)$ is chosen so that the exact solution is $\omega(\tau, z) = \cos(\tau + z) + e^\tau + z^2$.

The numerical results for $p = 3$ and $h = k = 0.05, 0.025$ of the Taylor collocation method are presented in Table 4.

Table 4: Comparison between the approximate and the exact solution for Example 3

(τ, z)	$N = M = 20$,$p = 3$	$N = M = 40$, $p = 3$
$(0.1, 0.1)$	$1.32e-05$	$1.66e-06$
$(0.2, 0.2)$	$2.84e-05$	$3.73e-06$
$(0.3, 0.3)$	$4.72e-05$	$6.54e-06$
$(0.4, 0.4)$	$7.13e-05$	$1.05e-05$
$(0.5, 0.5)$	$1.03e-04$	$1.62e-05$
$(0.6, 0.6)$	$1.46e-04$	$2.44e-05$
$(0.7, 0.7)$	$2.07e-04$	$3.64e-05$
$(0.8, 0.8)$	$2.98e-04$	$5.47e-05$
$(0.9, 0.9)$	$4.44e-04$	$8.47e-05$
$(1.0, 1.0)$	$1.04e-03$	$1.89e-04$
CPU time/sec	62.96	1207.40

Example 4. Consider the linear nonhomogeneous Goursat problem

$$\frac{\partial^2 u(t,s)}{\partial s \partial t} = -\frac{\partial u(t,s)}{\partial t} - \frac{\partial u(t,s)}{\partial s} - u(t,s) + f(t,s), \quad (t,s) \in [0,1] \times [0,1],$$

with initial conditions $u(t, 0) = e^t$ and $u(0, s) = e^{2s}$. This equation is equivalent to the nonlinear two-dimensional VIE of the first kind [22]:

$$\frac{1}{9}(e^{\tau+z} - e^{\tau+4z} - e^{7\tau+z} + e^{7\tau+4z}) = \int_0^\tau \int_0^z 2e^{\tau+z}\omega^3(t,s)dsdt.$$

It is also equivalent to the following linear 2D-VIE of the second kind:

$$u(\tau, z) = g(\tau, z) - \int_0^\tau u(t,z)dt - \int_0^z u(\tau,s)ds - \int_0^\tau \int_0^z u(t,s)dsdt,$$

where $u = \omega^3$ and $g(\tau, z)$ is chosen such that the exact solution is $u(\tau, z) = e^{\tau+2z}$.

In Table 5, the numerical results for $p = 3$ and $N = M = 64$ of the Taylor collocation method are compared with the numerical results obtained by using 2D-CPs [20] and 2D-BPFs [22].

Table 5: Comparison of the absolute errors of Example 4

$(2^{-i}, 2^{-i})$	2D-BPFs [22]	2D-CPs [20]	Present method
$i = 1$	$1.0e-1$	$3.5e-5$	$6.1e-6$
$i = 2$	$4.6e-2$	$2.0e-6$	$2.6e-6$
$i = 3$	$2.9e-2$	$1.5e-5$	$1.3e-6$
$i = 4$	$2.3e-2$	$1.2e-5$	$7.2e-7$
$i = 5$	$2.0e-2$	$5.9e-5$	$3.7e-7$
$i = 6$	$3.1e-2$	$9.6e-5$	$1.9e-7$

6 Conclusion

In this paper, a collocation approach was presented that involves utilizing Taylor polynomials to find the solution to a two-dimensional linear VIE of the second kind, which is a conversion of a hyperbolic linear PDE Goursat problem. The method's convergence and error were investigated, and several numerical examples were provided to demonstrate its efficiency and accuracy. The results showed that the method is convergent with a high level of precision, and the numerical results match the theoretical estimates. It is recognized that this approach can be expanded and used to solve three-dimensional Goursat problems in linear hyperbolic equations of the third-order

$$\frac{\partial^3 \omega}{\partial \tau_1 \partial \tau_2 \partial \tau_3} + \sum_{i,j=1, i<j}^{3} \psi_{i,j} \frac{\partial \omega}{\partial \tau_i \partial \tau_j} + \sum_{i=1}^{3} \psi_i \frac{\partial \omega}{\partial \tau_i} + \psi \omega = F,$$

where $\psi_{i,j}, \psi_i, , i < j, i, j = 1, 2, 3, \psi$ and F are given real functions.

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