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Research Article

Wavelet approximation with Chebyshev wavelets

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Abstract

In this paper, we study wavelet approximation of the Chebyshev polynomials of the first, second, third, and fourth kinds. We estimate the wavelet approximation of a function f having bounded first derivatives.

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1 Introduction

In recent years, wavelets have found their way into many different fields of science and engineering, particularly, in signal analysis, time-frequency analysis, and fast algorithms. Wavelets allow an accurate representation of several functions. The wavelet approximation technique is a new tool for finding and analyzing unexpected seismic signal processing changes. There is scarcely any area of numerical analysis where Chebyshev polynomials do not drop in like surprise visitors, and indeed there are now a number of subjects in which these polynomials take a significant position in modern developments including orthogonal polynomials, polynomial approximation, numerical integration, and spectral methods for partial differential equations (see [4, 6]).

In Table 1, we have Chebyshev polynomials on the trivial $[-1, 1]$.

Table 1: Chebyshev polynomials

Chebyshev Polynomials	Shifted Chebyshev	w (weight)	w^* (shifted weight)
$T_n(x) = \cos\ n\ \theta$	$T_n^*(x) = T_n(2x-1)$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{\sqrt{x-x^2}}$
$U_n(x) = \frac{\sin\ (n+1)\ \theta}{\sin\ \theta}$	$U_n^*(x) = U_n(2x-1)$	$\sqrt{1-x^2}$	$\sqrt{x-x^2}$
$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos(\frac{\theta}{2})}$	$V_n^*(x) = V_n(2x-1)$	$\sqrt{\frac{1+x}{1-x}}$	$\sqrt{\frac{x}{1-x}}$
$W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin(\frac{\theta}{2})}$	$W_n^*(x) = W_n(2x-1)$	$\sqrt{\frac{1-x}{1+x}}$	$\sqrt{\frac{1-x}{x}}$

With above definition, we obtain the fundamental recurrence relation:

$$W_{n+1}(x) = 2xW_n(x) - W_{n-1}(x), \quad n \geq 2, \quad W_0(x) = 1, \quad W_1(x) = 2x + 1.$$

In Table 2, we have Chebyshev wavelets, for $k \in \mathbb{N}$, $m \geq 0$, and $n = 0, 1, 2, \dots, 2^k$.

Table 2: Chebyshev wavelets

Sequences Chebyshev Polynomials	Wavelet Chebyshev
$T_{n,m}$	$\begin{cases} 2^{\frac{k}{2}} T_m(2^k t - n), & t \in [\frac{n}{2^k}, \frac{n+1}{2^k}], \\ 0 & \text{otherwise.} \end{cases}$
$U_{n,m}$	$\begin{cases} 2^{\frac{k}{2}} U_m(2^k t - n), & t \in [\frac{n}{2^k}, \frac{n+1}{2^k}], \\ 0 & \text{otherwise.} \end{cases}$
$V_{n,m}$	$\begin{cases} 2^{\frac{k}{2}} V_m(2^k t - n), & t \in [\frac{n}{2^k}, \frac{n+1}{2^k}], \\ 0 & \text{otherwise.} \end{cases}$
$W_{n,m}$	$\begin{cases} 2^{\frac{k}{2}} W_m(2^k t - n), & t \in [\frac{n}{2^k}, \frac{n+1}{2^k}], \\ 0 & \text{otherwise.} \end{cases}$

In the following, we suppose $\psi_{n,m} = T_{n,m}, U_{n,m}, V_{n,m}, W_{n,m}$. A function $f \in L^2[0,1)$ is expanded by wavelet series as

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t),$$

where

$$c_{n,m} = \int_0^1 f(t) \psi_{n,m}(t) \omega^*_{n,m}(t) dt,$$

and $\omega_{n,m}$ is the weight function of pseudo Chebyshevs. Also,

$$\int_0^1 \psi_n(x) \psi_m(x) \omega^*_{n,m}(t) dx = L,$$

where

$$L = \begin{cases} \frac{\pi}{2}, & m = n,\ \psi = T, U, \\ 0, & \text{m} \neq n,\ \psi = T, W, U, V, \\ \pi, & m = n,\ \psi = V, W, \end{cases}$$

$$\omega_{n,m}(t) = \begin{cases} \frac{1}{\sqrt{1-t^2}} & if\ \psi = T, \\ \sqrt{1-t^2} & if\ \psi = U, \\ \sqrt{\frac{1+t}{1-t}} & if\ \psi = V, \\ \sqrt{\frac{1-t}{1+t}} & if\ \psi = W. \end{cases}$$

Definition 1 (Multiresolution analysis)**.** A sequence of closed subspaces $\{V_j\}$ of $L^2(\mathbb{R})$ (inner product space), $j \in Z$, is called a multiresolution in $L^2(\mathbb{R})$ if it satisfies the following conditions:

(i) $V_j \subset V_{j+1}$;

(ii) $f(x) \in V_j \iff f(2x) \in V_{j+1}$;

(iii) $f(x) \in V_0 \iff (x+1) \in V_0$;

(iv) $\cup_{-\infty}^{\infty} V_j$ is dense in $L^2((\mathbb{R})$, and $\cap_{-\infty}^{\infty} V_j = \{0\}$;

(v) There exists a function $\phi \in V_0$ such that the collection $\{\phi(x–k) : \ k \in \mathbb{Z}\}$ is a Riesz basis of V_0.

Definition 2. Let $P_n(f)$ be the orthogonal projection of $L^2([0,1])$ onto V_n. Then

$$P_n f = \sum_{-\infty}^{\infty} a_{n,k}\phi_{n,k}, \quad n = 1, 2, 3, \dots,$$

where

$$a_{n,k} = \langle f, \phi_{n,k}\rangle.$$

Thus

$$P_n(f) = \sum_{-\infty}^{\infty} \langle f, \phi_{n,k}\rangle \phi_{n,k}, \quad n = 1, 2, 3, \dots,$$

Definition 3. The wavelet approximation of Chebyshev polynomial is defined by

$$E_n(f) = \|f–P_n(f)\|_2 = \int_a^1 (f(t)–P_n(f)(t))^2 dt = O(\phi(n)),$$

where

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}V_{n,m}(t)$$

or

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}W_{n,m}(t).$$

If $E_n(f) \to 0$ as $n \to \infty$, then $E_n(f)$ is called the best approximation of f (see [1, 2, 3]).

We conclude this section by a list of known lemmas.

Lemma 1. [5] Let $f \in L^2([0,1])$ be a continuous function such that for constant $P > 0$, $|f"(x)| \leq P$. If $f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} V_{n,m}(t)$ is expanded in terms of Chebyshev polynomial of the third kind. Then the Chebyshev wavelet approximation $E_{2^k,l}(t)$ of f by $(2^k, l)$th partial sum

$$s_{2^{k-2},M}(t) = \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t), \tag{1}$$

of its Chebyshev wavelet series in $L^2_{w*}[0,1]$ is given by

$$\|f - s_{2^{k-2},M}\|_2^2 = O(\frac{1}{2^{2k}(M-1)^{\frac{3}{2}}}). \quad M \geq 1 \quad (M \textit{ is resolution})$$

Lemma 2. [5] Let $f \in L^2([0,1])$ be a continuous function such that for constant $P > 0$, $|f"(x)| \leq P$. If $f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} W_{n,m}(t)$ is expanded in terms of the Chebyshev polynomial of the fourth kind. Then the Chebyshev wavelet approximation $E_{2^k,l}(t)$ of f by $(2^k, l)$th partial sum

$$s_{2^{k-2},M}(t) = \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(t), \tag{2}$$

of its Chebyshev wavelet series in $L^2_{w*}[0,1]$ is given by

$$E^2_{2^k-1,l} = O(\frac{1}{M!2^{M(k+1)}}).$$

2 Main results

In this section, we consider wavelet approximation in Chebyshev polynomials of first, second, third, and fourth kinds and Haar wavelet. We estimate the wavelet approximation of a function f having bounded first derivatives.

Theorem 1. Let $f \in L^2([0,1])$ be a continuous function and let for constant $P > 0$, $|f'(x)| \leq P$.

(i) If $f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} T_{n,m}(t)$ is expanded in terms of Chebyshev polynomial of the first kind. Then the Chebyshev wavelet approximation $E_{2^k,l}(t)$ of f by $(2^k, l)$th partial sum is

$$s_{2^k-1,l} = \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=l}^{\infty} c_{n,m} T_{n,m},$$

and

$$E_{2^k,l}(f) = \|f - s_{2^k,l}\|_2^2 = O(\frac{P}{(l+1)^{\frac{3}{2}}}).$$

(ii) If $f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} U_{n,m(t)}$ is expanded in terms of Chebyshev polynomial of the second kind. Then the Chebyshev wavelet approximation $E_{2^{k-1},l}(t)$ of f by $(2^{k-1}, l)$th partial sum is

$$s_{2^k-1,l} = \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=l}^{\infty} c_{n,m} U_{n,m},$$

and

$$E_{2^k-1,l}(f) = \|f - s_{2^{k-1},l}\|_2 = O(\frac{P^{\frac{1}{2}}}{2^{\frac{k}{4}}-1}(\frac{2l+4}{(l+1)(l+3)})^{\frac{1}{2}}).$$

(iii) If $f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} V_{n,m}(t)$ is expanded in terms of Chebyshev polynomial of the third kind. Then the Chebyshev wavelet approximation $E_{2^k,l}(t)$ of f by $(2^k, l)$ th partial sum is

$$s_{2^k-1,l} = \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=0} c_{n,m} V_{n,m},$$

and

$$E^2_{2^k,l}(f) = \|f - s_{2^k,l}\|_2^2 = O(2^{-\frac{k}{2}-1}(\ln \frac{l}{(l+1)(l+3)} - \frac{1}{l+1} - \frac{27}{8}\frac{1}{l+2} - \frac{1}{l})).$$

(iv) If $f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} W_{n,m}(t)$ is expanded in terms of Chebyshev polynomial of the fourth kind. Then the Chebyshev wavelet approximation $E_{2^k,l}(t)$ of f by $(2^k, l)$th partial sum is

$$s_{2^k-1,l} = \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=0} c_{n,m} W_{n,m},$$

and

$$E^2_{2^k-1,l} = O(2^{-\frac{3k}{2}-3}(\frac{1}{4(l+1)} + \frac{1}{l} + \frac{1}{l+3} + \frac{5}{3}\ln\frac{l}{l+3})).$$

Proof. (i) We have

$$\begin{aligned}
c_{n,m} &= < f(t), T_{n,m} >_{w_1} \\
&= 2^{\frac{k}{2}} \int_0^1 f(t) T_m(2^k t - n) w_1(kt^k - n) dt \\
&= 2^{\frac{k}{2}} \int_0^\pi f(\frac{\cos\theta + n}{2^k}) T_m(\cos\theta) \frac{1}{\sqrt{1-\cos^2\theta}} \sin\theta d\theta \\
&= 2^{\frac{k}{2}} \int_0^\pi f(\frac{\cos\theta + n}{2^k}) \cos(m\theta) d\theta \\
&= 2^{\frac{k}{2}} [(f(\frac{\cos\theta + n}{2^k})(-\frac{1}{m}\sin m\theta))|_0^\pi \\
&\quad + \frac{1}{2^k m} \int_0^\pi f'(\frac{\cos\theta + n}{2^k}) \sin\theta \sin(m\theta) d\theta] \\
&= 2^{\frac{k}{2}} [\frac{1}{2^k m} \int_0^\pi f'(\frac{\cos\theta + n}{2^k}) \sin\theta \sin(m\theta) d\theta] \\
&\leq 2^{\frac{k}{2}} [\frac{P}{2^k m} \int_0^\pi \sin\theta \sin(m\theta) d\theta] \\
&\leq 2^{\frac{k}{2}} (-\frac{P}{2^k m}) [-\frac{1}{m} \sin\theta \cos(m\theta)|_0^\pi - \frac{1}{m} \int_0^\pi \cos(m\theta)\cos(\theta) d\theta] \\
&\leq \frac{P}{2^{\frac{k}{2}} m^2} (\int_0^\pi \cos(m\theta)\cos(\theta) d\theta).
\end{aligned}$$

Therefore

$$\begin{aligned}
|c_{n,m}| &= \frac{P}{2^{\frac{k}{2}} m^2} | \int_0^\pi \cos(m\theta)\cos(\theta) d\theta| \\
&\leq \frac{P}{2^{\frac{k}{2}} m^2} \int_0^\pi |\cos(m\theta)| |\cos(\theta)| d\theta \\
&\leq \frac{P\pi}{2^{\frac{k}{2}} m^2} \\
|c_{n,m}|^2 &\leq \frac{P^2 \pi^2}{2^k m^4}.
\end{aligned}$$

For $T_{n,m}$

$$\begin{aligned}
\|T_{n,m}\|_2^2 &= 2^k \int_0^1 |T_m(2^k t - n)|^2 w_1(2^k t - n) dt \\
&\leq 2^k \int_{-1}^1 T_m(2^k t - n)^2 w_1(2^k t - n) dt \\
&= \frac{\pi}{2} 2^k \\
&= \pi 2^{k-1},
\end{aligned}$$

and

$$\begin{aligned}
E^2_{2^k-1,l} &= \|f - s_{2^k-1,l}\|_2^2 \\
&= \|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}T_{n,m} - \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{l} c_{n,m}T_{n,m}\|_2^2 \\
&= \|\sum_{n=0}^{2^k-1}\sum_{m=l+1}^{\infty} c_{n,m}T_{n,m}\|_2^2 \\
&= \sum_{n=0}^{2^k-1}\sum_{m=l+1}^{\infty} |c_{n,m}|^2\|T_{n,m}\|_2^2 \\
&\leq \frac{\pi}{2}\sum_{n=0}^{2^k-1}\sum_{m=l+1}^{\infty} \frac{P^2\pi^2}{m^4} \\
&= \frac{P^2\pi^3}{2}\sum_{n=0}^{2^k-1}\sum_{m=l+1}^{\infty} \frac{1}{m^4}.
\end{aligned}$$

It is follows that

$$E_{2^k-1,l} = O(\frac{P}{(l+1)^{\frac{3}{2}}}).$$

(ii) We have

$$\begin{aligned}
\|U_{n,m}\|_2^2 &= 2^k\int_a^1 |U_m(2^kt-n)|^2 w_2(2^kt-n)dt \\
&= 2^k\int_a^1 U_m(2^kt-n)^2 w_2(2^kt-n)dt \\
&= \frac{\pi}{2}2^k \\
&= \pi 2^{k-1}.
\end{aligned}$$

Also,

$$\begin{aligned}
c_{n,m} &= < f(t), U_{n,m} >_w \\
&= 2^{\frac{k}{2}}\int_0^1 f(t)U_m(2^kt-n)w_1(2^kt-n)dt \\
&= 2^{\frac{k}{2}}\int_0^{\pi} f(\frac{\cos\theta+n}{2^k})U_m(\cos\theta)\sin\theta\sin\theta d\theta \\
&= 2^{\frac{k}{2}}\int_0^{\pi} f(\frac{\cos\theta+n}{2^k})\frac{\sin(m+1)\theta}{\sin\theta}\sin^2\theta d\theta
\end{aligned}$$

$$
\begin{aligned}
&= 2^{\frac{k}{2}} \int_0^\pi f(\frac{\cos\theta + n}{2^k}) \sin(m+1)\theta \sin\theta d\theta \\
&= 2^{\frac{k}{2}} \int_0^\pi f(\frac{\cos\theta + n}{2^k}) \frac{1}{2}(\cos(m\theta) - \cos(m+2)\theta) d\theta \\
&= 2^{\frac{k}{2}-1} \int_0^\pi f(\frac{\cos\theta + n}{2^k})(\cos(m\theta) - \cos(m+2)\theta) d\theta \\
&= 2^{\frac{k}{2}-1} [f(\frac{\cos\theta + n}{2^k})(\frac{1}{m}\sin(m\theta) - \frac{1}{m+2}\sin(m+2)]_0^\pi \\
&\quad + \frac{1}{2^{\frac{k}{2}+1}} \int_0^\pi f'(\frac{\cos\theta + n}{2^k}) \sin\theta (\frac{1}{m}\sin(m\theta) - \frac{1}{m+2}\sin(m+2)) d\theta \\
&\leq \frac{P}{2^{\frac{k}{2}+1}} \int_0^\pi \sin\theta (\frac{1}{m}\sin(m\theta) - \frac{1}{m+2}\sin(m+2)) d\theta \\
&= \frac{P}{2^{\frac{k}{2}+1}} [\sin\theta (\frac{1}{m^2}\cos m\theta + \frac{1}{(m+2)^2}\cos(m+2)\theta]_0^\pi \\
&\quad + \frac{P}{2^{\frac{k}{2}+1}} \int_0^\pi \cos\theta (\frac{1}{m^2}\cos m\theta + \frac{1}{(m+2)^2}\cos(m+2)\theta) d\theta.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|c_{n,m}| &\leq \frac{P}{2^{\frac{k}{2}+1}} |\int_0^\pi \cos\theta (\frac{1}{m^2}\cos m\theta + \frac{1}{(m+2)^2}\cos(m+2)\theta) d\theta| \\
&\leq \frac{P}{2^{\frac{k}{2}+1}} \int_0^\pi |\cos\theta| (\frac{1}{m^2}|\cos m\theta| + \frac{1}{(m+2)^2}|\cos(m+2)|(\theta) d\theta \\
&\leq \frac{P}{2^{\frac{k}{2}+1}} \int_0^\pi (\frac{1}{m^2} + \frac{1}{(m+2)^2}) d\theta
\end{aligned}
$$

and

$$
\begin{aligned}
E^2_{2^k-1,l} &= \|f - s_{2^k-1,l}\|_2^2 \\
&= \|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{n,m}U_{n,m} - \sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{l} c_{n,m}T_{n,m}\|_2^2 \\
&= \|\sum_{n=0}^{2^{k-1}}\sum_{m=l+1}^{\infty} c_{n,m}U_{n,m}\|_2^2 \\
&= \sum_{n=0}^{2^{k-1}}\sum_{m=l+1}^{\infty} |c_{n,m}|^2 \|U_{n,m}\|_2^2 \\
&\leq \frac{P}{2^{\frac{k}{2}+1}} \sum_{n=0}^{2^{k-1}}\sum_{m=l+1}^{\infty} (\frac{1}{m^2} + \frac{1}{(m+2)^2})
\end{aligned}
$$

$$\leq \frac{P}{2^{\frac{k}{2}+1}} \sum_{n=0}^{2^{k-1}} (\frac{2l+4}{(l+1)(l+3)})$$
$$\leq \frac{P}{2^{\frac{k}{2}+1}} \sum_{n=0}^{2^{k-1}} (\frac{2l+4}{(l+1)(l+3)}).$$

Therefore

$$E_{2^k-1,l} = \|f - s_{2^k-1,l}\|_2 = O(\frac{P^{\frac{1}{2}}}{2^{\frac{k}{4}}-1}(\frac{2l+4}{(l+1)(l+3)})^{\frac{1}{2}}).$$

(iii) We have

$$\begin{aligned}
c_{n,m} &= < f(t), V_{n,m} >_w \\
&= 2^{\frac{k}{2}} \int_0^1 f(t) V_m(2^k t - n) w_1(2^k t - n) dt \\
&= -2^{\frac{k}{2}} \int_0^\pi f(\frac{\cos\theta + n}{2^k}) \frac{\cos(m+\frac{1}{2})\theta}{\cos(\frac{\theta}{2})} \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \sin\theta d\theta \\
&= -2^{\frac{k}{2}} \int_0^\pi f(\frac{\cos\theta + n}{2^k}) \frac{\cos(m+\frac{1}{2})\theta}{\cos(\frac{\theta}{2})} \sqrt{\frac{2\cos^2(\frac{\theta}{2})}{2\sin^2(\frac{\theta}{2})}} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} d\theta \\
&= -2^{\frac{k}{2}} \int_0^\pi f(\frac{\cos\theta + n}{2^k}) 2\cos(m+\frac{1}{2})\theta \cos\frac{\theta}{2} d\theta \\
&= -2^{\frac{k}{2}} \int_0^\pi f(\frac{\cos\theta + n}{2^k}) (\cos(m+1)\theta + \cos m\theta) d\theta \\
&= -2^{\frac{k}{2}} [f(\frac{\cos\theta + n}{2^k})(\frac{1}{m+1}\sin(m+1)\theta + \frac{1}{m}\sin m\theta)]_0^\pi \\
&\quad +2^{\frac{k}{2}} \int_0^\pi f'(\frac{\cos\theta + n}{2^k}) \frac{\sin\theta}{2^k} (\frac{1}{m+1}\sin(m+1)\theta + \frac{1}{m}\sin m\theta) d\theta \\
&= 2^{\frac{k}{2}} \int_0^\pi f'(\frac{\cos\theta + n}{2^k}) \frac{\sin\theta}{2^k} (\frac{1}{m+1}\sin(m+1)\theta + \frac{1}{m}\sin m\theta) d\theta \\
&\leq 2^{-\frac{k}{2}} P \int_0^\pi \sin\theta (\frac{1}{m+1}\sin(m+1)\theta + \frac{1}{m}\sin m\theta)) d\theta \\
&= 2^{-\frac{k}{2}} P [(\frac{1}{m+1}\sin\theta\sin(m+1)\theta + \frac{1}{m}\sin\theta\sin m\theta)]_0^\pi \\
&\quad -2^{-\frac{k}{2}-1} P \int_0^\pi \cos\theta (\frac{1}{m+1}\sin(m+1)\theta + \frac{1}{m}(\sin m\theta) d\theta \\
&= 2^{-\frac{k}{2}-1} P \int_0^\pi (\frac{1}{m+1}\cos\theta\sin(m+1)\theta - \frac{1}{m}\cos\theta\sin m\theta) \\
&= 2^{-\frac{k}{2}-1} P \int_0^\pi (\frac{1}{2(m+1)}(\sin m\theta + \sin(m+2)\theta)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2m}(\sin(m-1)\theta+\sin(m+1)\theta))d\theta\\
&=2^{-\frac{k}{2}-1}P[-\frac{1}{2m(m+1)}\cos m\theta-\frac{1}{(m+1)(m+2)}\cos(m+2)\theta\\
&+\frac{1}{2m(m-1)}\cos(m-1)\theta+\frac{1}{m(m+1)}\cos(m+1)\theta]_0^\pi\\
&=2^{-\frac{k}{2}-1}P[-\frac{1}{2m(m+1)}(\cos m\pi-1)\\
&-\frac{1}{(m+1)(m+2)}(\cos(m+2)\pi-1)\\
&+\frac{1}{2m(m-1)}(\cos(m-1)\pi-1)+\frac{1}{m(m+1)}(\cos(m+1)\pi-1)],
\end{aligned}
$$

and

$$
\begin{aligned}
|c_{n,m}|&=2^{-\frac{k}{2}-1}P|-\frac{1}{2m(m+1)}\cos(m\pi-1)\\
&-\frac{1}{(m+1)(m+2)}(\cos(m+2)\pi-1)\\
&+\frac{1}{2m(m-1)}(\cos(m-1)\pi-1)+\frac{1}{m(m+1)}(\cos(m+1)\pi-1)|\\
&\leq 2^{-\frac{k}{2}-1}P(\frac{1}{m(m+1)}+\frac{2}{(m+1)(m+2)}+\frac{1}{m(m-1)}+\frac{2}{m(m+1)})\\
|c_{n,m}|^2&=2^{-k-2}P^2(\frac{1}{(m^2(m+1)^2}+\frac{4}{(m+1)^2(m+2)^2}+\frac{1}{m^2(m-1)^2}\\
&+\frac{4}{m^2(m+1)^2}+\frac{2}{m(m+2)(m+1)^2}+\frac{1}{m^2(m+1)(m-1)}\\
&+\frac{2}{m^2(m+1)^2}+\frac{2}{m(m+1)(m-1)(m+2)}\\
&+\frac{4}{m(m+2)(m+1)^2}+\frac{2}{m^2(m-1)(m+1)})\\
&=2^{-k-2}P^2(\frac{4}{(m+1)^2(m+2)^2}+\frac{1}{m^2(m-1)^2}\\
&+\frac{6}{m^2(m+1)^2}+\frac{6}{m(m+2)(m+1)^2}\\
&+\frac{3}{m^2(m+1)(m-1)}+\frac{2}{m(m+1)(m-1)(m+2)})\\
&=-10\frac{1}{m}+4\frac{1}{m^2}+10\frac{1}{m+1}+8\frac{1}{(m+1)^2}-\frac{1}{6}\frac{1}{m-1}\\
&+\frac{1}{(m-1)^2}-\frac{23}{2}\frac{1}{m+2}.
\end{aligned}
$$

For $V_{n,m}$, we have

$$\begin{aligned}
\|V_{n,m}\|_2^2 &= 2^{-\frac{k}{2}} \int_0^1 |V_m(2^k t - n)|^2 w(t) dt \\
&\leq 2^{-\frac{k}{2}} \int_{-1}^1 V_m(2^k t - n)^2 \sqrt{\frac{1+t}{1-t}} dt \\
&= 2^{-\frac{k}{2}} \frac{\pi}{2} \\
&= 2^{-\frac{k}{2}-1}\pi.
\end{aligned}$$

Therefore

$$\begin{aligned}
E^2_{2^k-1,l} &= \|f - s_{2^k-1,l}\|_2^2 \\
&= \| \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{n,m} V_{n,m} - \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=0}^{l} c_{n,m} V_{n,m} \|_2^2 \\
&= \| \sum_{n=0}^{2^k-1} \sum_{m=l+1}^{\infty} c_{n,m} V_{n,m} \|_2^2 \\
&= \sum_{n=0}^{2^k-1} \sum_{m=l+1}^{\infty} |c_{n,m}|^2 \|V_{n,m}\|_2^2 \\
&\leq 2^{-\frac{k}{2}-1}\pi \sum_{m=l+1}^{\infty} (-10\frac{1}{m} + 4\frac{1}{m^2} + 10\frac{1}{m+1} \\
&\quad +8\frac{1}{(m+1)^2} - \frac{1}{m-1} + \frac{1}{(m-1)^2} - 10\frac{1}{m+2}. \\
&= 2^{-\frac{k}{2}-1}\pi(-10\ln(l+1) - \frac{1}{2(l+1)} + 10\ln(l+2) \\
&\quad -\frac{27}{8}\frac{1}{l+2} + \ln l - \frac{1}{l} - \ln(l+3)). \\
&= 2^{-\frac{k}{2}-1}\pi(10\ln\frac{l}{(l+1)(l+3)} - \frac{1}{l+1} - \frac{27}{8}\frac{1}{l+2} - \frac{1}{l}).
\end{aligned}$$

Therefore

$$E^2_{2^k-1,l} = O(2^{-\frac{k}{2}-1}(\ln\frac{l}{(l+1)(l+3)} - \frac{1}{l+1} - \frac{27}{8}\frac{1}{l+2} - \frac{1}{l}).$$

(iv) We have

$$\begin{aligned}
c_{n,m} &= < f(t), W_{n,m} >_w \\
&= 2^{\frac{k}{2}} \int_0^1 f(t) W_m(2^k t - n) \sqrt{\frac{1-t}{1+t}} dt
\end{aligned}$$

$$
\begin{aligned}
&= -2^{\frac{k}{2}}\int_0^{\pi} f(\frac{\cos\theta+n}{2^k})\frac{\sin(m+\frac{1}{2})\theta}{\sin(\frac{\theta}{2})}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sin\theta d\theta\\
&= -2^{\frac{k}{2}}\int_0^{\pi} f(\frac{\cos\theta+n}{2^k})\frac{\sin(m+\frac{1}{2})\theta}{\sin(\frac{\theta}{2})}\sqrt{\frac{2\sin^2(\frac{\theta}{2})}{2\cos^2(\frac{\theta}{2})}}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta\\
&= -2^{\frac{k}{2}}\int_0^{\pi} f(\frac{\cos\theta+n}{2^k})2\sin(m+\frac{1}{2})\theta\sin\frac{\theta}{2}d\theta\\
&= -2^{\frac{k}{2}}\int_0^{\pi} f(\frac{\cos\theta+n}{2^k})(-\cos(m+1)\theta+\cos m\theta)d\theta\\
&= -2^{\frac{k}{2}}[f(\frac{\cos\theta+n}{2^k})(-\frac{1}{m+1}\sin(m+1)\theta+\frac{1}{m}\sin m\theta)]_0^{\pi}\\
&\quad +2^{\frac{k}{2}}\int_0^{\pi} f'(\frac{\cos\theta+n}{2^k})\frac{\sin\theta}{2^k}(-\frac{1}{m+1}\sin(m+1)\theta+\frac{1}{m}\sin m\theta)d\theta\\
&= 2^{\frac{k}{2}}\int_0^{\pi} f'(\frac{\cos\theta+n}{2^k})\frac{\sin\theta}{2^k}(-\frac{1}{m+1}\sin(m+1)\theta+\frac{1}{m}\sin m\theta)d\theta\\
&\leq 2^{-\frac{k}{2}}P\int_0^{\pi}\sin\theta(\frac{1}{m+1}\sin(m+1)\theta+\frac{1}{m}\sin m\theta))d\theta\\
&= 2^{-\frac{k}{2}}P[(\frac{1}{m+1}\sin\theta\sin(m+1)\theta+\frac{1}{m}\sin\theta\sin m\theta)]_0^{\pi}\\
&\quad -2^{-\frac{k}{2}-1}P\int_0^{\pi}\cos\theta(\frac{1}{m+1}\sin(m+1)\theta+\frac{1}{m}(\sin m\theta)d\theta\\
&= 2^{-\frac{k}{2}-1}P\int_0^{\pi}(\frac{1}{m+1}\cos\theta\sin(m+1)\theta-\frac{1}{m}\cos\theta\sin m\theta)\\
&= 2^{-\frac{k}{2}-1}P\int_0^{\pi}(\frac{1}{2(m+1)}(\sin m\theta+\sin(m+2)\theta)\\
&\quad -\frac{1}{2m}(\sin(m-1)\theta+\sin(m+1)\theta))d\theta\\
&= 2^{-\frac{k}{2}-1}P[-\frac{1}{2m(m+1)}\cos m\theta-\frac{1}{(m+1)(m+2)}\cos(m+2)\theta\\
&\quad +\frac{1}{2m(m-1)}\cos(m-1)\theta+\frac{1}{m(m+1)}\cos(m+1)\theta]_0^{\pi}\\
&= 2^{-\frac{k}{2}-1}P[-\frac{1}{2m(m+1)}\cos m\pi-\frac{1}{(m+1)(m+2)}\cos(m+2)\pi\\
&\quad +\frac{1}{2m(m-1)}\cos(m-1)\pi+\frac{1}{m(m+1)}\cos(m+1)\pi].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|c_{n,m}| = 2^{-\frac{k}{2}-1}P|&-\frac{1}{2m(m+1)}\cos m\pi-\frac{1}{(m+1)(m+2)}\cos(m+2)\pi\\
&+\frac{1}{2m(m-1)}\cos(m-1)\pi+\frac{1}{m(m+1)}\cos(m+1)\pi|
\end{aligned}
$$

$$
\begin{aligned}
&\leq 2^{-\frac{k}{2}-1}P[\frac{3}{2m(m+1)}+\frac{1}{(m+1)(m+2)}+\frac{1}{2m(m-1)}]\\
&\leq 2^{-\frac{k}{2}-1}P(\frac{1}{2m}+\frac{1}{2(m+1)}+\frac{1}{m-1}-\frac{1}{m+2})\\
|c_{n,m}|^2 &= 2^{-k-2}P^2(\frac{1}{4m^2}+\frac{1}{4(m+1)^2}+\frac{1}{(m-1)^2}+\frac{1}{(m+2)^2}\\
&+\frac{1}{2m(m+1)}+\frac{1}{m(m-1)}+\frac{1}{m(m+2)}+\frac{1}{(m+1)(m-1)}\\
&+\frac{1}{(m+1)(m+2)}+\frac{2}{(m-1)(m+2)})\\
&= 2^{-k-2}P^2(\frac{1}{4m^2}+\frac{1}{4(m+1)^2}+\frac{1}{(m-1)^2}+\frac{1}{(m+2)^2}\\
&+\frac{5}{3}\frac{1}{m-1}-\frac{5}{3}\frac{1}{m+2}),
\end{aligned}
$$

$$
\begin{aligned}
E^2_{2^k-1,l} &= \|f-s_{2^k-1,l}\|_2^2\\
&= \|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_{n,m}V_{n,m}-\sum_{n=2^{k-1}+1}^{\infty}\sum_{m=0}^{l}c_{n,m}W_{n,m}\|_2^2\\
&= \|\sum_{n=0}^{2^k-1}\sum_{m=l+1}^{\infty}c_{n,m}W_{n,m}\|_2^2\\
&= \sum_{n=0}^{2^k-1}\sum_{m=l+1}^{\infty}|c_{n,m}|^2\|W_{n,m}\|_2^2\\
&\leq 2^{-\frac{3k}{2}-3}\pi P^2\sum_{m=l+1}^{\infty}(\frac{1}{4m^2}+\frac{1}{4(m+1)^2}+\frac{1}{(m-1)^2}+\frac{1}{(m+2)^2}\\
&+\frac{5}{3}\frac{1}{m-1}-\frac{5}{3}\frac{1}{m+2})\\
&= 2^{-\frac{3k}{2}-3}\pi P^2(\frac{1}{4(l+1)}+\frac{1}{l}+\frac{1}{l+3}+\frac{5}{3}\ln\frac{l}{l+3}).
\end{aligned}
$$

Therefore

$$
E^2_{2^k-1,l} = O(2^{-\frac{3k}{2}-3}(\frac{1}{4(l+1)}+\frac{1}{l}+\frac{1}{l+3}+\frac{5}{3}\ln\frac{l}{l+3})).
$$

$\square$

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