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Research Article

Fitted scheme for singularly perturbed time delay reaction-diffusion problems

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Abstract

In this article, we constructed a numerical scheme for singularly perturbed time-delay reaction-diffusion problems. For the discretization of the time derivative, we used the Crank-Nicolson method and a hybrid scheme, which is a combination of the fourth-order compact difference scheme and the central difference scheme on a special type of Shishkin mesh in the spatial direction. The proposed scheme is shown to be second-order accurate in time and fourth-order accurate with a logarithmic factor in space. The uniform convergence of the proposed scheme is discussed. Numerical investigations are carried out to demonstrate the efficacy and uniform convergence of the proposed scheme, and the obtained numerical results reveal the better performance of the present scheme, as compared with some existing methods in the literature.

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Keywords: Singular perturbation; Time-delay, Parabolic differential equation, Reaction-diffusion problem, Hybrid scheme.

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1 Introduction

Delay differential equations play a crucial role in the mathematical modeling of various practical phenomena and are widely applicable in fields such as biosciences, control theory, economics, material science, medicine, robotics, etc. [17, 16, 15]. If we restrict the delay differential equations to a class in which the highest order term is multiplied by a small parameter ε, then we call it singularly perturbed delay differential equations of the retarded type. Nowadays, there has been a growing interest in the study of singularly perturbed delay differential equations due to their occurrence in many practical models, for instance, Hutchinson's equation which is a model for the evolution of the population in mathematical ecology[1] and, the Wazewska-Czyzeska and Lasota equation that describes the survival of red blood cells in animals [27].

The solutions of singularly perturbed time-delayed problems are fundamentally different from those problems without time delay because small lags can have large effects. The solution of delay differential equations requires knowledge of both the current state and the state at a certain time in the past. Finding the solution to singularly perturbed delay differential equations using classical numerical methods fails to give stable and accurate results because of the presence of the perturbation parameter ε.

Many researchers have treated time-dependent singularly perturbed parabolic partial differential equations,with or without time-delay: to mention some [3, 4, 14, 12, 2, 10, 20, 21, 22, 23, 28]. However, time-delayed reaction diffusion problems have not been extensively investigated. Authors in [13] solved singularly perturbed time-delay parabolic partial differential equations of the reaction-diffusion type. The problem is discretized by a hybrid scheme on a generalized Shishkin mesh in the spatial direction and the implicit Euler scheme on a uniform mesh in the time direction, and to increase the order of convergence in the time direction, Richardson extrapolation is applied. The authors in [12] studied a similar problem with [13]. The scheme uses the Crank-Nicolson scheme for the time variable, and the spatial variable is discretized by the tension spline scheme on a non-uniform Shishkin mesh.

Recently, in [19] Crank-Nicolson method for the time derivative on the uniform mesh and the central difference scheme for the spatial derivative on the Shishkin-type meshes is used for the problem in [13], and to enhance the order of convergence Richardson extrapolation technique is used. Also, in [7] singularly perturbed delay parabolic reaction-diffusion problem with mixed type boundary condition is solved. The problem is discretized by the implicit Euler method on a uniform mesh in the time and the extended cubic B-spline collocation method on a Shishkin mesh in the space variable.

In real-world applications, higher-order methods are preferred to their lower-order counterparts since they provide better accuracy at a lower computing cost. This paper aims to design a uniformly convergent scheme with a higher order of convergence in both space and time variables or singularly perturbed time delay reaction-diffusion problems. To discretize the problem, a similar approach to [14] is applied, that is, the Crank-Nicolson scheme for the discretization of the time derivative, and for the spatial variable, a hybrid scheme on a special type of Shishkin mesh is used, which is gradually condensing starting from the center of the domain to both the right and left boundary layers.

In this article, we considered the following singularly perturbed time-delay parabolic partial differential equation of the reaction-diffusion type of the form:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\varepsilon,x}\right)u(x,t) = -b(x,t)u(x,t-\tau) + f(x,t), (x,t) \in D \tag{1}$$

subject to the initial and interval conditions:

$$u(x,t) = \phi_b(x,t), \text{ for } (x,t) \in \Gamma_b, \tag{1a}$$

and boundary conditions:

$$u(0,t) = \phi_l(t) \quad \text{on } \Gamma_l = \{(0,t) : 0 \le t \le T\}, \tag{1b}$$

$$u(1,t) = \phi_r(t) \quad \text{on } \Gamma_r = \{(1,t) : 0 \le t \le T\}, \tag{1c}$$

where

$$\mathcal{L}_{\varepsilon,x}u(x,t) = -\varepsilon u_{xx}(x,t) + a(x,t)u(x,t),$$

$0 < \varepsilon << 1$ and $\tau > 0$ is the delay parameter. Also, $D = \Omega_x \times \Omega_t = (0,1) \times (0,T]$ and $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$, $\Gamma_b = [0,1] \times [-\tau, 0]$, Γ_l and Γ_r are the left and the right sides of the rectangular domain D corresponding to $x = 0$ and $x = 1$, respectively. The functions $a(x,t), b(x,t), f(x,t), \phi_b, \phi_l$ and ϕ_r are assumed to be sufficiently smooth and bounded and satisfying,

$$a(x,t) \ge \alpha > 0, b(x,t) \ge \beta > 0, (x,t) \in \overline{D}.$$

For the existence and uniqueness of the solution of (1), see [1]. Under the above assumptions the solution of (1) exhibits boundary layers along $x = 0$ and $x = 1$.

The article is organized as follows: In Section 2, we provide some properties of the analytical solution and its derivatives. In Section 3, temporal semi-discretization, spatial discretization, and the derivation of the scheme are discussed. In Section 4, we investigate the uniform convergence of the fully discrete scheme. Numerical results and discussion are presented in Section 5. Finally, in Section 6, the conclusion of the paper is provided.

Notation : Throughout this paper, C denotes a generic positive constant that is independent of ε and mesh sizes. Also $||.||$ denotes the standard supremum norm, which is defined as $||f|| = \underset{(x,t)\in D}{Sup} |f(x,t)|$, for a function f defined on some domain D.

2 Analytical results

In this section, the analytical aspects of the solution of problem (1) and its derivatives are studied.

Lemma 1 (Maximum principle)**.** Let $\Phi(x,t) \in C^{2,1}(\overline{D})$ and assume that $\Phi(x,t) \geq 0$ on Γ.
Then $\left(\frac{\partial}{\partial t} + \mathcal{L}_{\varepsilon,x}\right)\Phi(x,t) \geq 0, \forall(x,t) \in D$ imply that $\Phi(x,t) \geq 0, \forall(x,t) \in \overline{D}$.

Proof. Let the point $(x^*,t^*) \in \overline{D}$, such that $\Phi(x^*,t^*) = \min \Phi(x,t)$ and assume that $\Phi(x^*,t^*) < 0$. Clearly, $(x^*,t^*) \notin \Gamma$ these implies that $(x^*,t^*) \in D$.

The function $\Phi(x,t)$ attains its minimum at (x^*,t^*), then $\Phi_x = 0, \Phi_t = 0$ and $\Phi_{xx} \geq 0$ at (x^*,t^*) . Therefore, from (1), it is easy to observe that

$$(\frac{\partial}{\partial t} + \mathcal{L}_{\varepsilon,x})\Phi(x,t) \leq 0,$$

which is a contradiction as $(\frac{\partial}{\partial t} + \mathcal{L}_{\varepsilon,x})\Phi(x,t) \geq 0$. Hence $\Phi(x,t) \geq 0$ for all $(x,t) \in \overline{D}$. □

Lemma 2. The solution u of (1) satisfies the following bounds,

$$||\frac{\partial^{i+j} u}{\partial x^i \partial t^j}|| \leq C\varepsilon^{-i/2}, \qquad \text{for } 0 \leq i + 2j \leq 8. \tag{2}$$

Proof. See [13]. □

In the proof of the error analysis, sharper bounds of the solution and its derivatives are required so that the solution u is decomposed into a regular component v, and a singular component w, as follows:

$$u = v + w,$$

for more detail see [1, 18].

Lemma 3. The solution u of (1) satisfies the following bounds,

$$||\frac{\partial^{i+j} v}{\partial x^i \partial t^j}|| \leq C(1 + \varepsilon^{2-i/2}), \tag{3}$$

$$||\frac{\partial^{i+j} w}{\partial x^i \partial t^j}|| \leq C\varepsilon^{-i/2}(\exp(-\sqrt{\alpha}x/\sqrt{\varepsilon}) + \exp(-\sqrt{\alpha}(1-x)/\sqrt{\varepsilon})), \tag{4}$$

$$\text{for} \quad (x,t) \in D, 0 \leq i + 2j \leq 8.$$

Proof. The proof of this Lemma is described in [1]. □

3 Numerical scheme

3.1 Temporal semi-discretization

On the time domain $[0, T]$, we use uniform mesh with step size Δt, such that $\Omega_t^M = \{t_j : t_j = j\Delta t, \Delta t = \frac{T}{M}, \text{ for } j = 0, 1, ..., M\}$, $T = y\tau$, $\tau = m_\tau \Delta t$, where y and m_τ are positive integer, and M is the number of mesh intervals in $[0, T]$. To descretize the time variable for problem (1), we use Crank-Nicolson method, given by

$$\begin{cases} u^{-j}(x) = \phi_b(x, -t_j), \text{ for } j = 0, 1, 2, ..., m_\tau, 0 \leq x \leq 1, \\ \left(I + \frac{\Delta t}{2}\mathcal{L}_{\varepsilon,x}\right)u^{j+1}(x) = \left(I - \frac{\Delta t}{2}\mathcal{L}_{\varepsilon,x}\right)u^j(x) + \frac{\Delta t}{2}\left[F^j(x) + F^{j+1}(x)\right], \\ u^{j+1}(0) = \phi_l, u^{j+1}(1) = \phi_r, \text{ for } j = 1, 2, ..., M-1, \end{cases} \tag{5}$$

where $F^j(x) = f(x, t_j) - b(x, t_j)u(x, t_{j-m_\tau})$ and, $u^{j+1}(x) = u(x, t_{j+1})$ is the semi-discrete approximation to the exact solution $u(x, t)$ of (1) at the time level $t_j = j\Delta t$. The local truncation error of the semi-discrete method (5) is given by

$$e_{j+1} = u(x, t_{j+1}) - \widetilde{u}^{j+1}(x),$$

where $\tilde{u}^j(x)$ is the solution evaluated after one step of the semi-discrete scheme (5) taking the exact value $u(x, t_j)$ instead of u^j as the initial data. Re-write (5) in the form

$$\begin{cases} \left(I + \frac{\Delta t}{2}\mathcal{L}_{\varepsilon,x}\right)\widetilde{u}^{j+1}(x) = g^j(x), x \in \Omega_x, \\ \widetilde{u}^{j+1}(0) = \phi_l, \quad \widetilde{u}^{j+1}(1) = \phi_r, \end{cases} \tag{6}$$

where $g^j(x) = \frac{\Delta t}{2}\left(F^j(x) + F^{j+1}(x) - \mathcal{L}_{\varepsilon,x}u^j(x)\right) + u(x, t_j)$.

For the stability of the Crank-Nicolson method, one can refer to [6, 8].

Lemma 4. Suppose that $\left|\frac{\partial^i}{\partial t^i}u(x, t)\right| \leq C, \quad (x, t) \in \overline{D}, \quad i = 0, 1, 2, 3,$ the local error associated to scheme (5) satisfies:

$$||e_{j+1}|| \leq C_1(\Delta t)^3, \quad j = 1, 2, ..., M. \tag{7}$$

Proof. Using Taylor's series expansion,

$$\begin{aligned} \frac{u(x, t_{j+1}) - u(x, t_j)}{\Delta t} &= u_t(x, t_{j+1/2}) + O((\Delta t)^2), \\ &= -\mathcal{L}_{\varepsilon,x}u(x, t_{j+1/2}) + F(x, t_{j+1/2}) + O((\Delta t)^2), \end{aligned} \tag{8}$$

where $F(x, t_{j+1/2}) = -b(x, t_{j+1/2})u(x, t_{j+1/2-m_\tau}) + f(x, t_{j+1/2})$ and

$$
\begin{aligned}
f(x, t_{j+1/2}) &= \frac{f(x, t_{j+1}) + f(x, t_j)}{2} + O((\Delta t)^2), \\
a(x, t_{j+1/2}) &= \frac{a(x, t_{j+1}) + a(x, t_j)}{2} + O((\Delta t)^2), \\
b(x, t_{j+1/2}) &= \frac{b(x, t_{j+1}) + b(x, t_j)}{2} + O((\Delta t)^2).
\end{aligned}
$$

From the above, we obtain

$$
\mathcal{L}_{\varepsilon,x} u(x, t_{j+1/2}) = \mathcal{L}_{\varepsilon,x} \frac{u(x, t_{j+1}) + u(x, t_j)}{2} + O((\Delta t)^2).
$$

Clearly, the local error $||e_{j+1}||$ is the solution of the following BVP:

$$
\begin{cases}
\left(I + \frac{\Delta t}{2} \mathcal{L}_{\varepsilon,x}\right) e_{j+1} = O((\Delta t)^3), \\
e_{j+1}(0) = 0, \quad e_{j+1}(1) = 0.
\end{cases} \tag{9}
$$

Next, using the maximum principle for the operator $\left(I + \frac{\Delta t}{2} \mathcal{L}_{\varepsilon,x}\right)$ proves the result, for more detail please refer [6]. □

All the results in the above insure the second order convergence of the scheme (5). The global error of the time semi-discretization is given by

$$
E_j = u(x, t_j) - u^j(x) = \sum_{k=1}^{j} e_k.
$$

Theorem 1 (Global error estimate)**.** The global error estimate at t_j is

$$
||E_j|| \leq C(\Delta t)^2, \quad j = 1, 2, ..., M.
$$

Proof. see [11]. □

3.2 Spatial discretization

In this section, we discretized problem (5), using a hybrid scheme. First, we define a special type of Shishkin mesh to discretize the domain $\overline{\Omega}_x$, then the required scheme can be constructed.

3.2.1 Shishkin mesh

For $N \geq 2^k$, for integer $k \geq 2$, the fitted mesh $\overline{\Omega}_x^N$ is constructed by dividing the domain $\overline{\Omega}_x = [0,1]$ in to three subintervals such that

$$\overline{\Omega}_x^N = [0,\sigma] \cup [\sigma, 1-\sigma] \cup [1-\sigma, 1].$$

Consequently, the mesh points and the transition parameter σ are defined as described in [14, 26]. Let, $L = L(N)$ satisfying,

$$\ln(\ln N) \leq L \leq lnN \quad \text{and} \quad \exp(-L) \leq \frac{L}{N}.$$

Then, the location of the transition point is

$$\sigma = \min\left\{\frac{1}{4}, \eta\sqrt{\varepsilon}L\right\}, \tag{10}$$

where $\eta \geq 4/\sqrt{\alpha}$. Moreover, mesh points in the spatial direction is given by

$$x_i = \begin{cases} \frac{4\sigma}{N} i, & \text{for } i = 1,2,...,\frac{N}{4}, \\ \sigma + r(\frac{i}{N} - 1/4)^3 + 4\sigma(\frac{i}{N} - 1/4), & \text{for } \frac{N}{4}+1, \frac{N}{4}+2, ..., \frac{N}{2} \\ 1 - x_{N-i}, & \text{for } i = \frac{N}{2}+1, \frac{N}{2}+2, ..., N. \end{cases} \tag{11}$$

where the coefficient r is determined from $x_{N/2} = 1/2$.
Let $h_{\max} = \max_{\forall i} h_i$, where $h_i = x_i - x_{i-1}, i = 1,2,...,N$, from (11) it is clear that the maximum mesh width h_{max} always correspond to $h_{\frac{N}{2}}$ and $h_{\frac{N}{2}+1}$ of the domain $\overline{\Omega}_N$, that is,

$$h_{\max} = h_{\frac{N}{2}} = h_{\frac{N}{2}+1} \leq CN^{-1}.$$

In the coarse part $[\sigma, 1-\sigma]$, the mesh width satisfies the following,

$$h_{i+1} \leq CN^{-1}, \tag{12}$$
$$|h_{i+1} - h_i| \leq CN^{-2}, \tag{13}$$

for more detail see [26].

3.2.2 Derivation of the scheme

We use a hybrid scheme which is a combination of the central difference scheme and the fourth-order compact difference scheme to approximate the semi-discrete problem (5), where the coefficients are determined to make the scheme exact for polynomials up to degree four and satisfy the normalization condition, $q_i^- + q_i^c + q_i^+ = 1,$ for $i = 1, 2, ..., N-1$.

$$\mathcal{L}_\varepsilon^N \widetilde{U}_i^{j+1} \equiv r_{i,j}^- \widetilde{U}_{i-1}^{j+1} + r_{i,j}^c \widetilde{U}_i^{j+1} + r_{i,j}^+ \widetilde{U}_{i+1}^{j+1} = Q(g_i^j) = q_i^- g_{i-1}^j + q_i^c g_i^j + q_i^+ g_{i+1}^j, \tag{14}$$

where

$$r_{i,j}^- = \frac{\Delta t}{2}\widetilde{r}_{i,j}^- + q_i^-, \qquad r_{i,j}^c = \frac{\Delta t}{2}\widetilde{r}_{i,j}^c + q_i^c, \qquad r_{i,j}^+ = \frac{\Delta t}{2}\widetilde{r}_{i,j}^+ + q_i^+,$$

and the coefficients $\widetilde{r}_{i,j}^-$, $\widetilde{r}_{i,j}^c$, $\widetilde{r}_{i,j}^+$,q_i^-, q_i^c and q_i^+ will be determined later. The development of the method is based on computing the local truncation error as follows:

$$\begin{aligned} T_{i,\widetilde{u}^{j+1}} &= \mathcal{L}_\varepsilon^N(\widetilde{u}_i^{j+1}) - \mathcal{L}_\varepsilon^N(\widetilde{U}_i^{j+1}) \\ &= \mathcal{L}_\varepsilon^N(\widetilde{u}_i^{j+1}) - Q\bigg(\big(I + \frac{\Delta t}{2}\mathcal{L}_{\varepsilon,x}\big)\widetilde{u}_i^{j+1}\bigg) \\ &= r_{i,j}^- \widetilde{u}_{i-1}^{j+1} + r_{i,j}^c \widetilde{u}_i^{j+1} + r_{i,j}^+ \widetilde{u}_{i+1}^{j+1} - q_i^-(\widetilde{u}_{i-1}^{j+1} + \frac{\Delta t}{2}\mathcal{L}_{\varepsilon,x}\widetilde{u}_{i-1}^{j+1}) \\ &- q_i^c(\widetilde{u}_i^{j+1} + \frac{\Delta t}{2}\mathcal{L}_{\varepsilon,x}\widetilde{u}_i^{j+1}) - q_i^+(\widetilde{u}_{i+1}^{j+1} + \frac{\Delta t}{2}\mathcal{L}_{\varepsilon,x}\widetilde{u}_{i+1}^{j+1}). \end{aligned} \tag{15}$$

Let's denote the local truncation error corresponds to the discretization of the spatial variable by $T_{i,\widetilde{u}}^x$, and $T_{i,\widetilde{u}^{j+1}}$ represent, $T_{i,\widetilde{u}^{j+1}} = \frac{\Delta t}{2} T_{i,\widetilde{u}}^x$

$$T_{i,\widetilde{u}}^x = T_i^0 \widetilde{u} + T_i^1 \widetilde{u}' + ... + T_i^6 \widetilde{u}^6 + O(h_{\max}^5), \tag{16}$$

where

$$\begin{aligned} T_i^0 &= (\widetilde{r}_{i,j}^- + \widetilde{r}_{i,j}^c + \widetilde{r}_{i,j}^+) - (q_i^- a_{i-1}^{j+1} + q_i^c a_i^{j+1} + q_i^+ a_{i+1}^{j+1}), \\ T_i^1 &= (h_{i+1}\widetilde{r}_{i,j}^+ - h_i \widetilde{r}_{i,j}^-) - (h_{i+1} q_i^+ a_{i+1}^{j+1} - h_i q_i^- a_{i-1}^{j+1}), \\ T_i^2 &= (h_{i+1}^2 \widetilde{r}_{i,j}^+ + h_i^2 \widetilde{r}_{i,j}^-) - 2\varepsilon - (h_{i+1}^2 q_i^+ a_{i+1}^{j+1} + h_i^2 q_i^- a_{i-1}^{j+1}), \\ &\vdots \\ T_i^k &= \frac{h_{i+1}^k}{k!}\widetilde{r}_{i,j}^+ + (-1)^k \frac{h_i^k}{k!}\widetilde{r}_{i,j}^- + q_i^+\left(\frac{h_{i+1}^{k-2}}{(k-2)!}\varepsilon - \frac{h_{i+1}^k}{k!}a_{i+1}^{j+1}\right) \\ &+ (-1)^k q_i^- \left(\frac{h_i^{k-2}}{(k-2)!}\varepsilon - \frac{h_i^k}{k!}a_{i-1}^{j+1}\right), \text{ for } k = 3, 4, 5, 6. \end{aligned}$$

The truncation error is said to be of order p if $T_{i,\widetilde{u}}^x = O(h_{\max}^p)$ as $h_{\max} \to 0$, for $i = 1, 2, ..., N-1$. The method is constructed under the conditions,

$$
\begin{aligned}
&T_i^k = 0, \quad k = 0,1,2,\\
&T_i^k = O(h_{\max}^4), \quad k = 3,4.
\end{aligned} \tag{17}
$$

Therefore, the local truncation error $T^x_{i,\widetilde{u}}$ can be written in the form,

$$
\begin{aligned}
T^x_{i,\widetilde{u}} = T_i^3 \widetilde{u}''' + T_i^4 \widetilde{u}^4 + \left(\frac{1}{5!}(h_{i+1}^5 \widetilde{r}^+_{i,j} - h_i^5 \widetilde{r}^-_{i,j}) + \frac{\varepsilon}{3!}(h_{i+1}^3 q_i^+ - h_i^3 q_i^-)\right)\widetilde{u}^5\\
+ \left(\frac{1}{6!}(h_{i+1}^6 \widetilde{r}^+_{i,j} + h_i^6 \widetilde{r}^-_{i,j}) + \frac{\varepsilon}{4!}(h_{i+1}^4 q_i^+ + h_i^4 q_i^-)\right)\widetilde{u}^6.
\end{aligned} \tag{18}
$$

As a result, to fulfill the condition stated in (17), the coefficients are determined as follows:

$$
\begin{aligned}
\widetilde{r}^-_{i,j} &= \left(\frac{-2\varepsilon}{h_i(h_i + h_{i+1})} + q_i^- a_{i-1}^{j+1}\right),\\
\widetilde{r}^+_{i,j} &= \left(\frac{-2\varepsilon}{h_{i+1}(h_i + h_{i+1})} + q_i^+ a_{i-1}^{j+1}\right),\\
\widetilde{r}^c_{i,j} &= \left(\frac{2\varepsilon}{h_i h_{i+1}} + q_i^c a_i^{j+1}\right), \quad 1 \le i \le N-1.
\end{aligned} \tag{19}
$$

The coefficients q_i^+, q_i^c and q_i^-, $i = 1, 2, ..., N-1$ are defined in two different ways,
(i) For the inner layer region, i.e $(0,\sigma) \cup (1-\sigma, 1)$, the coefficients are given by

$$
\begin{aligned}
q_i^- &= \frac{1}{6} - \frac{h_{i+1}^2}{6h_i(h_i + h_{i+1})},\\
q_i^+ &= \frac{1}{6} - \frac{h_i^2}{6h_{i+1}(h_i + h_{i+1})},\\
q_i^c &= \frac{h_i^2 + h_{i+1}^2 + 3h_i h_{i+1}}{6h_i h_{i+1}}
\end{aligned} \tag{20}
$$

(ii) For the outer layer region $[\sigma, 1-\sigma]$, depending on the maximum step length(h_{max}) and ε, the coefficients are defined in two different cases. Define $\widetilde{a} = (a + \frac{2}{\Delta t})$, γ is a positive constant independent of ε and Δt,
Case-1 When $\gamma h_{\max}^2 ||\widetilde{a}|| \le \varepsilon$, the coefficients q_i^+, q_i^c and q_i^-, are defined by (20)
Case-2 When $\gamma h_{\max}^2 ||\widetilde{a}|| \ge \varepsilon$, the coefficients q_i^+, q_i^c and q_i^-, are given by

$$
q_i^- = 0, \quad q_i^c = 1, \quad q_i^+ = 0. \tag{21}
$$

Remark 1. The coefficients q_i^+, q_i^c and q_i^- are determined by the fourth order compact difference scheme in the boundary layer region, and in the regular region when $\gamma h_{\max}^2||\widetilde{a}|| \leq \varepsilon$. While in the regular region these coefficients are determined by central difference scheme when $\gamma h_{\max}^2||\widetilde{a}|| \geq \varepsilon$. For more detail see [3, 14, 8, 5, 24, 25].

3.2.3 Stability and error analysis

To estimate the $\varepsilon-$ uniform convergence of the proposed scheme, we decompose the approximate solution of (14) into regular and singular components,

$$\widetilde{U} = \widetilde{V} + \widetilde{W}.$$

Now, the truncation error of the method (14) can be decomposed as

$$|T_{i,\widetilde{u}^{j+1}}| \leq |T_{i,\widetilde{v}^{j+1}}| + |T_{i,\widetilde{w}^{j+1}}|, \tag{22}$$

where $|T_{i,\widetilde{v}^{j+1}}|$ and $|T_{i,\widetilde{w}^{j+1}}|$ are the errors corresponding to the regular component $\widetilde{v}^{j+1}$ and the singular component $\widetilde{w}^{j+1}$, respectively.

Lemma 5. Let $N \geq N_0$, where N_0 is the smallest positive integer such that

$$\frac{4\eta^2}{3}(||a|| + 2/\Delta t) < \frac{N_0^2}{{L^*}^2}, \tag{23}$$

where $L^* = L(N_0)$. Then for any $N \geq N_0$, the coefficients

$r_{i,j}^- < 0, \;\; r_{i,j}^c > 0, \;\; r_{i,j}^+ < 0 \;\;$ and $\;\; r_{i,j}^- + r_{i,j}^c + r_{i,j}^+ \geq 0, \;$ for $\; i = 1, 2, ..., N-1.$

Proof. From (20) and (21) it is clear that $q_i^- \geq 0, \;\; q_i^c > 0,$ and $q_i^+ \geq 0$, then to show $r_{i,j}^- < 0$, $r_{i,j}^+ < 0$ and $r_{i,j}^c > 0$, we have

$$\begin{aligned}
r_{i,j}^- &= \frac{\Delta t}{2}\left(\frac{-2\varepsilon}{h_i(h_i + h_{i+1})} + q_i^-\left(a_{i-1}^{j+1} + \frac{2}{\Delta t}\right)\right),\\
r_{i,j}^+ &= \frac{\Delta t}{2}\left(\frac{-2\varepsilon}{h_{i+1}(h_i + h_{i+1})} + q_i^+\left(a_{i+1}^{j+1} + \frac{2}{\Delta t}\right)\right),\\
r_{i,j}^c &= \frac{\Delta t}{2}\left(\frac{2\varepsilon}{h_i h_{i+1}} + q_i^c\left(a_i^{j+1} + \frac{2}{\Delta t}\right)\right).
\end{aligned} \tag{24}$$

Using (20) and (21), the mesh width and the condition (23) proves $r_{i,j}^- < 0$, $r_{i,j}^+ < 0$ and $r_{i,j}^c > 0$, then from (24), we obtain
$r_{i,j}^- + r_{i,j}^c + r_{i,j}^+ > 0, \quad$ for $i = 1, 2, ..., N-1.$ □

Remark 2. From Lemma 5 it is clear that the matrix associated with the discrete operator $\mathcal{L}_\varepsilon^N$ defined in (14) is an irreducible M-matrix and so has a positive inverse. The operator $\mathcal{L}_\varepsilon^N$ in (14) satisfies the following maximum principle on $D^{N,M}$. Hence, the scheme (14) is uniformly stable in the maximum norm.

Lemma 6. Let Φ satisfies $\Phi \geq 0$ on $\Gamma^{N,M}$. Then $\mathcal{L}_\varepsilon^N \Phi \geq 0$ on $D^{N,M}$ implies that $\Phi \geq 0$ at each point of $D^{N,M}$.

Proof. The proof follows using Lemma 5 and Remark 2. □

Lemma 7. Let $\widetilde{u}^{n+1}$ be the solution of (5) and $\widetilde{U}^{n+1}$ be the solution of discrete scheme (14). Then the global error satisfies

$$||\widetilde{u}^{n+1}(x_i) - \widetilde{U}_i^{n+1}|| \leq C\Delta t(L/N)^4. \tag{25}$$

Proof. For uniform mesh i.e $\sigma = 1/4$, classical analysis can be used to prove the convergence of the scheme. So, we only consider the case $\sigma = \sigma_0\sqrt{\varepsilon}L$.
Case-1 Inner region $[0,\sigma] \cup [1-\sigma, 1]$, $h_i = h_{i+1} = \sigma_0\sqrt{\varepsilon}N^{-1}L$, from the truncation error in (18), we have

$$\begin{aligned} T_{i,\widetilde{u}}^x \leq C\Delta t\varepsilon\Bigg(&|h_{i+1} - h_i|(h_{i+1} + h_i)^2\left|\left|\frac{d^5\widetilde{u}^{n+1}}{dx^5}\right|\right|_{[x_{i-1},x_{i+1}]} \\ &+ (h_i^4 + h_{i+1}^4)\left|\left|\frac{d^6\widetilde{u}^{n+1}}{dx^6}\right|\right|_{[x_{i-1},x_{i+1}]}\Bigg). \end{aligned} \tag{26}$$

Using Lemma 3, we can get

$$T_{i,\widetilde{v}}^x \leq C\Delta t(L/N)^4 \quad \text{and} \quad T_{i,\widetilde{w}}^x \leq C\Delta t(L/N)^4 \tag{27}$$

Therefore, (22) gives

$$T_{i,\widetilde{u}}^x \leq C\Delta t(L/N)^4.$$

Case-2 Outer region $[\sigma, 1-\sigma]$, determined by the relation between $h_{\max}$ and ε
(i) When $\gamma h_{\max}^2||\widetilde{a}|| \leq \varepsilon$

$$\begin{aligned} T_{i,\widetilde{u}}^x \leq C\Delta t\varepsilon\Bigg(&|h_{i+1} - h_i|(h_{i+1} + h_i)^2\left|\left|\frac{d^5\widetilde{u}^{n+1}}{dx^5}\right|\right|_{[x_{i-1},x_{i+1}]} \\ &+ (h_i^4 + h_{i+1}^4)\left|\left|\frac{d^6\widetilde{u}^{n+1}}{dx^6}\right|\right|_{[x_{i-1},x_{i+1}]}\Bigg). \end{aligned} \tag{28}$$

Using (3), (12) and (13), we obtain the bound of the truncation error with respect to the regular component $\widetilde{v}$ is,

$$T_{i,\widetilde{v}}^x \leq C\Delta t N^{-4}. \tag{29}$$

Similarly for the singular component $\widetilde{w}$ using (4), (12) and (13), we obtain the bound of the truncation error with respect to singular component $\widetilde{w}$,

$$\begin{aligned} T^x_{i,\widetilde{w}} &\leq C\varepsilon^{-2}\Delta t N^{-4}(\exp(-\sqrt{\alpha}x/\sqrt{\varepsilon}) + \exp(-\sqrt{\alpha}(1-x)/\sqrt{\varepsilon})) \\ &\leq C\Delta t(L/N)^4, \end{aligned} \tag{30}$$

since

$$\big(\exp(-\sqrt{\alpha}x/\sqrt{\varepsilon}) + \exp(-\sqrt{\alpha}(1-x)/\sqrt{\varepsilon})\big) \leq C\exp(-a(x,t)L),$$

and

$$\exp(-L) \leq L/N.$$

(ii) When $\gamma h^2_{\max}||\widetilde{a}|| \geq \varepsilon$, Assume $L^{-4} \leq C\Delta t$,

$$T^x_{i,\widetilde{u}} \leq C\Delta t\varepsilon\left(|h_{i+1}-h_i|\left|\left|\frac{d^3\widetilde{u}^{n+1}}{dx^3}\right|\right|_{[x_{i-1},x_{i+1}]} + (h_i^2+h_{i+1}^2)\left|\left|\frac{d^4\widetilde{u}^{n+1}}{dx^4}\right|\right|_{[x_{i-1},x_{i+1}]}\right). \tag{31}$$

Using Lemma 3, (12) and (13), we obtain the bound of the truncation error with respect to regular component $\widetilde{v}$ and layer component $\widetilde{w}$ as follows

$$T^x_{i,\widetilde{v}} \leq C\varepsilon\Delta t N^{-2}.$$

Since $\gamma h^2_{\max} > \varepsilon$,

$$T^x_{i,\widetilde{v}} \leq CN^{-4} \tag{32}$$

and

$$\begin{aligned} T^x_{i,\widetilde{w}} \leq C\varepsilon\Delta t\left|\left|\frac{d^2w}{dx^2}\right|\right|_{[x_{i-1},x_{i+1}]} &\leq C\Delta t\big(\exp(-\sqrt{\alpha}x/\sqrt{\varepsilon}) + \exp(-\sqrt{\alpha}(1-x)/\sqrt{\varepsilon})\big) \\ &\leq C\Delta t(L/N)^4. \end{aligned} \tag{33}$$

Hence, from (32), (33) and (22), we get

$$T^x_{i,\widetilde{u}} \leq C\varepsilon\Delta t(L/N)^4. \tag{34}$$

Therefore, taking the estimate in case (i) and case (ii) then the truncation error estimate of the scheme (14) is given by

$$T^x_{i,\widetilde{u}} \leq C\varepsilon\Delta t(L/N)^4. \tag{35}$$

Finally, combining the uniform stability result and (35), concludes the proof. □

4 The full discrete scheme

A combination of time and the spatial semi-discretization gives the full discrete scheme. Let U_i^j be the numerical approximation for $u(x_i,t_j)$ then the fully discrete scheme on the mesh $D^{N,M}$ is of the form

$$\begin{cases} U^{-j}(x_i) = \phi_b(x_i, -t_j), \text{ for } j = 0, 1, 2, ..., m_\tau, \text{ and } 0, 1, 2, ..., N, \\ \mathcal{L}_\varepsilon^{N,M} U_i^{j+1} \equiv r_{i,j}^- U_{i-1}^{j+1} + r_{i,j}^c U_i^{j+1} + r_{i,j}^+ U_{i+1}^{j+1} = q_i^- G_{i-1}^j + q_i^c G_i^j + q_i^+ G_{i+1}^j, \\ U_0^{j+1} = \phi_l(t_{j+1}), \quad U_N^{j+1} = \phi_r(t_{j+1}), \\ \text{for} \quad i = 1, 2, ..., N-1 \quad \text{and } j = 1, 2, ..., M-1. \end{cases} \tag{36}$$

Where

$$\mathcal{L}_{\varepsilon,x}^N U_i^{j+1} = -\mathcal{L}_{\varepsilon,x}^N U_i^j - 2\frac{U_i^{j+1} - U_i^j}{\Delta t} + F^j(x_i) + F^{j+1}(x_i),$$
$$G_i^j = U_i^j + \frac{\Delta t}{2}\Big[F^j(x_i) + F^{j+1}(x_i) - \mathcal{L}_{\varepsilon,x}^N U_i^j\Big],$$

and the coefficients $r_{i,j}^-, r_{i,j}^c, r_{i,j}^+, q_i^-, q_i^c$ and q_i^+, are described in subsection 3.2.2.
The next theorem is for the $\varepsilon-$ uniform convergence of the proposed scheme (36).

Theorem 2. Let u be the exact solution of (1) and U_i^j be the approximate solution of (36). Under the hypothesis of Lemma (5). Then the global $\varepsilon-$ uniform error estimate of the scheme (36) satisfies the following bound

$$||u(x_i, t_j) - U_i^j||_{\overline{D}^{N,M}} \le C\Big((\Delta t)^2 + (L/N)^4\Big), \quad 1 \le i \le N \text{ and } 1 \le j \le M,$$

where L is as described in section 3.2.1.

Proof. Case-1 For $t \le \tau$, the delay term $u(x, t - \tau)$ is not dependent on ε, since the right hand side of problem (1) is $-b(x,t)\phi(x,t-\tau) + f(x,t)$. Let $\overline{D}_\tau^{N,M} = \Omega_x^N \times \Omega_\tau^M$, where Ω_τ^M denotes the uniform mesh elements of $[0, \tau]$, then the global error estimate on $\overline{D}_\tau^{N,M}$ can be computed as follows

$$\begin{aligned} ||u(x_i, t_j) - U_i^j||_{\overline{D}_\tau^{N,M}} \le ||u(x_i, t_j) - \widetilde{u}_i^j||_{\overline{D}_\tau^{N,M}} + ||\widetilde{u}_i^j - \widetilde{U}_i^j||_{\overline{D}_\tau^{N,M}} \\ + ||\widetilde{U}_i^j - U_i^j||_{\overline{D}_\tau^{N,M}}. \end{aligned} \tag{37}$$

The combination of (7) and Lemma 3.2.3 result,

$$||u(x_i, t_j) - U_i^j||_{\overline{D}_\tau^{N,M}} \le ||\widetilde{U}_i^j - U_i^j||_{\overline{D}_\tau^{N,M}} + C\Delta t\Big(\big(\Delta t\big)^2 + \big(L/N\big)^4\Big). \tag{38}$$

Taking the stability of the fully discrete method, we conclude that

$$||\widetilde{U}_i^j - U_i^j||_{\overline{D}_\tau^{N,M}} \le ||u(x_i, t_{j-1}) - U_i^{j-1}||_{\overline{D}_\tau^{N,M}}. \tag{39}$$

Now from (38) and (39) results a recurrence relation for the global error, so that

$$||u(x_i,t_j)-U_i^j||_{\overline{D}^N} \le C\bigg((\Delta t)^2+(L/N)^4\bigg),\ \ 1\le i\le N \text{ and } 1\le j\le M. \quad (40)$$

Case-2 When $t \ge \tau$, in this case the delay term $u(x, t-\tau)$ depends on ε. The proof is done following the approach given in [1, 25]. The solution $U(x_i, t_j)$ found on $\overline{D}_{\tau}^{N,M}$ is denoted by $U_\tau(x_i,t_j)$.To determine the estimate of the global error on $\overline{D}_{2\tau}^{N,M} = \Omega^N \times \Omega_{2\tau}^M$, where $\Omega_{2\tau}^M$ denotes the uniform mesh with m_τ mesh elements used on $[\tau, 2\tau]$,

$$\begin{aligned}|\mathcal{L}_\varepsilon^{N,M}(u(x_i,t_j)-U_i^j)| = {}& q_i^- b_{i-1}^j\big(U_\tau(x_{i-1},t_{j-m_\tau})-u(x_{i-1},t_{j-m_\tau})\big)+q_i^- E_j\\ &+q_i^c b_i^j\big(U_\tau(x_i,t_{j-m_\tau})-u(x_i,t_{j-m_\tau})\big)+q_i^c E_j\\ &+q_i^+ b_{i+1}^j\big(U_\tau(x_{i+1},t_{j-m_\tau})-u(x_{i+1},t_{j-m_\tau})\big)+q_i^+ E_j\\ &+T_{i,\widetilde{u}^j},\end{aligned} \quad (41)$$

where E_j and $T_{i,\widetilde{u}^j}$ are the local error associated to the Crank-Nicolson method and the method in (14) respectively. Then using the bound (40),

$$\begin{aligned}|\mathcal{L}_\varepsilon^{N,M}(u(x_i,t_j)-U_i^j)| &\le ||E_j|| + |T_{i,\widetilde{u}^j}| + ||u(x_i,t_j)-U_i^j||_{\overline{D}^N}\\ &\le C\bigg((\Delta t)^2+(L/N)^4\bigg),\\ &\quad 1\le i\le N \text{ and } 1\le j\le M.\end{aligned}$$

By introducing barrier functions and applying the discrete maximum principle over $\overline{D}_{2\tau}^{N,M}$, we found

$$||u(x_i,t_j)-U_i^j||_{\overline{D}_{2\tau}^{N,M}} \le C\bigg((\Delta t)^2+(L/N)^4\bigg),\ \ 1\le i\le N \text{ and } 1\le j\le M,$$

Lastly, by applying an induction argument we can get the required estimate. □

5 Numerical results and discussion

To test the performance of the proposed scheme, we applied it to three different problems of the form (1). The maximum absolute errors (point-wise) and the rate of the convergence are calculated and tabulated. For those problems whose exact solution is unknown, the maximum absolute error is calculated using the formula:

$$E_\varepsilon^{N,M} = \max_{0\le i,j\le N,M} |U^{N,M}(x_i,t_j) - U^{2N,2M}(x_{2i},t_{2j})|.$$

where $U^{N,M}(x_i, t_j)$ denote the numerical solution obtained by N mesh intervals in the spatial direction and M mesh intervals in the time direction, such that $M = T/\Delta t$. The corresponding rate of convergence rate is given by the formula:

$$R_\varepsilon^{N,M} = \log 2\left(\frac{E_\varepsilon^{N,M}}{E_\varepsilon^{2N,2M}}\right).$$

Also, the $\varepsilon-$ uniform maximum point -wise error $E^{N,M}$ is given by the formula:

$$E^{N,M} = \max_{\varepsilon} E_\varepsilon^{N,M}.$$

and the corresponding $\varepsilon-$uniform rate of convergence $R^{N,M}$ is given by the formula:

$$R^{N,M} = \log 2\left(\frac{E^{N,M}}{E^{2N,2M}}\right).$$

Example 1. Consider the following reaction-diffusion problem [9]:

$$\begin{aligned}
&u_t - \varepsilon u_{xx} + \frac{1}{2}u(x,t) = -2e^{-1}u(x,t-1) + f(x,t), \quad (x,t) \in (0,1) \times (0,2],\\
&u(x,t) = e^{-\left(t+\frac{x}{\sqrt{\varepsilon}}\right)} + e^{-\left(t+\frac{(1-x)}{\sqrt{\varepsilon}}\right)}, \quad (x,t) \in [0,1] \times [-1,0],\\
&u(0,t) = e^{-t} + e^{-\left(t+\frac{1}{\sqrt{\varepsilon}}\right)}, \;\; u(1,t) = e^{-\left(t+\frac{1}{\sqrt{\varepsilon}}\right)} + e^{-t}, \; t \in [0,2].
\end{aligned}$$

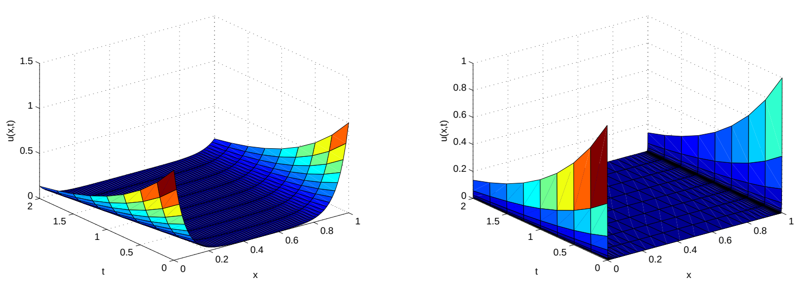
The exact solution of problem (1) is $u(x,t) = e^{-\left(t+\frac{x}{\sqrt{\varepsilon}}\right)} + e^{-\left(t+\frac{(1-x)}{\sqrt{\varepsilon}}\right)}$.

Table 1: Computed $E^{N,\Delta t}$ and $R^{N,\Delta t}$ for Example 1

$\varepsilon \downarrow$	$N \rightarrow 64$ $\Delta t \rightarrow 0.25$	128 0.25/2	256 $0.25/2^2$	512 $0.25/2^3$	1024 $0.25/2^4$
2^0	7.4762e-04	1.5843e-5	3.7666e-5	9.3469e-06	2.3224e-06
	2.2385	2.0725	2.0107	2.0027	
2^{-2}	1.0284e-03	2.4731e-04	6.1734e-5	1.5411e-5	3.8514e-6
	2.0560	2.0022	2.0021	2.0005	
2^{-4}	7.8737e-4	1.9537e-4	4.8763e-5	1.2199e-5	3.0495e-6
	2.0108	2.0024	1.9990	2.0001	
2^{-6}	5.6837e-4	1.4103e-4	3.5236e-5	8.8022e-6	2.2004e-6
	2.0108	2.0009	2.0011	2.0001	
2^{-8}	5.6713e-4	1.4042e-4	3.5088e-5	8.7639e-6	2.1905e-6
	2.0139	2.0007	2.0013	2.0003	
2^{-10}	5.8228e-04	1.4152e-04	3.5095e-05	8.7677e-06	2.1907e-06
	2.0407	2.0117	2.0010	2.0008	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	5.8228e-04	1.4152e-04	3.5095e-05	8.7677e-06	2.1907e-06
	2.0407	2.0117	2.0010	2.0008	
$\mathbf{E^{N,\,t}}$	**5.8228e-04**	**1.4152e-04**	**3.5095e-05**	**8.7677e-06**	**2.1907e-06**
$\mathbf{R^{N,\,t}}$	**2.0407**	**2.0117**	**2.0010**	**2.0008**	

Table 2: Computed $E^{N,\Delta t}$ and $R^{N,\Delta t}$ for Example 1, to show the order of convergence of space variable

$\varepsilon \downarrow$	$N \to 32$ $\Delta t \to 0.25$	64 $0.25/4$	128 $0.25/4^2$	256 $0.25/4^3$	512 $0.25/4^4$
2^{0}	7.4762e-04	3.7666e-05	2.3324e-06	1.4570e-07	9.1115e-09
	4.3110	4.0134	4.0007	3.9992	
2^{-2}	1.0284e-03	6.1735e05	3.8514e-06	2.4071e-07	1.5047e-08
	4.0582	4.0026	4.0000	3.9998	
2^{-4}	7.8752e-04	4.8772e-05	3.0501e-06	1.9063e-07	1.1915e-08
	4.0132	3.9991	4.0000	3.9999	
2^{-6}	5.7001e-04	3.5282e-05	2.2070e-06	1.3793e-07	8.6217e-09
	4.0140	3.9988	4.0001	3.9999	
2^{-8}	5.9289e-04	3.6752e-05	2.2962e-06	1.4374e-07	8.9838e-09
	4.0119	4.0005	3.9977	4.0000	
2^{-10}	1.0075e-03	6.2749e-05	3.9321e-06	2.4595e-07	1.7804e-08
	4.0050	3.9962	3.9989	3.7881	
2^{-12}	1.1750e-03	7.2942e-05	4.5593e-06	2.8753e-07	1.7959e-08
	4.0098	3.9999	3.9870	4.0009	
2^{-14}	1.1750e-03	7.2942e-05	4.5593e-06	2.8753e-07	1.7959e-08
	4.0098	3.9999	3.9870	4.0009	
2^{-16}	1.1750e-03	7.2942e-05	4.5593e-06	2.8753e-07	1.7959e-08
	4.0098	3.9999	3.9870	4.0009	
$\mathbf{E^{N,\,t}}$	**1.1750e-03**	**7.2942e-05**	**4.5593e-06**	**2.8753e-07**	**1.7959e-08**
$\mathbf{R^{N,\,t}}$	**4.0098**	**3.9962**	**3.9870**	**3.7881**	



(a) $\varepsilon = 2^{-8}, N = 64, \Delta t = 0.25$ (b) $\varepsilon = 2^{-16}, N = 64, \Delta t = 0.25$

Figure 1: Surface plot of numerical solution of Example 1

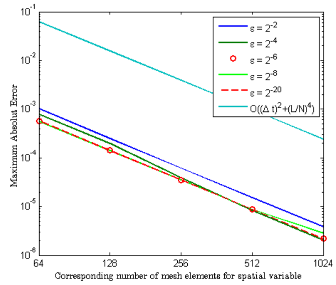


Figure 2: Log-log plot of Example(1)

Example 2. Consider the following reaction-diffusion problem [1, 12, 13]:

$$u_t - \varepsilon u_{xx} + \frac{1+x^2}{2} u(x,t) = -u(x,t-1) + t^3, \quad (x,t) \in (0,1) \times (0,2],$$

with

$$u(x,t) = 0, (x,t) \in [0,1] \times [-1,0], \quad u(0,t) = 0, u(1,t) = 0, \quad t \in (0,2].$$

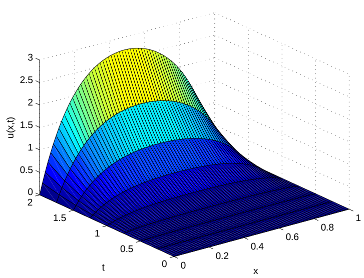
Table 3: Computed $E^{N,\Delta t}$ and $R^{N,\Delta t}$ for Example 2

Method	$\varepsilon \downarrow$	$N \rightarrow$**128** **t** $\rightarrow$ **0.25**	**256** **0.25/2**	**512** **0.25/2^2**	**1024** **0.25/2^3**	**2048** **0.25/2^4**
present	10^{0}	2.7387e-04	6.7253e-05	1.6808e-05	4.2019e-06	1.0505e-06
		2.0258	20004	2.0001	2.0000	
	10^{-2}	1.6900e-02	4.2067e-03	1.0448e-03	2.6493e-04	9.2519e-05
		2.0062	2.0095	1.9795	1.5178	
	10^{-4}	1.9293e-02	4.8267e-03	1.2067e-03	3.0157e-04	9.3848e-05
		1.9989	2.0000	2.0005	1.6841	
	10^{-6}	1.9357e-02	4.8428e-03	1.2067e-03	3.0275e-04	9.3848e-05
		1.9989	2.0048	1.9999	1.6895	
	10^{-8}	1.9358e-02	4.8432e-03	1.2111e-03	3.0280e-04	9.3833e-05
		1.9989	1.9998	1.9999	1.6902	
	10^{-10}	1.9358e-02	4.8432e-03	1.2111e-03	3.0280e-04	9.3833e-05
		1.9989	1.9998	1.9999	1.6902	
	10^{-12}	1.9358e-02	4.8432e-03	1.2111e-03	3.0280e-04	9.3833e-05
		1.9989	1.9998	1.9999	1.6902	
	$\mathbf{E^{N,\,t}}$	**1.9358e-02**	**4.8432e-03**	**1.2111e-03**	**3.0280e-04**	**9.3833e-05**
	$\mathbf{R^{N,\,t}}$	**1.9989**	**1.9998**	**1.9999**	**1.6902**	
[12]	$\mathbf{E^{N,\,t}}$	2.39e-01	1.28e-01	6.60e-02	3.35e-02	1.71e-02
[13]	$\mathbf{E^{N,\,t}}$	2.78e-01	1.42e-03	7.13e-02	3.58e-02	1.79e-02

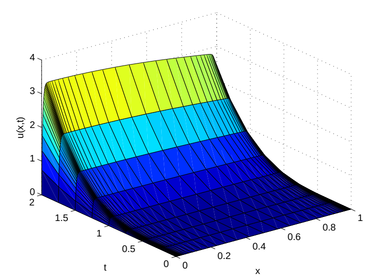
Example 3. Consider the following reaction-diffusion problem [13, 12]:

Table 4: Computed $E^{N,\Delta t}$ and $R^{N,\Delta t}$ for Example 2, to show the order of convergence of space variable

$\varepsilon \downarrow$	$N \rightarrow$ **32** **t** $\rightarrow$ **0.25**	**64** **0.25/4**	**128** $\mathbf{0.25/4^2}$	**256** $\mathbf{0.25/4^3}$	**512** $\mathbf{0.25/4^4}$
10^0	3.3960e-04	2.1010e-05	1.3131e-06	8.2068e-08	5.1299e-09
	4.0147	4.0001	4.0000	3.9998	
10^{-2}	2.1214e-02	1.3267e-03	8.2918e-05	5.1824e-06	3.2390e-07
	3.9991	4.0000	4.0000	4.0000	
10^{-4}	2.4127e-02	1.5092e-03	9.4330e-05	5.8957e-06	3.6848e-07
	3.9988	3.9999	4.0000	4.0000	
10^{-6}	2.4192e-02	1.5136e-03	9.4608e-05	5.9131e-06	3.6957e-07
	3.9984	3.9999	4.0000	4.0000	
10^{-8}	2.4192e-02	1.5137e-03	9.4615e-05	5.9136e-06	3.6960e-07
	3.9984	3.9999	4.0000	4.0000	
10^{-10}	2.4189e-02	1.5137e-03	9.4615e-05	5.9135e-06	3.6970e-07
	3.9982	3.9999	4.0000	3.9996	
$\mathbf{E^{N,\,t}}$	**2.4192e-02**	**1.5137e-03**	**9.4615e-05**	**5.9136e-06**	**3.6970e-07**
$\mathbf{R^{N,\,t}}$	**3.9982**	**3.9999**	**4.0000**	**3.9996**	



(a) $\varepsilon = 10^{-2}, N = 64, \Delta t = 0.25$



(b) $\varepsilon = 10^{-4}, N = 64, \Delta t = 0.25$

Figure 3: Surface plot for numerical solution for Example 2

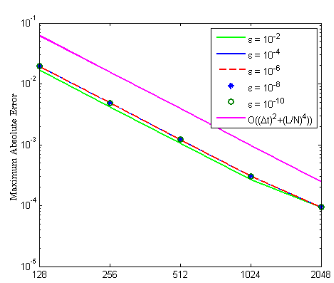


Figure 4: Log-log plot of Example 2

$$u_t - \varepsilon u_{xx} = -2e^{-1}u(x, t-1), \quad (x,t) \in (0,1) \times (0,2],$$

with

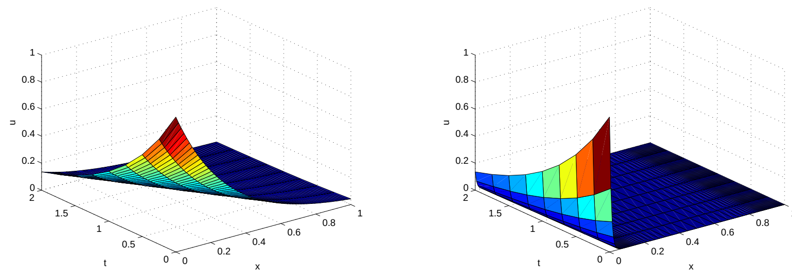
$$\begin{aligned} &u(0,t) = e^{-t}, \quad u(1,t) = e^{-(t+1/\sqrt{\varepsilon})}, \quad t \in (0,2], \\ &u(x,t) = e^{-(t+1/\sqrt{\varepsilon})}, \quad (x,t) \in [0,1] \times [-1,0]. \end{aligned} \tag{42}$$

$u(x,t) = e^{-(t+x/\sqrt{\varepsilon})}$ is the exact solution of the problem.

Table 5: Computed $E^{N,\Delta t}$ and $R^{N,\Delta t}$ for Example 3

Method	$\varepsilon \downarrow$	$N \to$128 $\Delta t \to 0.25$	256 $0.25/2$	512 $0.25/2^2$	1024 $0.25/2^3$	2048 $0.25/2^4$
present	10^0	3.9168e-05	8.2891e-05	1.9790e-05	4.9090e-06	1.2254e-06
		2.2404	2.0665	2.0112	2.0022	
	10^{-2}	6.7565e-04	1.6837e-04	4.2083e-05	1.0517e-05	2.6290e-06
		2.0047	2.0003	2.0005	2.0001	
	10^{-4}	6.8750e-04	1.6903e-04	4.2110e-05	1.0518e-05	2.6291e-06
		2.0241	2.0051	2.0013	2.0002	
	10^{-6}	6.8750e-04	1.6903e-04	4.2110e-05	1.0518e-05	2.6291e-06
		2.0241	2.0051	2.0013	2.0002	
	10^{-8}	6.8750e-04	1.6903e-04	4.2110e-05	1.0518e-05	2.6291e-06
		2.0241	2.0051	2.0013	2.0002	
	10^{-10}	6.8750e-04	1.6903e-04	4.2110e-05	1.0518e-05	2.6291e-06
		2.0241	2.0051	2.0013	2.0002	
	10^{-12}	6.8750e-04	1.6903e-04	4.2110e-05	1.0518e-05	2.6291e-06
		2.0241	2.0051	2.0013	2.0002	
	$\mathbf{E^{N,\,t}}$	**6.8750e-04**	**1.6903e-04**	**4.2110e-05**	**1.0518e-05**	**2.6291e-06**
	$\mathbf{R^{N,\,t}}$	**2.0241**	**2.0051**	**2.0013**	**2.0002**	
[12]	$\mathbf{E^{N,\,t}}$	9.29e-03	4.43e-03	2.17e-03	1.07e-03	5.32e-04
[13]	$\mathbf{E^{N,\,t}}$	1.49e-02	7.75e-03	3.95e-03	2.00e-03	1.00e-03

The fitted mesh technique applied in these computations, described in subsection (3.2.1), is uniform in the inner region, whereas in the outer region the mesh gradually condenses more and more from the center to both left and right ends of the interval. In Example 3 because of the boundary values there is no boundary layer on Γ_r, even though we use the same mesh technique for this computation, it is possible to reshuffle the mesh appropriate to each specific example, for further description of this special characteristic, see [1]. In Tables 1 and 2, 3 and 5, for examples 1,2 and 3 respectively, the maximum absolute error and rate of convergence for different values of perturbation parameter ε and mesh numbers are presented, in all the case we observed that the computed $\varepsilon-$ uniform errors $E^{N,\Delta t}$ decreases monotonically as the number of mesh points increases, this ensures that the proposed scheme converges $\varepsilon-$ uniformly. The orders of convergence $R^{N,\Delta t}$ is also independent of the perturbation parameter ε. To visualize the numerical order



(a) $\varepsilon = 10^{-2}, N = 64, \Delta t = 0.25$ (b) $\varepsilon = 10^{-4}, N = 64, \Delta t = 0.25$

Figure 5: Solution graph for Example 3

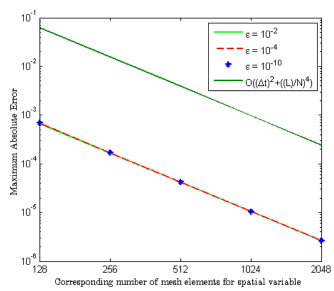


Figure 6: Log-log plot of Example 3

of convergence, the maximum point-wise errors are plotted in log-log scale in figures 2, 4 and 6. The effect of the perturbation parameter on the boundary layer behavior of the solution is also shown in figures 1, 3 and 5. The numerical result displayed in tables 2 and 4 reflects the actual theoretical order of convergence of the spatial variability of the proposed scheme.

6 Conclusion

Singularly perturbed time delay parabolic partial differential equation of reaction-diffusion type of the form (1) is considered. To get an approximate solution to this problem, we used a hybrid scheme which is a combination of a fourth-order compact difference scheme and a central difference scheme based on a special type of Shishkin mesh in space variable, and the Crank-Nicolson method on uniform mesh in the time variable. The proposed scheme is $\varepsilon-$ uniform convergent of order $O((\Delta t)^2 + (L/N)^4)$. Further, numerical experiments are carried out to demonstrate the performance of the proposed scheme. The numerical results clearly show the high accuracy and order of convergence of the proposed scheme as compared to results available in the literature.

References

[1] Ansari, A., Bakr, S. and Shishkin, G. *A parameter-robust finite difference method for singularly perturbed delay parabolic partial differential equations*, J. Comput. Appl. Math. 205 (2007), 552–566.

[2] Babu, G. and Bansal, K. *A high order robust numerical scheme for singularly perturbed delay parabolic convection diffusion problems*, J. Appl. Math. Comput. 68 (2021), 363–389.

[3] Clavero, C. and Gracia, J.L. *High order methods for elliptic and time dependent reaction–diffusion singularly perturbed problems*, Appl. Math. Comput. 168 (2005), 1109–1127.

[4] Clavero, C. and Gracia, J.L. *On the uniform convergence of a finite difference scheme for time dependent singularly perturbed reaction-diffusion problems*, Appl. Math. Comput. 216 (2010), 1478–1488.

[5] Clavero, C. and Gracia, J.L. *A high order hodie finite difference scheme for 1d parabolic singularly perturbed reaction–diffusion problems*, Appl. Math. Comput. 218 (2012), 5067–5080.

[6] Clavero, C., Gracia, J. and Jorge, J. *High-order numerical methods for one-dimensional parabolic singularly perturbed problems with regular layers*, Numer. Methods Partial Differential Equations 21 (2005), 149–169.

[7] Gelu, F.W. and Duressa, G.F. *A uniformly convergent collocation method for singularly perturbed delay parabolic reaction-diffusion problem*, Abstr. Appl. Anal. 2021 (2021).

[8] Govindarao, L. and Mohapatra, J. *A second order numerical method for singularly perturbed delay parabolic partial differential equation*, Eng. Comput.

[9] Govindarao, L., Mohapatra, J. and Das, A. *A fourth-order numerical scheme for singularly perturbed delay parabolic problem arising in population dynamics*, J. Appl. Math. Comput. 63 (2020), 171–195.

[10] Govindarao, L. and Mohapatra, J. *Numerical analysis and simulation of delay parabolic partial differential equation involving a small parameter*, Eng Comput 37 (2019), 289–312.

[11] Kadalbajoo, M.K. and Awasthi, A. *Crank–nicolson finite difference method based on a midpoint upwind scheme on a non-uniform mesh*

for time-dependent singularly perturbed convection–diffusion equations, Int. J. Comput. Math. 85 (2008), 771–790.

[12] Kumar, P.M.M. and Kanth, A.R. *Computational study for a class of time-dependent singularly perturbed parabolic partial differential equation through tension spline*, Comput. Appl. Math. 39 (2020), 1–19.

[13] Kumar, S. and Kumar, M. *High order parameter-uniform discretization for singularly perturbed parabolic partial differential equations with time delay*, Comput. Math. Appl. 68 (2014), 1355–1367.

[14] Kumar, M. and Rao, S.C.S. *High order parameter-robust numerical methodfor time dependent singularly perturbed reaction–diffusion problems*, Computing 90 (2010), 15–38.

[15] Longtin, A. and Milton, J.G. *Complex oscillations in the human pupil light reflex with mixed and delayed feedback*, Math. Biosci. 90 (1988), 183–199.

[16] Mackey, M.C. and Glass, L. *Oscillation and chaos in physiological control systems*, Science 197 (1977), 287–289.

[17] Mallet-Paret, J. and Nussbaum, R.D. *A differential-delay equation arising in optics and physiology*, SIAM J. Math. Anal. 20 (1989), 249–292.

[18] Miller, J. ORiordan, E. and Shishkin, G. *Fitted numerical methods for singular perturbation problems*, rev. edn, 2012.

[19] Mohapatra, J. and Govindarao, L. *A fourth order optimal numerical approximation and its convergence for singularly perturbed time delayed parabolic problems*, Iranian Journal of Numerical Analysis and Optimization (2021).

[20] Negero, N.T. and Duressa, G.F. *Uniform convergent solution of singularly perturbed parabolic differential equations with general temporal-lag*, Iran. J. Sci. Technol. Trans. A Sci. 46 (2022), 507–524.

[21] Priyadarshana, S., Mohapatra, J. and Govindrao, L. *An efficient uniformly convergent numerical scheme for singularly perturbed semilinear parabolic problems with large delay in time*, J. Appl. Math. Comput. 68 (2021), 2617–2639.

[22] Priyadarshana, S., Mohapatra, J. and Pattanaik, S.R.*Parameter uniform optimal order numerical approximations for time-delayed parabolic convection diffusion problems involving two small parameters*, J. Comput. Appl. Math. 41 (2022).

[23] Sahu, S.R. and Mohapatra, J. *Numerical investigation of time delay parabolic differential equation involving two small parameters*, Eng. Comput. 38 (2021), 2882–2899.

[24] Salama, A. and Al-Amery, D. *Asymptotic-numerical method for singularly perturbed differential difference equations of mixed-type*, J. Appl. Math. Inform. 33 (2015), 485–502.

[25] Salama, A. and Al-Amery, D. *A higher order uniformly convergent method for singularly perturbed delay parabolic partial differential equations*, Int. J. Comput. Math. 94 (2017), 2520–2546.

[26] Vulanović, R. *A higher-order scheme for quasilinear boundary value problems with two small parameters*, Computing 67 (2001), 287–303.

[27] Wazewska-Czyzewska, M. and Lasota, A. *Mathematical models of the red cell system*, Matematyta Stosowana 6 (1976), 25–40.

[28] Yadav, S. and Rai, P. *A higher order scheme for singularly perturbed delay parabolic turning point problem*, Eng. Comput. 38 (2020), 819–851.

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