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Using a LDG method for solving an inverse source problem of the time-fractional diffusion equation

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Abstract

In this paper, we apply a local discontinuous Galerkin (LDG) method to solve some fractional inverse problems. In fact, we determine a time-dependent source term in an inverse problem of the time-fractional diffusion equation. The method is based on a finite difference scheme in time and a LDG method in space. A numerical stability theorem as well as an error estimate is provided. Finally, some numerical examples are tested to confirm theoretical results and to illustrate effectiveness of the method. It must be pointed out that proposed method generates stable and accurate numerical approximations without using any regularization methods which are necessary for other numerical methods for solving such ill-posed inverse problems.

Keywords: Local discontinuous Galerkin method; Inverse source problem; Time-fractional diffusion equation.

1 Introduction

Recently, studying problems involving fractional order partial differential equations (PDEs) has attracted a lot of interests of scientists and engineers, see e.g. [1, 5–12, 15, 18–20, 22, 24–30, 33–35]. One of the fractional

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PDEs which has been very popular is the fractional diffusion equation (FDE), see e.g. [1, 6–12, 15, 18–20, 22, 24–30, 33–35]. Nowadays instead of the classical diffusion equations, FDEs have attracted wide attentions since a FDE is a generalization of a diffusion equation which can be used to describe an anomalous diffusion phenomenon such as super-diffusion or sub-diffusion. By replacing the standard time derivative with a time fractional derivative in a diffusion equation, a time-fractional diffusion equation is obtained. FDEs can be applied in modelling of some problems in porous flows, rheology and mechanical systems, models of a variety of biological processes, control and robotics, transport in fusion plasmas, and many other areas of applications. Analytical or numerical studying of the direct problems corresponding to the time-fractional diffusion equations have been carried out extensively in the recent years, see e.g. [6, 7, 9, 11, 12, 15, 30] as well as references cited therein [27, 28] for some subjects related to the obtaining some uniqueness and existence results, establishing maximum principle, finding analytical solutions, applying some numerical methods such as finite element methods or finite difference methods.

If in an initial or initial-boundary value problem some parts of data such as boundary data, or initial data, or source term or even some coefficients of the main equation may not be given, we have encountered an inverse problem. Inverse problems are appeared in many practical situations, and for solving them we need to some additional measured data, see e.g. [1,8,10,16–25,27–29, 32–35] and references cited therein. In some inverse problems, we need to find space-dependent source term [27, 32] or time-dependent source term [28] or even space- and time-dependent source term [23]. In this paper, we focus our attention on finding the time-dependent source term in a fractional inverse problem which is one of the interesting and novel inverse problems. One of the pioneering scientists in this subject is Murio [18–20]. In the following we mention some of works which have been published after that. An inverse problem for determining the order of fractional derivative and the diffusion coefficient in a FDE has been considered in [1] where a uniqueness result has been also obtained. A backward problem for the time-fractional diffusion equation has been solved by a quasi-reversibility regularization method in [13]. In [34, 35] some Cauchy problems based on the time-fractional diffusion equation have been investigated on a bounded domain and on a strip domain, respectively. Qian [22] has investigated a modified kernel method for solving an inverse fractional diffusion equation. An inverse source problem for a FDE has been solved in [33]. Rundell et al. [10] have been investigated some nonlinear fractional inverse problems. Recently, some interesting works have been carried out by Wei et al., see e.g. [27–29]. In [10] known theoretical results and computational techniques for FDEs have been provided.

In [28] the following initial-boundary value problem for the time-fractional diffusion equation has been considered

$$\begin{cases} D_t^\alpha u = u_{xx} + f(x)p(t), \; 0 < x < 1, \; 0 < t < T, \\ u(0,t) = k_0(t), \qquad\qquad\qquad\qquad 0 \leqslant t \leqslant T, \\ u(1,t) = k_1(t), \qquad\qquad\qquad\qquad 0 \leqslant t \leqslant T, \\ u(x,0) = \phi(x), \qquad\qquad 0 \leqslant x \leqslant 1, \end{cases} \tag{1}$$

where D_t^α is the Caputo fractional derivative of order α, i.e.,

$$D_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \qquad 0 < \alpha < 1, \tag{2}$$

in which Γ is the Gamma function. For $\alpha = 1$, we have $D_t^\alpha u = u_t$. We assume that k_0, k_1 and ϕ are given functions. Problem (1) will be a direct (forward) problem whenever f and p are known functions. Based on the availability of f or p, there are some inverse problems. If p is known and f is unknown we need to an extra condition such as

$$u(x,T) = g(x), \qquad 0 \leq x \leq 1.$$

We have investigated numerical solution of this inverse problem in [32]. In another inverse problem, f is known but p is unknown and an over-determination condition such as

$$u(x^*,t) = g(t), \qquad 0 \leq t \leq T, \tag{3}$$

will be needed where $x^* \in (0,1)$ is an interior measurement location. It must be pointed out that although these two inverse problems are very similar they are completely different. The inverse source problem which we consider here is to determine (p,u) based on problem (1) and condition (3). Using the main equation in (1) and Eq. (3), we have

$$p(t) = \frac{1}{f(x^*)} \left(D_t^\alpha g(t) - \frac{\partial^2 u(x^*,t)}{\partial x^2} \right). \tag{4}$$

We need (4) just for theoretical purposes.

The above mentioned inverse source problem is an ill-posed problem [28]. Sakamoto et al. [24] have given a stability estimate for the inverse source problem and mentioned that the uniqueness of p is guaranteed if $f \neq 0$, but they did not present any numerical method. This inverse source problem has been solved numerically by Wei et al. [28] using a regularized method based on the boundary element discretization for recovering a stable approximation to p. In [28], a numerical method has been applied to various examples in particular some examples with none-smooth data which lead to the none-smooth solutions. Therefore, we decided to apply the discontinuous Galerkin method to the above mentioned inverse source problem. To the best of our knowledge, for the inverse source problem with or without fractional operators, the development of the DG methods remains limited and there are a few results, see e.g. [32] where a space-dependent source term is determined in a time-fractional diffusion equation by using a local discontinuous Galerkin method. Of course, some DG methods have been applied successfully for the forward fractional diffusion equation. For example, Hesthaven et al. [6, 30] have been solved some space-fractional diffusion equations using a local discontinuous Galerkin method in a semi-discrete regime and Yang Liu et al. [14] have developed a LDG methos combined with WSGD approximation for a time fractional subdiffusion equation.

In the present paper, we aim to extend application of the discontinuous Galerkin method to some inverse source problems and for this purpose, we present a fully-discrete local discontinuous Galerkin method for solving the inverse source problem of the time-fractional diffusion equation. In the proposed fully-discrete method, we apply a LDG method for the space variable and the time-fractional derivative is discretized by using a backward difference scheme. The rest of the paper is organized as follows. In Section 2, some preliminaries are prepared. Construction of the proposed method for the inverse source problem (1)-(3) as well as stability and convergence theorems are dealt with in Section 3. Section 4 is devoted to some numerical experiments to illustrate the accuracy and capability of the method. Finally, the paper is concluded with a brief conclusion.

2 Preliminaries

In this section, some notations are defined and some auxiliary results are prepared. At first, we decompose interval $[0, 1]$ to some cells (subintervals) as follows

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 1,$$

and set $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, with the cell lengths $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, for $j = 1, \ldots, N$ and $h = \max_{1\le j\le N} \Delta x_j$. We denote by $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ the values of u at $x_{j+\frac{1}{2}}$, from the right cell I_{j+1} and from the left cell I_j. $[u]_{j+\frac{1}{2}}$ is used to denote $u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}$, that is the jump of u at the cell interfaces. For any integer k, we define the piecewise-polynomial space V_h^k as the space of polynomials of degree up to k in each cell I_j, thus

$$V_h^k = \left\{ v \in L^2[0, 1] : \ v|_{I_j} \in \mathbb{P}^k(I_j), \ j = 1, \ldots, N \right\}.$$

In order to investigate the convergence of the method, we need to define projections $\mathbf{P}$ and $\mathbf{P}^{\pm}$ as follows

$$\int_{I_j} (\mathbf{P}\omega(x) - \omega(x))v(x)dx = 0, \quad \forall v \in \mathbb{P}^k(I_j), \quad \forall j,$$

$$\int_{I_j} (\mathbf{P}^{\pm}\omega(x) - \omega(x))v(x)dx = 0, \quad \forall v \in \mathbb{P}^{k-1}(I_j), \quad \forall j,$$

$$\mathbf{P}^{\pm}\omega(x^{+}_{j\mp\frac{1}{2}}) = \omega(x_{j\mp\frac{1}{2}}).$$

It is well-known that these projections satisfy the following inequality [2,31]

$$\| \omega^e \| + h \| \omega^e \|_\infty + h^{\frac{1}{2}} \| \omega^e \|_{\tau_h} \leq Ch^{k+1},$$

where $\omega^e = \mathbf{P}\omega - \omega$ or $\omega^e = \mathbf{P}^{\pm}\omega - \omega$, the positive constant C, solely depending on ω, is independent of h, and τ_h denotes the set of boundary points of all cells I_j. In the following, we use C to denote a positive constant which is independent of h and may have a different value in each occurrence. v_x denotes the piecewise derivative with respect to x, and the norm $\| \cdot \|$ denotes the usual norm of the space $L^2[0,1]$.

3 Construction of the method

In this section, we construct a numerical scheme for solving problem (1)-(3). Let M be a positive integer, $\Delta t = T/M$ be the time step size, and $t_n = n\Delta t,\ n = 0, 1, \ldots, M$ denote the time mesh points. An approximation to time-fractional derivative (2) can be obtained by a simple quadrature formula given as [9,11,12],

$$D_t^\alpha u(x,t_n) = \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{n-1} b_i \frac{u(x,t_{n-i}) - u(x,t_{n-i-1})}{\Delta t} + O((\Delta t)^{2-\alpha}), \quad (5)$$

where $b_i = (i+1)^{1-\alpha} - i^{1-\alpha}$.
Following the LDG regime [6,30], we must rewrite (1) as the following first-order system of equations

$$q = u_x, \qquad D_t^\alpha u(x,t) - q_x = f(x)p(t). \quad (6)$$

Let $u_h^n, q_h^n \in V_h^k$ be the approximation of $u(\cdot,t_n), q(\cdot,t_n)$ respectively, and $p^n = p(t_n), g^n = g(t_n)$. Using (5), we establish the necessary weak forms corresponding to (6) and then define a fully-discrete local discontinuous Galerkin scheme as follows: find $u_h^n, q_h^n \in V_h^k$, such that for all test functions $v, w \in V_h^k$,

$$\begin{cases} \int_\Omega u_h^n v dx + \beta \left(\int_\Omega q_h^n v_x dx - \sum_{j=1}^{N} ((\hat{q}_h^n v^-)_{j+\frac{1}{2}} - (\hat{q}_h^n v^+)_{j-\frac{1}{2}}) \right) = \\ \qquad \beta p^n \int_\Omega f(x) v dx + \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega u_h^{n-i} v + b_{n-1} \int_\Omega u_h^0 v dx, \\ \int_\Omega q_h^n w dx + \int_\Omega u_h^n w_x dx - \sum_{j=1}^{N} ((\hat{u}_h^n w^-)_{j+\frac{1}{2}} - (\hat{u}_h^n w^+)_{j-\frac{1}{2}}) = 0, \\ u_h^n(x^*) = g^n, \end{cases} \tag{7}$$

where $\Omega = [0,1]$ and $\beta = (\Delta t)^\alpha \Gamma(2-\alpha)$ and without lose of generality we assume that x^* is a grid point. The "hat" terms in (7) in the cell boundary terms from integration by parts are the so-called "numerical fluxes", which are single valued functions defined on the interfaces and should be selected carefully for ensuring the numerical stability of the scheme. The choice for the numerical fluxes is not unique and among several choices, we can take $\hat{u}_h^n = (u_h^n)^-$ and $\hat{q}_h^n = (q_h^n)^+$ or $\hat{u}_h^n = (u_h^n)^+$ and $\hat{q}_h^n = (q_h^n)^-$. In fact, it is important to take $\hat{u}_h^n$ and $\hat{q}_h^n$ from opposite sides [3,4].

Before investigating theoretical aspects of the method, we are going to explain details of the method somewhat more. We set

$$u_h^n(x) = u_h(x, n\Delta t) = \sum_{i=1}^{N_p} \delta_i^n \Phi_i(x), \qquad q_h^n(x) = q_h(x, n\Delta t) = \sum_{i=1}^{N_p} \gamma_i^n \Phi_i(x),$$

where N_p is the total number of basis functions and

$$F = \left(\int_\Omega f_h(x) \Phi_1(x) dx \ \cdots \ \int_\Omega f_h(x) \Phi_{N_p}(x) dx \right)^T,$$

$$\Phi = \left(\Phi_1(x^*) \ \cdots \ \Phi_{N_p}(x^*) \right), \quad Z = (0 \ \cdots \ 0).$$

Setting $\delta^n = (\delta_1^n \ \cdots \ \delta_{N_p}^n)^T$ and $\gamma^n = (\gamma_1^n \ \cdots \ \gamma_{N_p}^n)^T$, scheme (7) leads to the following iteration scheme

$$\begin{cases} K_{11}\delta^n + K_{12}\gamma^n = \beta p^n F + \sum_{i=1}^{n-1} (b_{i-1} - b_i) K_{22} \delta^{n-i}, \\ K_{21}\delta^n + K_{22}\gamma^n = 0, \\ \Phi \delta^n = g^n, \end{cases}$$

where $n = 1, \ldots, M$ and

$$(K_{11})_{lr} = \int_\Omega \Phi_l(x) \Phi_r(x) dx, \qquad K_{22} = K_{11},$$

$$(K_{12})_{lr} = \beta \int_\Omega \Phi_l(x)(\Phi_r(x))_x dx - \beta \sum_{j=1}^{N} \Big(\Phi_l(x^-_{j+\frac{1}{2}})\Phi_r(x^-_{j+\frac{1}{2}}) - \Phi_l(x^-_{j-\frac{1}{2}})\Phi_r(x^+_{j-\frac{1}{2}}) \Big),$$

$$(K_{21})_{lr} = \int_\Omega \Phi_l(x)(\Phi_r(x))_x dx - \sum_{j=1}^{N} \Big(\Phi_l(x^+_{j+\frac{1}{2}})\Phi_r(x^-_{j+\frac{1}{2}}) - \Phi_l(x^+_{j-\frac{1}{2}})\Phi_r(x^+_{j-\frac{1}{2}}) \Big).$$

For solving the direct problem, we have

$$M \begin{pmatrix} \delta^n \\ \gamma^n \end{pmatrix} = \begin{pmatrix} \beta p^n F + \sum_{i=1}^{n-1}(b_{i-1} - b_i)K_{11}\delta^{n-i} \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{11} \end{pmatrix}, \tag{8}$$

where matrix K_{11} is nonsingular and block diagonal which every block is a $k \times k$ (k is degree of basis polynomials) matrix. Using the Schur complement, linear system (8) has a unique solution iff $\det(K_{11} - K_{12}K_{11}^{-1}K_{21}) \neq 0$. For solving the inverse problem, we have

$$\begin{pmatrix} K_{11} & K_{12} & -\beta F \\ K_{21} & K_{11} & Z^T \\ \Phi & Z & 0 \end{pmatrix} \begin{pmatrix} \delta^n \\ \gamma^n \\ p^n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n-1}(b_{i-1} - b_i)K_{11}\delta^{n-i} \\ 0 \\ g^n \end{pmatrix}, \tag{9}$$

where we put g^n_δ instead of g^n since data are usually obtained by measurement tools and have some noises. By solving the nonsymmetric linear system (9) using the BiCGStab method, we can obtain u^n_h and p^n.
Just for convenience and without lose of generality, we deal with the case $g = 0$ in the theoretical analysis. In order to examine the stability property of the scheme (7), we express following result.

Theorem 3. *Assume that the second derivative of u at $x = x^*$ is bounded and f is a continuous function on $[0, 1]$. For periodic or compactly supported boundary conditions, fully-discrete LDG scheme (7) is unconditionally stable, and the numerical solution u^n_h satisfies*

$$\|u^n_h\| \leq \|u^0_h\| + \kappa, \quad n = 1, \ldots, M,$$

where κ is a constant depending on β, f and u_{xx} at $x = x^*$.

Proof. We can rewrite scheme (7) as follows

$$\begin{aligned}
&\int_\Omega u_h^n v dx + \beta\left(\int_\Omega q_h^n v_x dx - \sum_{j=1}^{N}((\hat{q}_h^n v^-)_{j+\frac{1}{2}} - (\hat{q}_h^n v^+)_{j-\frac{1}{2}})\right)\\
&-\sum_{j=1}^{N}((\hat{u}_h^n w^-)_{j+\frac{1}{2}} - (\hat{u}_h^n w^+)_{j-\frac{1}{2}}) + \int_\Omega q_h^n w dx + \int_\Omega u_h^n w_x dx\\
&= \beta p^n \int_\Omega f(x) v dx + \sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_\Omega u_h^{n-i} v dx + b_{n-1}\int_\Omega u_h^0 v dx,
\end{aligned}$$

With taking test functions $v = u_h^n$, $w = \beta q_h^n$, for periodic or compactly supported boundary conditions, we can obtain

$$\begin{aligned}
&\beta\left(\int_\Omega q_h^n v_x dx - \sum_{j=1}^{N}((\hat{q}_h^n v^-)_{j+\frac{1}{2}} - (\hat{q}_h^n v^+)_{j-\frac{1}{2}})\right) + \int_\Omega u_h^n w_x dx\\
&-\sum_{j=1}^{N}((\hat{u}_h^n w^-)_{j+\frac{1}{2}} - (\hat{u}_h^n w^+)_{j-\frac{1}{2}}) = \beta\int_\Omega q_h^n (u_h^n)_x dx\\
&-\beta\sum_{j=1}^{N}\left(((q_h^n)^+(u_h^n)^-)_{j+\frac{1}{2}} - ((q_h^n)^+(u_h^n)^+)_{j-\frac{1}{2}}\right) + \beta\int_\Omega u_h^n (q_h^n)_x dx\\
&-\beta\sum_{j=1}^{N}\left(((u_h^n)^-(q_h^n)^-)_{j+\frac{1}{2}} - ((u_h^n)^-(q_h^n)^+)_{j-\frac{1}{2}}\right) = 0,
\end{aligned}$$

then

$$\begin{aligned}
\int_\Omega u_h^n v dx + \int_\Omega q_h^n w dx = &\sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_\Omega u_h^{n-i} u_h^n dx + b_{n-1}\int_\Omega u_h^0 u_h^n dx\\
&+ \beta p^n \int_\Omega f(x) u_h^n dx.
\end{aligned}\tag{10}$$

For $n = 1$ and using Eq. (4), we can get

$$\begin{aligned}
\| u_h^1 \|^2 + \beta \| q_h^1 \|^2 &= \int_\Omega u_h^0 u_h^1 dx + \beta p^1 \int_\Omega f(x) u_h^1 dx\\
&= \int_\Omega u_h^0 u_h^1 dx - \beta \int_\Omega \frac{f}{f^*}\left(u_h^1\right)_{xx}^* u_h^1 dx\\
&= \int_\Omega \left(u_h^0 - \beta\frac{f}{f^*}\left(u_h^1\right)_{xx}^*\right) u_h^1 dx\\
&\le \frac{1}{2}\left(\| u_h^0 - \beta\frac{f}{f^*}\left(u_h^1\right)_{xx}^* \|^2 + \| u_h^1 \|^2\right)\\
&\le \frac{1}{2}\left(\left(\| u_h^0 \| + \beta \| \frac{f}{f^*}\left(u_h^1\right)_{xx}^* \|\right)^2 + \| u_h^1 \|^2\right),\\
&\le \frac{1}{2}\left(\left(\| u_h^0 \| + \kappa\right)^2 + \| u_h^1 \|^2\right),
\end{aligned}$$

therefore

$$\| u_h^1 \| \leq \| u_h^0 \| + \kappa.$$

Next, we suppose the following inequalities hold

$$\| u_h^m \| \leq \| u_h^0 \| + \kappa, \qquad m = 1, \dots, K.$$

For $n = K + 1$, in Eq. (10), and using

$$\sum_{i=1}^{l} (b_{i-1} - b_i) + b_l = 1,$$

we can obtain

$$\| u_h^{l+1} \| \leq \| u_h^0 \| + \kappa.$$

□

In order to examine the convergence of the scheme (7), we express following result.

Theorem 4. *Let $u(\cdot, t_n)$ be the exact solution of the problem (1)-(3), which is sufficiently smooth with bounded derivatives, and u_h^n be the numerical solution of the fully-discrete LDG scheme (7). There holds the following error estimate*

$$\|u(\cdot, t_n) - u_h^n\| \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}} + c(\Delta t)^{\alpha}),$$

where C is a constant depending on α, u, and T, and c is a constant depending on f and u_{xx} at $x = x^*$.

Proof. Obviously for all test functions $v, w \in V_h^k$, we have

$$\begin{aligned}
&\int_\Omega u(x,t_n) v dx + \int_\Omega u(x,t_n) w_x dx - \sum_{j=1}^{N} ((u(x,t_n)w^-)_{j+\frac{1}{2}} - (u(x,t_n)w^+)_{j-\frac{1}{2}}) \\
&\quad + \beta \left(\int_\Omega q(x,t_n) v_x dx - \sum_{j=1}^{N} ((q(x,t_n)v^-)_{j+\frac{1}{2}} - (q(x,t_n)v^+)_{j-\frac{1}{2}}) \right) \\
&\quad - \sum_{i=1}^{n-1} (b_{i-1} - b_i) \int_\Omega u(x, t_{n-i}) v dx - b_{n-1} \int_\Omega u(x,t_0) v dx \\
&\quad + \int_\Omega q(x,t_n) w dx + \beta \int_\Omega \gamma^n(x) v dx - \beta p(x,t_n) \int_\Omega f(x) v dx = 0.
\end{aligned} \tag{11}$$

Subtracting equation (7) from (11), we can obtain the error equation

$$
\begin{aligned}
&\int_\Omega e_u^n v dx + \beta\left(\int_\Omega e_q^n v_x dx - \sum_{j=1}^{N}(((e_q^n)^+ v^-)_{j+\frac{1}{2}} - ((e_q^n)^+ v^+)_{j-\frac{1}{2}})\right)\\
&+\int_\Omega e_q^n w dx + \int_\Omega e_u^n w_x dx - \sum_{j=1}^{N}(((e_u^n)^- w^-)_{j+\frac{1}{2}} - ((e_u^n)^- w^+)_{j-\frac{1}{2}})\\
&-b_{n-1}\int_\Omega e_u^0 v dx - \sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_\Omega e_u^{n-i} v dx + \beta\int_\Omega \gamma^n(x) v dx\\
&-\beta p(x,t_n)\int_\Omega f(x) v dx = 0.
\end{aligned}
$$

where

$$
\begin{aligned}
e_u^n &= u(x,t_n) - u_h^n = \mathbf{P}^- e_u^n - (\mathbf{P}^- u(x,t_n) - u(x,t_n)),\\
e_q^n &= q(x,t_n) - q_h^n = \mathbf{P} e_q^n - (\mathbf{P} q(x,t_n) - q(x,t_n)).
\end{aligned} \tag{12}
$$

Using Eq. (4), we have

$$
\begin{aligned}
&\int_\Omega e_u^n v dx + \beta\left(\int_\Omega e_q^n v_x dx - \sum_{j=1}^{N}(((e_q^n)^+ v^-)_{j+\frac{1}{2}} - ((e_q^n)^+ v^+)_{j-\frac{1}{2}})\right)\\
&+\int_\Omega e_q^n w dx + \int_\Omega e_u^n w_x dx - \sum_{j=1}^{N}(((e_u^n)^- w^-)_{j+\frac{1}{2}} - ((e_u^n)^- w^+)_{j-\frac{1}{2}})\\
&-b_{n-1}\int_\Omega e_u^0 v dx - \sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_\Omega e_u^{n-i} v dx + \beta\int_\Omega \gamma^n(x) v dx\\
&+\beta\int_\Omega \frac{f}{f^*} u_{xx}(x^*,t_n) v dx = 0.
\end{aligned} \tag{13}
$$

Using Eq. (12), the error equation (13) can be rewritten as follows

$$
\begin{aligned}
&\int_\Omega \mathbf{P}^- e_u^n v dx + \beta\left(\int_\Omega \mathbf{P} e_q^n v_x dx - \sum_{j=1}^{N}(((\mathbf{P} e_q^n)^+ v^-)_{j+\frac{1}{2}} - ((\mathbf{P} e_q^n)^+ v^+)_{j-\frac{1}{2}})\right)\\
&+\int_\Omega \mathbf{P} e_q^n w dx + \int_\Omega \mathbf{P}^- e_u^n w_x dx - \sum_{j=1}^{N}(((\mathbf{P}^- e_u^n)^- w^-)_{j+\frac{1}{2}} - ((\mathbf{P}^- e_u^n)^- w^+)_{j-\frac{1}{2}})\\
&+\beta\int_\Omega \frac{f}{f^*} u_{xx}(x^*,t_n) v dx = b_{n-1}\int_\Omega \mathbf{P}^- e_u^0 v dx + \sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_\Omega \mathbf{P}^- e_u^{n-i} v dx\\
&+\int_\Omega (\mathbf{P}^- u(x,t_n) - u(x,t_n)) v dx + \beta\left(\int_\Omega (\mathbf{P} q(x,t_n) - q(x,t_n)) v_x dx\right.\\
&\left.-\sum_{j=1}^{N}(((\mathbf{P} q(x,t_n) - q(x,t_n))^+ v^-)_{j+\frac{1}{2}} - ((\mathbf{P} q(x,t_n) - q(x,t_n))^+ v^+)_{j-\frac{1}{2}})\right)\\
&+\int_\Omega (\mathbf{P} q(x,t_n) - q(x,t_n)) w dx + \int_\Omega (\mathbf{P}^- u(x,t_n) - u(x,t_n)) w_x dx
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{j=1}^{N}(((\mathbf{P}^{-}u(x,t_n)-u(x,t_n))^{-}w^{-})_{j+\frac{1}{2}}-((\mathbf{P}^{-}u(x,t_n)-u(x,t_n))^{-}w^{+})_{j-\frac{1}{2}})\\
&-\beta\int_{\Omega}\gamma^{n}(x)vdx-b_{n-1}\int_{\Omega}(\mathbf{P}^{-}u(x,t_0)-u(x,t_0))vdx\\
&-\sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_{\Omega}(\mathbf{P}^{-}u(x,t_{n-i})-u(x,t_{n-i}))vdx.
\end{aligned}
$$

With taking the test functions $v=u_h^n$, $w=\beta q_h^n$, for periodic or compactly supported boundary conditions, we derive

$$
\begin{aligned}
&\int_{\Omega}\left(\mathbf{P}^{-}e_u^n\right)^2dx+\beta\int_{\Omega}\left(\mathbf{P}e_q^n\right)^2dx+\beta\int_{\Omega}\frac{f}{f^*}u_{xx}(x^*,t_n)\mathbf{P}^{-}e_u^ndx\\
&=b_{n-1}\int_{\Omega}\mathbf{P}^{-}e_u^0\mathbf{P}^{-}e_u^ndx+\sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_{\Omega}\mathbf{P}^{-}e_u^{n-i}\mathbf{P}^{-}e_u^ndx\\
&-\beta\int_{\Omega}\gamma^{n}(x)\mathbf{P}^{-}e_u^ndx+\beta\sum_{j=1}^{N}\left(((\mathbf{P}q(x,t_n)-q(x,t_n))^{+}\left[\mathbf{P}^{-}e_u^n\right]\right)_{j-\frac{1}{2}}\\
&+\int_{\Omega}(\mathbf{P}^{-}u(x,t_n)-u(x,t_n))\mathbf{P}^{-}e_u^ndx-b_{n-1}\int_{\Omega}(\mathbf{P}^{-}u(x,t_0)-u(x,t_0))\mathbf{P}^{-}e_u^ndx\\
&-\sum_{i=1}^{n-1}(b_{i-1}-b_i)\int_{\Omega}(\mathbf{P}^{-}u(x,t_{n-i})-u(x,t_{n-i}))\mathbf{P}^{-}e_u^ndx.
\end{aligned}
$$

For $n=1$, we have

$$
\begin{aligned}
&\int_{\Omega}\left(\mathbf{P}^{-}e_u^1\right)^2dx+\beta\int_{\Omega}\left(\mathbf{P}e_q^1\right)^2dx+\beta\int_{\Omega}\frac{f}{f^*}u_{xx}(x^*,t_1)\mathbf{P}^{-}e_u^1dx\\
&=\int_{\Omega}\mathbf{P}^{-}e_u^0\mathbf{P}^{-}e_u^1dx+\int_{\Omega}(\mathbf{P}^{-}u(x,t_0)-u(x,t_0))\mathbf{P}^{-}e_u^1dx\\
&-\beta\int_{\Omega}\gamma^{1}(x)\mathbf{P}^{-}e_u^1dx+\beta\sum_{j=1}^{N}\left(((\mathbf{P}q(x,t_1)-q(x,t_1))^{+}\left[\mathbf{P}^{-}e_u^1\right]\right)_{j-\frac{1}{2}}\\
&-\int_{\Omega}(\mathbf{P}^{-}u(x,t_1)-u(x,t_1))\mathbf{P}^{-}e_u^1dx.
\end{aligned}
$$

Using the following facts

$$
\parallel P^{-}e_u^0\parallel\leq Ch^{k+1},\quad ab\leq\varepsilon a^2+\frac{1}{4\varepsilon}b^2, \tag{14}
$$

we obtain

$$
\begin{aligned}
\parallel\mathbf{P}^{-}e_u^1\parallel^2+\beta\parallel\mathbf{P}e_q^1\parallel^2\leq{}&(\parallel\mathbf{P}^{-}e_u^0\parallel+\beta\parallel\gamma^1(x)\parallel+\parallel\mathbf{P}^{-}u(x,t_1)-u(x,t_1)\parallel\\
&+\parallel\mathbf{P}^{-}u(x,t_0)-u(x,t_0)\parallel+c\beta)\parallel\mathbf{P}^{-}e_u^1\parallel\\
&+\frac{\beta}{4\varepsilon}\sum_{j=1}^{N}\left((\mathbf{P}q(x,t_1)-q(x,t_1))^{+}\right)^2_{j-\frac{1}{2}}+\beta\varepsilon\sum_{j=1}^{N}\left[\mathbf{P}^{-}e_u^1\right]^2_{j-\frac{1}{2}}
\end{aligned}
$$

$$\leq C(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}}+c(\Delta t)^{\alpha})^2+\varepsilon \parallel \mathbf{P}^-e_u^1 \parallel^2 \\ +\beta\varepsilon\sum_{j=1}^{N}\left[\mathbf{P}^-e_u^1\right]_{j-\frac{1}{2}}^2.$$

If we choose ε very small, we conclude that

$$\parallel \mathbf{P}^-e_u^1 \parallel^2 +\beta \parallel \mathbf{P}e_q^1 \parallel^2 \leq C(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}}+c(\Delta t)^{\alpha})^2.$$

Now, we suppose the following inequalities hold

$$\|\mathbf{P}^-e_u^m\| \leq C(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}}+c(\Delta t)^{\alpha}), \quad m=1,2,\ldots,l.$$

We need to prove $\|\mathbf{P}^-e_u^{l+1}\| \leq C(h^{k+1}+(\Delta t)^2+(\Delta t)^{\frac{\alpha}{2}}h^{k+\frac{1}{2}}+c(\Delta t)^{\alpha})$. Letting $n=l+1$, we have

$$\int_\Omega \left(\mathbf{P}^-e_u^{l+1}\right)^2dx+\beta\int_\Omega\left(\mathbf{P}e_q^{l+1}\right)^2dx+\beta\int_\Omega\frac{f}{f^*}u_{xx}(x^*,t_{l+1})\mathbf{P}^-e_u^{l+1}dx \\ =b_l\int_\Omega \mathbf{P}^-e_u^0\mathbf{P}^-e_u^{l+1}dx+\sum_{i=1}^{l}(b_{i-1}-b_i)\int_\Omega \mathbf{P}^-e_u^{l+1-i}\mathbf{P}^-e_u^{l+1}dx \\ -\beta\int_\Omega\gamma^{l+1}(x)\mathbf{P}^-e_u^{l+1}dx+\beta\sum_{j=1}^{N}\left(((\mathbf{P}q(x,t_{l+1})-q(x,t_{l+1}))^+\left[\mathbf{P}^-e_u^{l+1}\right]\right)_{j-\frac{1}{2}} \\ +\int_\Omega(\mathbf{P}^-u(x,t_{l+1})-u(x,t_{l+1}))\mathbf{P}^-e_u^{l+1}dx \\ -b_l\int_\Omega(\mathbf{P}^-u(x,t_0)-u(x,t_0))\mathbf{P}^-e_u^{l+1}dx \\ -\sum_{i=1}^{l}(b_{i-1}-b_i)\int_\Omega(\mathbf{P}^-u(x,t_{l+1-i})-u(x,t_{l+1-i}))\mathbf{P}^-e_u^{l+1}dx.$$

Then by using (14) and

$$\sum_{i=1}^{l}(b_{i-1}-b_i)+b_l=1,$$

we can obtain

$$\parallel \mathbf{P}^-e_u^{l+1} \parallel^2 +\beta \parallel \mathbf{P}e_q^{l+1} \parallel^2 \leq (b_l \parallel \mathbf{P}^-e_u^0 \parallel + \parallel \mathbf{P}^-u(x,t_{l+1})-u(x,t_{l+1}) \parallel \\ +\beta \parallel \gamma^1(x) \parallel +b_l \parallel \mathbf{P}^-u(x,t_0)-u(x,t_0) \parallel +c\beta) \parallel \mathbf{P}^-e_u^{l+1} \parallel \\ +\sum_{i=1}^{l}(b_{i-1}-b_i) \parallel \mathbf{P}^-e_u^{l+1-i} \parallel\parallel \mathbf{P}^-e_u^{l+1} \parallel +\beta\varepsilon\sum_{j=1}^{N}\left[\mathbf{P}^-e_u^{l+1}\right]_{j-\frac{1}{2}}^2 \\ +\frac{\beta}{4\varepsilon}\sum_{j=1}^{N}\left((\mathbf{P}q(x,t_{l+1})-q(x,t_{l+1}))^+\right)_{j-\frac{1}{2}}^2$$

$$
\begin{aligned}
&\leq C_1 b_l (h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}} + c(\Delta t)^{\alpha}) \parallel \mathbf{P}^- e_u^{l+1} \parallel \\
&+\beta\varepsilon \sum_{j=1}^{N} \left[\mathbf{P}^- e_u^{l+1}\right]_{j-\frac{1}{2}}^2 + \sum_{i=1}^{l} (b_{i-1} - b_i) C_2 (h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}} \\
&+c(\Delta t)^{\alpha}) \parallel \mathbf{P}^- e_u^{l+1} \parallel \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}} + c(\Delta t)^{\alpha})^2 \\
&+\varepsilon \parallel \mathbf{P}^- e_u^{l+1} \parallel^2 + \beta\varepsilon \sum_{j=1}^{N} \left[\mathbf{P}^- e_u^{l+1}\right]_{j-\frac{1}{2}}^2 .
\end{aligned}
$$

Choosing a small ε, we derive

$$
\|\mathbf{P}^- e_u^{l+1}\| \leq C(h^{k+1} + (\Delta t)^2 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}} + c(\Delta t)^{\alpha}).
$$

□

4 Numerical examples

In this section, we carry out some numerical tests to confirm theoretical results and to investigate the efficiency of the proposed method. The maximum time is $T = 1$ otherwise it will be specified. The space and time step sizes are $h = 1/N$ and $\Delta t = T/M$, respectively. We use the relative root mean square error, i.e.,

$$
\varepsilon(p) = \left(\sum_{n=1}^{M} (p_h^n - p(t_n))^2 / \sum_{n=1}^{M} p(t_n)^2 \right)^{1/2}, \tag{15}
$$

for checking the accuracy of the numerical solutions. In all of tests, we take $k = 2$, i.e., we consider piecewise polynomials of degree two as the basis functions in the LDG regime. For dealing with the sensitivity of the solution with respect to the data, we use the following noisy data

$$
g^{\delta}(t_n) = g(t_n)(1 + \delta \, \mathrm{rnd}(n)), \quad n = 0, 1, \ldots,
$$

where g is the exact data and $\mathrm{rnd}(n)$ is a random number uniformly distributed in $[-1, 1]$ and the magnitude δ indicates a relative noise level.

Example 1 We consider the inverse source problem of the time-fractional diffusion equation (1)-(3) with the exact solution $u(x,t) = e^{-t}\cos(2\pi x)$. Setting Δt very small and using the usual L^2 and L^{∞} error norms, we show in Table 1 that the order of convergence of the proposed method is about three as we expected (according to the obtained error estimate since $k = 2$). Since the exact p of this problem is not accessible, in Fig 1. we show p_h^n for $\alpha = 0.5, x^{\star} = 0.75$ and $N = 8, 16$. The errors in L^2-norm and L^{∞}-norm for piecewise $P^k, k = 1, 2, 3$ polynomials for $\alpha = 0.1, x^\star = 0.25$ are presented in Fig 2.

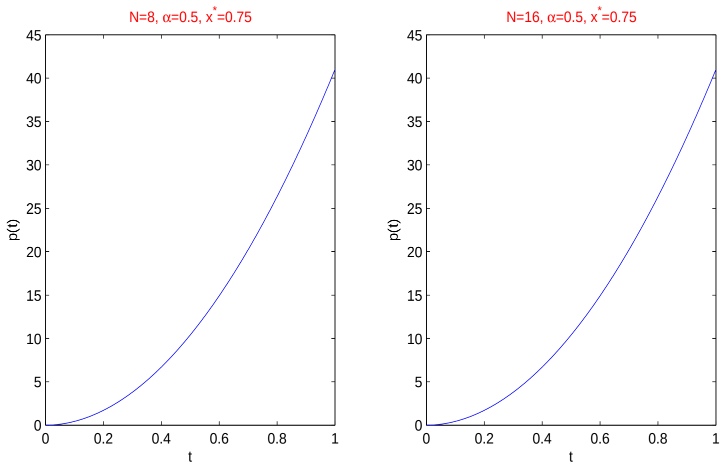

Figure 1: Numerical approximations to p for Example 1 for $N = 8$ (left) and $N = 16$ (right).

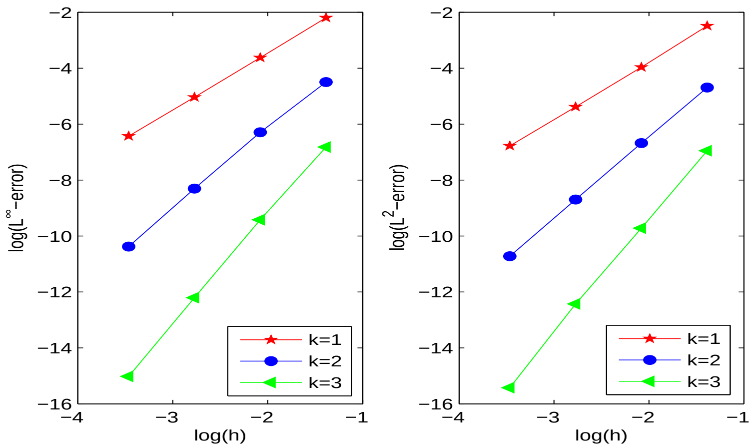

Figure 2: log(error) as a function of log(h) for $\alpha = 0.1, x^\star = 0.25$ when using piecewise $P^k, k = 1, 2, 3$ polynomials for Example 1.

Example 2. Let the exact solution for problem (1)-(3) be $u(x,t) = t^2 \sin\left(\frac{\pi}{2}x\right)$. Therefore, $\phi(x) = u(x,0) = 0$, $k_0(t) = u(0,t) = 0$, $k_1(t) = u(1,t) = t^2$, $f(x) = \sin\left(\frac{\pi}{2}x\right)$, and $p(t) = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + \frac{\pi^2}{4}t^2$. We take

Table 1: Accuracy test for Example 1 for different α with $x^* = 0.5$.

	N	L^2 error	Order	L^∞ error	Order
$\alpha = 0.3$	5	0.002067	-	0.002780	-
	10	0.000280	2.9	0.000376	2.9
	15	0.000079	3.1	0.000112	3.0
	20	0.000034	2.9	0.000047	3.0
$\alpha = 0.5$	5	0.002068	-	0.002781	-
	10	0.000280	2.9	0.000376	2.9
	15	0.000079	3.1	0.000112	3.0
	20	0.000034	2.9	0.000047	3.0
$\alpha = 0.7$	5	0.002069	-	0.002783	-
	10	0.000280	2.9	0.000377	2.9
	15	0.000079	3.1	0.000112	3.0
	20	0.000034	2.9	0.000047	3.0
$\alpha = 1$	5	0.002070	-	0.002784	-
	10	0.000281	2.9	0.000377	2.9
	15	0.000079	3.1	0.000112	3.0
	20	0.000034	2.9	0.000047	3.0

$x^* = 0.5$, then $g(t) = \sin(\frac{\pi}{4})t^2$. p_h^n for $\alpha = 0.5$ and $\alpha = 0.95$ with various noise levels $\delta = 5\%,\ 10\%,\ 15\%$ are plotted in Fig. 3. Corresponding relative root mean square errors are $\varepsilon(p) = 8.5762 \times 10^{-5},\ 8.6406 \times 10^{-5},\ 8.7104 \times 10^{-5}$ for $\alpha = 0.5$ and $\varepsilon(p) = 2.030156 \times 10^{-3},\ 3.077799 \times 10^{-3},\ 4.153917 \times 10^{-3}$ for $\alpha = 0.95$. In Table 2, we compare the relative root mean square errors for the

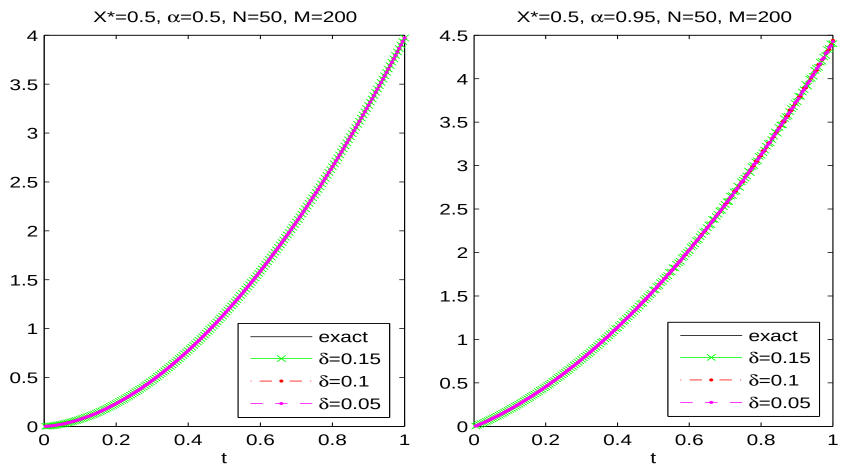

Figure 3: Numerical approximations to p for Example 2.

proposed method in [28] ($\varepsilon_1(p)$ for no regularization method and $\varepsilon_2(p)$ with a regularization method) with the proposed LDG method ($\varepsilon_3(p)$). In Table 3 the relative errors for different x^* with $N = 50$ are compared. The proposed method could generate more satisfactory results without any regularization method. To verify the role of the final time T, we depict p_h^n for $\alpha = 0.95$ with $T = 100, 10000$ and $\delta = 5\%,\ 10\%,\ 15\%$ in Fig. 4. We can see that the dependency of the results to final time T is almost unimportant, even when T is very large, i.e., $T = 10000$. Without any regularization method, the trace of the loss of stability does not appear.

Table 2: Comparison approximation solutions of Example 2 with the literature for various α.

		$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
$\delta = 5\%$	$\varepsilon_1(p)$	0.0773	0.0880	0.1230	0.2651	0.8079
	$\varepsilon_2(p)$	0.0331	0.0362	0.0393	0.0413	0.0431
	$\varepsilon_3(p)$	2.162×10^{-6}	1.9319×10^{-5}	8.5698×10^{-5}	3.33882×10^{-4}	0.001419157
$\delta = 10\%$	$\varepsilon_1(p)$	0.1545	0.1738	0.2436	0.5296	1.6159
	$\varepsilon_2(p)$	0.0634	0.0633	0.0662	0.0721	0.0812
	$\varepsilon_3(p)$	2.1600×10^{-6}	1.9334×10^{-5}	8.6232×10^{-5}	3.51023×10^{-4}	0.002105562
$\delta = 15\%$	$\varepsilon_1(p)$	0.2322	0.2611	0.3658	0.7948	2.4241
	$\varepsilon_2(p)$	0.0952	0.0943	0.0948	0.0721	0.1248
	$\varepsilon_3(p)$	2.1660×10^{-6}	1.9356×10^{-5}	8.6557×10^{-5}	3.96662×10^{-4}	0.002444287

Table 3: Comparison approximation solutions of Example 2 with the literature for various x^*.

x^*	$\varepsilon_1(p)$	$\varepsilon_2(p)$	$\varepsilon_3(p)$
0.1	1.0010	0.1801	1.024501×10^{-3}
0.2	0.9047	0.0661	8.97953×10^{-4}
0.3	0.9197	0.0767	8.03529×10^{-4}
0.4	0.9323	0.0694	7.96885×10^{-4}
0.5	0.9222	0.0839	7.50627×10^{-4}
0.6	0.9071	0.0761	7.92723×10^{-4}
0.7	0.9147	0.1026	7.78528×10^{-4}
0.8	0.9256	0.0974	7.55676×10^{-4}

Example 3. We test a none-smooth problem corresponding to (1)-(3), with $\phi(x) = u(x, 0) = \sin(2\pi x)$, $k_0(t) = u(0, t) = 0$, $k_1(t) = u(1, t) = 0$, $f(x) = x^2$, and

$$p(t) = \begin{cases} 2t + \alpha, & t \in [0, 0.5], \\ -2t + 2 + \alpha, & t \in (0.5, 1]. \end{cases}$$

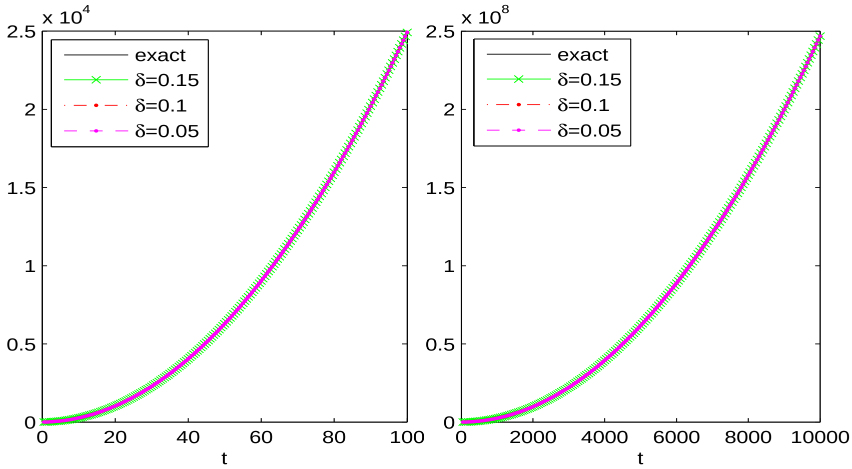

Figure 4: Numerical approximations to p for Example 2 for $T = 100$ (left) and $T = 10000$ (right).

Since the exact solution of this problem is not accessible, we first solve a direct problem using a suitable LDG method to obtain the input data g then we solve the inverse problem using our method. In [28], the direct problem has been solved by using an implicit finite difference (FD) method but since p is not a smooth function we expect to have a none-smooth solution and we decided to solve the direct problem using a LDG method too. We have to point out that since in our method we face the sparse systems, the computational complexity of both methods, i.e., FD and LDG is almost equal. p_h^n for $\alpha = 0.5,\ 0.95$ with noise levels $\delta = 5\%,\ 10\%,\ 15\%$ are presented in Fig 5. Without applying any regularization methods, our results are in good agreement with the results of [28]. The corresponding relative root mean square errors are $\varepsilon(p) = 3.8300 \times 10^{-7},\ 6.2300 \times 10^{-7},\ 9.4800 \times 10^{-7}$ for $\alpha = 0.5$ and $\varepsilon(p) = 1.03320 \times 10^{-4},\ 2.33279 \times 10^{-4},\ 1.03320 \times 10^{-4}$ for $\alpha = 0.95$, which are considerably better than reported in [28].

Example 4. We test a discontinuous problem corresponding to (1)-(3), with $\phi(x) = u(x, 0) = \sin(2\pi x)$, $k_0(t) = u(0, t) = 0$, $k_1(t) = u(1, t) = 0$, $f(x) = x^2$, and

$$p(t) = \begin{cases} 1,\ t \in [0.25, 0.75], \\ 0,\ t \in [0, 0.25) \cup (0.75, 1]. \end{cases}$$

Since the exact solution of this problem is not accessible, we first solve a direct problem using a suitable LDG method to obtain the input data g then we solve the inverse problem using our method. In [28], the direct problem has been solved by using an implicit finite difference method but since p is not a continuous function we expect to have a discontinuous solution and we decided to solve the direct problem using a LDG method too. We

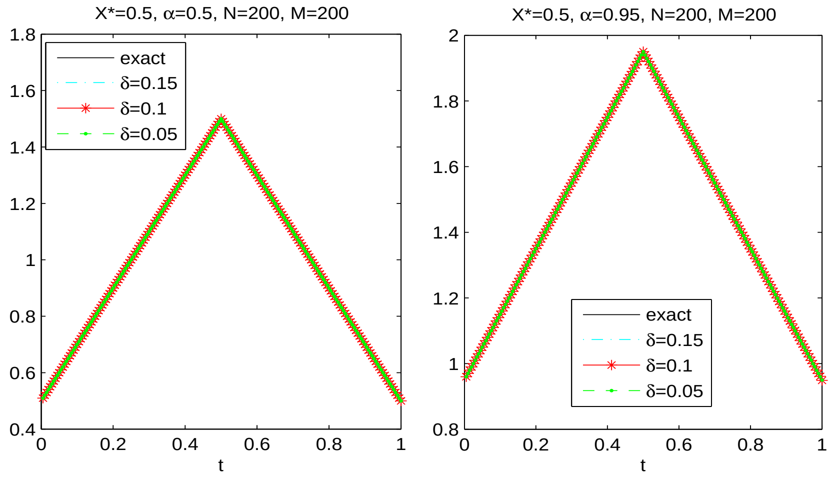

Figure 5: Numerical approximations to p for Example 3 with various α.

have to point out that since in our method we face the sparse systems, the computational complexity of both methods, i.e., FD and LDG is almost equal. p_h^n for $\alpha = 0.5,\ 0.95$ with noise levels $\delta = 5\%,\ 10\%,\ 15\%$ are plotted in Fig 6. Without applying any regularization methods, our results are in good agreement with the results of [28]. The corresponding relative root mean square errors are $\varepsilon(p) = 4.1200 \times 10^{-7},\ 6.2300 \times 10^{-7},\ 8.2700 \times 10^{-7}$ for $\alpha = 0.5$ and $\varepsilon(p) = 8.8388 \times 10^{-5},\ 1.61535 \times 10^{-4},\ 2.58527 \times 10^{-4}$ for $\alpha = 0.95$, which are considerably better than reported in [28].

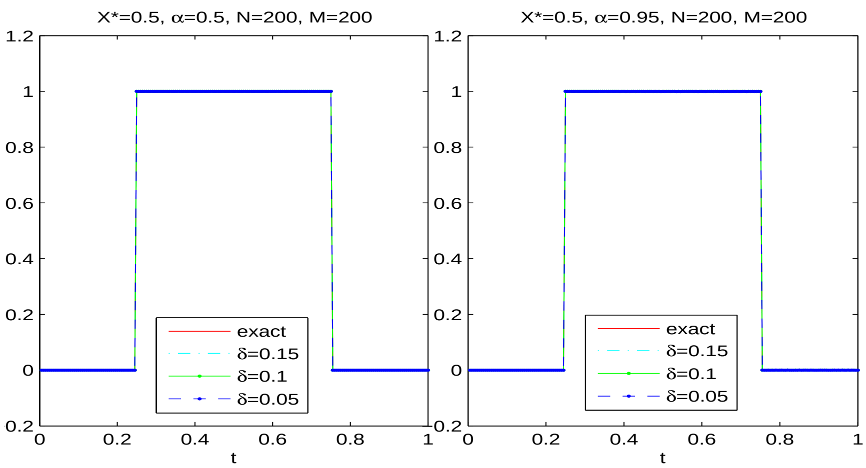

Figure 6: Numerical approximations to p for Example 4 with various α.

5 Conclusion

In this paper, an inverse source problem for the time-fractional diffusion equation was solved numerically by using a local discontinuous Galerkin method. In fact, we could extend a fully-discrete LDG finite element method for solving a class of time-fractional inverse problem. By applying this method without using any regularization methods, we could obtain stable and accurate numerical approximations to the time-dependent source term using an additional data in an interior measurement location. The numerical stability and convergence of the proposed method have been investigated and theoretically proven. Various numerical examples with smooth or none-smooth data and maybe solutions have been verified to demonstrate the effectiveness and robustness of the proposed method. This outstanding and promising method can be further applied to another one-dimensional or higher dimensional inverse problems which can be considered for the future works.

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استفاده از یک روش LDG برای حل مساله منبع وارون از نوع معادله انتشار کسری-زمانی

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چکیده : در این مقاله یك روش گالرکین ناپیوسته موضعی (LDG) را برای حل برخی مسائل وارون کسری به کار می بریم. در واقع جمله منبع وابسته به زمان را در یك مساله وارون از نوع معادله انتشار کسری-زمانی تعیین می کنیم. این روش بر اساس یك طرح تفاضل متناهی در زمان و یك روش (LDG) در مکان است. یک قضیه پایداری عددی به علاوه یك تخمین خطا مهیا می شود. در پایان، چند مثال عددی به منظور تائید نتایج نظری و نشان دادن اثربخشی روش، آزمایش می شوند. باید اشاره کرد که روش پیشنهادی بدون استفاده از روشهای منظم سازی که برای سایر روش های عددی در حل چنین مسائل وارون بدطرح ضروری هستند، تقریب های عددی پایدار و دقیقی را تولید می کند.

کلمات کلیدی : روش گالرکین ناپیوسته موضعی؛ مسئله منبع وارون؛ معادله انتشار کسری-زمانی.