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Research Article

Using homotopy analysis method to find the eigenvalues of higher order fractional Sturm–Liouville problems

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Abstract

We utilize the homotopy analysis method to find eigenvalues of fractional Sturm–Liouville problems. Inasmuch as very few papers have been devoted to estimating eigenvalues of these kind of problems, this work enjoys a particular significance in many different branches of science. The convergence of the homotopy analysis method is also considered on the high order fractional Sturm–Liouville problem. The numerical results acknowledge the ability of the proposed method. Eigenvalues are computed within a couple of minutes CPU time at core i3, 2.7 GHz PC.

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Keywords: Homotopy analysis method; Eigenvalues; Fractional Sturm-Liouville problems.

1 Introduction

In this paper, the following special class of nth order fractional Sturm–Liouville eigenvalue problems are considered:

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$$\sum_{j=1}^{n} q_j(x) D^{\alpha_j} y(x) + q_0(x) y(x) = \lambda w(x) y(x), \quad a \le x \le b, \tag{1}$$

where $w(x)$ and $q_j(x)$, $j = 0, 1, 2, \ldots, n$, are integrable functions over $[a, b]$. The (left-sided) Caputo fractional derivative D^{α_j} of order $\alpha_j \in (j-1, j)$, $j = 0, 1, 2, \ldots, n$, is defined by

$$D^{\alpha_j} y(x) = \frac{1}{\Gamma(j-\alpha_j)} \int_0^x \frac{y^{(j)}(t)}{(x-t)^{\alpha_j+1-j}} dt.$$

Note that n should be an even number denoted by $2r$. The separate boundary conditions of problem (1) are as follows:

$$y^{(i)}(a) = 0, \tag{2}$$

$$y^{(i)}(b) = 0, \tag{3}$$

for $i \in S' \subset S := \{0, 1, 2, \ldots, 2r-1\}$ in which S' has r members. Problems (1), (2), and (3) generally have arisen from linear fractional equations (see [3, 4, 14, 8]).

Definition 1. The left-sided Riemann–Liouville fractional integral operator of order α is defined by

$$J^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t) dt, \tag{4}$$

where $y \in L^1[0, T]$ and $\alpha \in \mathbb{R}^+$.

Some useful properties of the operator J^{α} are found in [18, 19, 16]. It must be mentioned that the left-sided Caputo fractional derivative (4) is originally defined via the left-sided Riemann–Liouville fractional integral (4) [12], as follows

$$D^{\alpha} y(x) = J^{n-\alpha} y^{(n)}(x) = y(x), \quad x > 0,$$

where $\alpha \in \mathbb{R}^+$, $n = [\alpha] + 1$ and $y \in L^1[0, T]$.

Fractional Sturm–Liouville problems (FSLP) are interesting problems from a practical point of view. It turns out that such fractional equations have seen many applications in science and engineering problems such as capturing the dynamical behaviors of amorphous materials, for example, polymer and porous media [15] or the superior modeling of the anomalous diffusion in materials with memory, for instant, viscoelastic materials for which the mean square variance grows faster (superdiffusion) or slower (subdiffusion) than in a Gaussian process, see [9]. The interested reader may wish to examine the great variety of works on the subject such as [1, 2, 7, 5, 6]

Therefore, many scientists resort to mathematicians to deal with this kind of problem. The homotopy analysis method (HAM), Adomian decomposition method (see [11, 4, 14, 20]), fractional differential transform method (see [10]), iterative approximation method (see [17]), and variational method (see [12]) were implemented on second-order FSLP.

To the best of our knowledge, few studies have been carried out to find the eigenvalues of high order FSLP (see [11]). So, we generalize the homotopy analysis method on this particular problem. By taking polynomials as basis functions, we approximate the homotopy series solution that satisfies initial conditions (2). Then, by imposing boundary conditions (3) on this series solution, a system of equations will appear for which its determinant should be zero to have a nontrivial solution. In spite of using fewer terms to find the eigenvalues of the problem (1), we gain more accurate results compared with [11]. Moreover, our method is easily applied in solving an arbitrary high-order fractional Sturm–Liouville problems. On the other hand, the convergence of the series solution of the HAM for high order FSLP is proved which demonstrates the efficiency of this proposed method.

This paper is organized as follows: In Section 2 some basic definitions of fractional calculus are presented. The HAM is mentioned, in detail, in Section 3. The convergence of the HAM on high order FSLP is established in Section 4. The numerical results are presented in Section 5 to illustrate the efficiency of the proposed method. The last section is devoted to including discussions and conclusions.

2 Homotopy analysis method

To illustrate the basic concept of the HAM, we consider the following general nonlinear functional operator:

$$N[u(x)] = 0, \tag{5}$$

where N is a nonlinear operator, x denotes an independent variable, and $u(x)$ is an unknown function. For the sake of simplicity, we ignore all boundary or initial conditions, which can be treated similarly. Through generalizing the traditional homotopy method Liao in [13], the following the so-called zero-order deformation equation was constructed:

$$(1-p)L[\phi(x;p) - u_0(x)] = p\hbar H(x)N[\phi(x;p)], \tag{6}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $H(x)$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x)$ is an initial guess of $u(x)$, and $\phi(x;p)$ is an unknown function. It is important, that one has a great deal of freedom to choose auxiliary operators in the HAM. Obviously, when $p = 0$ and $p = 1$, the following relations, respectively, hold:

$$\phi(x;0) = u_0(x), \quad \phi(x;1) = u(x).$$

Thus, as p increases from 0 to 1, the solution $\phi(x;p)$ varies from the initial guess to the solution $u(x)$. Expanding $\phi(x;p)$ in Taylor series with respect to p, one has

$$\phi(x;p) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)p^m, \tag{7}$$

where

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x;p)}{\partial^m p}\Big|_{p=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, series (7) converges at $p = 1$, and one has

$$u(x) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x), \tag{8}$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao (see [13]). The governing equation can be deduced from the zero-order deformation equation. Define the vector

$$\overrightarrow{u}_m = \{u_0(x), u_1(x), \ldots, u_n(x)\}.$$

Differentiating (6), m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth order deformation equation

$$L[u_m(x) - \chi_m u_{m-1}(x)] = \hbar H(x) R_m(\overrightarrow{u}_{m-1}), \tag{9}$$

where

$$R_m(\overrightarrow{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x;p)]}{\partial p^{m-1}}\Big|_{p=0}, \tag{10}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1. \end{cases} \tag{11}$$

In equation (1) we choose the initial guess as follows:

$$u_0(x) = c_1\mu_1(x) + c_2\mu_2(x) + \cdots + c_r\mu_r(x)$$

where $\mu_i(x), i = 1, 2, \ldots, r$, are chosen in such a way that $u_0(x)$ satisfies the boundary condition (2). By implementing the HAM we can consider N first terms of series (8)

$$w_N(x) = u_0(x) + \sum_{m=1}^{N} u_m(x),$$

as an approximation of the solution of equation (1). It is worth to note that, according to our main problem (1) including the parameter λ and process of the HAM consisting of the parameter $\hbar$, the approximate solution $w_N(x)$ enjoys these two parameters, that is, λ and $\hbar$. In fact , we have

$$w_N(x;\lambda,\hbar) = u_0(x) + \sum_{m=1}^{N} u_m(x;\lambda,\hbar). \tag{12}$$

By imposing the boundary condition (3) on the approximate solution (12), we get

$$w_N^{(i)}(b;\lambda,\hbar) = u_0^{(i)}(b) + \sum_{m=1}^{N} u_m^{(i)}(b;\lambda,\hbar),$$

where $i \in S'$. Therefore, we encounter with a system of r equations and r unknown coefficients $c_1, c_2, \ldots, c_r$. In order to have a nontrivial solution, the determinant of this system of equation including parameters λ and $\hbar$ should be equal to zero. So, by plotting this implicit function of λ and $\hbar$, so-called $\hbar$-curve, we can determine eigenvalues that correspond to each horizontal plateaus.

3 Convergence of the HAM in FSLP

In the following theorem, we consider the convergence of the HAM on fractional Sturm–Liouvill problem.

Theorem 1. If the following series

$$Z(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x), \tag{13}$$

is convergent in which $u_m(x)$ is obtained from (9), (10), and (11), then the series (13) can be a solution of (1) and (2).

Proof. Since series (13) is convergent, we have

$$\lim_{m\to\infty} u_m(x) = 0, \tag{14}$$

by the necessary condition of convergent series.

Now, by using (11), we have

$$\begin{aligned}\sum_{m=1}^{n} \left[u_m(x) - \chi_m u_{m-1}(x)\right] &= u_1(x) + (u_2(x) - u_1(x)) + \cdots + (u_n(x) - u_{n-1}(x)) \\ &= u_n(x).\end{aligned}$$

Then by virtue of (14), we get

$$\sum_{m=1}^{\infty}\big[u_m(x)-\chi_m u_{m-1}(x)\big]=\lim_{n\to\infty}\big[u_m(x)-\chi_m u_{m-1}(x)\big]=\lim_{n\to\infty}u_n(x)=0.$$

According to above relation and linearity of the auxiliary operator L, we can write

$$L\bigg(\sum_{m=1}^{\infty}[u_m(x)-\chi_m u_{m-1}(x)]\bigg)=\sum_{m=1}^{\infty}L\bigg([u_m(x)-\chi_m u_{m-1}(x)]\bigg)=0.$$

So, by means of equation (9), we have

$$\sum_{m=1}^{\infty}L\bigg([u_m(x)-\chi_m u_{m-1}(x)]\bigg)=\hbar H(x)\sum_{m=1}^{\infty}\big[R_m(\overrightarrow{u}_{m-1})\big]=0.$$

For as much as, $\hbar\neq 0$ and $H(x)\neq 0$, the following equality is obtained

$$\sum_{m=1}^{\infty}[R_m(\overrightarrow{u}_{m-1})]=0. \tag{15}$$

Consequently, from (10), one can write

$$\begin{aligned}R_m[\overrightarrow{u}_{m-1}]&=\frac{1}{m!}\frac{\partial^{m-1}}{\partial p^{m-1}}\bigg[\sum_{j=1}^{n}q_j(x)D^{\alpha_j}[\sum_{i=1}^{\infty}u_i(x)p^i]+q_0(x)[\sum_{i=1}^{\infty}u_i(x)p^i]\\&\qquad-\lambda w(x)[\sum_{i=1}^{\infty}u_i(x)p^i]\bigg]\\&=\sum_{j=1}^{n}q_j(x)D^{\alpha_j}[u_{m-1}(x)]+q_0(x)u_{m-1}(x)-\lambda w(x)u_{m-1}(x).\end{aligned}$$

Then,

$$\begin{aligned}\sum_{m=1}^{\infty}R_m[\overrightarrow{u}_{m-1}]&=\sum_{m=0}^{\infty}\big[\sum_{j=1}^{n}q_j(x)D^{\alpha_j}[u_{m-1}]+q_0(x)u_{m-1}-\lambda w(x)u_{m-1}\big]\\&=\sum_{j=1}^{n}q_j(x)D^{\alpha_j}[\sum_{m=1}^{\infty}u_{m-1}]+q_0[\sum_{m=1}^{\infty}u_{m-1}]-\lambda w(x)[\sum_{m=1}^{\infty}u_{m-1}]\\&=\sum_{j=1}^{n}q_j(x)D^{q}[Z(x)]+q_0[Z(x)]-\lambda w(x)[Z(x)].\end{aligned}\tag{16}$$

Regarding to (15) and (16), we have

$$\sum_{j=1}^{n}q_j(x)D^{q}[Z(x)]+q_0[Z(x)]-\lambda w(x)[Z(x)]=0.$$

Moreover, from the initial guess $u_0(x)$ and the initial condition (2), we obtain

$$u_m^{(i)}(a) = u_0^{(i)}(a) = 0, \quad i \in S',$$

which implies

$$Z(a) = u_0(a) + \sum_{m=0}^{\infty} u_m(a) = u_0(a) = 0.$$

Therefore, $Z(x)$ is an analytic solution of the differential equation (1). □

Remark 1. We consider $\{x^{1+i\alpha_n}, x^{3+i\alpha_n}, \dots, x^{2r-1+i\alpha_n}\}$, in which $r \in \mathbb{N}$ is an arbitrary constant, as a basis function of the HAM in problem (1) in which $\alpha_1 = \alpha_2 = \alpha_{n-1} = 0$, with subject to initial condition (2) for which even order of derivatives are just assumed. Then, we can write the series solution (8) as a linear combination of basis functions with coefficients $a_{i,1}, a_{i,2}, \dots, a_{i,r}$, $i = 0, \dots, m$, as follows:

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{i=0}^{m} a_{i,1}x^{1+i\alpha_n} + a_{i,2}x^{3+i\alpha_n} + \cdots + a_{i,r}x^{2r-1+i\alpha_n} \\
&= \lim_{t\to\infty}\sum_{m=0}^{t-1}\sum_{i=0}^{m} a_{i,1}x^{1+i\alpha_n} + a_{i,2}x^{3+i\alpha_n} + \cdots + a_{i,r}x^{2r-1+i\alpha_n} \\
&= \lim_{t\to\infty}(a_{0,1}x + a_{0,2}x^3 + \cdots + a_{0,r}x^{2r-1}) \\
&\quad + (a_{0,1}x + \cdots + a_{0,r}x^{2r-1} + a_{1,1}x^{1+\alpha_n} + \cdots + a_{1,r}x^{2r-1+\alpha_n}) \\
&\quad + \cdots + (a_{0,1}x + \cdots + a_{0,r}x^{2r-1} + \cdots + a_{t-1,1}x^{1+(t-1)\alpha_n} \\
&\qquad\qquad + \cdots + a_{t-1,r}x^{2r-1+(t-1)\alpha_n}) \\
&= \lim_{t\to\infty} t(a_{0,1}x + a_{0,2}x^3 + \cdots + a_{0,r}x^{2r-1}) \\
&\quad + (t-1)(a_{1,1}x^{1+\alpha_n} + \cdots \quad + a_{1,r}x^{2r-1+\alpha_n}) \\
&\quad + \cdots + (a_{t-1,1}x^{1+(t-1)\alpha_n} + \cdots + a_{t-1,r}x^{2r-1+(t-1)\alpha_n}) \\
&= \lim_{t\to\infty}\sum_{k=0}^{t-1}(t-k)(a_{k,1}x^{1+k\alpha_n} + \cdots + a_{k,r}x^{2r-1+k\alpha_n}).
\end{aligned}$$

In order to establish the convergence of above series we should have

$$\lim_{t\to\infty}\left|\frac{(t-k-1)a_{k+1,j}x^{1+(k+1)\alpha_n}}{(t-k)a_{k,j}x^{1+(k\alpha_n)}}\right| < 1$$

for $j = 1, \dots, r$. It is obvious that the $\lim_{t\to\infty}|\frac{t-k-1}{t-k}| \le 1$ for all $k < t$. So, $|\frac{a_{k+1,j}}{a_{k,j}}x^{\alpha_n}| < 1$.

4 Numerical examples

To illustrate the proposed approach, some examples are presented in this section. We use Mathematica software to calculate eigenvalues of following problems.

Example 1. Consider the following fourth-order FSLP:

$$D^{\alpha}[y(x)] = \lambda y(x), \qquad x \in (0,1), \tag{17}$$

where $3 < \alpha \leqslant 4$, subject to the boundary conditions

$$y(0) = y''(0) = 0, \tag{18}$$

and

$$y(1) = y''(1) = 0. \tag{19}$$

We assume that the solution of (17) can be expressed by the set of the following basis functions:

$$\{x, x^3, x^{1+\alpha}, x^{3+\alpha}, x^{1+2\alpha}, x^{3+2\alpha}, \dots\}.$$

According to the boundary condition (18), we can choose the initial approximation of the solution in the form $u_0(x) = c_1x + c_2x^3$. It is obvious that one must choose the auxiliary linear operator

$$L[\phi(x;p)] = D^{\alpha}\phi(x;p).$$

From (17), we define the following nonlinear operator:

$$N[\phi(x;p)] = D^{\alpha}\phi(x;p) - \lambda\phi(x;p). \tag{20}$$

We have the zero order deformation equation (6) with the initial conditions

$$\phi(0;p) = 0, \quad \phi''(0;p) = 0.$$

From (10) and (20), we have

$$\begin{aligned} R_m(\overrightarrow{u}_{m-1}) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}[D^{\alpha}(u_0 + pu_1 + \cdots) - \lambda(u_0 + pu_1 + \cdots)]}{\partial p^{m-1}} \\ &= D^{\alpha}u_{m-1}(x) - \lambda u_{m-1}(x). \end{aligned}$$

Now, the solution of the $m-$th-order deformation equation (9), for $m \geq 1$ becomes

$$u_m(x) = \chi_m u_{m-1}(x) + \hbar J^{\alpha}[H(x)R_m(\overrightarrow{u}_{m-1})].$$

According to the rule of solution expression denoted by (8) and from (9), the auxiliary function $H(x)$ is uniquely determined with $H(x) = 1$.

The first few terms of HAM series solution for $\alpha = 3.7$ are as follows:

$$
\begin{aligned}
u_1 &= c_1[(1+\hbar)x - 0.013788\hbar\lambda x^{1+\alpha}] + c_2[(1+\hbar)x^3 - 0.002166\hbar\lambda x^{3+\alpha}],\\
u_2 &= c_1[(1+\hbar)^2 x + (-0.027757 - 0.0275757\hbar\lambda x^{1+\alpha} + 0.000010\hbar^2\lambda^2 x^{1+2\alpha})]\\
&\quad + c_2[(1+\hbar^2)x^3 + (-0.004332 - 0.004332h)\hbar\lambda x^{3+\alpha} + 6.40592\times 10^{-7}\hbar^2\lambda^2 x^{3+2\alpha}],\\
&\ \vdots
\end{aligned}
$$

Consequently the $N-$th order approximate solution of the HAM, $w_N(x)$, is in the form

$$w_N(x) = \sum_{m=0}^{N} u_m(x) = c_1\zeta_N(x) + c_2\eta_N(x). \tag{21}$$

By imposing the boundary condition (19) on (21), we get

$$
\begin{aligned}
c_1\zeta_N(1;\lambda,\hbar) + c_2\eta_N(1;\lambda,\hbar) = 0,\\
c_1\zeta''_N(1;\lambda,\hbar) + c_2\eta''_N(1;\lambda,\hbar) = 0.
\end{aligned}
$$

In order to have nontrivial eigenfunctions, we just need to solve

$$\det\begin{pmatrix} \zeta_N(1;\lambda,\hbar) & \eta_N(1;\lambda,\hbar)\\ \zeta''_N(1;\lambda,\hbar) & \eta''_N(1;\lambda,\hbar)\end{pmatrix} = 0.$$

The above equation, which depends on λ and $\hbar$, is used to plot the $\hbar$-curve.

We can observe that in the plot of λ as a function of $\hbar$, several horizontal plateaus occur, each of which corresponds to an eigenvalue of the problem.

The results of Table 1 has been obtained by choosing $\hbar = -1$. The exact eigenvalues of this problem for the integer case $\alpha = 4$, are known $\lambda_k = (k\pi)^4$, $k \geq 1$ (see [11]). The obtained results of HAM are compared with those calculated in [11]. As we can observe in Figure 2 the nth eigenfunction has exactly, $n-1$ zeros in the mentioned interval.

Table 1: Comparison results of first four eigenvalues of Example 1

λ_k	$\alpha = 3.7$		$\alpha = 3.9$		$\alpha = 3.9999$		$\alpha = 4$	
	HAM	[11]	HAM	[11]	HAM	[11]	HAM	[11]
λ_1	91.412292	91.412293	93.533242	93.533230	97.404401	97.404401	97.409091	97.409091
λ_2	944.796194	944.795695	1324.156313	1324.156357	1558.290698	1558.290698	1558.545456	1558.545456
λ_3	4544.318681	4544.336950	6456.132806	6456.132485	7888.515088	7888.515134	7890.136374	7890.136374
λ_4	12012.668230	12012.483491	19613.918138	19613.888783	24930.699029	24930.689877	24936.727305	24936.727305
CPU time	$45s$		$42s$		$47s$		$22s$	

Example 2. Consider the following sixth-order FSLP:

$$D^\alpha[y(x)] + \sum_{j=0}^{4} q_j(x)y^{(j)}(x) + \lambda y(x) = 0, \qquad x \in (0,5),$$

subject to the boundary conditions

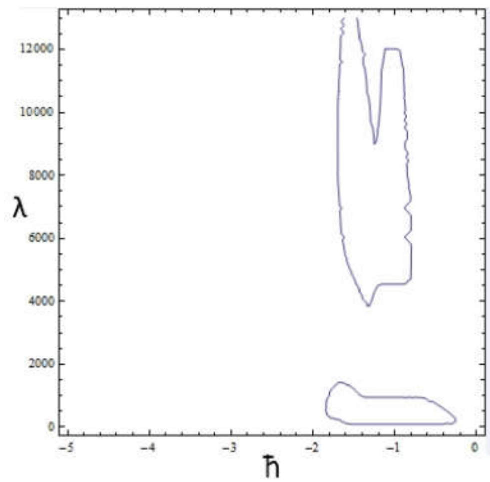

Figure 1: $\hbar$-curve according to $\alpha = 3.7$

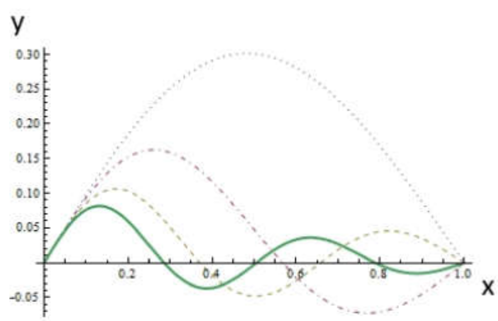

Figure 2: dotted line: Eigenfunction corresponding to λ_1; dot dashed line: Eigenfunction corresponding to λ_2; dashed line: Eigenfunction corresponding to λ_3; thick lines: Eigenfunction corresponding to λ_4 of Example 1

$$y(0) = y''(0) = y^{(4)}(0) = 0,$$
$$y(5) = y''(5) = y^{(4)}(5) = 0.$$

where $5 < \alpha < 6$, $q_j(x)$, $0 \leq j \leq 5$ are given by $q_0(x) = -r_3(x)$, $q_1(x) = r_2'(x)$, $q_2(x) = r_2(x) - r_1''(x)$, $q_3(x) = -2r_1'(x)$, $q_4(x) = -r_1(x)$, $r_1(x) = 0.03x^2$, $r_2(x) = 0.0003x^4 - 0.08$, $r_3(x) = 10^{-6}x^6 - 0.0014x^2$. Figure 4 shows the first four eigenfunctions of this example.

Example 3. Consider the following tenth order FSLP:

$$-D^{\alpha}y(x) = \lambda y(x)$$

Table 2: Comparison results of first four eigenvalues of Example 2

λ_k	$\alpha = 5.7$		$\alpha = 5.8$		$\alpha = 5.9$		$\alpha = 5.99$	
	HAM	[11]	HAM	[11]	HAM	[11]	HAM	[11]
λ_1	0.147973	0.144551	0.117816	0.114673	0.094189	0.091268	0.076819	0.074124
λ_2	4.575315	4.572390	4.563104	4.560000	4.515779	4.512994	4.461465	4.458254
λ_3	44.714848	44.712872	45.154858	45.152024	46.234038	46.230890	47.596156	47.592932
λ_4	208.439520	0.699270	225.477660	1.826100	242.79754	1.923205	359.596203	2.595920
CPU time	304s		308s		327s		307s	

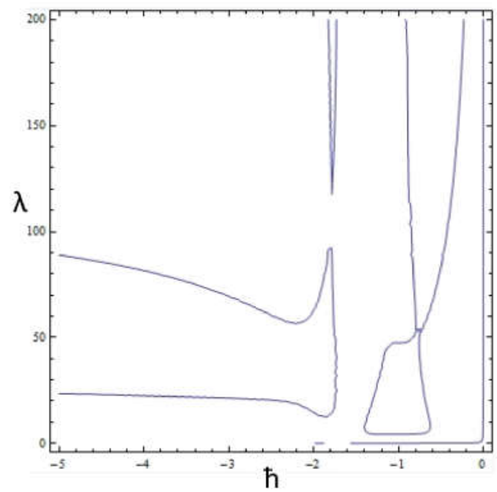

Figure 3: $\hbar$-curve according to $\alpha = 5.99$

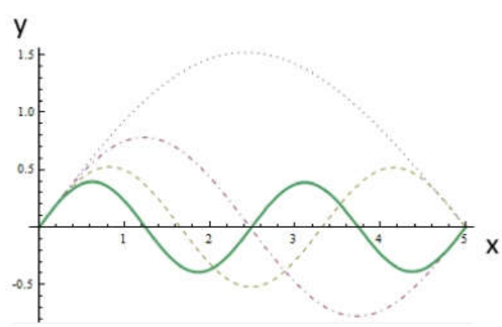

Figure 4: dotted line: Eigenfunction corresponding to λ_1; dot dashed line: Eigenfunction corresponding to λ_2; dashed line: Eigenfunction corresponding to λ_3; thick lines: Eigenfunction corresponding to λ_4 of Example 2

subject to the boundary conditions

$$y(0) = y''(0) = y^{(4)}(0) = y^{(6)}(0) = y^{(8)}(0) = 0,$$
$$y(\pi) = y''(\pi) = y^{(4)}(\pi) = y^{(6)}(\pi) = y^{(8)}(\pi) = 0,$$

where $9 < \alpha \leqslant 10$. The kth exact eigenvalue of this problem is known to be k^{10}.

Table 3: Comparison results of first six eigenvalues of Example 3

λ_k	$\alpha = 9.5$	$\alpha = 9.7$	$\alpha = 9.9$	$\alpha = 9.99$	$\alpha = 9.9999$	exact $\lambda = k^{10}$ for $\alpha = 10$
λ_1	1.748599	1.306696	1.075690	1.006707	1.000066	1
λ_2	728.928601	869.080768	974.250744	1018.989895	1023.949824	1024
λ_3	42282.627032	46348.445365	54048.023602	58515.254210	59043.62485	59049
λ_4	525209.338050	712914.190485	923496.5577796	1035311.328463	1048443.414051	1048576
λ_5	5168605.499169	6378765.105038	8421530.523672	9620208.712720	9765096.934552	9765625
λ_6	21875877.180515	36191783.299387	52015671.224970	59033451.785699	60657173.019026	60466176
CPU time	$73s$	$78s$	$77s$	$70s$	$81s$	$31s$

In our proposed method, we use fewer polynomial terms comparing with another procedure (see [11]) and get more accurate results. For instance, we just calculate $N = 13$ and $N = 7$ terms of approximate solutions, in Examples 1 and 2, respectively; While the author in [11] considered $N = 18$ and $N = 20$ terms in aforementioned examples. Moreover, we obtain as the same number eigenvalues as Hajji, Al-Mdallal, and Allan reported in [11].

5 Conclusion

In this paper, the HAM has been applied to high order fractional Sturm–Liouville problems. The main core of this paper is about to find the eigenvalues and eigenfunctions of these kind of problems. The proposed method has more convenient than the existing methods, since, in spite of using fewer terms to approximate the solution, we obtained more accurate numerical results. Moreover, it is readily used to solve complex fractional Sturm–Liouville problems of an arbitrary high-order. This leads to fewer computations and also more efficiency of HAM in comparison with the one proposed in [11]. Besides, the HAM on high order FSLPs is validated by verifying its convergence.

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