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Research Article

Modal spectral Tchebyshev Petrov–Galerkin stratagem for the time-fractional nonlinear Burgers' equation

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Abstract

Herein, we construct an explicit modal numerical solver based on the spectral Petrov–Galerkin method via a specific combination of shifted Chebyshev polynomial basis for handling the nonlinear time-fractional Burger-type partial differential equation in the Caputo sense. The process reduces the problem to a nonlinear system of algebraic equations. Solving this algebraic equation system will yield the approximate solution's unknown coefficients. Many relevant properties of Chebyshev polynomials are reported, some connection and linearization formulas are reported and proved, and

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all elements of the obtained matrices are evaluated neatly. Also, convergence and error analyses are established. Various illustrative examples demonstrate the applicability and accuracy of the proposed method and depict the absolute and estimated error figures. Besides, the current approach's high efficiency is proved by comparing it with other techniques in the literature.

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Keywords: Time-fractional Burgers' equation; Chebyshev polynomials; Petrov–Galerkin method; Convergence analysis.

1 Introduction

The orthogonal polynomials of the shifted first-kind Chebyshev polynomials (S1KCPs) are defined on the interval $[-1, 1]$. Due to their advantageous characteristics and value in spectral approaches, they play a vital part in the numerical solution of partial differential equations (PDEs).

Using an orthogonal polynomial-based series expansion, spectral methods [11, 25, 19, 20, 10] are numerical approaches that try to approximate the solution of a PDE. These techniques are particularly well adapted to the orthogonality, clustering, and exponential convergence of the S1KCPs.

The S1KCPs were denoted as $T_n(x) = \cos(n\,\theta)$, where n is the degree of the polynomial and $\theta = \cos^{-1}(x)$ is the angle between the x-axis and the point $(x, 0)$ on the unit circle.

The clustering property of the S1KCPs is one of their main benefits. The Chebyshev polynomials are unique among orthogonal polynomial families in grouping at the ends of the interval $[-1, 1]$. This clustering property is beneficial when approximating functions with boundary layers or steep gradients at the endpoints.

Another feature of spectral methods based on Chebyshev polynomials is their exponential convergence. By truncating the series expansion involving the Chebyshev polynomials, spectral methods approximate the solution of a PDE. As the degree of the truncated series increases, the approximation approaches the real solution exponentially. Chebyshev spectral methods are particularly efficient and precise due to their rapid convergence feature.

Chebyshev spectral methods for numerically solving PDEs require modeling the solution as a series expansion with S1KCPs as basis functions. By projecting the PDE onto the basis functions and solving a sequence of algebraic equations, the series expansion coefficients are determined.

The number of basis functions (polynomial degree) chosen depends on the required accuracy and the issue features. Higher degrees provide more precise results but also require more computational resources. In practice, it is critical to balance precision and efficiency.

Chebyshev spectral methods, such as [7, 9, 10], have been effectively used for various fractional PDEs, such as elliptic, parabolic, and hyperbolic equations. They have proven especially useful in problems with smooth solutions, periodic boundary conditions, and unbounded domains.

Finally, the S1KCPs [27] are valuable tools for numerically solving PDEs. They are well-suited for spectral approaches due to their orthogonality, clustering, and exponential convergence qualities. These polynomials can obtain accurate and efficient approximations of PDE solutions, making them a powerful tool in computational mathematics and engineering. Chebyshev polynomials have recently been widely employed to solve several forms of differential problems; for example, see [41, 8, 1, 38, 42, 17, 2, 6, 16, 26].

We can guarantee that knowledge of the properties and applications of S1KCPs in the numerical solution of PDEs is generally established and found in various references and textbooks on numerical methods for PDEs. Among the significant works in this field are [14, 35, 13].

The nonlinear time-fractional equation Burger's equation is a PDE with a fractional derivative in time that incorporates nonlinear convection and diffusion factors. It is a variation of the traditional Burger's equation, a simple model for various physical phenomena, such as fluid flow and traffic movement. Incorporating fractional derivatives in time allows the system to include nonlocal and memory effects.

The general form of the nonlinear time-fractional Burger's equation is given as [18]:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \sigma\, u\, \frac{\partial\, u}{\partial\, x} - \kappa\, \frac{\partial^{2}\, u}{\partial\, x^{2}} = s(x,t), \quad 0 < \alpha < 1, \tag{1}$$

$$\begin{aligned} u(x,0) &= g^{1}(x), \quad 0 < x \leq 1, \\ u(0,t) &= g^{2}(t), \quad u(1,t) = g^{3}(t), \quad 0 < t \leq 1, \end{aligned} \tag{2}$$

where $u(x,t)$ is the unknown function representing the dependent variable, t is time, x is the spatial variable, σ is a constant coefficient that controls the strength of the convection term, κ is a constant coefficient controlling the diffusion term, and $s(x,t)$ is the source term.

Due to the presence of both nonlinear and fractional variables, solving the nonlinear time-fractional Burger's equation analytically is difficult. Numerical approaches, on the other hand, can be used to approximate its solutions. Many techniques, such as finite difference methods, finite element methods, and spectral methods, can be used to deal with fractional derivatives and nonlinearity.

Numerical approaches for solving the nonlinear time-fractional Burger's equation frequently involve griding the spatial domain and using time-stepping methods to approximate the temporal derivatives. Fractional difference operators and fractional integral transformations can approximate fractional derivatives. For further methods that studied other types of fractional differential equations, see [5, 22, 29, 23].

Furthermore, the behavior of the nonlinear time-fractional Burger's equation can exhibit fascinating phenomena, such as the generation of shock waves, solitons, and other nonlinear waves. Memory effects are introduced by the fractional derivative in time, which might alter the transmission and evolution of these nonlinear structures.

Scientists are constantly researching the properties, analytical solutions, and numerical approaches of the nonlinear time-fractional Burger's equation. It is used in a variety of domains, such as fluid dynamics, heat transfer, and nonlinear wave phenomena, where the inclusion of fractional derivatives in time allows a more precise representation of the system dynamics. The interested reader can see the recent diverse numerical methods used to solve Burgers' problem in [15, 31, 44, 36, 24].

The Petrov–Galerkin method [34, 27] is a numerical technique for solving PDEs. It is a variation of the more general Galerkin approach, which aims to approximate a PDE solution by projecting it onto a finite-dimensional trial function subspace. To improve the accuracy and stability of the approximation, the Petrov–Galerkin approach incorporates an additional weighting function known as the test function or the Petrov function.

The Petrov–Galerkin approach [43] is especially effective for PDEs with specific properties, such as those with convection-dominated terms or singularities. It solves the usual Galerkin method's difficulties in capturing precise solutions in these challenging settings.

The Petrov–Galerkin approach [28] has been effectively used for various PDEs, including convection-diffusion, Navier-Stokes, and advection-dominated situations. It provides a versatile framework for dealing with difficult PDEs and is more accurate and stable than the Galerkin technique.

Overall, the Petrov–Galerkin approach is a strong numerical methodology that extends the Galerkin method to meet the difficulties given by specific types of PDEs. It improves the accuracy and stability of the approximation by introducing Petrov functions as extra weighting functions, making it an important tool in the field of numerical PDE solving.

The structure of this article is as follows: The theory of fractional calculus and the relevant properties of Chebyshev polynomials are briefly introduced in section 2. A numerical spectral Petrov–Galerkin technique for solving the time-fractional Burgers' type equation is constructed in section 3. Section 4 discusses the convergence and error analysis of the method. Some numerical examples are given in section 5 to illustrate the theoretical conclusions. Section 6 contains conclusions.

2 Preliminaries and essential relations

2.1 The fractional derivative in the Caputo sense

Definition 1. [32] The Caputo fractional derivative of order s is defined as

$$D_x^s u(x) = \frac{1}{\Gamma(m-s)} \int_0^x (x-y)^{m-s-1} u^{(m)}(y) dy, \quad s>0, \quad x>0, \tag{3}$$

where $m-1 \leqslant s < m, \quad m \in \mathbb{N}$.

The following properties are satisfied by the operator D_x^s for $m-1 \leqslant s < m, \quad m \in \mathbb{N}$,

$$D_x^s c = 0, \text{ (c is a constant)} \tag{4}$$

$$D_x^s\, x^m = \begin{cases} 0, & \text{if } m \in \mathbb{N}_0 \text{ and } m < \lceil s \rceil, \\ \frac{\Gamma(m+1)}{\Gamma(m-s+1)}\, x^{m-s}, & \text{if } m \in \mathbb{N}_0 \text{ and } m \geq \lceil s \rceil, \end{cases} \tag{5}$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and the notation $\lceil \alpha \rceil$ denotes the ceiling function.

2.2 An account on the S1KCPs

Let $T_j^*(x)$ be the S1KCPs defined in the interval $[0,1]$ by $T_j^*(x) = T_j(2\,x-1)$. These polynomials can be defined as [8, 38]

$$T_m^*(x) = m \sum_{k=0}^{m} \frac{(-1)^{m-k}\, 2^{2\,k}\, (m+k-1)!}{(m-k)!\,(2\,k)!}\, x^k, \quad m>0, \tag{6}$$

satisfying the following orthogonality relation with respect to the weight function $\hat{w}(x) = \frac{1}{\sqrt{x\,(1-x)}}$ [8, 38]:

$$\int_0^1 \hat{w}(x)\, T_m^*(x)\, T_n^*(x)\, dx = h_m\, \delta_{m,n}, \tag{7}$$

where

$$h_m = \begin{cases} \pi, & \text{if } m=0, \\ \frac{\pi}{2}, & \text{if } m>0, \end{cases} \tag{8}$$

and

$$\delta_{m,n} = \begin{cases} 1, & \text{if } m=n, \\ 0, & \text{if } m \neq n. \end{cases} \tag{9}$$

The recurrence relation of $T_m^*(x)$ is

$$T^*_{m+1}(x) = 2\,(2\,x-1)\,T^*_m(x) - T^*_{m-1}(x), \tag{10}$$

where $T^*_0(x) = 1$ and $T^*_1(x) = 2\,x-1$.

Moreover, the inversion formula is [8, 38]

$$x^j = 2^{1-2\,j}\,(2\,j)! \sum_{p=0}^{j} \frac{\epsilon_p}{(j-p)!(j+p)!}\, T^*_p(x), \quad j \geq 0, \tag{11}$$

where

$$\epsilon_m = \begin{cases} \frac{1}{2}, & \text{if } m = 0, \\ 1, & \text{otherwise.} \end{cases} \tag{12}$$

The following relation between $T^*_i(x)$ and $U^*_i(x)$ is correct

$$D\,T^*_i(x) = 2i\,U^*_{i-1}(x), \qquad \text{for all } i \geq 1. \tag{13}$$

The following linearization formula is valid

$$T^*_r(x)\,U^*_s(x) = \frac{1}{2}\left(U^*_{s+r}(x) + U^*_{s-r}(x)\right), \qquad \text{for all } r, s \geq 0. \tag{14}$$

Corollary 1. [1] For every positive integer q, the qth derivative of $T^*_j(x)$ can be expressed in terms of their original polynomials as

$$D^q\,T^*_j(x) = \sum_{\substack{p=0 \\ (j+p+q)\ \text{even}}}^{j-q} d_{j,p,q}\; T^*_p(x), \tag{15}$$

where

$$d_{j,p,q} = \frac{j\,2^{2\,q}\,\epsilon_p\,(q)_{\frac{1}{2}(j-p-q)}}{\left(\frac{1}{2}(j-p-q)\right)!\,\left(\frac{1}{2}(j+p+q)\right)_{1-q}}, \tag{16}$$

and ϵ_p is defined in (12).

Lemma 1. [4] For all nonnegative integers m and n, the following linearization formula holds for the S1KCPs:

$$T^*_m(x)\,T^*_n(x) = \frac{1}{2}\left(T^*_{m+n}(x) + T^*_{|m-n|}(x)\right). \tag{17}$$

Lemma 2. Let j and i be any two nonnegative integers. The moments' formula for the S1KCPs are given by

$$x^j\,T_i^*(x)=\sum_{s=i-j}^{i+j}\frac{\Gamma(2j+1)}{4^j\Gamma(i+j-s+1)\Gamma(-i+j+s+1)}\,T_s^*(x). \tag{18}$$

Proof. Multiplying both sides of (11) with $T_i^*(x)$ and direct use of Lemma 1, we get the desired result. □

Remark 1. The following relation is satisfied:

$$\int_0^1 \hat{w}(x)\,U_i^*(x)\,T_r^*(x)\,dx=\sigma_{i,r}, \tag{19}$$

where $U_i^*(x)$ is the shifted Chebyshev polynomials of the second kind and

$$\sigma_{i,r}=\begin{cases}\pi, & \text{if } (i-r)\,\text{even},\ i\geq r,\\ 0, & \text{otherwise.}\end{cases} \tag{20}$$

Remark 2. The following relation is satisfied

$$\int_0^1 x^2\,\hat{w}(x)\,T_i^*(x)\,T_s^*(x)\,dx=\kappa_{i,s}, \tag{21}$$

where

$$\kappa_{i,s}=\frac{\pi}{32\,\epsilon_i\epsilon_s}\begin{cases}1, & \text{if } |s-i|=2,\\ 4, & \text{if } |s-i|=1,\\ 6, & \text{if } s-i=0,\ i>1,\\ 3, & \text{if } s=i=0,\\ 7, & \text{if } s=i=1,\\ 0, & \text{otherwise.}\end{cases} \tag{22}$$

3 Petrov–Galerkin approach for the time-fractional Burgers' equation

In this section, we consider the following time-fractional Burgers' equation [33]:

$$\frac{\partial^\alpha\chi(x,t)}{\partial t^\alpha}+\chi(x,t)\,\frac{\partial\chi(x,t)}{\partial x}-\Psi\,\frac{\partial^2\chi(x,t)}{\partial x^2}=S(x,t),\quad 0<\alpha<1, \tag{23}$$

subject to the following initial and boundary conditions:

$$\begin{aligned} &\chi(x,0) = g(x), \quad 0 < x \leq 1, \\ &\chi(0,t) = \zeta_1(t), \quad \chi(1,t) = \zeta_2(t), \quad 0 < t \leq 1, \end{aligned} \tag{24}$$

where Ψ is the kinematic viscosity and $S(x,t)$ is the source term.

Now, to proceed with our proposed Petrov–Galerkin approach, we will use the following transformation:

$$\chi(x,t) := u(x,t) + \Upsilon(x,t), \tag{25}$$

where

$$\Upsilon(x,t) = (1-x)\ (\chi(0,t) - \chi(0,0)) + x\ (\chi(1,t) - \chi(1,0)) + \chi(x,0), \tag{26}$$

to convert (23) governed by the conditions (24) into the following modified equation:

$$\begin{aligned} &\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\, \frac{\partial u(x,t)}{\partial x} + u(x,t)\, \frac{\partial \Upsilon(x,t)}{\partial x} \\ &\qquad + \Upsilon(x,t)\, \frac{\partial u(x,t)}{\partial x} - \Psi\, \frac{\partial^2 u(x,t)}{\partial x^2} = F(x,t), \quad 0 < \alpha < 1, \end{aligned} \tag{27}$$

governed by the following homogeneous conditions:

$$\begin{aligned} &u(x,0) = 0, \quad 0 < x < 1, \\ &u(0,t) = u(1,t) = 0, \quad 0 < t \leq 1, \end{aligned} \tag{28}$$

where

$$F(x,t) = S(x,t) - \frac{\partial^\alpha \Upsilon(x,t)}{\partial t^\alpha} - \Upsilon(x,t)\, \frac{\partial \Upsilon(x,t)}{\partial x} + \Psi\, \frac{\partial^2 \Upsilon(x,t)}{\partial x^2}. \tag{29}$$

Therefore, instead of solving (23) governed by (24), we can solve the modified equation (27) governed by the homogeneous conditions (28).

3.1 Trial functions

Consider the following basis functions:

$$\lambda_i^*(x) = T_{i+2}^*(x) - T_i^*(x),$$

$$\phi_j^*(t) = t\, T_j^*(t). \tag{30}$$

Remark 3. The polynomials $\lambda_i^*(x)$ can be written alternatively in the following form:

$$\lambda_i^*(x) = -8\, x\, (1-x)\, U_i^*(x). \tag{31}$$

The orthogonality relations of $\lambda_i^*(x)$ and $\phi_j^*(t)$ are given by

$$\int_0^1 \lambda_m^*(x)\, \lambda_n^*(x)\, \frac{1}{(x\,(1-x))^{3/2}}\, dx = 8\,\pi\,\delta_{n,m}, \tag{32}$$

and

$$\int_0^1 \phi_m^*(t)\, \phi_n^*(t)\, \frac{1}{t^2\, \sqrt{t\,(1-t)}}\, dx = h_m\, \delta_{m,n}, \tag{33}$$

where h_m is defined in (8).

Lemma 3. For all nonnegative integers m and n, the following linearization formula holds:

$$\begin{aligned} \lambda_m^*(x)\, \frac{d\,\lambda_i^*(x)}{d\,x} &= -i\, U_{i-m-3}^*(x) + (2\,i+2)\, U_{i-m-1}^*(x) - (i+2)\, U_{i-m+1}^*(x) \\ &\quad + i\, U_{i+m-1}^*(x) - (2\,i+2)\, U_{i+m+1}^*(x) + (i+2)\, U_{i+m+3}^*(x). \end{aligned} \tag{34}$$

Proof. We express $D\,\lambda_i^*(x)$ as a combination of $U_i^*(x)$ via (13). Then we linearize $\lambda_m^*(x)\, D\,\lambda_i^*(x)$ using (14), we get the desired result. □

Lemma 4. For all nonnegative integers m and n, the following linearization formula holds:

$$\phi_n^*(t)\, \phi_j^*(t) = \frac{t^2}{2}\, \left(T_{j+n}^*(t) + T_{j-n}^*(t)\right). \tag{35}$$

Proof. The proof of this lemma is a direct result of Lemma 1. □

Theorem 1. The first and second derivatives of $\psi_m^*(x)$ can be expressed explicitly as

$$\begin{aligned} \frac{d\,\lambda_i^*(x)}{d\,x} &= \sum_{j=0}^{i+2} \varsigma_{j,i}\, T_j^*(x), \\ \frac{d^2\,\lambda_i^*(x)}{d\,x^2} &= \sum_{j=0}^{i+1} \tau_{j,i}\, T_j^*(x), \end{aligned} \tag{36}$$

where

$$\varsigma_{j,i} = 4 \begin{cases} j+1, & \text{if } i+1=j, \\ 2, & \text{if } (i-j)\, odd\,, j>0, \\ 1, & \text{if } (i-j)\, odd\,, j=0, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tau_{j,i} = \delta_j \begin{cases} 8\left(3\,i\,(i+2)-j^2+4\right), & \text{if } (i-j)\, even\,, i \geq j, \\ 0, & \text{otherwise}, \end{cases}$$

$$\delta_j = \begin{cases} 1, & \text{if } j=0, \\ \frac{1}{2}, & \text{otherwise}. \end{cases} \tag{37}$$

Proof. Relations (36) can be deduced after using Corollary 1 when $q = 1, 2$ along with $\lambda_i^*(x)$ defined in (30), then collecting like terms and rearranging the summations. □

3.2 Petrov–Galerkin solution for the time-fractional Burgers' equation

Now, one may set

$$\begin{aligned} \Theta^N(\Omega) &= \text{span}\{\lambda_i^*(x)\,\phi_j^*(t) : i,j = 0,1,\ldots,N\}, \\ \Lambda^N(\Omega) &= \{u \in \Theta^N(\Omega) : u(x,0) = u(0,t) = u(1,t) = 0\}, \end{aligned} \tag{38}$$

where $\Omega = [0,1]^2$. Then any function $u^N(x,t) \in \Lambda^N(\Omega)$ may be written as

$$u^N(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij}\, \lambda_i^*(x)\, \phi_j^*(t). \tag{39}$$

The application of Petrov–Galerkin technique [39] is used to find $u^N(x,t) \in \Lambda^N$ such that

$$\left(\frac{\partial^\alpha u^N(x,t)}{\partial t^\alpha}, T_r^*(x)\, T_s^*(t)\right)_{\omega(x,t)} + \left(u^N(x,t)\, \frac{\partial u^N(x,t)}{\partial x}, T_r^*(x)\, T_s^*(t)\right)_{\omega(x,t)}$$

$$
\begin{aligned}
&+\left(u^N(x,t)\,\frac{\partial\,\Upsilon^N(x,t)}{\partial\,x}, T_r^*(x)\,T_s^*(t)\right)_{\omega(x,t)}\\
&+\left(\Upsilon^N(x,t)\,\frac{\partial\,u^N(x,t)}{\partial\,x}, T_r^*(x)\,T_s^*(t)\right)_{\omega(x,t)}\\
&-\Psi\left(\frac{\partial^2\,u^N(x,t)}{\partial\,x^2}, T_r^*(x)\,T_s^*(t)\right)_{\omega(x,t)}\\
&\quad=\left(F(x,t), T_r^*(x)\,T_s^*(t)\right)_{\omega(x,t)},\quad 0\le r,s\le N,
\end{aligned}
\tag{40}
$$

where $T_r^*(x)\,T_s^*(t)$ is the test function and $\omega(x,t)=\hat{w}(x)\,\hat{w}(t)$.

Therefore, (40) can be written after using the definition of $u^N(x,t)$ (39) as

$$
\begin{aligned}
&\sum_{i=0}^{N}\sum_{j=0}^{N}c_{ij}\,g_{i,r}\,b_{j,s}+\sum_{m=0}^{N}\sum_{n=0}^{N}\sum_{i=0}^{N}\sum_{j=0}^{N}c_{mn}\,c_{ij}\,d_{m,i,r}\,h_{n,j,s}\\
&\quad+\sum_{m=0}^{N}\sum_{n=0}^{N}\sum_{i=0}^{N}\sum_{j=0}^{N}c_{mn}\,a_{ij}\,d_{m,i,r}\,h_{n,j,s}+\sum_{m=0}^{N}\sum_{n=0}^{N}\sum_{i=0}^{N}\sum_{j=0}^{N}b_{mn}\,c_{ij}\,d_{m,i,r}\,h_{n,j,s}\\
&\quad-\Psi\sum_{i=0}^{N}\sum_{j=0}^{N}c_{ij}\,Z_{i,r}\,Q_{j,s}=F_{r,s},\quad 0\le r,s\le N,
\end{aligned}
\tag{41}
$$

where a_{ij} and b_{ij} are determined from the following relations:

$$
\begin{aligned}
a_{ij}&=\frac{1}{8\,\pi\,h_j}\int_0^1\int_0^1\frac{\partial\,\Upsilon^N(x,t)}{\partial\,x}\,\lambda_i^*(x)\,\phi_j^*(t)\,\frac{1}{(x\,(1-x))^{3/2}}\,\frac{1}{t^2\,\sqrt{t\,(1-t)}}dx\,dt,\\
b_{ij}&=\frac{1}{8\,\pi\,h_j}\int_0^1\int_0^1\Upsilon^N(x,t)\,\lambda_i^*(x)\,\phi_j^*(t)\,\frac{1}{(x\,(1-x))^{3/2}}\,\frac{1}{t^2\,\sqrt{t\,(1-t)}}dx\,dt.
\end{aligned}
\tag{42}
$$

Also,

$$
\begin{aligned}
g_{i,r}&=\left(\lambda_i^*(x), T_r^*(x)\right)_{\hat{w}(x)}, & b_{j,s}&=\left(\frac{d^\alpha\,\phi_j^*(t)}{d\,t^\alpha}, T_s^*(t)\right)_{\hat{w}(t)},\\
d_{m,i,r}&=\left(\lambda_m^*(x)\,\frac{d\,\lambda_i^*(x)}{d\,x}, T_r^*(x)\right)_{\hat{w}(x)}, & h_{n,j,s}&=\left(\phi_n^*(t)\,\phi_j^*(t), T_s^*(t)\right)_{\hat{w}(t)},\\
Z_{i,r}&=\left(\frac{d^2\,\lambda_i^*(x)}{d\,x^2}, T_r^*(x)\right)_{\hat{w}(x)}, & Q_{j,s}&=\left(\phi_j^*(t), T_s^*(t)\right)_{\hat{w}(t)},
\end{aligned}
$$

and $F_{r,s} = (f(x,t), T_r^*(x)\, T_s^*(t))_{\omega(x,t)}$.

Theorem 2. The elements $g_{i,r}$, $b_{j,s}$, $d_{m,i,r}$, $h_{n,j,s}$, $Z_{i,r}$, and $Q_{j,s}$ are given by

$$g_{i,r} = \begin{cases} -\frac{\pi}{2\epsilon_i}, & \text{if } \; r-i=0, \\ \frac{\pi}{2}, & \text{if } \; r-i=2, \, , \\ 0, & \text{otherwise}, \end{cases}$$

$$b_{j,s} = \pi\,(-1)^j\,\Gamma\left(\frac{3}{2}-\alpha\right)\;{}_4F_3\left(\left.\begin{matrix} 2,-j,j,\frac{3}{2}-\alpha \\ \\ \frac{1}{2},-s-\alpha+2,s-\alpha+2 \end{matrix}\right|1\right),$$

$$d_{m,i,r} = -i\,\sigma_{i-m-3,r} + (2\,i+2)\,\sigma_{i-m-1,r} - (i+2)\,\sigma_{i-m+1,r} + i\,\sigma_{i+m-1,r} - (2\,i+2)\,\sigma_{i+m+1,r} + (i+2)\,\sigma_{i+m+3,r},$$

$$h_{n,j,s} = \frac{1}{2}\left(\kappa_{j+n,s} + \kappa_{j-n,s}\right),$$

$$Z_{i,r} = \begin{cases} 4\,\pi\,\left(3\,i\,(i+2) - r^2 + 4\right), & \text{if } (i-r)\, even,\; i \geq r, \\ 0, & \text{otherwise}, \end{cases}$$

$$Q_{j,s} = \begin{cases} \frac{\pi}{8\epsilon_i}, & \text{if } s-j=1, \\ \frac{\pi}{8\epsilon_s}, & \text{if } j-s=1, \\ \frac{\pi}{4\epsilon_j}, & \text{if } s-j=0, \\ 0, & \text{otherwise}. \end{cases} \tag{43}$$

Proof. The elements $g_{i,r}$ can be obtained after using the orthogonality relation (7) and the definition of $\lambda_i^*(x)$ defined in (30), then collecting like terms.

To find the elements $b_{j,s}$, based on the definition of Caputo's fractional derivative (3) and the power form of $T_j^*(x)$ (6), one can write

$$\begin{aligned} b_{j,s} &= \left(\frac{d^\alpha\,\phi_j^*(t)}{d\,t^\alpha}, T_s^*(t)\right)_{\hat{w}(t)} \\ &= \sum_{k=0}^{j} \frac{j 2^{2k}(k+1)!(-1)^{j-k}(j+k-1)!}{(2k)!(j-k)!(-\alpha+k+1)!} \int_0^1 T_s^*(t) t^{-\alpha+k+1}\,\hat{w}(t)\,dt \end{aligned}$$

$$= \sum_{k=0}^{j} \frac{\sqrt{\pi} j 4^k (-1)^{j-k} \Gamma(k+2)\Gamma(j+k)\Gamma\left(k-\alpha+\frac{3}{2}\right)}{\Gamma(2k+1)\Gamma(j-k+1)\Gamma(k-s-\alpha+2)\Gamma(k+s-\alpha+2)}. \quad (44)$$

The last relation can be summed to give the following result:

$$b_{j,s} = \pi\,(-1)^j\,\Gamma\left(\frac{3}{2}-\alpha\right)\ {}_4F_3\left(\begin{array}{c|}2,-j,j,\frac{3}{2}-\alpha\\ \\ \frac{1}{2},-s-\alpha+2,s-\alpha+2\end{array}\,1\right). \quad (45)$$

The elements $d_{m,i,r}$ can be obtained after using Lemma 3 along with Remark 1.

Similarly, the elements $h_{n,j,s}$ can be obtained after using Lemma 4 along with Remark 2.

The elements $Z_{i,r}$ can be obtained after using Theorem 1 along with the orthogonality relation (7), collecting like terms, and rearranging the summations.

Finally, the elements $Q_{j,s}$ can be obtained after using Lemma 2 along with the orthogonality relation (7) and doing some computations. □

Remark 4. The inner product of $(u(x,t),v(x,t))_{\omega(x,t)}$, $(u(x),v(x))_{\hat{w}(x)}$, and $(u(t),v(t))_{\hat{w}(t)}$ are defined as

$$\begin{aligned}(u(x,t),v(x,t))_{\omega(x,t)} &= \int_0^1\int_0^1 u(x,t)\,v(x,t)\,\omega(x,t)\,dx\,dt,\\ (u(x),v(x))_{\hat{w}(x)} &= \int_0^1 u(x)\,v(x)\,\hat{w}(x)\,dx,\\ (u(t),v(t))_{\hat{w}(t)} &= \int_0^1 u(t)\,v(t)\,\hat{w}(t)\,dt. \quad (46)\end{aligned}$$

Remark 5. The $(N+1)\times(N+1)$ nonlinear system of equations in (41) in the unknown expansion coefficient c_{ij} can be solved through a suitable numerical solver such as Newton's iterative technique.

4 Error bound

Herein, we give an upper bound of the absolute errors (AEs) using Lagrange interpolation polynomials.

Algorithm 1: Coding algorithm for the proposed technique

Input Ψ, $g(x)$, $\zeta_1(t)$, $\zeta_2(t)$, α, and $S(x,t)$.

Step 1. Using transformation (25) to convert the nonlinear TFCE (23)–(24), into modified equation (27)–(28)

Step 2. Assume an approximate solution $u^N(x,t)$ as in (39).

Step 3. Apply Petrov–Galerkin method to obtain the system in (41).

Step 4. Use Theorem 2 to get the elements of $g_{i,r}$, $b_{j,s}$, $d_{m,i,r}$, $h_{n,j,s}$, $Z_{i,r}$, and $Q_{j,s}$.

Step 5. Use *FindRoot* command with initial guess $\{c_{ij} = 10^{-i-j},\ i,j : 0,1,\dots,N\}$, to solve the system (41) to get c_{ij}.

Output $u^N(x,t)$.

Let $u^N(x,t) \in \Lambda^N(\Omega)$ be the best approximation of $u(x,t)$; then, the definition of the best approximation enables us to write the following inequality:

$$||u(x,t) - u^N(x,t)||_\infty \leq ||u(x,t) - v^N(x,t)||_\infty, \quad \text{for all } v^N(x,t) \in \Lambda^N(\Omega). \tag{47}$$

Moreover, the previous inequality is also true if $\hat{u}^N$ denotes the interpolating polynomial for $u(x,t)$ at points (x_i,t_j), where x_i are the roots of $\lambda_i^*(x)$, while t_j are the roots of $\phi_j^*(t)$.

Using similar steps as in [12, 40], one has

$$\begin{aligned} u(x,t) - v^N(x,t) = {} & \frac{\partial^{N+1} u(\eta,t)}{\partial x^{N+1}\,(N+1)!} \prod_{i=0}^{N}(x-x_i) + \frac{\partial^{N+1} u(x,\mu)}{\partial t^{N+1}\,(N+1)!} \prod_{j=0}^{N}(t-t_j) \\ & - \frac{\partial^{2N+2} u(\hat{\eta},\hat{\mu})}{\partial x^{N+1}\,\partial t^{N+1}\,((N+1)!)^2} \prod_{i=0}^{N}(x-x_i) \prod_{j=0}^{N}(t-t_j), \end{aligned} \tag{48}$$

where $\eta, \hat{\eta}, \mu, \hat{\mu} \in [0,1]$.

Now,

$$\begin{aligned} ||u(x,t) - v^N(x,t)||_\infty \leq {} & \max_{(x,t)\in\Omega} \left| \frac{\partial^{N+1} u(\eta,t)}{\partial x^{N+1}} \right| \frac{||\prod_{i=0}^{N}(x-x_i)||_\infty}{(N+1)!} \\ & + \max_{(x,t)\in\Omega} \left| \frac{\partial^{N+1} u(x,\mu)}{\partial t^{N+1}} \right| \frac{||\prod_{j=0}^{N}(t-t_j)||_\infty}{(N+1)!} \end{aligned}$$

$$- \max_{(x,t)\in\Omega} \left| \frac{\partial^{2N+2}\, u(\hat{\eta},\hat{\mu})}{\partial\, x^{N+1}\,\partial\, t^{N+1}} \right| \frac{||\prod_{i=0}^{N}(x-x_i)||_\infty \, ||\prod_{j=0}^{N}(t-t_j)||_\infty}{((N+1)!)^2}. \tag{49}$$

Since u is a smooth function on Ω, then there exist three constants ℓ_1, ℓ_2, and ℓ_3 such that

$$\max_{(x,t)\in\Omega} \left| \frac{\partial^{N+1}\, u(x,t)}{\partial\, x^{N+1}} \right| \le \ell_1, \quad \max_{(x,t)\in\Omega} \left| \frac{\partial^{N+1}\, u(x,\mu)}{\partial\, t^{N+1}} \right| \le \ell_2,$$
$$\max_{(x,t)\in\Omega} \left| \frac{\partial^{2N+2}\, u(\hat{\eta},\hat{\mu})}{\partial\, x^{N+1}\,\partial\, t^{N+1}} \right| \le \ell_3. \tag{50}$$

To minimize the factor $||\prod_{i=0}^{N}(x-x_i)||_\infty$, let us use the one-to-one mapping $x = \frac{1}{2}\,(z+1)$ between the intervals $[-1,1]$ and $[0,1]$ to deduce that

$$\begin{aligned}
\min_{x_i\in[0,1]} \max_{x\in[0,1]} \left| \prod_{i=0}^{N}(x-x_i) \right| &= \min_{z_i\in[-1,1]} \max_{z\in[-1,1]} \left| \prod_{i=0}^{N} \frac{1}{2}\,(z-z_i) \right| \\
&= \left(\frac{1}{2}\right)^{N+1} \min_{z_i\in[-1,1]} \max_{z\in[-1,1]} \left| \prod_{i=0}^{N}(z-z_i) \right| \\
&= \left(\frac{1}{2}\right)^{N+1} \min_{z_i\in[-1,1]} \max_{z\in[-1,1]} \left| \frac{\lambda_{N-1}(z)}{\bar{\lambda}_N} \right|,
\end{aligned} \tag{51}$$

where $\bar{\lambda}_N = 2^N$ is the leading coefficient of $\lambda_{N-1}(z) = T_{N+1}(z) - T_{N-1}(z)$ and z_i are the roots of $\lambda_{N-1}(z)$.

Also, the factor $||\prod_{j=0}^{N}(t-t_j)||_\infty$ may be minimized by using the one-to-one mapping $t = \frac{1}{2}\,(\bar{t}+1)$ between the intervals $[-1,1]$ and $[0,1]$ to deduce that

$$\min_{t_j\in[0,1]} \max_{t\in[0,1]} \left| \prod_{j=0}^{N}(t-t_j) \right| = \left(\frac{1}{2}\right)^{N+1} \min_{\bar{t}_j\in[-1,1]} \max_{\bar{t}\in[-1,1]} \left| \frac{\phi_N(\bar{t})}{\hat{\phi}_N} \right|, \tag{52}$$

where $\hat{\phi}_N = 2^{N-2}$ is the leading coefficient of $\phi_N(\bar{t}) = \left(\frac{\bar{t}+1}{2}\right) T_N(\bar{t})$ and $\bar{t}_j$ are the roots of $\phi_N(\bar{t})$.

Since

$$\max_{z\in[-1,1]} |\lambda_{N-1}(z)| = |T_{N+1}(1)| + |T_{N-1}(1)| = 2,$$

$$\max_{\bar{t}\in[-1,1]} |\phi_{N+1}(\bar{t})| = |\phi_{N+1}(1)| = 1, \tag{53}$$

then, inequalities (50), (51), (52), and (53) enable us to get the following desired result:

$$||u(x,t)-u^N(x,t)||_\infty \leq \ell_1\, \frac{(\frac{1}{2})^N}{\bar{\lambda}_N\,(N+1)!} + \ell_2\, \frac{(\frac{1}{2})^{N+1}}{\hat{\phi}_N\,(N+1)!} + \ell_3\, \frac{(\frac{1}{2})^{2\,N+1}}{\bar{\lambda}_N\,\hat{\phi}_N\,((N+1)!)^2}, \tag{54}$$

which represents an upper bound of the AE.

5 Illustrative examples

Example 1. [33, 37] Consider the time-fractional Burgers' equation of the form

$$\frac{\partial^\alpha\, u(x,t)}{\partial\, t^\alpha} + u(x,t)\, \frac{\partial\, u(x,t)}{\partial\, x} - 2\, \frac{\partial^2\, u(x,t)}{\partial\, x^2} = g(x,t), \tag{55}$$

subject to the following initial and boundary conditions:

$$\begin{aligned} &u(x,0) = 0, \quad 0 < x \leq 1, \\ &u(0,t) = u(1,t) = 0, \quad 0 < t \leq 1, \end{aligned} \tag{56}$$

where $g(x,t) = \frac{2}{\Gamma(3-\alpha)}\, t^{2-\alpha} \sin(\pi\, x) + \pi\, t^4 \sin(\pi\, x) \cos(\pi\, x) + 2\, \pi^2\, t^2 \sin(\pi\, x)$, and $u(x,t) = t^2 \sin(\pi\, x)$ is the exact solution of this problem.

Table 1 gives a comparison of AE between our method and the method in [33] at $\alpha = 0.7$ and $\alpha = 0.8$. Figure 1 shows the maximum absolute error (MAE) at different values of N when $\alpha = 0.5$. Table 2 gives a comparison of L_∞-error between our method and methods in [37, 33] at different values of α. Figure 2 shows the AE (left) and approximate solution (right) at $\alpha = 0.9, N = 14$.

Example 2. [33] Consider the time-fractional Burgers' equation of the form

$$\frac{\partial^\alpha\, u(x,t)}{\partial\, t^\alpha} + u(x,t)\, \frac{\partial\, u(x,t)}{\partial\, x} - \frac{\partial^2\, u(x,t)}{\partial\, x^2} = g(x,t), \tag{57}$$

subject to the following initial and boundary conditions:

$$\begin{aligned} &u(x,0) = 0, \quad 0 < x \leq 1, \\ &u(0,t) = t^2, \quad u(1,t) = -t^2, \quad 0 < t \leq 1, \end{aligned} \tag{58}$$

Table 1: Comparison of AE for Example 1

	$\alpha = 0.7$		$\alpha = 0.8$	
x	Method in [33]	Our method at $M = 14$	Method in [33]	Our method at $N = 14$
0.1	2.18701×10^{-5}	5.55112×10^{-17}	8.10773×10^{-7}	5.55112×10^{-17}
0.2	4.20505×10^{-5}	1.11022×10^{-16}	1.57153×10^{-6}	1.11022×10^{-16}
0.3	5.88630×10^{-5}	2.22045×10^{-16}	2.22713×10^{-6}	1.11022×10^{-16}
0.4	7.06958×10^{-5}	3.33067×10^{-16}	2.71560×10^{-6}	0
0.5	7.61346×10^{-5}	2.22045×10^{-16}	2.97247×10^{-6}	0
0.6	7.41772×10^{-5}	3.33067×10^{-16}	2.94204×10^{-6}	0
0.7	6.44989×10^{-5}	4.44089×10^{-16}	2.59370×10^{-6}	1.11022×10^{-16}
0.8	4.76900×10^{-5}	3.33067×10^{-16}	1.93833×10^{-6}	1.11022×10^{-16}
0.9	2.53573×10^{-5}	5.55112×10^{-17}	1.03759×10^{-6}	5.55112×10^{-17}

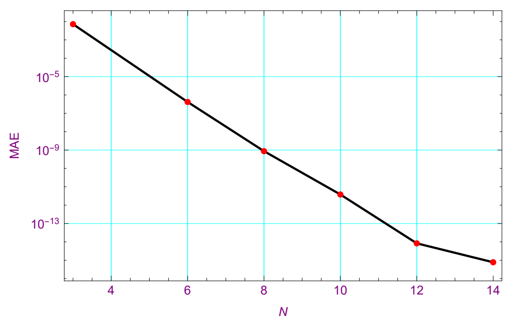

Figure 1: The MAE of Example 1 at $\alpha = 0.5$.

Table 2: Comparison of L_∞-error of Example 1

α	Method in [37] at $N = 2^7$, $M = 2^{12}$	Method in [33] at $N = 2^7$, $M = 2^{12}$	Our method at $N = 14$
0.2	5.47402×10^{-5}	4.76511×10^{-5}	3.33067×10^{-16}
0.3	5.46367×10^{-5}	4.75486×10^{-5}	3.33067×10^{-16}
0.4	5.44879×10^{-5}	4.74015×10^{-5}	2.22045×10^{-16}
0.5	5.42862×10^{-5}	4.72029×10^{-5}	7.77156×10^{-16}

where $g(x,t) = \frac{2}{\Gamma(3-\alpha)}\, t^{2-\alpha} \cos(\pi\, x) - \pi\, t^4 \sin(\pi\, x) \cos(\pi\, x) + \pi^2\, t^2 \cos(\pi\, x)$, and $u(x,t) = t^2 \cos(\pi\, x)$ is the exact solution of this problem.

Table 3 reports the AE at $N = 12$ and $N = 14$ when $\alpha = 0.5$. Figure 3 shows the MAE at different values of N when $\alpha = 0.9$. Table 4 gives a comparison of L_∞-error between our method and method in [33] at different

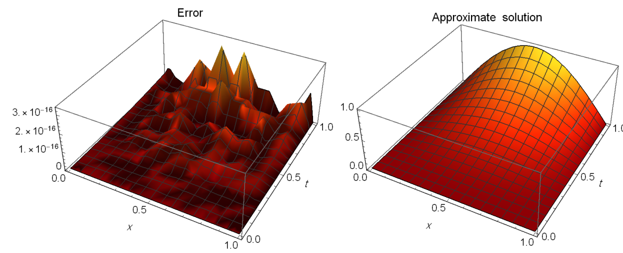

Figure 2: The AE (left) and approximate solution (right) for Example 1 at $\alpha = 0.9, N = 14$.

values of α. Figure 4 shows the exact and approximate solutions at $\alpha = 0.4, N = 14$.

Table 3: The AE of Example 2 at $\alpha = 0.5$

	$N=12$			$N=14$		
x	$t=0.1$	$t=0.5$	$t=0.9$	$t=0.1$	$t=0.5$	$t=0.9$
0.1	5.05238×10^{-16}	1.26565×10^{-14}	4.10366×10^{-14}	6.50521×10^{-19}	6.93889×10^{-18}	4.16334×10^{-17}
0.2	1.22949×10^{-15}	3.07462×10^{-14}	9.96425×10^{-14}	2.60209×10^{-18}	1.38778×10^{-17}	5.55112×10^{-17}
0.3	1.43115×10^{-15}	3.57214×10^{-14}	1.14991×10^{-13}	8.67362×10^{-19}	2.77556×10^{-17}	1.11022×10^{-16}
0.4	1.04387×10^{-15}	2.59792×10^{-14}	8.32667×10^{-14}	1.95156×10^{-18}	1.73472×10^{-17}	1.38778×10^{-17}
0.5	1.88161×10^{-19}	1.16453×10^{-17}	2.50372×10^{-17}	2.69252×10^{-19}	1.10588×10^{-17}	5.19085×10^{-17}
0.6	1.04431×10^{-15}	2.60071×10^{-14}	8.32112×10^{-14}	2.81893×10^{-18}	3.81639×10^{-17}	1.24901×10^{-16}
0.7	1.43071×10^{-15}	3.56937×10^{-14}	1.15047×10^{-13}	8.67362×10^{-19}	1.38778×10^{-17}	0
0.8	1.22949×10^{-15}	3.07601×10^{-14}	9.95871×10^{-14}	2.60209×10^{-18}	2.77556×10^{-17}	0
0.9	5.05238×10^{-16}	1.26427×10^{-14}	4.10644×10^{-14}	6.50521×10^{-19}	2.08167×10^{-17}	9.71445×10^{-17}

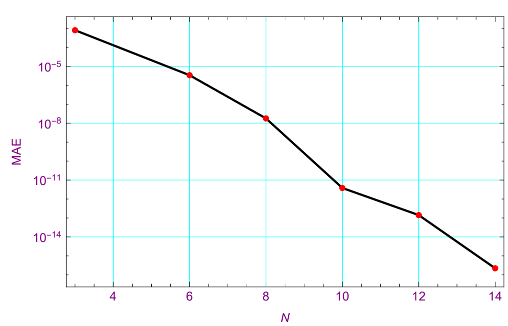

Figure 3: The MAE of Example 2 at $\alpha = 0.9$.

Table 4: Comparison of L_∞-error of Example 2

	Method in [33]		
α	$N=2^7,\, M=2^{12}$	$M=2^7,\, N=2^{11}$	Our method at $N=14$
0.2	1.06427×10^{-5}	9.66513×10^{-6}	4.44089×10^{-16}
0.3	1.06304×10^{-5}	9.60898×10^{-6}	4.44089×10^{-16}
0.4	1.06126×10^{-5}	9.53096×10^{-6}	3.33067×10^{-16}
0.5	1.05886×10^{-5}	9.42186×10^{-6}	3.33067×10^{-16}

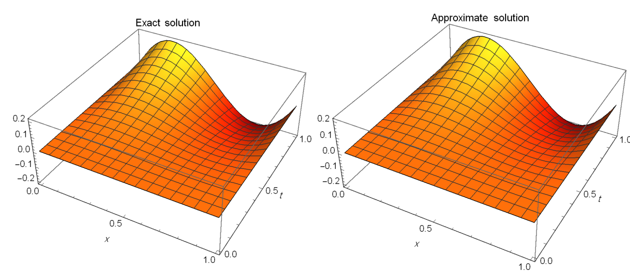

Figure 4: The exact and approximate solutions of Example 2 at $\alpha = 0.4, N = 14$.

Example 3. [33] Consider the time-fractional Burgers' equation of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\,\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t), \tag{59}$$

subject to the following initial and boundary conditions:

$$\begin{aligned} &u(x,0) = 0, \quad 0 < x \le 1,\\ &u(0,t) = t^2, \quad u(1,t) = e\,t^2, \quad 0 < t \le 1, \end{aligned} \tag{60}$$

where $g(x,t) = \frac{2}{\Gamma(3-\alpha)}\, e^x\, t^{2-\alpha} + t^4\, e^{2x} - t^2\, e^x$ and $u(x,t) = t^2\, e^x$ is the exact solution of this problem.

Table 5 gives a comparison of AE between our method and method in [33] at $\alpha = 0.9$. Table 6 gives a comparison of L_∞-error between our method and methods in [33] at different values of α. Figure 5 shows the AE (left) and approximate solution (right) at $\alpha = 0.3, N = 12$.

Table 5: Comparison of AE for Example 3 at $\alpha = 0.9$

	Method in [33]		
x	$N = 200, \Delta t = 0.0005$	$N = 80, \Delta t = 0.001$	Our method at $N = 12$
0.1	6.18496×10^{-9}	2.23568×10^{-7}	5.55112×10^{-17}
0.2	7.14787×10^{-9}	3.99993×10^{-7}	1.11022×10^{-16}
0.3	3.49621×10^{-9}	5.30327×10^{-7}	2.22045×10^{-16}
0.4	3.99695×10^{-9}	6.14693×10^{-7}	2.22045×10^{-16}
0.5	1.42437×10^{-8}	6.52199×10^{-7}	2.22045×10^{-16}
0.6	2.56063×10^{-8}	6.40791×10^{-7}	0
0.7	3.55155×10^{-8}	5.76962×10^{-7}	2.22045×10^{-16}
0.8	3.98360×10^{-8}	4.55240×10^{-7}	0
0.9	3.17808×10^{-8}	2.67253×10^{-7}	0

Table 6: Comparison of L_∞-error of Example 3

	Method in [33]		
α	$N = 2^7$, $M = 2^{11}$	$M = 2^7$, $N = 2^4$	Our method at $N = 12$
0.2	5.69465×10^{-7}	1.97043×10^{-5}	2.22045×10^{-16}
0.3	5.66088×10^{-7}	1.99857×10^{-5}	2.22045×10^{-16}
0.4	5.61387×10^{-7}	2.03716×10^{-5}	2.22045×10^{-16}
0.5	5.54967×10^{-7}	2.09231×10^{-5}	2.22045×10^{-16}

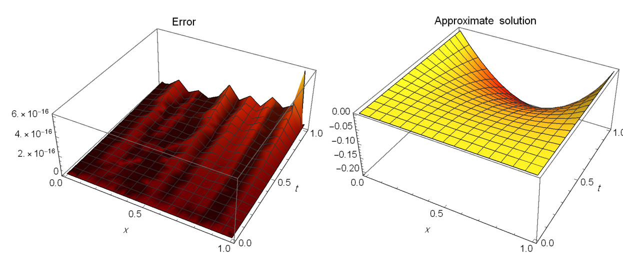

Figure 5: The AE (left) and approximate solution (right) for Example 3 at $\alpha = 0.3, N = 12$.

Example 4. Consider the following time-fractional Burgers' equation of two dimensional

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} + u(x,y,t)\left(\frac{\partial u(x,y,t)}{\partial x} + \frac{\partial u(x,y,t)}{\partial y}\right) - \frac{\partial^2 u(x,y,t)}{\partial x^2} - \frac{\partial^2 u(x,y,t)}{\partial y^2} = g(x,y,t), \tag{61}$$

subject to the following initial and boundary conditions:

$$\begin{aligned} &u(x,y,0) = 0, \quad 0 < x,y \leq 1,\\ &u(0,y,t) = t^2 e^{-y}, \quad u(1,y,t) = t^2 e^{1-y}, \quad 0 < y,t \leq 1,\\ &u(x,0,t) = t^2 e^{x}, \quad u(x,1,t) = t^2 e^{x-1}, \quad 0 < x,t \leq 1, \end{aligned} \tag{62}$$

where $g(x,t) = \frac{2\, e^{x-y}}{\Gamma(3-\alpha)}\, t^{2-\alpha} - 2\, t^2\, e^{x-y}$ and $u(x,t) = t^2\, e^{x-y}$ is the exact solution of this problem.

Table 7 illustrates the AE at different values of t when $N = 6$ and $\alpha = 0.2$. Figure 6 shows the AE at different values of t at $N = 6$ and $\alpha = 0.2$.

Table 7: The AE of Example 4 at $\alpha = 0.2$

$x = y$	$t = 0.3$	$t = 0.6$	$t = 0.9$
0.2	5.50355×10^{-14}	2.19933×10^{-13}	4.94896×10^{-13}
0.4	4.41578×10^{-14}	1.7732×10^{-13}	4.01347×10^{-13}
0.6	4.40518×10^{-14}	1.75651×10^{-13}	3.92865×10^{-13}
0.8	5.50272×10^{-14}	2.19789×10^{-13}	4.94157×10^{-13}

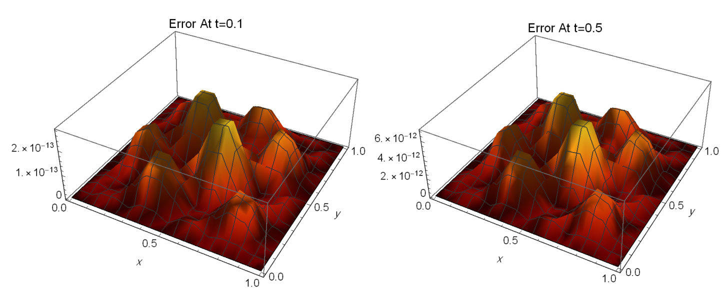

Figure 6: The AE for Example 4 at $\alpha = 0.2, N = 6$.

6 Concluding remarks

In this study, using a special combination of shifted Chebyshev polynomial bases, we built an explicit modal numerical solution based on the spectral Petrov–Galerkin technique to handle the nonlinear time-fractional Burger-type PDE in the Caputo sense. The procedure reduces the issue to a set of nonlinear algebraic equations. Numerous important Chebyshev polynomial characteristics were reported, along with various connection and linearization equations that were mentioned and verified. All components of the resultant matrices were also elegantly assessed. Additionally, studies of convergence and error were created. The applicability and accuracy of the suggested technique were illustrated through a number of illustrative instances, which also show the absolute and anticipated error rates. As an expected future work, we aim to employ the developed theoretical results in this paper along with suitable spectral methods to treat some other problems, for instance, [21, 3, 30]. All codes were written and debugged by *Mathematica* 11 on an HP Z420 Workstation, Processor: Intel (R) Xeon(R) CPU E5-1620 - 3.6 GHz, 16 GB RAM DDR3, and 512 GB storage.

Conflict of Interests

The authors have no conflicts of interest to declare.

Data Availability

No data associated with this research.

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