Iranian Journal of Numerical Analysis and Optimization
Vol. 9, No. 1, (2019), pp 1–16
DOI:10.22067/ijnao.v9i1.53910

Transformation to a fixed domain in LP modelling for a class of optimal shape design problems

H.H. Mehne* and M.H. Farahi

Abstract

A class of optimal shape design problems is studied in this paper. The boundary shape of a domain is determined such that the solution of the underlying partial differential equation matches, as well as possible, a given desired state. In the original optimal shape design problem, the variable domain is parameterized by a class of functions in such a way that the optimal design problem is changed to an optimal control problem on a fixed domain. Then, the resulting distributed control problem is embedded in a measure theoretical form, in fact, an infinite-dimensional linear programming problem. The optimal measure representing the optimal shape is approximated by a solution of a finite-dimensional linear programming problem. The method is evaluated via a numerical example.

Keywords: Approximation; Optimal shape design; Linear programming; Measure theory.

1 Introduction

An optimal shape design (OSD) problem is concerned with the optimization of a performance index depending on the shape of some region. Many-faceted problems naturally arise in engineering applications with the goal of designing a specific structure in an optimal sense. Typical applications are design of a nozzle [12], airfoil boundary [4, 10], and spacecraft shape [6] with respect to specific optimality conditions.

---

*Corresponding author
Received 16 February 2016; revised 3 July 2016; accepted 11 November 2017
H.H. Mehne
Aerospace Research Institute, Tehran, 14665-834, Iran. e-mail: hmehne@ari.ac.ir

M.H. Farahi
Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran. e-mail: farahi@math.um.ac.ir

There are several methods for a numerical solution of an OSD problem, for example, gradient based methods [2], the Newton method [9, 15], sequential quadratic method [19], and fictitious domain method [8]. In the above mentioned methods, computation of solution is a time-consuming task, because they need to solve many boundary value problems. Moreover, these methods are iterative and they need to have an initial guess for solution.

The main focus of this article is to find an appropriate formulation of the optimal shape design problem that is attractive for consistent numerical computation. The advantages of the proposed method lie in the fact that the method is not iterative, that it is self-starting, and that it does not need to solve the corresponding boundary value problem. Moreover, in the modeling of the OSD problem, differentiability of the cost function being a limiting condition in the above mentioned methods, is reduced to a measurability condition. In the methods of [9], the control and state functions are discretized and the linearity of variational forms is an inherent condition, while it is not a restrictive condition in the proposed method, where a general variational form is studied.

In the present work, the variable domain is first parameterized by a class of functions in such a way that the OSD problem changes to an optimal control problem accompanied with partial differential equations on a fixed domain. The converted problem can be solved by extending the measure theoretical method proposed by Rubio [16]. The methods of this type were frequently and successfully applied to different OSD problems. In [3] the problem of finding the upper wall of a nozzle was discussed where the goal is to match the velocity of flow to a constant prescribed value in a given subregion near exit. It also includes details about the measure theoretical approach to solve OSD problems. The same problem for pressure of the flow was given in [12] with more numerical simulations to study the sensitivity analysis and stability of the method. In [4] the problem of finding the optimum shape for a low speed airfoil has been solved by the method. A problem concerning with the control of thermoelastic deformation was also presented in [13], where the extension of the method of [3] has been used to solve it.

The main difference of the present work with the above mentioned researches is that the optimization occurs in a completely unknown two-dimensional area and that there is no restriction to any special form of the problem. Such a generalization of the domain is also presented in [7], which is based on the domain discretization, while in the present paper, the transformation of domain is used.

2 The optimal shape design problem

The set of admissible shapes in the definition of an OSD problem is the set of all possible shapes in which we seek for the best. In our problem definition, let consider a class of OSD problems where the admissible shapes are described by two-dimensional domains with a free boundary:

$$\tilde{\Omega} = \{\tilde{x} = (\tilde{x}_1, \tilde{x}_2)^T \in \mathbb{R}^2 \ \mid \ \tilde{x}_2 \in I, \ \tilde{x}_1 \in (0, u(\tilde{x}_2))\},$$

where $I = (0, 1)$. The set of admissible shapes is usually parameterized or discretized in order to reduce the search process from the class of admissible shapes to a set of admissible functions. So, we parameterize the above domains by a function $u \in \mathcal{U}_{ad}$, where

$$\mathcal{U}_{ad} = \{u \in C^1(I) \mid u(0) = u(1) = 1, \ u(\tilde{x}_2) \in [\beta_1, \beta_2], \ \frac{du}{d\tilde{x}_2}(\tilde{x}_2) \in [\alpha_1, \alpha_2]\},$$

where $\beta_1 > 0$, β_2, α_1, and α_2 are given.

By measuring the desirability of admissible shapes, let $\tilde{J}$ be a cost function depending on a function $\tilde{y}$ and its gradient as follows:

$$\tilde{J} = \int_{\tilde{\Omega}} \tilde{f}(\tilde{y}, \nabla \tilde{y}) d\tilde{x}, \tag{1}$$

where $\tilde{f}$ is a nonlinear real-valued measurable function and $\tilde{y}$ is defined by the solution of a second order elliptic boundary value problem on $\tilde{\Omega}$. The boundaries of the feasible domains are divided into the moving boundary part

$$\tilde{\Gamma}^M = \{(\tilde{x}_1, \tilde{x}_2)^T \in \mathbb{R}^2 \ \mid \ \tilde{x}_1 = u(\tilde{x}_2) \ \text{ for all } \tilde{x}_2 \in I\},$$

and the fixed boundary part $\tilde{\Gamma}^F$ with $\partial\tilde{\Omega} = \tilde{\Gamma}^M \cup \tilde{\Gamma}^F$ and $\tilde{\Gamma}^M \cap \tilde{\Gamma}^F = \emptyset$.

Now, the OSD problem is defined by

$$\min_{u \in \mathcal{U}_{ad}} \tilde{J}(u),$$

with the equality constraint

$$\int_{\tilde{\Omega}} \tilde{a}(\tilde{y}, \tilde{\eta}) d\tilde{x} = \int_{\partial\tilde{\Omega}} \tilde{l}(\tilde{\eta}) d\tilde{\Gamma} \qquad \text{for all } \tilde{\eta} \in \tilde{V},$$

which is a variational form of the underlying PDE, $\tilde{V} \subset H^1(\tilde{\Omega})$ is the space of corresponding test functions, and $H^1(\tilde{\Omega})$ is a Sobolev space of square integrable functions with square integrable first derivatives on $\tilde{\Omega}$. Functions $\tilde{a}$ and $\tilde{l}$ are assumed to be nonlinear in general. We use the notations of [9] in our article.

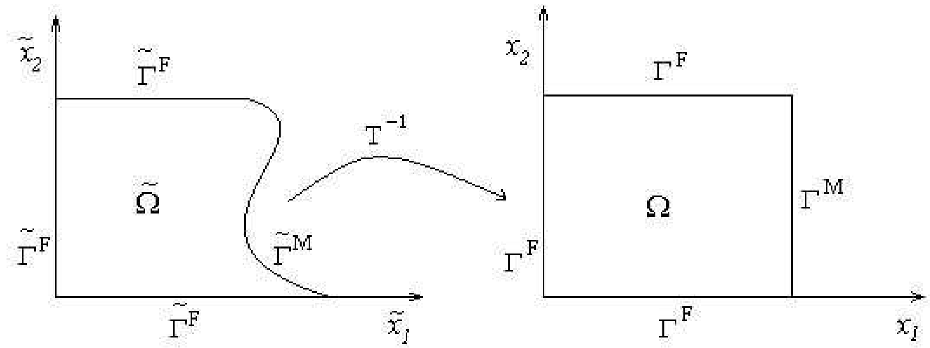

Figure 1: Transformation of the domain.

3 Transformation to a fixed domain

To cope with the variation of the domain, we transform the optimal shape design problem into an optimal control problem on a fixed domain. This routine approach has been used in literature before, for example [9, 13].

As sketched in Figure 1, the transformation defined by

$$\begin{aligned} T^{-1}: \tilde{\Omega} &\to \Omega \\ (\tilde{x}_1, \tilde{x}_2)^t &\mapsto (x_1, x_2)^t = \left(\frac{\tilde{x}_1}{u(\tilde{x}_2)}, \tilde{x}_2 \right)^t \end{aligned} \tag{2}$$

maps the moving domain $\tilde{\Omega}$ to the fixed reference area $\Omega = I \times I$.

So, we consider the problem over a fixed domain

$$\min_{u \in \mathcal{U}_{ad}} J(u, y),$$

with the equality constraint

$$\int_\Omega a(y, \eta, u, \frac{du}{dx_2})dx = l_\eta \qquad \text{for all } \eta \in V, \tag{3}$$

where l_η is a function depending only on the test function η. Here, a includes all terms depending on y, η, u, and $\frac{du}{dx_2}$.

By this transformation, the unknown geometrical factors are transformed from boundary to equations, that is, the equations must be solved on a fixed known domain while they contain unknown parameters. It will be shown in the next sections that this type of problems may be solved by the described method.

The transformation will also changes the integrand of functional (1) as a function f dependent on y, ∇y, u and $\frac{du}{dx_2}$:

$$J = \int_{\Omega} f(u, y, \nabla y, \frac{du}{dx_2})dx \tag{4}$$

Now, we choose the derivative of $u(x_2)$ as a control variable w in order to obtain the following control system:

$$\frac{du}{dx_2} = w, \qquad \text{a.e.} \quad x_2 \in I, \tag{5}$$

$$u(0) = u(1) = 1. \tag{6}$$

The differential form of this control system should be changed as a variational form consistent to (3). To do this, let define an admissible pair for (5)–(6).

Definition 1. *The pair $p = (w, u)$ is said admissible if the control function $w : I \to [\alpha_1, \alpha_2]$ is measurable and $u \in \mathcal{U}_{ad}$ is the response of the system (5)–(6) to w.*

As unknown factors in (3)-(4) are depending to u, then for a given admissible pair $p = (w, u)$, there exists a solution for (3) with a cost measured by (4). So, the problem will be finding optimal admissible pair in the set of all admissible pairs which is assumed nonempty and is denoted by $\mathcal{P}$. However to find the best admissible pair we have to determine characteristic properties of an admissible pair.

Let $\Upsilon = \Omega \times [\beta_1, \beta_2] \times [\alpha_1, \alpha_2] \times \mathcal{A}$, where $\mathcal{A} \subset \mathbb{R}^3$ is the set that $(y(x_1, x_2), \nabla y(x_1, x_2))$ gets its values in it. Moreover, assume that B is an open ball in $\mathbb{R}^2$ containing $[\beta_1, \beta_2] \times I$, and let $C^1(B)$ be the space of all real-valued continuous differentiable functions with continuous first partial derivatives on B. For $\varphi \in C^1(B)$ and $p = (w, u) \in \mathcal{P}$, define $\varphi^w \in C(\Upsilon)$ as follows:

$$\varphi^w(x_1, x_2, u(x_2), w(x_2), y, \nabla y) = \varphi_1(u(x_2), x_2)w(x_2) + \varphi_2(u(x_2), x_2). \tag{7}$$

In (7), φ_1, φ_2 denote the first partial derivatives of φ with respect to its variables. Admissibility of $p = (w, u)$ implies:

$$\begin{aligned}\int_0^1 & \varphi^w(x_1, x_2, u(x_2), w(x_2), y, \nabla y)dx_2 \\ &= \int_0^1 \{\varphi_1(u(x_2), x_2)w(x_2) + \varphi_2(u(x_2), x_2)\}dx_2 \\ &= \varphi(u(1), 1) - \varphi(u(0), 0) := \Delta_\varphi,\end{aligned}$$

for all $\varphi \in C^1(B)$, where $\varphi(u(1), 1)$ and $\varphi(u(0), 0)$ are known. Integrating the above equation with respect to x_2 on I, the integral form of (5)–(6) can be written as

$$\int_{\Omega} \varphi^w(x_1, x_2, u(x_2), w(x_2), y, \nabla y)dx = \Delta_\varphi \qquad \text{for all } \varphi \in C^1(B). \tag{8}$$

Therefore, the optimal control formulation of the OSD problem may be interpreted as the minimization of (4) over variational constraints (3) and (8).

4 Measure theoretical approach

By Definition 1, each shape corresponds to exactly one admissible pair $p = (w,u) \in \mathcal{P}$. This is an injection between $\mathcal{U}_{ad}$ and $\mathcal{P}$. So, the minimization of (1) over the class of shapes is equivalent to the minimization of (4) over $\mathcal{P}$. Now, we transfer the problem of minimization of (4) over $\mathcal{P}$ into another nonclassical problem. Let us define the following mapping:

$$\Lambda_p : F \to \int_\Omega F(x_1, x_2, u(x_2), w(x_2), y, \nabla y) dx, \qquad F \in C(\Upsilon),$$

where $C(\Upsilon)$ is the space of all continuous real-valued functions defined on Υ. As an injection, the transformation $p \mapsto \Lambda_p$ provides us to describe the set of all admissible pairs $\mathcal{P}$ as a subset of the set of all linear continuous mappings on $C(\Upsilon)$. Moreover, by the Riesz representation theorem (see [18]), corresponding to Λ_p, there is a positive Borel measure μ on Υ such that

$$\Lambda_p(F) = \mu(F), \qquad \text{for all } F \in C(\Upsilon).$$

Now, the problem of minimization of the functional (4) over $\mathcal{P}$ is enlarged to the minimization of

$$\mathcal{I} \; : \; \mu \to \mu(f) \tag{9}$$

over the set of all measures μ satisfying

$$\mu(F_\eta) = l_\eta, \qquad \text{for all } \eta \in V, \tag{10}$$

$$\mu(G_\varphi) = \Delta_\varphi, \qquad \text{for all } \varphi \in C^1(B), \tag{11}$$

where $F_\eta(x_1, x_2, u(x_2), w(x_2), y, \nabla y) = a(y, \eta, u, \frac{du}{dx_2})$ and $G_\varphi = \varphi^w$. In other words, (10) and (11) are a measure theoretical interpretation of (3) and (8) respectively.

We denote the set of all positive Borel measures on Υ satisfying (10)–(11) as Q. We also assume that $\mathcal{M}^+(\Upsilon)$ is the set of all positive Borel measures on Υ. If we consider the space $\mathcal{M}^+(\Upsilon)$ with weak*-topology, it can be seen from [16], that Q is compact. In the sense of this topology, the functional $\mathcal{I} : Q \to \mathbb{R}$ defined by (9) is a linear continuous functional on the compact set Q, thus it attains its minimum on Q (see [16, Theorem III.1]), and then the measure-theoretical problem, which consists of finding the minimum of the functional (9) over a subset of $\mathcal{M}^+(\Upsilon)$, possesses a minimizing solution, say μ^*, in Q.

The minimization problem of (9)–(11) is an infinite-dimensional LP problem, and we are mainly interested in approximating it. It is possible to approximate the solution of the problem (9)–(11) by the solution of a finite-dimensional linear program of sufficiently large dimension.

We first consider the minimization of (9) not only over the set Q but over a subset of it defined by requiring that only a finite number of constraints (10)–(11) are satisfied.

Consider equalities (10)–(11), let the sets of functions $\{\eta_i,\ i = 1, 2, 3, \dots\}$ and $\{\varphi_j,\ j = 1, 2, 3, \dots\}$ have dense linear combinations in V and $C^1(B)$, respectively. Now we can prove the following.

Proposition 1. *Let $Q(M_1, M_2)$ be a subset of $\mathcal{M}^+(\Upsilon)$ consisting of all measures satisfying*

$$\mu(F_{\eta_i}) = l_{\eta_i}, \qquad i = 1, 2, \dots, M_1,$$
$$\mu(G_{\varphi_j}) = \Delta_{\varphi_j}, \qquad j = 1, 2, \dots, M_2.$$

If $\theta(M_1, M_2) = \inf_{Q(M_1,M_2)} \mu(f)$ and $\theta = \inf_Q \mu(f)$, then $\theta(M_1, M_2) \to \theta$ as $M_1, M_2 \to \infty$.

Proof. See the proof of [3, Proposition 2]. □

The first stage of approximation is completed successfully. As the second stage, it is possible to develop a finite-dimensional linear program whose solution can be used to construct the solution of the infinite-dimensional linear program of minimizing (9) subject to (10)–(11). From [16, Theorem A.5], we can characterize a measure, say μ^*, in the set $Q(M_1, M_2)$ at which the function $\mu \to \mu(f)$ attains its minimum.

Proposition 2. *The measure μ^* in the set $Q(M_1, M_2)$ in which the function $\mu \to \mu(f)$ attains its minimum has the form*

$$\mu^* = \sum_{k=1}^{M_1+M_2} \varrho_k^* \delta(z_k^*) \tag{12}$$

with $z_k^* \in \Upsilon$ and $\varrho_k^* \geq 0, k = 1, 2, \dots, M_1 + M_2$.

Here $\delta(z)$ is the atomic measure concentrated at z, characterized by

$$\delta(z)(S) = \begin{cases} 1 & \text{if } z \in S, \\ 0 & \text{if } z \notin S, \end{cases}$$

for all $S \subset \Upsilon$; see [17]. This also implies that $\delta(z)(H) = H(z)$, where $H \in C(\Upsilon)$ and $z \in \Upsilon$. Now, the measure theoretical optimization problem is equivalent to a nonlinear optimization problem, in which unknowns are coefficients ϱ_k^* and supports $z_k^*, \text{for} k = 1, 2, \dots, M_1 + M_2$. It would be convenient if we could minimize the function $\mu \to \mu(f)$ only with respect to the coefficients ϱ_k^*, for$k = 1, 2, \ldots, M_1 + M_2$, in (12), which would be a linear programming problem. However, we do not know the support of the optimal measure. The answer lies in the approximation of this support by introducing a dense set in Υ.

Proposition 3. *Let E be a countable dense subset of Υ. Given $\epsilon > 0$, a measure $\lambda \in \mathcal{M}^+(\Upsilon)$ can be found such that*

$$\begin{aligned} |(\mu^* - \lambda)(f)| &\leq \epsilon, \\ |(\mu^* - \lambda)(F_{\eta_i})| &\leq \epsilon, \qquad i = 1, 2, \ldots, M_1, \\ |(\mu^* - \lambda)(G_{\varphi_j})| &\leq \epsilon, \qquad j = 1, 2, \ldots, M_2, \end{aligned}$$

and the measure λ has the form

$$\lambda = \sum_{k=1}^{M_1+M_2} \varrho_k^* \delta(z_k),$$

where the coefficients ϱ_k^* are the same as in the optimal measure (12) and $z_k \in E$.

Proof. See the proof of [16, Proposition III.3]. □

Thus, the infinite-dimensional linear programming (9) with restrictions defined by (10)–(11) can be approximated by another LP problem in which z_k, for $k = 1, 2, \ldots, N$, belongs to the known dense subset of Υ. To construct such a subset, we grid entire of Υ into $N \gg M_1 + M_2$ nodes $z_k = (x_1^k, x_2^k, u^k, w^k, y^k, \nabla y^k)$. This gird may be generated by dividing Ω, $[\beta_1, \beta_2]$, $[\alpha_1, \alpha_2]$, and $\mathcal{A}$ separately and rearranging nodes from 1 to N. For example, one could divide $[0, 1]$ for x_1 to n_1 points: $x_{11}, x_{12}, \ldots, x_{1n_1}$ which repeat in arrangement of z_k's.

Now, we have the following LP problem with $\varrho_1, \varrho_2, \ldots, \varrho_N$ as nonnegative decision variables:

$$\text{Minimize} \qquad \sum_{k=1}^{N} \varrho_k f(z_k)$$

subject to

$$\begin{aligned} \sum_{k=1}^{N} \varrho_k F_{\eta_i}(z_k) &= l_{\eta_i} \qquad i = 1, 2, \ldots, M_1, \\ \sum_{k=1}^{N} \varrho_k G_{\varphi_j}(z_k) &= \Delta_{\varphi_j}, \qquad j = 1, 2, \ldots, M_2, \end{aligned}$$

The solution of this LP problem determines nodes that make the optimal measure and finally the optimal shape. As the solution may be trivial (zero), it is necessary to prevent from trivial situation. To do this, we benefit from

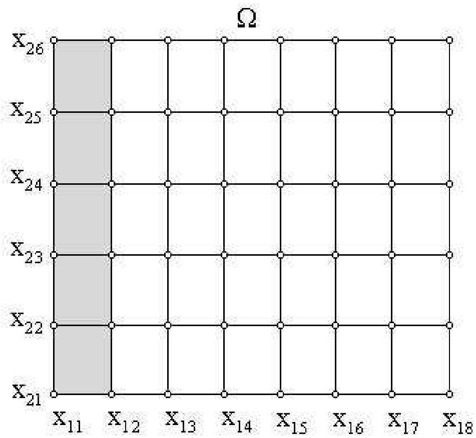

Figure 2: A typical grid on Ω.

theoretical properties of measure. Let consider a simple example in which Ω is covered with 48 nodes as given in Figure 2. If H is the characteristic function on the gray area in the picture, then the measure of H will be the area of band. On the other hand, this measure is equal to $\sum \varrho_k$, where the summation is on the set of all points with x_1^k equaling to x_{11}. So, we have a new constraint preventing the measure to be zero on this area:

$$\sum_{\{k:x_1^k=x_{11}\}} \varrho_k = x_{12} - x_{11}$$

This constraint forces the solution to be nonzero and preserving the properties of measure in the mentioned vertical band. We have to add similar constraints for all vertical and horizontal bands of this picture.

4.1 Extraction of optimal control

After solving the LP problem, we will find the optimal values for decision variables. The form of measure given in (12) is a combination of coefficients ϱ_k and support points z_k. This tel us that, if a coefficient is zero, then the corresponding support does not have a role in the optimal measure. Therefore, nonzero variables will make the optimal measure in a fashion which we discuss here. In fact, we use the effect of this optimal measure to find the approximate solution for the control function. First, define $\gamma_0 = 0$ and

$$\gamma_l = \sum_{k \in A_l} \varrho_k,$$

$$w_l = \frac{1}{|A_l|} \sum_{k \in A_l} w^k,$$

where

$$A_l = \{k : \varrho_k > 0,\ x_1^k = x_{1l}\}, \qquad l = 1, 2, \ldots, n_1.$$

Now the approximation for optimal control function $w(.)$ is a piecewise constant function that equals w_l on each interval $[\gamma_{l-1}, \gamma_l]$. The trajectory is then simply found by solving the differential equation (5) with the initial condition $u(0) = 1$. The resulting solution is a piecewise linear function.

We may also use numerical methods such as interpolation or fitting methods to make a smooth shape. Choosing the convenient method depends on design purposes. A numerical treatment is also given in [12] which may be useful in this connection; see also [1] for numerical smoothing and approximation methods.

5 Numerical example

In this section, a problem with applications in aerodynamic shape optimization is studied to validate the presented method from the computational standpoint.

The velocity $\tilde{v}(\tilde{x})$ at a point $\tilde{x} \in \tilde{\Omega}$ in a nonviscous incompressible potential flow (such as for air or water at moderate speed) may be approximated by

$$\tilde{v}(\tilde{x}) = \nabla \tilde{y}(\tilde{x}), \qquad \tilde{x} \in \tilde{\Omega},$$

where $\tilde{y}$ satisfies the second order partial differential equation on $\tilde{\Omega}$:

$$\Delta \tilde{y} = 0. \tag{13}$$

This type of potential flow occurs mainly in design of low speed airfoils, wings, and blades. The variational form of the above PDE is as follows (see [14]):

$$\int_{\tilde{\Omega}} \nabla \tilde{y} \nabla \tilde{\eta} d\tilde{x} = \int_{\partial \tilde{\Omega}} \tilde{\eta} \nabla \tilde{y} . \vec{n} d\tilde{\Gamma}, \qquad \text{for all } \tilde{\eta} \in \tilde{V}, \tag{14}$$

where $\vec{n}$ is the outward normal vector to $\tilde{\Gamma} = \partial \tilde{\Omega}$.

We also assume the following Dirichlet condition on the fixed boundary part

$$y|_{\tilde{\Gamma}^F} = 0. \tag{15}$$

So, the above variational form is changed to

$$\int_{\tilde{\Omega}} \nabla \tilde{y} \nabla \tilde{\eta} d\tilde{x} = \int_0^1 \tilde{\eta} \tilde{y}_1 d\tilde{x}_2, \qquad \text{for all } \tilde{\eta} \in \tilde{V},$$

where $\tilde{V} = \{\eta \in H^1(\Omega) : \eta|_{\tilde{\Gamma}^F} = 0\}$.

Now, the specific transformation (2) can be used on the variable domain. This leads to the following transformed bilinear form

$$\int_\Omega \left[(\frac{1}{u} + \frac{x_1^2 w^2}{u}) y_1 \eta_1 - x_1 w y_2 \eta_1 - x_1 w y_1 \eta_2 + u y_2 \eta_2 \right] dx$$
$$= \int_0^1 \eta(1, x_2) y_1(1, x_2) \sqrt{1 + w^2} dx_2.$$

The above integral is now defined on the fixed domain Ω, where coefficient functions depend on the parameter function $u \in \mathcal{U}_{ad}$ and its derivative w.

It is easy to check that $\tilde{y} = \sinh(\pi \tilde{x}_1) \sin(\pi \tilde{x}_2)$ is a solution of the boundary value problem (13), (15), or equivalently (14)–(15).

To make a comparison between exact and numerical solution, we follow an inverse scheme to construct an OSD problem with known exact solution. To do this, we choose a cost function as

$$\int_{\tilde{\Gamma}^M} \|\nabla \tilde{y} - V_0\| d\tilde{\Gamma},$$

which leads to a velocity matching problem. In other words, we want to find the moving boundary $\tilde{\Gamma}^M$ in such a way that the velocity of the flow matches, as well as possible, a given velocity V_0. For applications of this type of matching OSD, we may address, for example, the pressure distribution matching in airfoil section design (see [5]).

The above cost function finds the following format after transformation:

$$\int_0^1 \|\nabla y - V_0\| \sqrt{1 + w^2} dx_2$$

or

$$\int_\Omega \|\nabla y - V_0\| \sqrt{1 + w^2} dx. \tag{16}$$

In this example, for test functions of type η, polynomials of x_1 and x_2 with compact support on Ω of the following forms are chosen:

$$x_1^r (1 - x_2)^s, \quad x_2^r (1 - x_1)^s, \qquad r, s = 1, 2, \ldots.$$

For test functions of type φ, we use polynomials of u, functions depending only on x_2, and functions in $C^1(B)$ with compact support on I; see [3, Sec.5] for more details about test functions.

To illustrate the scheme, two specific moving boundaries are implemented:

a: If the moving part of the boundary $\tilde{\Gamma}^M$, is $\tilde{x_1} = 1$, then the the above solution gives a velocity vector as

$$\nabla y = (\pi \cosh(\pi) \sin(\pi \tilde{x}_2), \pi \sinh(\pi) \cos(\pi \tilde{x}_2))$$

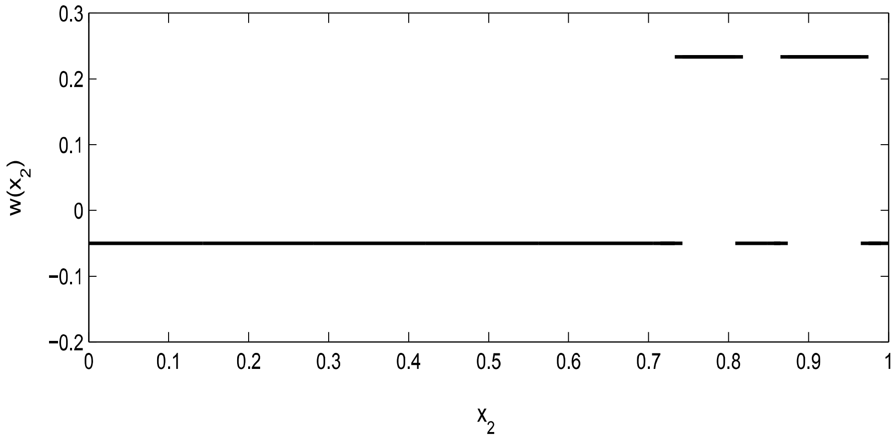

Figure 3: Resulting control function, part a.

on $\tilde{\Gamma}^M$. Now, if we let $V_0 = (\pi\cosh(\pi)\sin(\pi\tilde{x}_2), \pi\sinh(\pi)\cos(\pi\tilde{x}_2))$, then the OSD problem of minimizing (16) subject to (14)–(15) has the unique solution $\tilde{x}_1 = 1$ or equivalently $u = 1$. The solution of the corresponding linear programming problem is used to construct the control function w as shown in Figure 3. The corresponding optimal trajectory, u, is compared with the exact solution in Figure 4. It is clear that the numerical solution approximates the exact solution with a small error.

In this case, we choose $N = 30625$, $\beta_1 = 0.4$, $\beta_2 = 1.3$, $\alpha_1 = -0.8$, $\alpha_2 = 0.9$, $M_1 = 1$, and $M_2 = 12$. The LP problem has been solved by two phase revised simplex method, and the corresponding optimal objective was found as 0.01194.

b: If the moving part of the boundary $\tilde{\Gamma}^M$, is $\tilde{x_1} = 1+\sin(\pi\tilde{x_2})$, then the above solution gives a velocity distribution on $\tilde{\Gamma}^M$ as

$$\nabla y = (\pi\cosh(\pi(1+\sin(\pi\tilde{x_2})))\sin(\pi\tilde{x}_2), \pi\sinh(\pi(1+\sin(\pi\tilde{x_2})))\cos(\pi\tilde{x}_2)).$$

Now if we set

$$V_0 = (\pi\cosh(\pi(1+\sin(\pi\tilde{x}_2)))\sin(\pi\tilde{x}_2), \pi\sinh(\pi(1+\sin(\pi\tilde{x}_2)))\cos(\pi\tilde{x}_2)),$$

then the OSD problem of minimizing (16) over (14)–(15) has the unique solution $u = 1+\sin(\pi\tilde{x}_2)$. The solution of the corresponding linear programming problem is used to construct the control function w as shown in Figure 5. To enable a comparison, Figure 6 shows the resulting numerical shape and the exact solution of the problem that indicates the accuracy of results.

In this part of the example, we chose $N = 21875$, $\beta_1 = 0.62$, $\beta_2 = 1$, $\alpha_1 = -1.96$, $\alpha_2 = 1.96$, $M_1 = 1$, and $M_2 = 12$. The optimal objective of the corresponding LP problem was found as 2.1275×10^{-4}. Figure 7 shows the

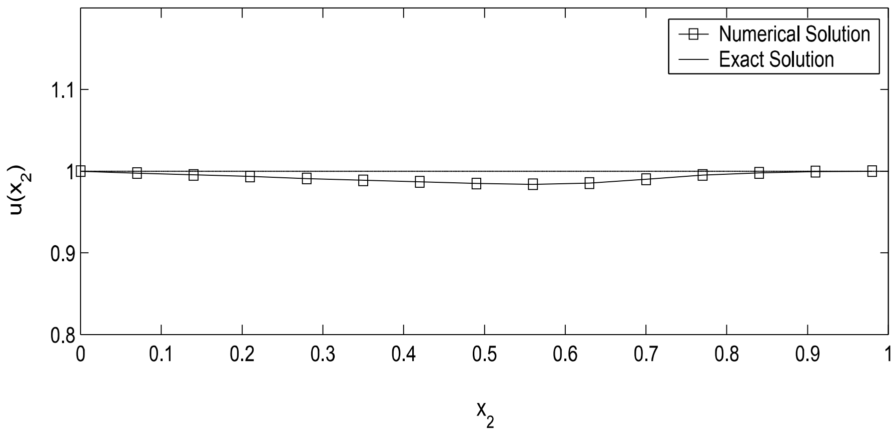

Figure 4: Numerical and exact solution for part a.

decreasing behavior of the objective function and the rate of convergence in the second phase of LP solving.

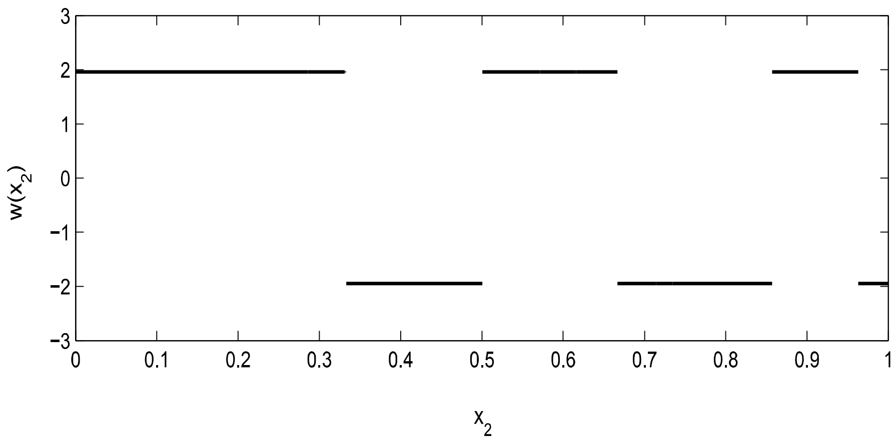

Figure 5: Resulting control function, part b.

6 Conclusions

The method of embedding admissible shapes into a subset of measures is extended to an LP formulation for a class of optimal shape design problems resulting in a numerical solution scheme. Recent works on this subject

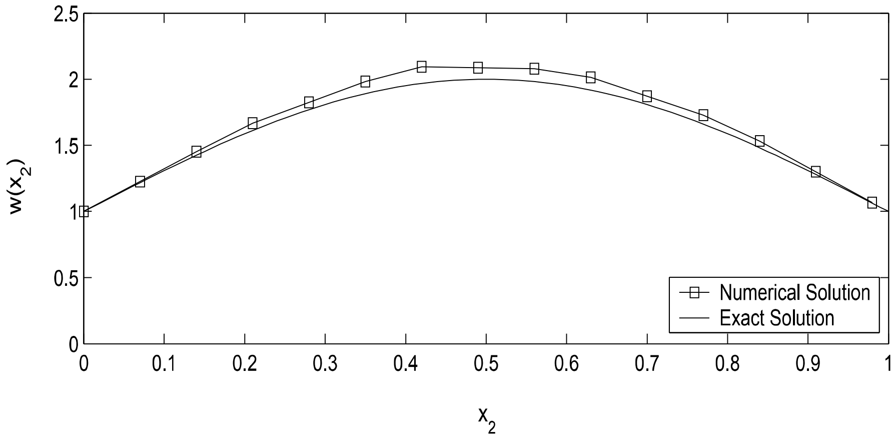

Figure 6: Numerical and exact solution for, part b.

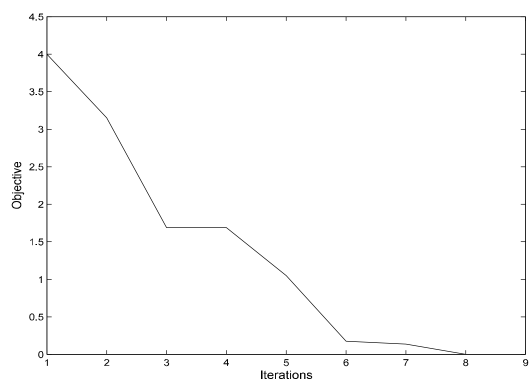

Figure 7: Objective function in iterations of the simplex method.

deal with optimization on a specified known part of the shape, while in the present paper, the optimization takes place on entirely unknown domain. The method is self starting by means that it does not need any initial guess of the solution to be started. It is also independent from type of partial differential equations and has been presented for a relatively general variational form of equations. Numerical examples were used to interpret the implementation of the method and to check its validity.

References

1. Burden R.L. and Faires, J.D. *Numerical Analysis*, (7th Edition), Brooks/Cole (ITP) 2000.

2. De Santis, D., Zahr, M.J. and Farhat, C. *Gradient based aerodynamic shape optimization using the FIVER embedded boundary method*, 54th AIAA Aerospace Sciences Meeting, AIAA SciTech, (2016), 1–11.

3. Farahi, M. H., Borzabadi, A. H., Mehne H. H. and Kamyad, A. V. *Measure theoretical approach for optimal shape design of a nozzle*, J. Appl. Math. Comput. 17 (2005), 315–328.

4. Farahi, M.H., Mehne, H.H. and Borzabadi, A.H. *Wing drag minimization by using measure theory*, Optim. Method. Softw. 21(2) (2006), 169–177.

5. Hacioğlu, A. and Özkol, İ. *Inverse airfoil design by using an accelerated genetic algorithm via distribution strategies*, Inverse. Probl. Sci. En. 13(6) (2005), 563–579.

6. Ghaedamini Harouni, A. and Mehne, H.H. *Multi-Disciplinary Multi-Objective Shape Optimization of Orion Type Re-entry Capsule*, Modares. Mech. Eng. 19(3) (2019), 665–675.

7. Jahromi, A.F. *A shape-measure method for solving free-boundary elliptic systems with boundary control function*, Iran. J. Numer. Anal. Optim. 3(2) (2013), 47–65.

8. Kurahashi, T. *Investigations on boundary temperature control analysis considering a moving body based on the adjoint variable and the fictitious domain finite element methods*, Int. J. Numer. Meth. Eng. 103(8) (2015), 582–597.

9. Laumen, M. *Newton's method for a class of optimal shape design problems*, SIAM J. Optim. 10(2) (2000), 503–533.

10. Li, D. and Hartmann, R. *Adjoint-based airfoil optimization with discretization error control*, Int. J. Numer. Meth. Fl. 77(1) (2015), 1–17.

11. Mehne, H.H. *On solving constrained shape optimization problems for finding the optimum shape of a bar cross-section*, Appl. Numer. Math. 58(8) (2008), 1129–1141

12. Mehne, H.H., Farahi M.H. and Esfahani, J.A. *Slot nozzle design with specified pressure in a given subregion by using embedding method*, Appl. Math. Comput. 168(2) (2005), 1258–1272.

13. Mehne, H.H., Farahi M.H. and Esfahani, J.A. *On an optimal shape design problem to control a thermoelastic deformation under a prescribed thermal treatment*, Appl. Math. Model. 31(4) (2007), 663–675.

14. Mikhailov, V.P. *Partial Differential Equations*, MIR publishers, Moscow 1978.

15. Nguyen, Q. and Yang, C. *Design of a longitudinal cooling fin with minimum volume by a modified Newton–Raphson method*, Appl. Therm. Eng. 98 (2016), 169–178.

16. Rubio, J. E. *Control and Optimization: the linear treatment of non-linear problems*, Manchester University Press, Manchester 1986.

17. Rudin, W. *Real and Complex Analysis* (3rd Edition), McGraw-Hill, New York 1987.

18. Rudin, W. *Functional Analysis* (2ed Edition), McGraw-Hill, New York 1991.

19. Zhang, Z., Liu, J., Tang, Y. and Meng, X. *Optimizing the shape of top piston ring face using inverse method*, Ind. Lubr. Tribol. 68(1) (2016), 9–15.

انتقال به دامنه ثابت در مدلسازی برنامه ریزی خطی برای رده ای از مسایل طراحی شکل بهینه

سیّدحامد هاشمی مهنه[1] *و محمّد هادی فراهی*[2]

[1] تهران، پژوهشگاه هوافضا
[2] دانشگاه فردوسی مشهد، دانشکده علوم ریاضی

دریافت مقاله ۲۷ بهمن ۱۳۹۴، دریافت مقاله اصلاح شده ۱۳ تیر ۱۳۹۵، پذیرش مقاله ۲۰ آبان ۱۳۹۶

چکیده : رده ای از مسائل طراحی شکل بهینه در این مقاله مطالعه می شوند. شکل دامنه به گونه ای تعیین می شود که پاسخ معادلات با مشتقات جزئی در ناحیه حاصل تا حد ممکن به یک حالت مطلوب نزدیک شود. بدین منظور، دامنه متغیر در مساله اولیه توسط دسته ای از توابع پارامتری می شود تا مساله طراحی شکل بهینه به یک مساله کنترل بهینه روی یک دامنه ثابت تبدیل شود. سپس این مساله به صورت یک مساله در فضای اندازه بیان می شود که خود یک مساله برنامه ریزی خطی با بعد نامتناهی است. در نهایت، اندازه بهینه که مبین شکل بهینه است، به کمک پاسخ یک مساله برنامه ریزی خطی با بعد متناهی تقریب زده می شود. روش پیشنهادی بر چند مثال عددی پیاده سازی و ارزیابی شده است.

کلمات کلیدی : تقریب؛ طراحی شکل بهینه؛ برنامهریزی خطی؛ نظریه اندازه.