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Research Article

The strict complementarity in linear fractional optimization

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Abstract

As an important duality result in linear optimization, the Goldman–Tucker theorem establishes strict complementarity between a pair of primal and dual linear programs. Our study extends this result into the framework of linear fractional optimization. Associated with a linear fractional program, a dual program can be defined as the dual of the equivalent linear program obtained from applying the Charnes–Cooper transformation to the given program. Based on this definition, we propose new criteria for primal and dual optimality by showing that the primal and dual optimal sets can be equivalently modeled as the optimal sets of a pair of primal and dual linear programs. Then, we define the concept of strict complementarity and establish the existence of at least one, called *strict complementary*, pair of primal and dual optimal solutions such that in every pair of complementary variables, exactly one variable is positive and the other is zero. We geometrically interpret the strict complementarity in terms of the relative interiors of two sets that represent the primal and dual optimal sets in higher dimensions. Finally, using this interpretation, we develop two approaches for finding a strict complementary solution in linear fractional optimization. We illustrate our results with two numerical examples.

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Keywords: Linear fractional optimization; Charnes–Cooper transformation; Duality; Strict complementarity.

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1 Introduction

A mathematical optimization problem is specified as a (primal) linear fractional program (LFP) when a linear fractional function (i.e., ratio of two affine functions) is optimized subject to a set of linear constraints on the given variables.[1] The linear fractional optimization frequently appears in a wide variety of real-world applications, including information theory, numerical analysis, game theory, cutting stock problems, shipping schedules, macroeconomic planning model, and so on. More details can be found in [1, 11, 24, 25] and references therein. It is also applied in the measurement of efficiency by data envelopment analysis; see, for example, [10, 15, 22, 26, 27, 28, 29, 30, 31] among others. Therefore, considerable research interest has been devoted to this branch of optimization.

The literature on the duality of linear fractional optimization associate various duals to the primal LFP. Chadha [7] suggested a dual in the form of a linear program (LP) and proved some duality statements directly. However, the constant scalars in the numerator and denominator of the primal objective function are assumed to be absent in their work. Chadha and Chadha [8] extended Chadha's results to the general case, where the constant scalars are taken into account. An interesting note regarding their impressive work is that their results can be deduced in an alternative way from the duality of linear optimization. In fact, an indirect approach for constructing a dual program is to transform the primal LFP into an equivalent problem that its dual can be constructed in the classical way; see [25]. By using this approach, it can be verified (as in Section 2) that the dual program proposed in [8] is nothing else than the dual of the equivalent LP resulting from applying the well-known transformation of Charnes and Cooper [9] to the primal LFP.

Though demonstrating the common complementary slackness condition between the primal LFP and its dual, Chadha and Chadha [8] did not investigate the strict complementarity between them. Furthermore, to the best of our knowledge, no other research exists on such investigation. Motivated by these, we extend an important duality result proved by Goldman and Tucker [12] from linear optimization to linear fractional optimization. The so-called Goldman–Tucker theorem establishes the strict complementarity between a pair of primal and dual LPs. It states that at least one, so-called strict complementary, pair of primal and dual optimal solutions exists such that the sum of each pair of complementary variables is positive. That is, in every pair of complementary variables, exactly one variable is positive and the other is zero; see, for example, [20] for more details on the theory and applications of strict complementarity in linear optimization.

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[1] If the given objective function is optimized with no restrictions on the values of its variables, then the optimization problem is called *unconstrained*. Useful information on approaches developed for solving unconstrained optimization problems can be found in [3, 16, 21], among others.

As a complementary to the work of Chadha and Chadha [8], this paper shows that the primal and dual optimal sets can be equivalently modeled as the optimal sets of a pair of primal and dual LPs. Using this fact, we propose new criteria for primal and dual optimality in terms of the belongingness of these LPs' objective vectors to the binding polyhedral cones at primal and dual feasible solutions. Then we define the strict complementary slackness condition for an LFP and demonstrate the existence of a strict complementary solution. We also show that any strict complementary solution induces unique optimal partitions for the sets of indices of nonnegative decision variables.

To deal with the problem of finding a strict complementary solution, we equivalently represent the primal and dual optimal sets by two nonnegative polyhedral sets in higher dimensions, which are described only by equality *defining constraints*.[2] Then we geometrically interpret the strict complementarity by proving that any pair of relative interior points of these polyhedral sets is a strict complementary solution, and vice versa. Based on this interpretation, we turn the problem under consideration to the equivalent problem of identifying a maximal element of a nonnegative polyhedral set. Exploiting the recent work of Mehdiloozad et al. [20], who have addressed the latter problem, we develop two linear optimization approaches for finding a strict complementary solution.

The remainder of this paper is organized as follows. Section 2 provides the necessary background needed for the rest of the paper. Section 3 proposes new criteria for primal and dual optimality and illustrates them with a numerical example. Section 4 establishes the strict complementarity for LFPs. Section 5 proposes an LP for finding a maximal element of a nonnegative polyhedral set and, thereby, develops two approaches for finding a strict complementary solution. Section 6 illustrates these approaches by a numerical example. Section 7 contains concluding remarks and suggestions for future research. Appendix A provides the GAMS (General Algebraic Modeling System) code of our proposed approaches.

2 Background

2.1 Notation

Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space, and let $\mathbb{R}^d_+$ denote its nonnegative orthant. We denote sets by uppercase calligraphic letters, vectors by boldface lowercase letters, and matrices by boldface uppercase letters. We

[2] A polyhedral set is said to be nonnegative if it is a subset of the nonnegative orthant of Euclidean space. By the "defining constraints" of such a polyhedral set, we refer to the constraints imposed other than the nonnegativity conditions.

denote the cardinality of a set $\mathcal{S}$ by $\operatorname{Card}(\mathcal{S})$. By convention, all vectors are column vectors. The superscript $\top$ denotes the transpose of a vector or matrix.

Vectors $\mathbf{0}$ and $\mathbf{1}$ are vectors all components of that are equal to 0 and 1, respectively. The dimensions of these vectors are clear from the context in which they are used. For simplicity, the notation $(\mathbf{a};\mathbf{b}) \in \mathbb{R}^{d+d'}$ is used to show the column vector obtained by adding vector $\mathbf{b} \in \mathbb{R}^{d'}$ below the vector $\mathbf{a} \in \mathbb{R}^{d}$. For vectors $\mathbf{a},\mathbf{b} \in \mathbb{R}^{d}$, the inequality $\mathbf{a} \geq \mathbf{b}$ (resp., $\mathbf{a} > \mathbf{b}$) means that $a_i \geq b_i$ (resp., $a_i > b_i$) for all $i = 1,\ldots,d$.

Matrix $\mathbf{0}$ is the matrix all components of that are equal to 0, and matrix $\mathbf{I}$ is the identity matrix. The dimensions of these matrices are clear from the context in which they are used. We denote the ith ($i = 1,\ldots,d$) row and the jth ($j = 1,\ldots,d'$) column of a $d \times d'$ matrix $\mathbf{A}$ by $\mathbf{a}^i$ and $\mathbf{a}_j$, respectively. In particular, we use the notation $\mathbf{e}_j$ to denote the jth column of the identity matrix of size $d \times d$, that is, $\mathbf{e}_j = \big(0,\ldots,\underset{j\text{th}}{1},\ldots,0\big)^\top \in \mathbb{R}^d$ for $j = 1,\ldots,d$.

Recall from [23] that the relative interior of a subset $\mathcal{X}$ of $\mathbb{R}^d$, denoted by $\operatorname{ri}(\mathcal{X})$, is defined as the interior we get when $\mathcal{X}$ is regarded as a subset of its affine hull, denoted by $\operatorname{aff}(\mathcal{X})$. Formally,

$$\operatorname{ri}(\mathcal{X}) = \big\{\mathbf{x}^{\mathrm{o}} \in \mathcal{X} \colon \mathcal{N}_\varepsilon(\mathbf{x}^{\mathrm{o}}) \cap \operatorname{aff}(\mathcal{X}) \subseteq \mathcal{X} \text{ for some } \varepsilon > 0\big\},$$

where $\mathcal{N}_\varepsilon(\mathbf{x}^{\mathrm{o}}) = \big\{\mathbf{x} \in \mathbb{R}^d \colon \|\mathbf{x} - \mathbf{x}^{\mathrm{o}}\| < \varepsilon\big\}$.

Recall also from [20] that any convex (and, in particular, polyhedral) subset of $\mathbb{R}^d_+$ is called a nonnegative convex (polyhedral) set. Additionally, any element of a nonnegative convex set is said to be maximal, if the number of its positive components is maximum. We denote the support of a nonnegative vector $\mathbf{a} \in \mathbb{R}^d_+$ by $\operatorname{supp}(\mathbf{a})$, that is, $\operatorname{supp}(\mathbf{a}) = \big\{i \in \{1,\ldots,d\} : a_i > 0\big\}$. We also denote by $\operatorname{me}(\mathcal{X})$ the set of all maximal elements of $\mathcal{X}$, that is, $\operatorname{me}(\mathcal{X}) = \underset{\mathbf{x}\in\mathcal{X}}{\operatorname{argmax}} \operatorname{Card}(\operatorname{supp}(\mathbf{x}))$.

2.2 Linear fractional program

A function of variables is said to be *linear fractional* if both its numerator and denominator are affine functions of the given variables. A mathematical optimization problem that optimizes a linear fractional objective function subject to a set of linear constraints is called as a linear fractional program (LFP). Formally, the general form[3] of the primal LFP is defined as

[3] The *standard* form of the primal LFP results from (1) by replacing the inequality sign "$\leq$" in (1b) by the equality sign; see [1, Section 1.3].

$$\max \quad f(\mathbf{x}) = \frac{\mathbf{c}^\top \mathbf{x} + \alpha}{\mathbf{d}^\top \mathbf{x} + \beta} \tag{1a}$$

subject to

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}, \tag{1b}$$

$$\mathbf{x} \geq \mathbf{0}, \tag{1c}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the vector of decision variables, $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{d} \in \mathbb{R}^n$ are, respectively, numerator and denominator vectors of objective function, $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ are objective scalars, $\mathbf{A}$ is an $m \times n$ constraint matrix, and $\mathbf{b} \in \mathbb{R}^m$ is the right-hand side vector.

Let $\mathcal{S}$ denote the feasible set of program (1), which is clearly a polyhedral set in $\mathbb{R}^n$. To ensure that the function f is well-defined on $\mathcal{S}$, it is assumed that its denominator maintains a constant sign on $\mathcal{S}$. Without loss of generality, we assume that $\mathbf{d}^\top \mathbf{x} + \beta > 0$ for all $\mathbf{x} \in \mathcal{S}$. Then the objective function f is both quasi-convex and quasi-concave over $\mathcal{S}$ and, therefore, every local maximum is a global maximum (see, e.g., [3]). To guarantee the occurrence of finite optimality for program (1), we also assume that $\mathcal{S}$ is regular (i.e., nonempty and bounded).

An effective approach for solving program (1) is to transform it into an equivalent LP by the well-known Charnes–Cooper transformation [9]. In fact, if we define $t = \frac{1}{\mathbf{d}^\top \mathbf{x} + \beta}$ and $\bar{\mathbf{x}} = t\mathbf{x}$, then multiplying both sides of (1b) by t converts program (1) to the following LP:

$$\max \quad \mathbf{c}^\top \bar{\mathbf{x}} + \alpha t \tag{2a}$$

subject to

$$\mathbf{A}\bar{\mathbf{x}} - \mathbf{b}t \leq \mathbf{0}, \tag{2b}$$

$$\mathbf{d}^\top \bar{\mathbf{x}} + \beta t = 1, \tag{2c}$$

$$\bar{\mathbf{x}} \geq \mathbf{0},\ t \geq 0. \tag{2d}$$

Let $\bar{\mathcal{S}}$ be the feasible set of program (2). Because $t > 0$ for all $(\bar{\mathbf{x}}; t) \in \bar{\mathcal{S}}$,[4] the following implication between the feasible solutions of programs (1) and (2) is established:

$$(\bar{\mathbf{x}}; t) \in \bar{\mathcal{S}} \quad \Rightarrow \quad \frac{1}{t}\bar{\mathbf{x}} \in \mathcal{S}, t > 0. \tag{3}$$

Especially, if $(\bar{\mathbf{x}}^*; t^*)$ is an optimal solution to program (2), then $\frac{1}{t^*}\bar{\mathbf{x}}^*$ is an optimal solution to program (1) (see, e.g., [1, p. 57]).

[4] Indeed, if $t = 0$ for some $(\bar{\mathbf{x}}; t) \in \bar{\mathcal{S}}$, then it follows from (2b)–(2c) that $\mathbf{A}\bar{\mathbf{x}} \leq \mathbf{0}$ and $\bar{\mathbf{x}} \neq \mathbf{0}$. This means that the vector $\bar{\mathbf{x}}$ is a recession direction of the feasible set $\mathcal{S}$, thereby contradicting the regularity assumption. Therefore, $t > 0$ for all $(\bar{\mathbf{x}}; t) \in \bar{\mathcal{S}}$.

2.3 Dual of linear fractional program

To state the dual to LFP (1), let the vector $\mathbf{y} \in \mathbb{R}^m$ be dual to (2b) and the scalar z be dual to (2c). Then, by the duality of linear optimization, the dual to LP (2) is stated as follows:[5]

$$\min \quad g(\mathbf{y}; z) = z \tag{4a}$$
subject to
$$\mathbf{A}^\top \mathbf{y} + \mathbf{d}z \geq \mathbf{c}, \tag{4b}$$
$$-\mathbf{b}^\top \mathbf{y} + \beta z = \alpha, \tag{4c}$$
$$\mathbf{y} \geq \mathbf{0},\ z \text{ sign free.} \tag{4d}$$

Observe that program (4) is nothing else than the LP introduced in [8] as the dual of program (1). We denote by $\mathcal{D}$ the feasible set of this program.

Throughout this paper, LP (4) is defined to be the dual of LFP (1). The next three theorems demonstrate the duality relationships between programs (1) and (4).

Theorem 1 (Weak duality)**.** [8] For any $\mathbf{x} \in \mathcal{S}$ and any $(\mathbf{y}; z) \in \mathcal{D}$, we have $f(\mathbf{x}) \leq g(\mathbf{y}; z)$.

Theorem 2 (Optimality criterion)**.** [8] If the feasible solutions $\mathbf{x} \in \mathcal{S}$ and $(\mathbf{y}; z) \in \mathcal{D}$ satisfy $f(\mathbf{x}) = g(\mathbf{y}; z)$, then they are optimal solutions to programs (1) and (4), respectively.

Theorem 3 (Strong duality)**.** [8] If $\mathbf{x}^*$ is an optimal solution to program (1), then there exists some optimal solution $(\mathbf{y}^*; z^*)$ to program (4) such that $f(\mathbf{x}^*) = g(\mathbf{y}^*; z^*)$.

The following result gives a necessary and sufficient optimality condition, called *complementary slackness condition* (CSC), in terms of the complementarity of the primal and dual feasible solutions.

Theorem 4. [8] Feasible solutions $\mathbf{x}^* \in \mathcal{S}$ and $(\mathbf{y}^*; z^*) \in \mathcal{D}$ are optimal if and only if they fulfill the following conditions:

$$\mathbf{v}^{*\top} \mathbf{x}^* = \mathbf{u}^{*\top} \mathbf{y}^* = 0, \tag{5}$$

where $\mathbf{u}^* = \mathbf{b} - \mathbf{A}\mathbf{x}^*$ and $\mathbf{v}^* = \mathbf{A}^\top \mathbf{y}^* + \mathbf{d}z^* - \mathbf{c}$.

From Theorem 4, the pairs (x_j, v_j), $j = 1, \ldots, n$, and (u_i, y_i), $i = 1, \ldots, m$, are called complementary variables.

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[5] Note that the inequality constraint $-\mathbf{b}^\top \mathbf{y} + \beta z \geq \alpha$ has been replaced with its equality form in program (4), because the optimal value of its corresponding dual variable in program (2), t, is always positive.

3 Our criteria for primal and dual optimality

In this section, we propose new criteria for the optimality of LFP (1) and its dual (4), and present their geometrical interpretations.

Denote by f^* the optimal objective value of program (1), and let $\mathbf{d}^* = f^*\mathbf{d}$ and $\beta^* = f^*\beta$. Then the set of all optimal solutions of program (1) can be defined by conditions (1b)–(1c) and the additional equality requiring that the objective function of program (1) to be equal to f^*. Equivalently, this set is stated as follows:

$$\mathcal{X}^* = \left\{\mathbf{x} \in \mathbb{R}^n \colon (\mathbf{c} - \mathbf{d}^*)^\top \mathbf{x} = -\alpha + \beta^*,\, \mathbf{A}\mathbf{x} \leq \mathbf{b},\, \mathbf{x} \geq \mathbf{0}\right\}.$$

Similarly, an equivalent statement of the optimal set of program (4) is

$$\left\{(\mathbf{y}; z^*) \in \mathbb{R}^{m+1} \colon \mathbf{b}^\top \mathbf{y} = -\alpha + \beta^*,\, \mathbf{A}^\top \mathbf{y} \geq \mathbf{c} - \mathbf{d}^*,\, \mathbf{y} \geq \mathbf{0}\right\},$$

where z^* denotes the optimal objective value of program (4) and is equal to f^*. Observe that the last components of all optimal solutions of program (4) are equal to z^*. Therefore, without losing anything, we can remove the last dimension of the optimal set of program (4) by projecting it onto the space of $\mathbf{y}$-variables. This results the following set:

$$\mathcal{Y}^* = \left\{\mathbf{y} \in \mathbb{R}^m \colon \mathbf{b}^\top \mathbf{y} = -\alpha + \beta^*,\, \mathbf{A}^\top \mathbf{y} \geq \mathbf{c} - \mathbf{d}^*,\, \mathbf{y} \geq \mathbf{0}\right\},$$

which will be loosely referred to as the optimal set of program (4).

It is clear that the nonempty optimal sets $\mathcal{X}^*$ and $\mathcal{Y}^*$ are polyhedral subsets of $\mathbb{R}^n_+$ and $\mathbb{R}^m_+$, respectively. The next result shows that these sets are interestingly the optimal sets of the LP

$$\begin{aligned} &\max \quad (\mathbf{c} - \mathbf{d}^*)^\top \mathbf{x} \\ &\text{subject to} \\ &(1b) - (1c), \end{aligned} \tag{6}$$

and its dual

$$\min \quad \mathbf{b}^\top \mathbf{y} \tag{7a}$$

subject to

$$\mathbf{A}^\top \mathbf{y} \geq \mathbf{c} - \mathbf{d}^*, \tag{7b}$$

$$\mathbf{y} \geq \mathbf{0}. \tag{7c}$$

Theorem 5. Let $\mathcal{F}_P^*$ and $\mathcal{F}_D^*$ be the optimal sets of programs (6) and (7), respectively. Then, $\mathcal{F}_P^* = \mathcal{X}^*$ and $\mathcal{F}_D^* = \mathcal{Y}^*$.

Proof. Let $\hat{\mathbf{x}} \in \mathcal{X}^*$ and $\hat{\mathbf{y}} \in \mathcal{Y}^*$. Then $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are, respectively, feasible solutions to LPs (6) and (7) such that $(\mathbf{c} - \mathbf{d}^*)^\top \hat{\mathbf{x}} = \mathbf{b}^\top \hat{\mathbf{y}}$. By the optimality criterion theorem of linear optimization, it follows that $\hat{\mathbf{x}} \in \mathcal{F}_P^*$ and $\hat{\mathbf{y}} \in \mathcal{F}_D^*$. Therefore, $\mathcal{X}^* \subseteq \mathcal{F}_P^*$ and $\mathcal{Y}^* \subseteq \mathcal{F}_D^*$.

Conversely, let $\hat{\mathbf{x}} \in \mathcal{F}_P^*$ and $\hat{\mathbf{y}} \in \mathcal{F}_D^*$. By the strong duality theorem of linear optimization, we have $(\mathbf{c}-\mathbf{d}^*)^\top \hat{\mathbf{x}} = \mathbf{b}^\top \hat{\mathbf{y}}$. Because $\mathcal{X}^* \neq \varnothing$ and $\mathcal{X}^* \subseteq \mathcal{F}_P^*$, the weak duality theorem of linear optimization implies that $-\alpha + \beta^* \leq \mathbf{b}^\top \hat{\mathbf{y}}$. Similarly, it follows from $\mathcal{Y}^* \neq \varnothing$ and $\mathcal{Y}^* \subseteq \mathcal{F}_D^*$ that $(\mathbf{c}-\mathbf{d}^*)^\top \hat{\mathbf{x}} \leq -\alpha + \beta^*$. Consequently, we have $(\mathbf{c}-\mathbf{d}^*)^\top \hat{\mathbf{x}} = \mathbf{b}^\top \hat{\mathbf{y}} = -\alpha + \beta^*$. Therefore, $\hat{\mathbf{x}} \in \mathcal{X}^*$ and $\hat{\mathbf{y}} \in \mathcal{Y}^*$, which, respectively, imply $\mathcal{F}_P^* \subseteq \mathcal{X}^*$ and $\mathcal{F}_D^* \subseteq \mathcal{Y}^*$. $\square$

Associated with any feasible solution of an LP, the *binding cone* is defined as the convex cone generated by the gradients of all constraints that are binding (active) at that solution. Recall from linear optimization that a feasible solution to an LP is optimal if and only if its corresponding binding cone includes the gradient of the objective function. Based on this fact, we apply Theorem 5 to provide necessary and sufficient geometrical conditions for feasible solutions of LFP (1) and its dual (4) to be optimal.

Let $\mathbf{x}^*$ be a feasible solution to LFP (1), and let $\mathcal{G}_P^*$ denote the union of the gradients of all binding constraints at $\mathbf{x}^*$, that is,

$$\mathcal{G}_P^* = \left\{ \left(\mathbf{a}^i\right)^\top \in \mathbb{R}^n : \mathbf{a}^i \mathbf{x}^* = b_i \right\} \cup \left\{ -\mathbf{e}_j \in \mathbb{R}^n : x_j^* = 0 \right\}.$$

Furthermore, denote by $\mathcal{B}_P^*$ the binding polyhedral cone generated by $\mathcal{G}_P^*$, that is, $\mathcal{B}_P^* = \operatorname{cone}\left(\mathcal{G}_P^*\right)$, where the operator "cone" denotes the conical hull. Because the feasible regions of programs (1) and (6) are equal, $\mathbf{x}^*$ is a feasible solution of LP (6). Therefore, the next corollary follows immediately from Theorem 5.

Corollary 1. Let $\mathbf{x}^* \in \mathcal{S}$. Then, $\mathbf{x}^* \in \mathcal{X}^*$ if and only if $\mathbf{c} - \mathbf{d}^* \in \mathcal{B}_P^*$.

Similarly, let $(\mathbf{y}^*; z^*)$ be a feasible solution to program (4). Additionally, assume that $\mathcal{B}_D^* = \operatorname{cone}\left(\mathcal{G}_D^*\right)$, where

$$\mathcal{G}_D^* = \left\{ \mathbf{a}_j \in \mathbb{R}^m : \mathbf{a}_j^\top \mathbf{y}^* = c_j - d_j^* \right\} \cup \left\{ \mathbf{e}_i \in \mathbb{R}^m : y_i^* = 0 \right\}.$$

Then we obtain the following corollary as a consequence of Theorem 5.

Corollary 2. Let $(\mathbf{y}^*; z^*) \in \mathcal{D}$. Then, $\mathbf{y}^* \in \mathcal{Y}^*$ if and only if $\mathbf{b} \in \mathcal{B}_D^*$.

We now present a numerical example verifying Corollaries 1 and 2.

Example 1. Consider the following LFP:

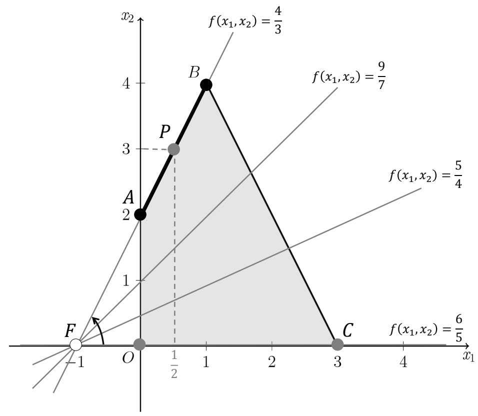


Figure 1: Multiple optimal solutions of program (8)

$$\max \quad f(x_1, x_2) = \frac{6x_1 + 3x_2 + 6}{5x_1 + 2x_2 + 5} \tag{8a}$$

subject to

$$2x_1 + x_2 \leq 6, \tag{8b}$$

$$-2x_1 + x_2 \leq 2, \tag{8c}$$

$$x_1, x_2 \geq 0. \tag{8d}$$

A graphical approach for finding optimal solution(s) of two-dimensional LFPs is to rotate the level-line around its focus point in positive direction (i.e., counterclockwise).[6] Figure 1 illustrates an application of this approach to program (8). The feasible region of program (8) in two dimensions x_1 and x_2 is the bounded polyhedral set $OABC$ (shaded in gray), and the focus point is $F = (-1, 0)$. Therefore, the optimal objective value of program (8) is $f^* = \frac{4}{3}$. Additionally, the set of all optimal solutions to program (8) is the segment AB, which is stated below as all convex combinations of the two extreme points A and B of the feasible region:

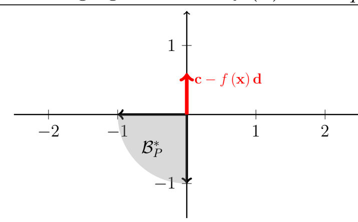
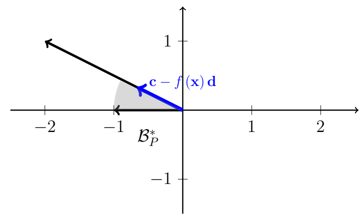
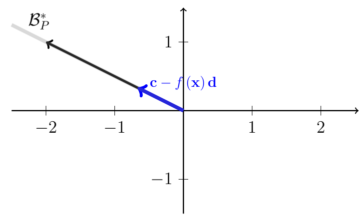
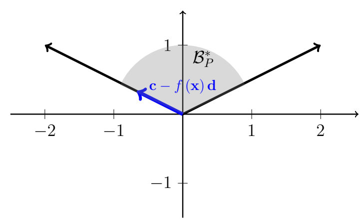
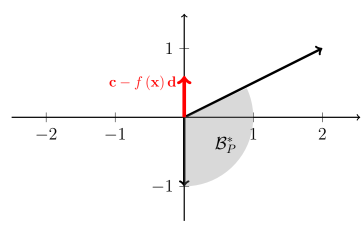
$$\mathcal{X}^* = \left\{ \mathbf{x}^\lambda \in \mathbb{R}^2 \colon \left(x_1^\lambda, x_2^\lambda\right) = \lambda\,(0, 2) + (1 - \lambda)\,(1, 4)\,,\ \lambda \in [0, 1] \right\}.$$

Table 1 presents the geometrical investigation of the proposed condition of primal optimality in Corollary 1 at four extreme points O, A, B, and C, and one nonextreme point P of the feasible region of program (8). As expected, the condition holds at the optimal points A, B, and P, but not at the nonoptimal points O and C.

The dual to program (8) is the following LP:

[6] More details on graphical solution of LFPs involving only two variables can be found in [1, Chapter 3].

Table 1: The proposed criterion of primal optimality for program (8)

	$\mathbf{x}$	$f(\mathbf{x})$	$\mathbf{c} - f(\mathbf{x})\,\mathbf{d}$	$\mathcal{G}_P^*$	Belongingness of $\mathbf{c} - f(\mathbf{x})\,\mathbf{d}$ to $\mathcal{B}_P^*$
O	$\begin{pmatrix}0\\0\end{pmatrix}$	$\frac{6}{5}$	$\begin{pmatrix}0\\\frac{3}{5}\end{pmatrix}$	$\begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}$	
A	$\begin{pmatrix}0\\2\end{pmatrix}$	$\frac{4}{3}$	$\begin{pmatrix}\frac{-2}{3}\\\frac{1}{3}\end{pmatrix}$	$\begin{pmatrix}-2\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}$	
P	$\begin{pmatrix}\frac{1}{2}\\3\end{pmatrix}$	$\frac{4}{3}$	$\begin{pmatrix}\frac{-2}{3}\\\frac{1}{3}\end{pmatrix}$	$\begin{pmatrix}-2\\1\end{pmatrix}$	
B	$\begin{pmatrix}1\\4\end{pmatrix}$	$\frac{4}{3}$	$\begin{pmatrix}\frac{-2}{3}\\\frac{1}{3}\end{pmatrix}$	$\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix}$	
C	$\begin{pmatrix}3\\0\end{pmatrix}$	$\frac{6}{5}$	$\begin{pmatrix}0\\\frac{3}{5}\end{pmatrix}$	$\begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$	

$$\min \quad z \tag{9a}$$
$$\text{subject to}$$
$$2y_1 - 2y_2 + 5z \geq 6, \tag{9b}$$
$$y_1 + \ \ y_2 + 2z \geq 3, \tag{9c}$$
$$-6y_1 - 2y_2 + 5z = 6, \tag{9d}$$
$$y_1, y_2 \geq 0,\ z \text{ sign free.} \tag{9e}$$

By Theorem 3, the optimal objective value of the dual program (9) is equal to that of the primal LFP (8), so $z^* = \frac{4}{3}$. By Theorem 4, $y_1^* = 0$ for any optimal solution $\left(y_1^*, y_2^*\right)$ of (9) because the point A is an optimal solution to program (8) for which the inequality constraint (8b) is strict. Additionally, both constraints (9b) and (9c) must be binding at optimality because point B with both positive components is an optimal solution to program (8). Taking these into account, it follows from the constraints of program (9) that $y_2^* = \frac{1}{3}$ for any optimal solution. Therefore, $\left(y_1^*, y_2^*, z^*\right) = \left(0, \frac{1}{3}, \frac{3}{4}\right)$ is the unique optimal solution of LP (9). The stated facts are observable from Figure 2, which draws the feasible region of program (9) in three dimensions y_1, y_2, and z as the section $LMNK$ of the two-dimensional hyperplane $\mathcal{H} = \left\{(\mathbf{y}; z) \in \mathbb{R}^3 \colon -6y_1 - 2y_2 + 5z = 6\right\}$.

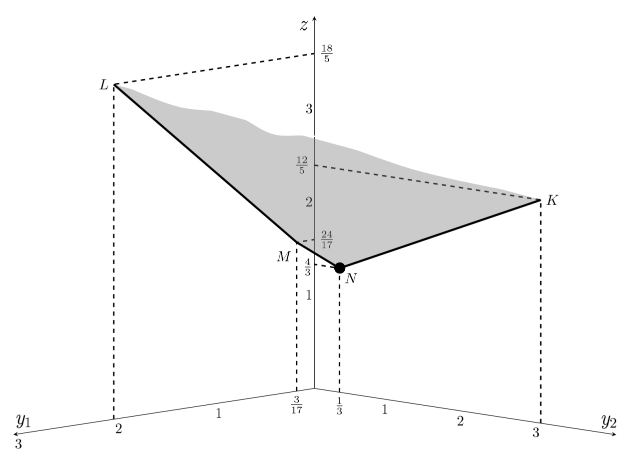


Figure 2: Unique optimal solution of program (9)

Observe that projecting the unique optimal solution of program (4) onto the space of $\mathbf{y}$-variables follows that $\mathcal{Y}^* = \left\{\left(0, \frac{1}{3}\right)^\top\right\}$. As an illustration to Theorem 5, Figure 3 shows the singleton set $\mathcal{Y}^*$ to be the optimal extreme point of the following LP:

$$\min \quad 6y_1 + 2y_2 \tag{10a}$$
subject to
$$2y_1 - 2y_2 \geq \frac{-2}{3}, \tag{10b}$$
$$y_1 + \; y_2 \geq \frac{1}{3}, \tag{10c}$$
$$y_1, y_2 \geq 0. \tag{10d}$$

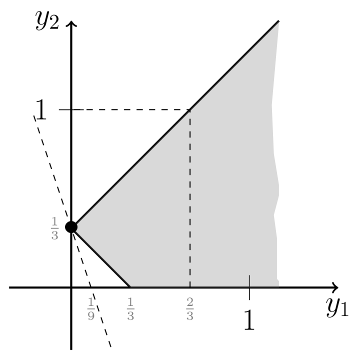
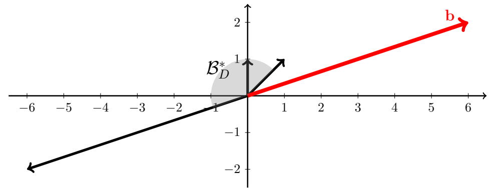
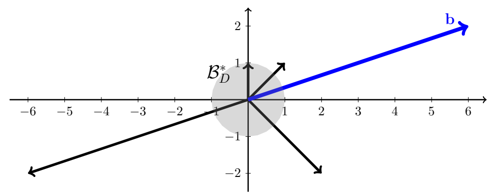


Figure 3: Representing $\mathcal{Y}^*$ as the unique optimal solution of program (10)

Table 2 geometrically investigates the proposed condition of dual optimality in Corollary 2 at the two extreme points M and N of the feasible region of program (9). While the condition is met by the optimal point N, it is not true at the nonoptimal point M. This verifies that the projection of the optimal point N onto the space of $\mathbf{y}$-variables is in $\mathcal{Y}^*$.

Remark 1. Corollary 1 suggests a geometrical approach for finding optimal solution(s) of program (1). It states that the optimality of any feasible point in $\mathcal{S}$ is equivalent to satisfying the condition given in Corollary 1. Therefore, taking into account the fact that the finite optimum must occur at some extreme points of $\mathcal{S}$, optimal solution(s) of program (1) can be found by examining the proposed condition only at extreme points of $\mathcal{S}$. (A similar approach for finding optimal solution(s) of program (4) can be devised based on Corollary 2.) It is important to note that this graphical approach does not require the enumeration of all extreme points of the feasible region.

Table 2: The proposed criterion of primal optimality for program (8)

	$\mathbf{y}$	z	$\mathcal{G}_D^*$	Belongingness of $\mathbf{b}$ to $\mathcal{B}_D^*$
M	$\begin{pmatrix} \frac{3}{17} \\ 0 \end{pmatrix}$	$\frac{24}{17}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ -2 \end{pmatrix}$	
N	$\begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}$	$\frac{4}{3}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -6 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}$	

4 Strict complementarity

4.1 Strict complementary solution

By Theorem 4, the CSC requires only that the product of each pair of complementary variables is zero at optimality. Therefore, not only one of the complementary variables must be zero at optimality, but also both are allowed to take simultaneously zero optimal values. It means that the CSC does not imply the positivity of pairwise sum of the complementary variables. If such positivity holds for a pair of optimal solutions for the primal LFP and its dual, then in every pair of complementary variables, exactly one variable is positive and the other is zero. Calling this property as *strict complementarity*, we present the following definition.

Definition 1. Feasible solutions $\mathbf{x}^{*s} \in \mathcal{S}$ and $(\mathbf{y}^{*s}; z^{*s}) \in \mathcal{D}$ satisfy the *strict complementary slackness condition* (SCSC), if they fulfill the following conditions in addition to the conditions given in (5):

$$\mathbf{x}^{*s} + \mathbf{v}^{*s} > \mathbf{0}, \quad \mathbf{u}^{*s} + \mathbf{y}^{*s} > \mathbf{0}. \tag{11}$$

It is clear that feasible solutions to programs (1) and (4) that satisfy the SCSC are optimal. We refer to such a pair of solutions as a *strict complementary solution* and denote it by $(\mathbf{x}^{*s}, \mathbf{y}^{*s})$. By the next result, we prove the existence of such a strict complementary solution.

Theorem 6. The following statements are true:

(i) Any strict complementary solution to LPs (6) and (7) is a strict complementary solution to programs (1) and (4).

(ii) There exists at least one strict complementary solution to LFP (1) and its dual (4).

Proof. Part (i) Let $(\mathbf{x}^{*s}, \mathbf{y}^{*s})$ be a strict complementary solution to LPs (6) and (7). Then, by the definition of strict complementary in linear optimization, $\mathbf{x}^{*s}$ and $\mathbf{y}^{*s}$ are, respectively, optimal solutions to these LPs, such that

$$\mathbf{v}^{*s\top}\mathbf{x}^{*s} = \mathbf{u}^{*s\top}\mathbf{y}^{*s} = 0, \quad \mathbf{x}^{*s} + \mathbf{v}^{*s} > \mathbf{0}, \quad \mathbf{u}^{*s} + \mathbf{y}^{*s} > \mathbf{0}, \tag{12}$$

where $\mathbf{u}^{*s}$ and $\mathbf{v}^{*s}$ are, respectively, the slack vectors added to the inequality constraints in $\mathcal{X}^*$ and $\mathcal{Y}^*$.

By Theorem 5, $\mathbf{x}^{*s}$ and $(\mathbf{y}^{*s}; z^{*s})$ are optimal solutions to programs (1) and (4), respectively. Furthermore, it follows from (12) that these solutions meet the SCSC in the sense of Definition 1. Therefore, $(\mathbf{x}^{*s}, \mathbf{y}^{*s})$ is a strict complementary solution to programs (1) and (4).

Part (ii) Because a finite optimum occurs for LPs (6) and (7), the Goldman–Tucker theorem implies the existence of a strict complementary solution to these LPs, which is a strict complementary solution to programs (1) and (4) by part (i) of the theorem. □

4.2 Optimal partitions

Let $(\mathbf{x}^{*s}, \mathbf{y}^{*s})$ be a strict complementary solution to programs (1) and (4). Then, the supports of vectors $\mathbf{x}^{*s}$ and $\mathbf{v}^{*s}$ are disjoint and their union is equal to the index set $\{1, \ldots, n\}$. Similarly, the supports of vectors $\mathbf{y}^{*s}$ and $\mathbf{u}^{*s}$ form a partition for the index set $\{1, \ldots, m\}$. Formally, we can write

$$\begin{aligned} &\operatorname{supp}(\mathbf{x}^{*s}) \cap \operatorname{supp}(\mathbf{v}^{*s}) = \varnothing, \quad \operatorname{supp}(\mathbf{x}^{*s}) \cup \operatorname{supp}(\mathbf{v}^{*s}) = \{1, \ldots, n\}; \\ &\operatorname{supp}(\mathbf{u}^{*s}) \cap \operatorname{supp}(\mathbf{y}^{*s}) = \varnothing, \quad \operatorname{supp}(\mathbf{u}^{*s}) \cup \operatorname{supp}(\mathbf{y}^{*s}) = \{1, \ldots, m\}. \end{aligned} \tag{13}$$

We call the above partitions as the *optimal partitions* induced for programs (1) and (4). By the next result, we show that these partitions, being independent from the given strict complementary solution, are unique.

Theorem 7. The optimal partitions induced for programs (1) and (4) as in (13) are the same across all strict complementary solutions and are, therefore, unique.

Proof. By contradiction, let $\left(\mathbf{x}^1, \mathbf{y}^1\right)$ and $\left(\mathbf{x}^2, \mathbf{y}^2\right)$ be two distinct strict complementary solutions such that $x^1_{\hat{j}} > 0$ and $x^2_{\hat{j}} = 0$ for some $\hat{j} \in \{1, \ldots, n\}$. Then, $v^1_{\hat{j}} = 0$ and $v^2_{\hat{j}} > 0$. Because both optimal sets of programs (1) and (4) are convex, $\frac{1}{2}(\mathbf{x}^1, \mathbf{y}^1) + \frac{1}{2}(\mathbf{x}^2, \mathbf{y}^2)$ must be an optimal solution such that $\frac{1}{2}(\mathbf{u}^1 + \mathbf{u}^2)$ and $\frac{1}{2}(\mathbf{v}^1 + \mathbf{v}^2)$ are, respectively, its primal and dual slack vectors. For this solution, we have the contradiction (with the CSC) that $\frac{1}{2}(x^1_{\hat{j}} + x^2_{\hat{j}}) > 0$ and $\frac{1}{2}(v^1_{\hat{j}} + v^2_{\hat{j}}) > 0$. Therefore, the optimal partition of $\{1, \ldots, n\}$ is unique. The uniqueness of the optimal partition of $\{1, \ldots, m\}$ follows from a similar argument. □

It is worth noting that the optimal partitions can be useful in situations, where knowing the positivity of a variable in some optimal solution of an LFP is concerned with. For example, while the slack-based measure (SBM) model of Tone [26] is used for the measurement of efficiency in the field of data envelopment analysis, the global reference set (peer group) of an inefficient decision making unit can be identified by the optimal partition of the index set of intensity vector.[7]

4.3 Geometrical interpretation

We begin this section by recalling the following definition from [4].

Definition 2. Let $\mathcal{S} \subset \mathbb{R}^d$. A subset $\mathcal{S}^+$ of $\mathbb{R}^{d+d'}$ is a *representing set* for $\mathcal{S}$, if its projection onto the space of $\mathbf{x}$-variables is exactly $\mathcal{S}$, that is, $\mathbf{x} \in \mathcal{S}$ if and only if there exists some $\mathbf{s} \in \mathbb{R}^{d'}$ such that $(\mathbf{x}; \mathbf{s}) \in \mathcal{S}^+$:

$$\mathcal{S} = \left\{ \mathbf{x} \in \mathbb{R}^d \colon\ (\mathbf{x}; \mathbf{s}) \in \mathcal{S}^+ \text{ for some } \mathbf{s} \in \mathbb{R}^{d'} \right\}.$$

By adding slack vectors to the inequality constraints of $\mathcal{X}^*$ and $\mathcal{Y}^*$, we define the following nonnegative polyhedral sets:

$$\mathcal{X}^{*+} = \left\{ (\mathbf{x}; \mathbf{u}) \in \mathbb{R}^{n+m} \colon (\mathbf{c} - \mathbf{d}^*)^\top \mathbf{x} = -\alpha + \beta^*,\ \mathbf{A}\mathbf{x} + \mathbf{u} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{u} \geq \mathbf{0} \right\},$$
$$\mathcal{Y}^{*+} = \left\{ (\mathbf{y}; \mathbf{v}) \in \mathbb{R}^{m+n} \colon \mathbf{b}^\top \mathbf{y} = -\alpha + \beta^*,\ \mathbf{A}^\top \mathbf{y} - \mathbf{v} = \mathbf{c} - \mathbf{d}^*,\ \mathbf{y} \geq \mathbf{0},\ \mathbf{v} \geq \mathbf{0} \right\}.$$

By Definition 2, $\mathcal{X}^{*+}$ and $\mathcal{Y}^{*+}$ are polyhedral representing sets for $\mathcal{X}^*$ and $\mathcal{Y}^*$, respectively. The next result shows that projecting the relative interiors of these representing sets gives the strict complementary solutions of programs (1) and (4).

Theorem 8. It follows that $(\mathbf{x}^{*s}, \mathbf{y}^{*s})$ is a strict complementary solution to LFP (1) and its dual (4) if and only if $(\mathbf{x}^{*s}; \mathbf{u}^{*s}) \in \operatorname{ri}(\mathcal{X}^{*+})$ and $(\mathbf{y}^{*s}; \mathbf{v}^{*s}) \in \operatorname{ri}(\mathcal{Y}^{*+})$.

[7] For more details on the concept of global reference set and its identification, the reader may refer to [17, 18, 19].

Proof. Let $(\mathbf{x}^{*s};\mathbf{u}^{*s}) \in \mathrm{ri}\,(\mathcal{X}^{*+})$ and let $\left(\mathbf{y}^{*s};\mathbf{v}^{*s}\right) \in \mathrm{ri}\,(\mathcal{Y}^{*+})$. By [20, Theorem 4.1], the relative interior of a nonnegative polyhedral set with equality defining constraints consists of its maximal elements. It follows that $(\mathbf{x}^{*s};\mathbf{u}^{*s}) \in \mathrm{me}\,(\mathcal{X}^{*+})$ and $\left(\mathbf{y}^{*s};\mathbf{v}^{*s}\right) \in \mathrm{me}\,(\mathcal{Y}^{*+})$. By the Goldman–Tucker theorem, $(\mathbf{x}^{*s},\mathbf{y}^{*s})$ is thus a strict complementary solution to LPs (6) and (7). By part (i) of Theorem 6, it follows that $(\mathbf{x}^{*s},\mathbf{y}^{*s})$ is a strict complementary solution to programs (1) and (4).

Conversely, let $(\mathbf{x}^{*s},\mathbf{y}^{*s})$ be a strict complementary solution to programs (1) and (4). Then, similar to the proof of Theorem 6, it can be proved that $(\mathbf{x}^{*s},\mathbf{y}^{*s})$ is a strict complementary solution to LPs (2) and (4). □

To illustrate the concept of strict complementarity, we return back to Example 1. Adding the nonnegative slack variables u_1 and u_2 to (8b) and (8c) obtains the following representing set for $\mathcal{X}^*$:

$$\mathcal{X}^{*+} = \left\{\left(\mathbf{x}^{\lambda};\mathbf{u}^{\lambda}\right) \in \mathbb{R}^4_+ \colon \left(x_1^{\lambda},x_2^{\lambda},u_1^{\lambda},u_2^{\lambda}\right) = (1-\lambda, 4-2\lambda, 4\lambda, 0)\,,\ \lambda \in [0,1]\right\}.$$

Similarly, adding the nonnegative slack variables v_1 and v_2 to (10b) and (10c) results the following representing set for $\mathcal{Y}^*$:

$$\mathcal{Y}^{*+} = \Big\{(\mathbf{y};\mathbf{v}) \in \mathbb{R}^4_+ \colon \left(y_1,y_2,v_1,v_2\right) = \Big(0,\frac{1}{3},0,0\Big)\Big\}.$$

Consider the midpoint $P = \left(\frac{1}{2},3\right)$ of the line segment AB in Figure 1. This point is associated with the vector $\left(\mathbf{x}^{\frac{1}{2}};\mathbf{u}^{\frac{1}{2}}\right) = \left(\left(\frac{1}{2},3\right)^{\top};(2,0)^{\top}\right)$, which is a maximal element and, therefore, a relative interior point of the set $\mathcal{X}^{*+}$. Furthermore, consider the point $\left(0,\frac{1}{3}\right)$ in Figure 3 that is associated with the single element $(\mathbf{y};\mathbf{v}) = \left(\left(0,\frac{1}{3}\right)^{\top};\mathbf{0}\right)$ of the set $\mathcal{Y}^{*+}$. Clearly, $\left(\mathbf{x}^{\frac{1}{2}},\mathbf{y}\right)$ is a strict complementary solution to programs (8) and (9).

Note that $\left(\mathbf{x}^{\lambda},\mathbf{y}\right)$ is a strict complementary solution for all $\lambda \in (0,1)$. However, this is not true for $\lambda = 0,1$. This is because in either of these two cases, pairwise sum of the complementary variables is not positive.

5 Finding a strict complementary solution

5.1 Finding a maximal element of a nonnegative polyhedral set

Consider the following nonempty polyhedral set in $\mathbb{R}^d$:

$$\mathcal{P} = \left\{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{P}\mathbf{x} + \mathbf{Q}\mathbf{y} + \mathbf{R}\mathbf{z} = \mathbf{t},\ \mathbf{x},\mathbf{y} \geq \mathbf{0},\ \mathbf{z} \text{ sign free}\right\},$$

where $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{y} \in \mathbb{R}^e$, and $\mathbf{z} \in \mathbb{R}^f$ are the vectors of variables, $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ are, respectively, matrices of coefficients of orders $c \times d$, $c \times e$, and $c \times f$, and $\mathbf{t} \in \mathbb{R}^c$ is a constant vector.

Mehdiloozad et al. [20] developed a general convex optimization program for finding a maximal element of a nonnegative convex set. As a consequence of their Theorem 3.2, the following result develops an LP for finding a maximal element of $\mathcal{P}$.

Theorem 9. Let $\left(\mathbf{x}^{1*}, \mathbf{x}^{2*}, \mathbf{y}^*, \mathbf{z}^*, w^*\right)$ be an optimal solution to the following LP:

$$\begin{aligned} &\max \quad \mathbf{1}^\top \mathbf{x}^1 \\ &\text{subject to} \\ &\mathbf{P}\left(\mathbf{x}^1 + \mathbf{x}^2\right) + \mathbf{Q}\mathbf{y} + \mathbf{R}\mathbf{z} = \mathbf{t}w, \\ &\mathbf{1} \geq \mathbf{x}^1 \geq \mathbf{0},\ \mathbf{x}^2, \mathbf{y} \geq \mathbf{0},\ \mathbf{z} \text{ sign free},\ w \geq 1. \end{aligned} \tag{15}$$

Then $\frac{1}{w^*}\left(\mathbf{x}^{1*} + \mathbf{x}^{2*}\right) \in \mathrm{me}\left(\mathcal{P}\right)$.

Proof. By [20, Definition 2.5], the characteristic cone of the nonnegative polyhedral set $\mathcal{P}$ is $\mathcal{C}_{\mathcal{P}} = \left\{x_{d+1}\binom{\mathbf{x}}{1} : \mathbf{x} \in \mathcal{P},\ x_{d+1} > 0\right\}$. To find a maximal element of $\mathcal{P}$, this cone is incorporated into the convex program proposed in [20, Theorem 3.2]. This leads to the following LP:

$$\begin{aligned} &\max \quad \mathbf{1}^\top \mathbf{x}^1 + w^1 \\ &\text{subject to} \\ &\mathbf{P}\left(\mathbf{x}^1 + \mathbf{x}^2\right) + \mathbf{Q}\mathbf{y} + \mathbf{R}\mathbf{z} = \mathbf{t}\left(w^1 + w^2\right), \\ &\mathbf{1} \geq \mathbf{x}^1 \geq \mathbf{0},\ \mathbf{x}^2, \mathbf{y} \geq \mathbf{0},\ \mathbf{z} \text{ sign free},\ 1 \geq w^1 \geq 0,\ w^2 \geq 0. \end{aligned} \tag{16}$$

Because the maximization linear program (16) is feasible and its objective function is upper bounded by $d+1$, it has a finite optimal solution, namely, $\left(\mathbf{x}^{1*}, \mathbf{x}^{2*}, \mathbf{y}^*, \mathbf{z}^*, w^{1*}, w^{2*}\right)$.

By the assumption, we have $\mathcal{P} \neq \varnothing$. Hence, it follows from [20, Theorem 3.2] that $w^{1*} = 1$. It is clear that program (15) is derived from program (16) by replacing w^1 with its optimal value and using the variable substitution $w = w^1 + 1$. This implies that any optimal solution of program (15) gives an optimal solution to program (16). Namely, if we define $\mathbf{x}^{1\prime} = \mathbf{x}^{1*}$, $\mathbf{x}^{2\prime} = \mathbf{x}^{2*}$, $\mathbf{y}' = \mathbf{y}$, $\mathbf{z}' = \mathbf{z}$, $w^{1\prime} = 1$, and $w^{2\prime} = w^* - 1$, then $\left(\mathbf{x}^{1\prime}, \mathbf{x}^{2\prime}, \mathbf{y}', \mathbf{z}', w^{1\prime}, w^{2\prime}\right)$ is an optimal solution to program (16). Therefore, the statement of the theorem follows from [20, Theorem 3.2]. $\square$

5.2 Our proposed approaches

Though Theorem 6 demonstrates the existence of a strict complementary solution for programs (1) and (4), it does not specify how to identify such a solution. To deal with this issue, we develop two approaches in this section by applying Theorem 9. The GAMS code of these approaches is provided in Appendix A.

5.2.1 First approach

From Theorem 8, any pair of relative interior points of $\mathcal{X}^{*+}$ and $\mathcal{Y}^{*+}$ determines a strict complementary solution to programs (1) and (4). By Theorem 9, we develop the following two LPs to find such relative interior points:

$$\begin{aligned}
&\max \quad \mathbf{1}^\top \mathbf{x}^1 + \mathbf{1}^\top \mathbf{u}^1 \\
&\text{subject to} \\
&\begin{bmatrix} (\mathbf{c}-\mathbf{d}^*)^\top & \mathbf{0}^\top \\ \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{x}^1+\mathbf{x}^2 \\ \mathbf{u}^1+\mathbf{u}^2 \end{pmatrix} = \begin{pmatrix} -\alpha+\beta^* \\ \mathbf{b} \end{pmatrix} w_P, \\
&\mathbf{1} \geq \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{u}^1 \end{pmatrix} \geq \mathbf{0}, \ \begin{pmatrix} \mathbf{x}^2 \\ \mathbf{u}^2 \end{pmatrix} \geq \mathbf{0}, \ w_P \geq 1.
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\max \quad \mathbf{1}^\top \mathbf{y}^1 + \mathbf{1}^\top \mathbf{v}^1 \\
&\text{subject to} \\
&\begin{bmatrix} \mathbf{b}^\top & \mathbf{0}^\top \\ \mathbf{A}^\top & -\mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{y}^1+\mathbf{y}^2 \\ \mathbf{v}^1+\mathbf{v}^2 \end{pmatrix} = \begin{pmatrix} -\alpha+\beta^* \\ \mathbf{c}-\mathbf{d}^* \end{pmatrix} w_D, \\
&\mathbf{1} \geq \begin{pmatrix} \mathbf{y}^1 \\ \mathbf{v}^1 \end{pmatrix} \geq \mathbf{0}, \ \begin{pmatrix} \mathbf{y}^2 \\ \mathbf{v}^2 \end{pmatrix} \geq \mathbf{0}, \ w_D \geq 1.
\end{aligned} \tag{18}$$

Let $\left(\mathbf{x}^{1*}, \mathbf{x}^{2*}, \mathbf{u}^{1*}, \mathbf{u}^{2*}, w_P^*\right)$ and $\left(\mathbf{y}^{1*}, \mathbf{y}^{2*}, \mathbf{y}^{1*}, \mathbf{y}^{2*}, w_D^*\right)$ be optimal solutions to programs (17) and (18), respectively. Then, it follows from Theorem 9 that

$$\left(\mathbf{x}^{\mathrm{ri}}; \mathbf{u}^{\mathrm{ri}}\right) = \frac{1}{w_P^*}\left(\mathbf{x}^{1*}+\mathbf{x}^{2*}; \mathbf{u}^{1*}+\mathbf{u}^{2*}\right) \in \mathrm{me}\left(\mathcal{X}^{*+}\right), \tag{19a}$$

$$\left(\mathbf{y}^{\mathrm{ri}}; \mathbf{v}^{\mathrm{ri}}\right) = \frac{1}{w_D^*}\left(\mathbf{y}^{1*}+\mathbf{y}^{2*}; \mathbf{v}^{1*}+\mathbf{v}^{2*}\right) \in \mathrm{me}\left(\mathcal{Y}^{*+}\right). \tag{19b}$$

Therefore, taking into account [20, Theorem 4.1] and Theorem 8, it follows from (19) that $\left(\mathbf{x}^{\mathrm{ri}}, \mathbf{y}^{\mathrm{ri}}\right)$ is a strict complementary solution to programs (1) and (4).

5.2.2 Second approach

Our first approach of identifying a strict complementary solution requires the knowledge of the optimal objective value of program (1). In this section, we propose an alternative approach that is exempt from this requirement. We exploit the fact that the optimality of feasible solutions to a pair of primal and dual LPs follows from the equality of their corresponding objective function values. Specifically, we consider the following set:

$$\begin{aligned}\mathcal{W}^* = \big\{(\bar{\mathbf{x}};\mathbf{y}) \in \mathbb{R}^{n+m} \colon & \mathbf{c}^\top \bar{\mathbf{x}} + \alpha t = z, \\ & \mathbf{A}\bar{\mathbf{x}} \le \mathbf{b}t,\ \mathbf{d}^\top \bar{\mathbf{x}} + \beta t = 1, \\ & -\mathbf{b}^\top \mathbf{y} + \beta z = \alpha,\ \mathbf{A}^\top \mathbf{y} + \mathbf{d}z \ge \mathbf{c}, \\ & \bar{\mathbf{x}} \ge \mathbf{0},\ t \ge 0,\ \mathbf{y} \ge \mathbf{0},\ z \text{ sign free}\big\}.\end{aligned} \tag{20}$$

By the projection lemma,[8] it follows that $\mathcal{W}^*$ is a nonnegative polyhedral set in $\mathbb{R}^{n+m}_+$. By adding slack vectors $\bar{\mathbf{u}}$ and $\mathbf{v}$ to the inequality constraints of this set, we obtain the following set:

$$\begin{aligned}\mathcal{W}^{*+} = \big\{(\bar{\mathbf{x}};\mathbf{y};\bar{\mathbf{u}};\mathbf{v}) \in \mathbb{R}^{2(n+m)} \colon & \mathbf{c}^\top \mathbf{x} + \alpha t - z = 0, \\ & \mathbf{A}\bar{\mathbf{x}} + \bar{\mathbf{u}} - \mathbf{b}t = \mathbf{0},\ \mathbf{d}^\top \mathbf{x} + \beta t = 1, \\ & -\mathbf{b}^\top \mathbf{y} + \beta z = \alpha,\ \mathbf{A}^\top \mathbf{y} + \mathbf{d}z - \mathbf{v} = \mathbf{c}, \\ & \mathbf{x}, \mathbf{v} \ge \mathbf{0},\ \mathbf{y}, \bar{\mathbf{u}} \ge \mathbf{0},\ t \ge 0,\ z \text{ sign free}\big\}.\end{aligned} \tag{21}$$

By Definition 2, $\mathcal{W}^{*+}$ is a polyhedral representing set for $\mathcal{W}^*$. Let $(\bar{\mathbf{x}};\mathbf{y};\bar{\mathbf{u}};\mathbf{v}) \in \mathcal{W}^{*+}$. Then $(\bar{\mathbf{x}};\mathbf{y};\bar{\mathbf{u}};\mathbf{v})$ satisfies (21) with some scalars t and z. The set $\mathcal{W}^*$ is defined by conditions (2b)–(2d) and (4b)–(4d) and the additional equality requiring that the objective function of LP (2) to be equal to the objective function of its dual (4). By the optimality criterion theorem of linear optimization, it follows that $(\bar{\mathbf{x}}, t)$ and $(\mathbf{y}, z)$ are optimal solutions to LPs (2) and (4), respectively. Additionally, $\bar{\mathbf{u}}$ and $\mathbf{v}$ are their corresponding slack vectors added to inequalities (2b) and (4b), respectively. Consequently, we have $\frac{1}{t}(\bar{\mathbf{x}};\bar{\mathbf{u}}) \in \mathcal{X}^{*+}$ and $(\mathbf{y};\mathbf{v}) \in \mathcal{Y}^{*+}$. Based on this, the next result shows that any maximal element of $\mathcal{W}^{*+}$ determines two relative interior points of $\mathcal{X}^{*+}$ and $\mathcal{Y}^{*+}$, and therefore a strict complementary solution to programs (1) and (4).

Theorem 10. Let $(\bar{\mathbf{x}}^{\mathrm{me}};\mathbf{y}^{\mathrm{me}};\bar{\mathbf{u}}^{\mathrm{me}};\mathbf{v}^{\mathrm{me}}) \in \mathrm{me}(\mathcal{W}^{*+})$ satisfy (21) with some scalars t^{me} and z^{me}. Then $\left(\frac{1}{t^{\mathrm{me}}}\bar{\mathbf{x}}^{\mathrm{me}}, \mathbf{y}^{\mathrm{me}}\right)$ is a strict complementary solution to programs (1) and (4).

[8] The projection lemma states that the projection of a polyhedral set onto the space of any subset of its characterizing variables is a polyhedral set; see, for example, [6, Corollary 2.4].

Proof. Let $(\bar{\mathbf{x}}^{\mathrm{me}}; \mathbf{y}^{\mathrm{me}}; \bar{\mathbf{u}}^{\mathrm{me}}; \mathbf{v}^{\mathrm{me}})$ be a maximal element of $\mathcal{W}^{*+}$ that satisfies (21) with some scalars t^{me} and z^{me}. Then $\frac{1}{t^{\mathrm{me}}}(\bar{\mathbf{x}}^{\mathrm{me}}; \bar{\mathbf{u}}^{\mathrm{me}}) \in \mathrm{me}(\mathcal{X}^{*+})$ and $(\mathbf{y}^{\mathrm{me}}; \mathbf{v}^{\mathrm{me}}) \in \mathrm{me}(\mathcal{Y}^{*+})$. Therefore, by [20, Theorem 4.1], the statement follows from Theorem 8. □

Based on Theorem 9, we develop the following LP to find a maximal element of $\mathcal{W}^{*+}$:

$$\begin{aligned}
&\max \quad \mathbf{1}^\top \bar{\mathbf{x}}^1 + \mathbf{1}^\top \bar{\mathbf{u}}^1 + \mathbf{1}^\top \mathbf{y}^1 + \mathbf{1}^\top \mathbf{v}^1 \\
&\text{subject to} \\
&\begin{bmatrix} \mathbf{c}^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{d}^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{b}^\top & \mathbf{0} \\ \mathbf{A}^\top & \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{pmatrix} \bar{\mathbf{x}}^1 + \bar{\mathbf{x}}^2 \\ \bar{\mathbf{u}}^1 + \bar{\mathbf{u}}^2 \\ \mathbf{y}^1 + \mathbf{y}^2 \\ \mathbf{v}^1 + \mathbf{v}^2 \end{pmatrix} + \begin{bmatrix} \alpha \\ -\mathbf{b} \\ \beta \\ 0 \\ \mathbf{0} \end{bmatrix} t + \begin{bmatrix} -1 \\ \mathbf{0} \\ 0 \\ \beta \\ \mathbf{d} \end{bmatrix} z = \begin{pmatrix} 0 \\ \mathbf{0} \\ 1 \\ \alpha \\ \mathbf{c} \end{pmatrix} w, \\
&\mathbf{1} \geq \begin{pmatrix} \bar{\mathbf{x}}^1 \\ \bar{\mathbf{u}}^1 \\ \mathbf{y}^1 \\ \mathbf{v}^1 \end{pmatrix} \geq \mathbf{0}, \quad \begin{pmatrix} \bar{\mathbf{x}}^2 \\ \bar{\mathbf{u}}^2 \\ \mathbf{y}^2 \\ \mathbf{v}^2 \end{pmatrix} \geq \mathbf{0}, \ t \geq 0, \ z \text{ sign free}, \ w \geq 1.
\end{aligned} \tag{22}$$

Let $\left(\bar{\mathbf{x}}^{1*}, \bar{\mathbf{x}}^{2*}, \bar{\mathbf{u}}^{1*}, \bar{\mathbf{u}}^{2*}, \mathbf{y}^{1*}, \mathbf{y}^{2*}, \mathbf{v}^{1*}, \mathbf{v}^{2*}, t^*, z^*, w^*\right)$ be an optimal solution to program (22), and define

$$(\bar{\mathbf{x}}^{\mathrm{me}}; \mathbf{y}^{\mathrm{me}}; \bar{\mathbf{u}}^{\mathrm{me}}; \mathbf{v}^{\mathrm{me}}) = \frac{1}{w^*}\left(\bar{\mathbf{x}}^{1*} + \bar{\mathbf{x}}^{2*}; \mathbf{y}^{1*} + \mathbf{y}^{2*}; \bar{\mathbf{u}}^{1*} + \bar{\mathbf{u}}^{2*}; \mathbf{v}^{1*} + \mathbf{v}^{2*}\right).$$

By Theorem 9, we have $(\bar{\mathbf{x}}^{\mathrm{me}}; \mathbf{y}^{\mathrm{me}}; \bar{\mathbf{u}}^{\mathrm{me}}; \mathbf{v}^{\mathrm{me}}) \in \mathrm{me}(\mathcal{W}^{*+})$. If $t^{\mathrm{me}} = \frac{t^*}{w^*}$, then it follows by Theorem 10 that $\left(\frac{1}{t^{\mathrm{me}}}\bar{\mathbf{x}}^{\mathrm{me}}, \mathbf{y}^{\mathrm{me}}\right) = \left(\frac{1}{t^*}\left(\bar{\mathbf{x}}^{1*} + \bar{\mathbf{x}}^{2*}\right), \frac{1}{w^*}\left(\mathbf{y}^{1*} + \mathbf{y}^{2*}\right)\right)$ is a strict complementary solution to programs (1) and (4).

6 Numerical example

In this section, we illustrate our proposed approaches of finding strict complementary solutions with a numerical example, taken from [1, 14].

Example 2. Consider the following LFP:

$$
\begin{aligned}
&\max \quad \frac{x_1 + 2x_2 + 3.5x_3 + \ \ x_4 + 1}{2x_1 + 2x_2 + 3.5x_3 + 3x_4 + 4} \\
&\text{subject to} \\
&2x_1 + \ \ x_2 + 3x_3 + 3x_4 \leq 10, \\
&\ \ x_1 + 2x_2 + \ \ x_3 + \ \ x_4 \leq 14, \\
&x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}
\tag{23}
$$

The dual of program (23) is the following LP:

$$
\begin{aligned}
&\min \quad z \\
&\text{subject to} \\
&\ \ 2y_1 + \ \ \ y_2 + \ \ \ 2z \geq 1, \\
&\ \ \ y_1 + \ 2y_2 + \ \ \ 2z \geq 2, \\
&\ \ 3y_1 + \ \ \ y_2 + 3.5z \geq 3.5, \\
&\ \ 3y_1 + \ \ \ y_2 + \ \ \ 3z \geq 1, \\
&-10y_1 - 14y_2 + 4z = 1, \\
&y_1, y_2 \geq 0, \ z \text{ sign free.}
\end{aligned}
\tag{24}
$$

Let us add primal slack variables u_1 and u_2 to program (23), and dual slack variables v_1, v_2, v_3, and v_4 to the inequality constraints of program (24), to turn their inequality constraints to equalities. We use our proposed approaches to find a strict complementary solution to the above pair of programs. Running the modified version of the GAMS code provided in Appendix A results that the joint optimal objective value of programs (23) and (24) is equal to 0.857. Furthermore, the strict complementary solution obtained from both of our proposed approaches is

$$
\begin{aligned}
&x_1^* = 0, \quad &&x_2^* = 1.071, \quad &&x_3^* = 1.2, \quad &&x_4^* = 0, \quad &&u_1^* = 0, \quad &&u_2^* = 0; \\
&v_1^* = 1.071 \quad &&v_2^* = 0 \quad &&v_3^* = 0, \quad &&v_4^* = 2.071, \quad &&y_1^* = 0.143 \quad &&y_2^* = 0.071.
\end{aligned}
$$

7 Concluding remarks

An indirect approach for establishing duality results in linear fractional optimization is based on applying the well-known transformation of Charnes and Cooper [9]. This approach converts a primal LFP into an equivalent LP and then defines the dual of the obtained LP as the dual of the primal LFP. An advantage of using this approach is that it allows exploiting the duality results of linear optimization for establishing duality statements in linear fractional optimization.

In this paper, we show that the dual program derived from the above approach is the same dual program suggested in [8]. Based on this version of duality, we provide new criteria for primal and dual optimality as our first contribution. We equivalently represent the primal and dual optimal sets as the optimal sets of a pair of primal and dual LPs. By this representation, it follows that a primal (resp., dual) feasible solution is optimal if and only if its binding polyhedral cone contains the objective vector of the corresponding primal (resp., dual) LP. This condition not only is (theoretically) necessary and sufficient for the optimality of any general LFP, but also is a new geometrical tool for solving two- and three-dimensional LFPs.

As our second contribution, we introduce the concept of strict complementarity into the framework of linear fractional optimization. We prove the existence of a strict complementary solution and show that all such solutions induce unique optimal partitions for the sets of indices of nonnegative variables. To geometrically interpret the strict complementarity, we equivalently represent primal and dual optimal sets by two nonnegative polyhedral sets that are described only by equality constraints. Then we prove that each pair of relative interior points of these representing sets is a strict complementary solution, and vice versa. By this result, we deal with the problem of identifying a strict complementary solution. Specifically, we turn this problem to the equivalent problem of identifying a maximal element of a nonnegative polyhedral set. Then, by applying the technique of finding a maximal element of a nonnegative polyhedral set, we develop two linear optimization approaches with different strategies for finding a strict complementary solution in linear fractional optimization.

Our first approach identifies a strict complementary solution by solving two LPs and requires knowing the optimal objective value of the given LFP. In contrast, our second approach involves solving a single (but larger) LP and does not need the per-knowledge of the optimal objective value. As our proposed approaches are linear optimization based, they allow for applying the ordinary simplex algorithm of linear optimization to identify a strict complementary solution in linear fractional optimization. Nonetheless, regarding the preference on using the proposed approaches, note that each of the LPs developed in our first approach has less number of constraints than the LP of our second approach. Therefore, the use of our first approach is particularly recommended in situations, where only primal or dual part of a strict complementary solution needs to be found. For example, while the SBM model of Tone [26] is used for the measurement of efficiency in the field of data envelopment analysis, the global reference set of an inefficient decision making unit can be identified by either the primal or dual part of a strict complementary solution. However, solving the primal SBM model is recommended because the number of its constraints are mostly less than that of its dual.

The approaches developed in our paper open up a number of further research avenues. First, it should be interesting to extend the results proposed in linear optimization literature on the use of strict complementarity for post-optimality analysis [5, 13] to linear fractional optimization. Second, an interesting method for finding a strict complementary solution in linear optimization is to apply the so-called Balinski–Tucker tableau [2]. From our contribution, it is found that this method can be used to generate (indirectly) a strict complementary solution in linear fractional optimization. Therefore, it is worth exploring modification of the Balinski–Tucker method so that a strict complementary solution is directly obtained.

Appendix A

The following computer program written in GAMS identifies a strict complementary solution for the primal LFP (8) and its dual (9). Making this program applicable for any LFP in the general form of (1) just requires modifying "Sets", "Table $A(i, j)$", "Parameters", "Alpha," and "Beta" in Lines 1–23.

```
1  Sets
2         i       row number of matrix A   /i1*i2/
3         j    column number of matrix A   /j1*j2/;
4
5  Table A(i,j)
6         j1      j2
7  i1     2       1
8  i2     -2      1;
9
10  Parameters
11       b/i1     6
12         i2     2/
13       c/j1     6
14         j2     3/
15       d/j1     5
16         j2     2/;
17
18  Scalars
19       Alpha
20       Beta;
21
22  Alpha=6;
23  Beta =5;
24
25  File ProgSC / Results.txt /;
26  Put ProgSC;
27
28  ************************************************************
29  *Stage 1: Solving program (3)
30
31  Free Variables
32       Theta;
33
34  Positive Variables
35       xbar(j)
36       t ;
37
38  Scalar
39       ThetaStar;
40
41  Equations
42       Obj
```

```
       Con1
       Con2;

     Obj..          Theta =E= Sum(j, c(j)*xbar(j)) + Alpha*t;
       Con1(i)..          Sum(j, a(i,j)*xbar(j))             =L= b(i)*t;
       Con2..             Sum(j,    d(j)*xbar(j)) + Beta*t =E= 1;

Model MainLFP      / Obj, Con1, Con2 /;

Put /'Finding the Optimal Obj. Value (ThetaStar)';
Put /'------------------------------------------'/;
Option LP=CONOPT;
Solve MainLFP using LP Maximizing Theta;
      Put 'Obj = ':>6; Put Theta.L:<10:3;
      ThetaStar=Theta.L;
Put /'------------------------------------------'/;

*End of Stage 1
************************************************************

************************************************************
*First approach: Solving program (18)

Positive Variables
     xbar1(j)
     xbar2(j)
     ubar1(i)
     ubar2(i)
     y1(i)
     y2(i)
     v1(j)
     v2(j)
     w1
     w2
     p;

Free variable
     q;

xbar2.up(j) = 1;
ubar2.up(i) = 1;
y2.up(i) = 1;
v2.up(j) = 1;
w2.up = 1;

Parameters
     XbarStar
     tStar
     UbarStar
     YStar(i)
     zStar
     VStar(j);

Equations
     ObjP
       ConP1
       ConP2
       ConP3
     ObjD
       ConD1
       ConD2
       ConD3;

     ObjP..          Theta =E= Sum(j, xbar2(j)) + Sum(i, ubar2(i)) + w2;
       ConP1(i)..          Sum(j, a(i,j)*(xbar1(j)+xbar2(j))) -  b(i)*p
       + ubar1(i)+ubar2(i) =E= 0;
       ConP2..             Sum(j, d(j)  *(xbar1(j)+xbar2(j))) +  Beta*p
```

```
      - w1+w2            =E= 0;
      ConP3..            Sum(j, c(j)  *(xbar1(j)+xbar2(j))) + Alpha*p
      - (w1+w2)*ThetaStar =E= 0;

    ObjD..          Theta =E= Sum(i, y2(i)) + Sum(j, v2(j)) + w2;
      ConD1(j)..         Sum(i, a(i,j)*(y1(i)+y2(i))) + d(j)*q - v1(j)
      - v2(j) - c(j)*(w1+w2) =E= 0;
      ConD2..           -Sum(i, b(i)  *(y1(i)+y2(i))) + Beta*q
      - Alpha*(w1+w2) =E= 0;
      ConD3..            q -  ThetaStar*(w1+w2) =E= 0;

Models  Primal_SCSC  / ObjP , ConP1, ConP2, ConP3 /
          Dual_SCSC  / ObjD , ConD1, ConD2, ConD3 / ;

Solve Primal_SCSC using LP Maximizing Theta;
      XbarStar(j) = (xbar1.L(j)+xbar2.L(j))/(w1.L+w2.L);
      tStar       = p.L/(w1.L+w2.L);
      UbarStar(i) = (ubar1.L(i)+ubar2.L(i))/(w1.L+w2.L);

Solve Dual_SCSC using LP Maximizing Theta;
      YStar(i) = (y1.L(i)+y2.L(i))/(w1.L+w2.L);
      zStar    = q.L/(w1.L+w2.L);
      VStar(j) = (v1.L(j)+v2.L(j))/(w1.L+w2.L);

Put / /'Finding a SC Solution via Approach I';
Put /'----------------------------------------------'/;
Put  '       Primal                Dual     '/;
Put  '----------------------------------------------'/;
Loop(j,
       Put 'x(':>5; Put ord(j):<>3:0; Put ')= ':3; Put (XbarStar(j)/tStar):<10:3;
       Put 'v(':>5; Put ord(j):<>3:0; Put ')= ':3; Put Vstar(j):<10:3;
       Put /;
    );
Put /;
Loop(i,
       Put 'u(':>5; Put ord(i):<>3:0; Put ')= ':3; Put (UbarStar(i)/tStar):<10:3;
       Put 'y(':>5; Put ord(i):<>3:0; Put ')= ':3; Put Ystar(i):<10:3;
       Put /;
    );
Put '----------------------------------------------'/ / /;

*End of First approach
************************************************************

************************************************************
*Second approach: Solving program (22)

Equations
     ObjPD
       ConPD ;

     ObjPD..     Theta =E= Sum(j, xbar2(j)) + Sum(i, ubar2(i)) + Sum(i, y2(i))
     + Sum(j, v2(j)) + w2;
       ConPD..          Sum(j, c(j)*(xbar1(j)+xbar2(j))) + Alpha*p - q =E= 0;

Model PD_SCSC    / ObjPD, ConP1, ConP2, ConD1, ConD2, ConPD/ ;

Solve PD_SCSC using LP Maximizing Theta;
      XbarStar(j) = (xbar1.L(j)+xbar2.L(j))/(w1.L+w2.L);
      UbarStar(i) = (ubar1.L(i)+ubar2.L(i))/(w1.L+w2.L);
      Ystar(i) = (y1.L(i)+y2.L(i))/(w1.L+w2.L);
      Vstar(j) = (v1.L(j)+v2.L(j))/(w1.L+w2.L);

Put / /'Finding a SC Solution via Approach II';
Put /'----------------------------------------------'/;
Put  '       Primal                Dual     '/;
Put  '----------------------------------------------'/;
```

```
Loop(j,
       Put 'x(':>5; Put ord(j):<>3:0; Put ')= ':3; Put (XbarStar(j)/tStar):<10:3;
       Put 'v(':>5; Put ord(j):<>3:0; Put ')= ':3; Put Vstar(j):<10:3;
       Put /;
     );
Put /;
Loop(i,
       Put 'u(':>5; Put ord(i):<>3:0; Put ')= ':3; Put (UbarStar(i)/tStar):<10:3;
       Put 'y(':>5; Put ord(i):<>3:0; Put ')= ':3; Put Ystar(i):<10:3;
       Put /;
     );
Put '------------------------------------------------'/ / /;

*End of Second approach
*************************************************************
```

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