Iranian Journal of Numerical Analysis and Optimization
Vol. 8, No. 1, (2018), pp 1–17
DOI:10.22067/ijnao.v8i1.60791

A computational method for solving weakly singular Fredholm integral equation in reproducing kernel spaces

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Abstract

In the present paper, we propose a method to solve a class of weakly singular Fredholm integral equations of the second kind in reproducing kernel spaces. The Taylor series of the unknown function is used to remove the singularity and bases of reproducing kernel spaces are used to solve this equation. Efficiency of the proposed method is investigated through various examples.

Keywords: Weakly singular kernel; Fredholm integral equations; Taylor series; Reproducing kernel space.

1 Introduction

The Fredholm integral equation with weakly singular kernel arises in different problems of mathematical physics such as potential problem, variational equilibrium, fracture mechanics, infrared radiation, and elastic contact problems [5,20,21]. Although in [11,16,17,22] authors obtained analytical solution for integral equations with weakly singular kernel in special cases, but generally this is not an easy task. So the numerical analysis standpoint plays major role for solving such equations.

In this paper, we consider Fredholm singular integral equation

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Received 5 December 2016; revised 23 March 2017; accepted 14 June 2017
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$$\mu(x)u(x) + \lambda(x)\int_{-1}^{1} \frac{k(x,y)u(y)}{(y-x)^{\alpha}}dy = f(x), \qquad |x| < 1 \text{ and } 0 < \alpha \leq 1, \quad (1)$$

in which $\mu(x) \neq 0$ and $\lambda(x) \neq 0$, for all $x \in [-1,1]$, and μ, λ, f, and k are known smooth functions and u is the solution of equation (1) to be determined. We assume that problem has a unique smooth solution u. Several authors considered the numerical solutions of Fredholm integral equations with weakly singular kernel. In [12] Jiang and Cui considered integral equation of first or third kind with weakly singular kernel of the form $k(x,y) = \frac{G(x,y)}{x^{\alpha}y^{\beta}}$; they solved the problem in reproducing kernel space $W^1[0,1]$. Chen and Zhou considered second kind Fredholm integral equation with Hilbert type singularity [8]. They used transform to remove singularity and solved problem in $W[0,2\pi]$ using reproducing kernel method. In [2] Babolian and Arzhang Hajikandi solved (1) with $k(x,y) = 1$. Du, Zhao, G., and Zhao, C. considered integro-differential equation with logarithmic kernel and Kalman kernel with boundary values [10]. They used smooth transform to remove singularity, solving the converted equation with reproducing kernel method in $W^3[0,1]$. Chen and Cheng in [7] used piecewise homotopy perturbation method (PHPM), for solving integro-differential equation with weakly singular kernel. In [3] authors solved (1), using Taylor series of the unknown function u to remove singularity, and then Taylor expansion of k together with Legendre polynomials as bases to implement Galerkin method. The Sinc-collocation method is studied by Maleknejad, Mollapourasl, and Ostadi, to solve nonlinear Fredholm integral equations with weakly singular kernel [15]. Beyrami, Lotfi, and Mahdiani solved Fredholm integral equation of the second kind with Cauchy kernel [6]; they removed singularity by smooth transform and used reproducing kernel Hilbert space (RKHS) method to solve problem in $W_o^3[0,1]$. Nili and Dastjerdi in [18] solved weakly singular Volterra–Hammerstein integral equation with operational Tau method. We use Taylor series expansion of $u(y)$, at point $y = x$, to remove singularity; then by use of reproducing kernel space $W^m[-1,1]$ we convert this equation to a system of linear equations. We demonstrate the method with convergence rate $O(h^m)$; so when we increase m and move from one reproducing kernel space to other, the rate of convergence will increase.

The rest of the paper is organized as follows. In section 2 we present definitions and some useful properties of reproducing kernel spaces. In section 3 we remove singularity, by use of reproducing kernel space to implement our method. Section 4 is devoted to error estimate and convergence analysis of our method. Section 5 contains some numerical examples illustrating the application of the proposed method. We end the paper with some conclusions.

2 Preliminaries and notations

In this section we briefly review reproducing kernel properties of the Hilbert space $W^m[a,b]$ and also fix notations used in this paper. The Hilbert function space, $W^m[a,b]$, is defined as the linear function space

$$W^m[a,b] = \{f | f, f^{(\prime)}, \ldots, f^{(m-1)} \text{are absolutely continuous}, f^{(m)} \in L^2[a,b]\},$$

which is equipped with the following inner product

$$\langle f, g\rangle_{W^m} = \sum_{i=0}^{m-1} f^{(i)}(a) g^{(i)}(a) + \int_a^b f^{(m)}(x) g^{(m)}(x) dx. \tag{2}$$

The inner product (2) induces the following Hilbert norm

$$\|f\|_{W^m} = \sqrt{\langle f, f\rangle_{W^m}}.$$

The following theorem presents an interesting property of the Hilbert space $W^m[a,b]$.

Theorem 1. [9] *The Hilbert function space $W^m[a,b]$ is a reproducing kernel space with the conjugate symmetric reproducing kernel $R_m(x,y)$, that is given by*

$$R_m(x,y) = \begin{cases} lR_m(x,y) = \sum_{i=1}^{2m} c_i(y) x^{i-1}, & x < y, \\ \\ rR_m(x,y) = \sum_{i=1}^{2m} d_i(y) x^{i-1}, & x \geq y, \end{cases}$$

in which coefficients $c_i(y)$ and $d_i(y)$ are the solutions of the following system of differential equations

$$\begin{cases} (-1)^m \frac{\partial^{2m} R(x,y)}{\partial x^{2m}} = \delta(x-y), \\ \frac{\partial^i R(a,y)}{\partial x^i} - (-1)^{m-i-1} \frac{\partial^{2m-i-1} R(a,y)}{\partial x^{2m-i-1}} = 0, \\ \frac{\partial^{2m-i-1} R(b,y)}{\partial x^{2m-i-1}} = 0, \qquad i = 0, 1, \ldots, m-1, \end{cases} \tag{3}$$

where $\delta(x-y)$ is the Dirac delta function.

Remark 1. [9] According to system (3), by use of the Dirac delta function properties, the equation

$$(-1)^m \frac{\partial^{2m} R(x,y)}{\partial x^{2m}} = \delta(x-y)$$

is convert to

$$\begin{cases} \left.\frac{\partial^i \ lR_m(x,y)}{\partial x^i}\right|_{x=y} = \left.\frac{\partial^i \ rR_m(x,y)}{\partial x^i}\right|_{x=y}, & i=0,1,\ldots,2m-2, \\ \left.\frac{\partial^{2m-1} \ lR_m(x,y)}{\partial x^{2m-1}}\right|_{x=y^+} - \left.\frac{\partial^{2m-1} \ rR_m(x,y)}{\partial x^{2m-1}}\right|_{x=y^-} = (-1)^m. \end{cases} \tag{4}$$

By solving the system (4) with boundary condition of (3) with Mathematica, the coefficients $c_i(y)$ and $d_i(y)$ are computed.

In particular, each function $f \in W^m[a,b]$ satisfies the following reproducing property

$$f(y) = \langle f(.), R_m(.,y)\rangle_{W^m} \qquad \forall y \in [a,b].$$

Theorem 2. [9]*Let $R(x,y)$ be a reproducing kernel of $W^m[a,b]$; then*

$$\frac{\partial^{i+j} R(x,y)}{\partial x^i \partial y^j} \in L^2[a,b], \qquad i+j = 2m-1$$

with respect to x and y and

$$\frac{\partial^{i+j} R(x,y)}{\partial x^i \partial y^j}, \qquad 0 \le i+j \le 2m-2,$$

are absolutely continuous functions in $[a,b]$, with respect to x and y; so we have

$$\frac{\partial^{i+j} R(x,y)}{\partial x^i \partial y^j} \in W^m[a,b], \qquad i+j = m-1.$$

Lemma 1. *Let f be a smooth function of order m on $[a,b]$; then, for $k = 1,2,\ldots,m$, we have $f \in W^k[a,b]$.*

Proof. Let f be a smooth function of order m on $[a,b]$; thus $f^{(k)}$, for $k = 0,1,\ldots,m$, are continuous functions on $[a,b]$. Therefore $f^{(k)} \in L^2[a,b]$, for $k = 1,2,\ldots,m$, beside the real constants M_k exist which we have

$$|f^{(k)}(x)| \le \ M_k \qquad \forall x \in [a,b];$$

trivially $f^{(k)}$, for $k = 0,1,\ldots,m-1$, are absolutely continuous functions, and proof is complete. □

3 Removing the singularity and implementation of our method

Throughout this section, we use the Taylor series to remove singularity of (1); then by use of reproducing kernel space property, we construct our method. With Taylor series of the unknown function $u(y)$, at $x \in [a, b]$, we have

$$\begin{aligned} u(y) &= u(x) + (y - x)u^{'}(x) + \frac{(y - x)^2}{2!}u^{''}(x) + \cdots + \frac{(y - x)^k}{k!}u^{(k)}(x) \\ &\quad + \frac{(y - x)^{k+1}}{(k + 1)!}u^{(k+1)}(\xi_{x,y}), \end{aligned} \tag{5}$$

in which $\xi_{x,y}$ is between x and y. Using (5), thus equation (1) converts to

$$\begin{aligned} \mu(x)u(x) + \lambda(x)\Bigg(&\sum_{i=0}^{k} \frac{u^{(i)}(x)}{i!} \int_{-1}^{1} k(x, y)(y - x)^{i-\alpha} dy \\ &+ \int_{-1}^{1} \frac{u^{(k+1)}(\xi_{x,y})}{(k + 1)!} k(x, y)(y - x)^{k+1-\alpha} dy\Bigg) = f(x), \end{aligned}$$

and singularity is removed; now we are in position to define operator $L : W^m[-1, 1] \to W^m[-1, 1]$,

$$\begin{aligned} L(u(.)) = \mu(.)u(.) + \lambda(.)\Bigg(&\sum_{i=0}^{k} \frac{u^{(i)}(.)}{i!} \int_{-1}^{1} k(., y)(y - .)^{i-\alpha} dy \\ &+ \int_{-1}^{1} \frac{u^{(k+1)}(\xi_{.,y})}{(k + 1)!} k(., y)(y - .)^{k+1-\alpha} dy\Bigg). \end{aligned}$$

By use of operator L, the equation (1) is convert to $Lu(x) = f(x)$. It is easy to prove that, for $0 \leq \alpha < 1$, operator L is a bounded linear operator on $[-1, 1]$ and that, for $\alpha = 1$, L is a bounded linear operator on $[-1 + \epsilon, 1 - \epsilon]$ for $\epsilon > 0$. Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of interval $[a, b]$, for $i = 1, 2, \ldots$; take $y = x_i$, in $R_m(x, y)$, and define $\varphi_i(.) = R_m(., x_i)$. Moreover, assume that $\psi_i = L^*\varphi_i$, where L^* is the adjoint operator of L.

Theorem 3. *Let $\{x_i\}_{i=1}^{\infty}$ be a dense set in $[-1, 1]$; then the functions $\psi_i \in W^m[-1, 1]$, and $\{\psi_i\}_{i=1}^{\infty}$ is complete in $W^m[-1, 1]$.*

Proof. By using the reproducing kernel properties of $W^m[-1, 1]$, we have

$$\begin{aligned}\psi_i = L^*\varphi_i &= \langle L^*\varphi_i(y), R_m(y,.)\rangle_{W^m} \\ &= \langle \varphi_i(y), L_y R_m(y,.)\rangle_{W^m} \\ &= \langle \varphi_i(y), \overline{L_y R_m(.,y)}\rangle_{W^m} \\ &= \overline{\langle \overline{L_y R_m(.,y)}, \varphi_i(y)\rangle_{W^m}} = L_y R_m(.,y)\Big|_{y=x_i}.\end{aligned}$$

Therefore, we get

$$\psi_i = \mu(x_i)R_m(.,x_i) + \lambda(x_i)\sum_{i=0}^{2m-1}\frac{1}{i!}\,\frac{\partial^i R_m(.,s)}{\partial s^i}\Big|_{s=x_i}\int_{-1}^{1} k(x_i,s)(x_i-s)^{i-\alpha}ds;$$

hence $\psi_i \in W^m[-1,1]$. Let $u \in W^m[-1,1]$, and $i \in \mathbb{N}$. Then we have

$$\begin{aligned}\langle u, \psi_i\rangle_{W^m} &= \langle u, L^*\varphi_i\rangle_{W^m} \\ &= \langle Lu, \varphi_i\rangle_{W^m} \\ &= Lu(x_i) = 0.\end{aligned}$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense in $[-1,1]$, we have $Lu = 0$. By the uniqueness of the solution of (1), we get $u = 0$. Hence, $\{\psi_i\}_{i=1}^{\infty}$ is complete in $W^m[-1,1]$. □

As a consequence of Theorem 3, we can expand each function as the following. Let $\{x_i\}_{i=1}^{\infty}$ be dense in $[-1,1]$, and let u be the solution of (1). Since $u \in W^m[-1,1]$, we get

$$u(x) = \sum_{i=0}^{\infty} a_i\psi_i(x).$$

In what follows, we use u_n as approximate solution of equation (1) which is defined by

$$u_{m,n}(x) = \sum_{i=0}^{n} a_i\psi_i(x), \qquad x \in [-1,1],$$

and by use of operator L, we get

$$\sum_{i=0}^{n} a_i\; L\psi_i(x) \simeq f(x), \tag{6}$$

and with inner product of equation (6), with φ_j, we have

$$\sum_{i=0}^{n} a_i\; \langle L\psi_i, \varphi_j\rangle_{W^m} \simeq \langle f, \varphi_j\rangle_{W^m}.$$

Thus by replacing $\simeq$ by $=$, the singular integral problem converts to the linear system of equations $Ba = F$, in which the coefficient matrix $B = [b_{i,j}] = [L\psi_i(x_j)]$, and the right hand side vector is $F = [F_j] = f(x_j)$.

4 Error estimate and convergence analysis

In this section first the convergence of the method is investigated, and then the error estimate of proposed method is studied.

Theorem 4. *Let $\{x_i\}_{i=1}^{\infty}$ be dense in $[-1,1]$, and let $\{\overline{\psi_i}\}_{i=1}^{\infty}$ be the orthonormal basis of $W^m[-1,1]$ produced by Gram–Schmidt process on $\{\psi_i\}_{i=1}^{\infty}$. Then $u_{m,n}$, the approximate solution of equation (1), and its derivatives are convergent uniformly to u; that is, the exact solution of equation (1), and its derivatives, respetively.*

Proof. With the Gram process we have

$$\psi_i(x) = \sum_{k=1}^{i} \beta_{ik}\overline{\psi_k(x)},$$

and hence we get

$$u(x) = \sum_{i=0}^{\infty}\sum_{k=1}^{i} a_i\beta_{ik}\overline{\psi_k(x)}.$$

Therefore the approximate solution of equation (1) is

$$u_{m,n}(x) = \sum_{i=0}^{n}\sum_{k=1}^{i} a_i\beta_{ik}\overline{\psi_k(x)},$$

and clearly we have $\lim_{n\to\infty} \|u - u_{m,n}\|_{W^m[-1,1]} = 0$. On the other hand, we have

$$u(x) = \langle u(.), R_m(., x)\rangle = \sum_{i=0}^{m-1} u^{(i)}(1)R_m^{(i)}(1, x) + \int_{-1}^{1} u^{(m)}(z)R_m^{(m)}(z, x)dz.$$

Differentiating j times with respect to x, we get

$$\begin{aligned} u^{(j)}(x) &= \sum_{i=0}^{m-1} u^{(i)}(1)\frac{\partial^j R_m^{(i)}(1, x)}{\partial x^j} + \int_{-1}^{1} u^{(m)}(z)\frac{\partial^j R_m^{(m)}(z, x)}{\partial x^j}dz \\ &= \langle u(y), \frac{\partial^j R_m(y, x)}{\partial x^j}\rangle_{W^m}. \end{aligned}$$

Thus we have

$$\begin{aligned}| u^{(i)}(x) - u^{(i)}_{m,n}(x) | &= \langle u(.) - u_{m,n}(.), \frac{\partial^i R_m(.,x)}{\partial x^i} \rangle_{W^m} \\ &\leq \| u - u_{m,n} \|_{W^m[-1,1]} \| \frac{\partial^i R_m(.,x)}{\partial x^i} \|_{W^m[-1,1]} .\end{aligned}$$

Since $\frac{\partial^i R_m(.,x)}{\partial x^i} \in W^m[-1,1]$ and $\| \frac{\partial^i R_m(.,x)}{\partial x^i} \|_{W^m[-1,1]}$, is continuous with respect to x, on $[-1,1]$. For some $M > 0$, we have

$$\| \frac{\partial^i R_m(.,x)}{\partial x^i} \|_{W^m[-1,1]} \leq M,$$

which completes the proof. □

Let $\{x_i\}_{i=1}^n$ be a subset of $[-1,1]$, and let $-1 < x_1 < x_2 < \cdots < x_n < 1$. We define $h_j := x_{j+1} - x_j, \ j = 1, 2, \ldots, n-1$, and also put $h := \max_{1 \leq j \leq n-1} h_j$.

Theorem 5. [4]*Let* $u \in W^m[-1,1]$ *be a smooth solution, and let* $u_{m,n}$ *be the approximate solution of the equation* (1)*. Then*

$$\|u - u_{m,n}\|_\infty \leq ch^m,$$

where c *is a constant.*

5 Numerical examples

We examine accuracy and convergence of the proposed method through four examples, which indicate efficiency of the method. The errors were defined as

$$E_{m,n} = \|u - u_{m,n}\|_\infty \simeq \max_{1 \leq i \leq n} | u(x_i) - u_{m,n}(x_i) | \qquad m = 1, 2, \ldots .$$

Example 1. [3] Consider the following integral equation

$$u(x) + \int_{-1}^{1} \frac{(1 + xy + 2y^3)u(y)}{(y-x)^{\frac{3}{5}}} dy = f(x),$$

where f is chosen such that

$$u(x) = x^5, \qquad x \in [-1,1],$$

is the exact solution. The numerical results are given in Table 1. In Table 1, for fix n, by increasing m, the absolute errors converge to zero. In Figures 1, for $m = 6, 8, 10$, when $n = 8$ the absolute errors of u and its derivatives are

Table 1: Values of $E_{m,n}$, for Example 1.

m	$n=8$	$n=16$	$n=32$
6	1.72675×10^{-1}	1.34164×10^{-1}	1.06141×10^{-1}
8	9.37476×10^{-4}	3.09832×10^{-4}	2.91592×10^{-5}
10	6.73973×10^{-6}	2.6006×10^{-6}	3.41363×10^{-6}

Table 2: Values of $E_{m,n}$, for Example 2.

m	$n=8$	$n=16$	$n=32$
4	2.21811×10^{-2}	2.89879×10^{-2}	7.6194×10^{-2}
6	4.92002×10^{-4}	7.14302×10^{-4}	1.95649×10^{-3}
8	2.02376×10^{-5}	1.07565×10^{-5}	2.91592×10^{-5}
10	3.13506×10^{-6}	1.2141×10^{-7}	2.02656×10^{-7}

shown.

Example 2. Consider the following integral equation

$$u(x)+(x+2)\int_{-1}^{1}\frac{e^{x+y}u(y)}{(y-x)}dy=f(x),$$

where f is chosen such that

$$u(x)=xe^{x},\qquad x\in[-1,1],$$

is the exact solution. The numerical results are given in Table 2. In Table 2, for fixed n, by increasing m, the absolute errors converge to zero. In Figures 2, for $m=6,8,10$, when $n=8$ the approximate solutions and its derivatives have high accuracies.

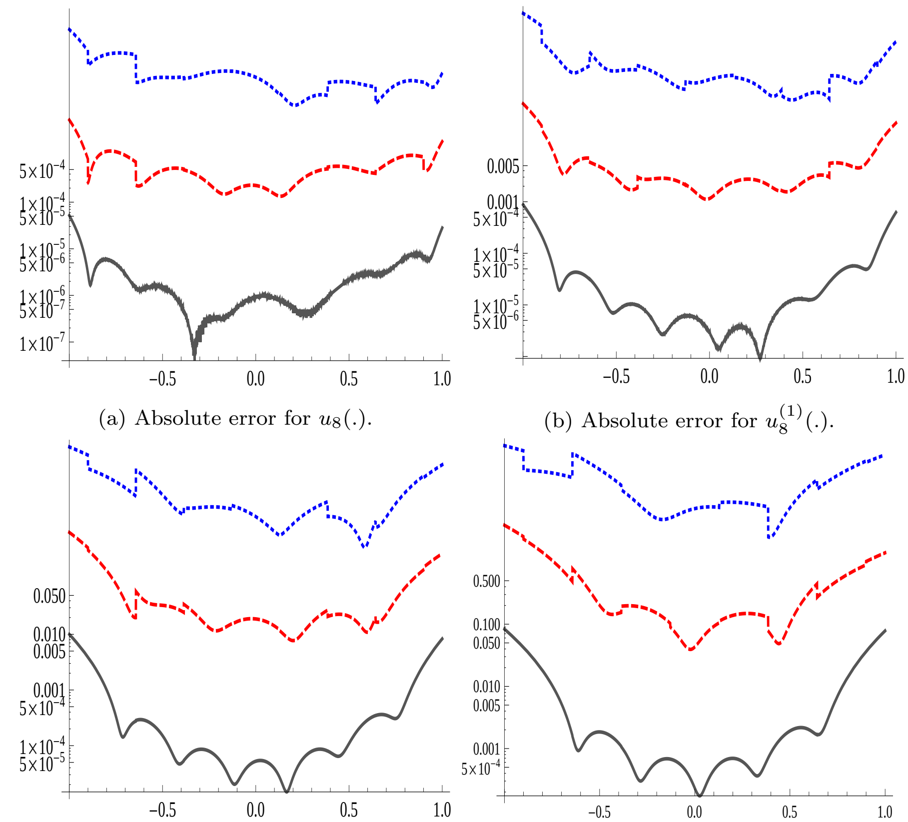
Example 3. $[1,3,13,14]$Consider the following integral equation

$$u(x)-\int_{0}^{1}\frac{u(y)}{\mid x-y\mid^{\frac{1}{2}}}dy=f(x),$$

where f is chosen such that

$$u(x)=x,\qquad x\in[0,1]$$

is the exact solution. For $m=4,6,8$, when $n=8$, the absolute errors of u and its derivatives are shown in Figures 3. In Table 3, the numerical results of proposed method are compared with $[1,3,13]$. where the Max Error is



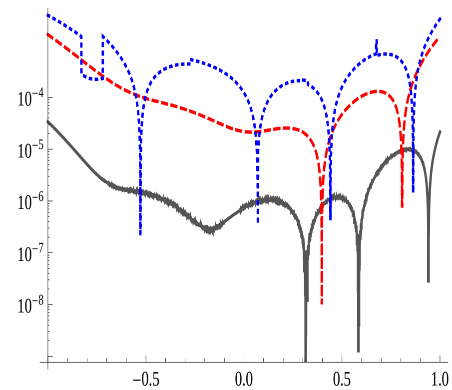
(a) Absolute error for $u_8(.)$.

(b) Absolute error for $u_8^{(1)}(.)$.

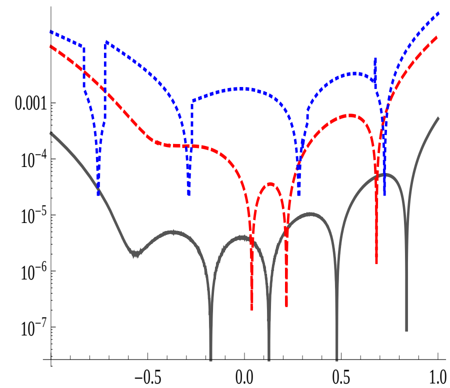
(c) Absolute error for $u_8^{(2)}(.)$.

(d) Absolute error for $u_8^{(3)}(.)$.

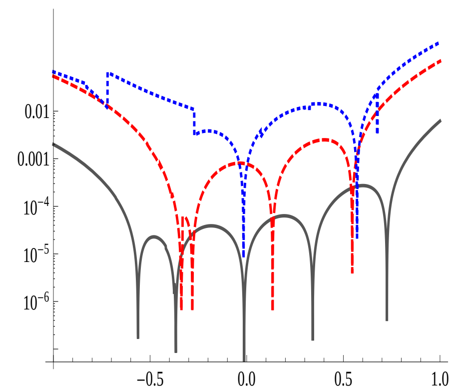
Figure 1: Example 1, $m = 6$(blue dotted), $m = 8$(red dash), $m = 10$(Gray line).



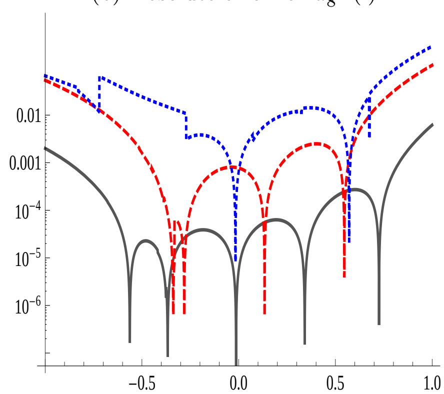
(a) Absolute error for $u_8(.)$.



(b) Absolute error for $u_8^{(1)}(.)$.



(c) Absolute error for $u_8^{(2)}(.)$.



(d) Absolute error for $u_8^{(3)}(.)$.

Figure 2: Example 2, $m = 6$(blue dotted), $m = 8$(red dash), $m = 10$(Gray line).

Table 3: The Max Error result of Example 3.

n	$n = 16$	$n = 32$
Babolian's method [3]	1.59×10^{-16}	
proposed method for $m = 10$	5.29781×10^{-11}	4.75211×10^{-11}
Lakestani's method [13]	1.27×10^{-6}	2.71×10^{-8}
Product integration method [1]	1.5×10^{-5}	9.39×10^{-7}
Lagrangian interpolant [1]	2.12×10^{-5}	1.94×10^{-6}

Table 4: The Max Error result of $u(.)$, Example 4.

m	$n = 8$	$n = 16$	$n = 32$
4	3.17985×10^{-4}	4.44036×10^{-4}	4.24008×10^{-4}
6	2.38288×10^{-6}	2.63628×10^{-6}	2.23585×10^{-6}
8	2.93877×10^{-9}	7.58042×10^{-9}	3.42293×10^{-8}

defined as Max Error $= \max_{0 \leq x \leq 1} \mid u(x) - u_{m,n}(x) \mid$.

Example 4. [13, 19]Consider the following integral equation

$$u(x) - \frac{1}{10} \int_0^1 \frac{u(y)}{\mid x - y \mid^{\frac{1}{3}}} dy = f(x),$$

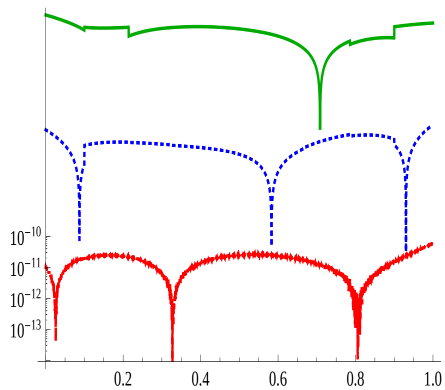
where f is chosen such that

$$u(x) = x^2(1 - x^2), \qquad x \in [0, 1],$$

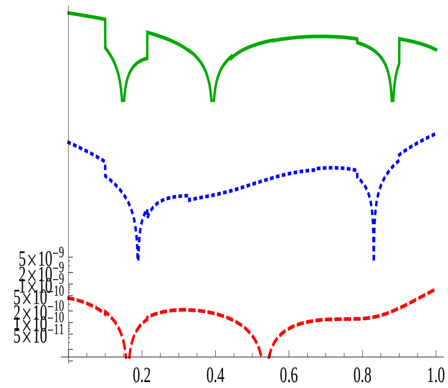
is the exact solution. For $m = 4, 6, 8$, when $n = 8$, the absolute errors of u and its derivatives are shown in Figures 4. In Tables 4, 5, and 6 the numerical results of proposed method are given. where the Max Error is defined as Max Error $= \max_{0 \leq x \leq 1} \mid u^{(i)}(x) - u^{(i)}_{m,n}(x) \mid$.

Table 5: The Max Error result of $u^{(1)}(.)$, Example 4.

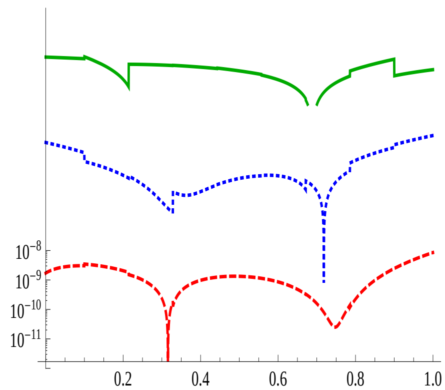
m	$n = 8$	$n = 16$	$n = 32$
4	2.10369×10^{-1}	7.59437×10^{-3}	7.61556×10^{-2}
6	8.89262×10^{-6}	2.43063×10^{-6}	1.37019×10^{-6}
8	3.66355×10^{-7}	2.21809×10^{-7}	1.20238×10^{-7}



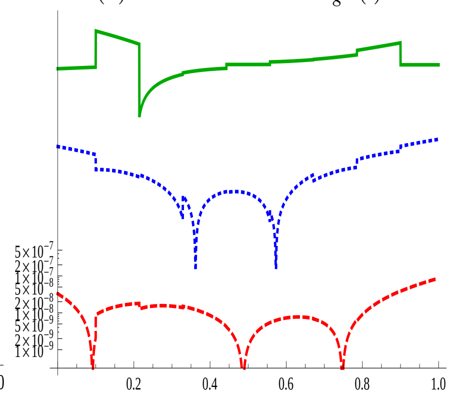
(a) Absolute error for $u_8(.)$.



(b) Absolute error for $u_8^{(1)}(.)$.



(c) Absolute error for $u_8^{(2)}(.)$.

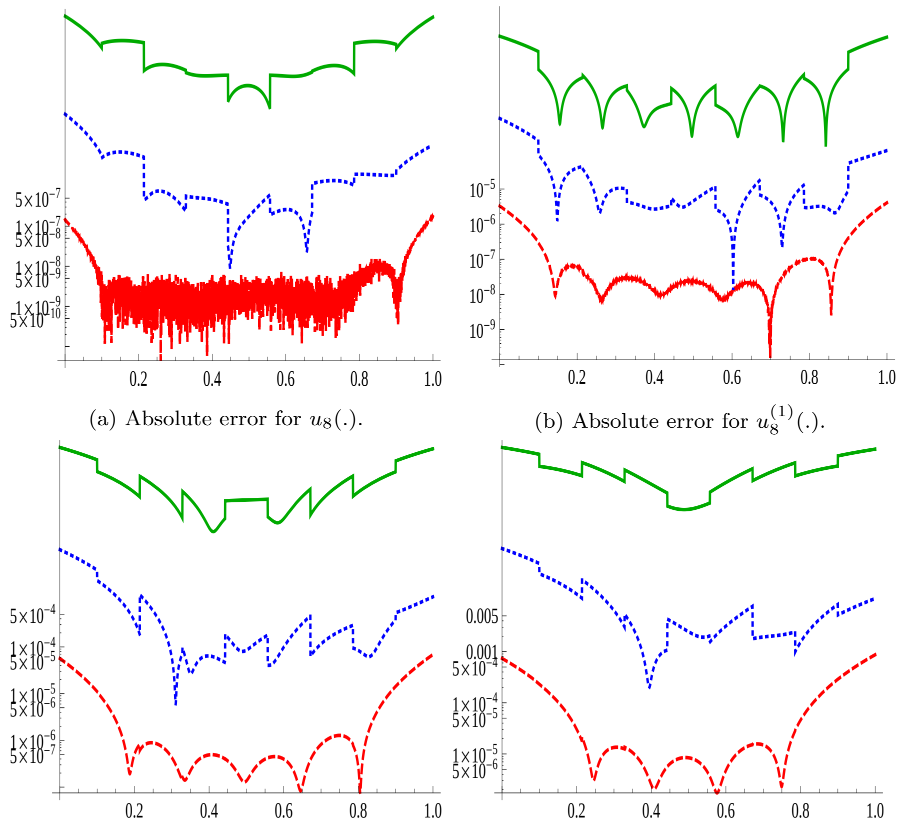


(d) Absolute error for $u_8^{(3)}(.)$.

Figure 3: Example 3, $m = 4$(green line), $m = 6$(blue dotted), $m = 8$(red dash).

Table 6: The Max Error result of $u^{(2)}(.)$, Example 4.

m	$n = 8$	$n = 16$	$n = 32$
4	4.6866×10^{-1}	3.45622×10^{-1}	3.75916×10^{-1}
6	1.18157×10^{-3}	1.25931×10^{-4}	1.20977×10^{-3}
8	1.3664×10^{-5}	2.79569×10^{-7}	1.67734×10^{-5}



(a) Absolute error for $u_8(.)$.

(b) Absolute error for $u_8^{(1)}(.)$.

(c) Absolute error for $u_8^{(2)}(.)$.

(d) Absolute error for $u_8^{(3)}(.)$.

Figure 4: Example 4, $m = 4$(green line), $m = 6$(blue dotted), $m = 8$(red dash).

6 Conclusion

Different problems have been solved by researchers using reproducing kernel Hilbert spaces (RKHS). They assume the unknown solution belongs to special fix W_2^m and use Gram–Schmidt process to implement RKHS method. In this paper, we attempted to solve Fredholm integral equations of the second kind with weakly singular kernel. We used the Taylor series to remove the singularity. According to the problem, we investigated the problem in reproducing kernel spaces $W^m[-1,1]$. In the proposed method by use of reproducing kernel space property, the problem was converted to a system of linear equations. In the proposed method we do not use Gram process, we use RKHS property and use W_2^m for different m. Though we suppose the unknown solution is smooth, but this is a very strong assumption, and it is enough for the solution to be in W_2^m for some m. In this method for fix m, when the number of bases n increases the coefficient matrix of the linear system will be ill-conditioned; so we increase m and move from one reproducing kernel to other instead, to find approximate solution with high accuracy. Several test examples were used to show the efficiency and applicability of the method.

Acknowledgements

Authors are grateful to there anonymous referees and editor for their constructive comments.

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روش محاسباتی برای حل معادله انتگرال فردهلم با هسته منفرد ضعیف در فضاهای هسته بازتولید

دانیال حامدزاده و اسمعیل بابلیان

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دریافت مقاله ۱۵ آذر ۱۳۹۵، دریافت مقاله اصلاح شده ۳ فروردین ۱۳۹۶، پذیرش مقاله ۲۴ خرداد ۱۳۹۶

چکیده : در این مقاله، روشی جدید برای حل دسته ای از معادلات انتگرال فردهلم نوع دوم با هسته منفرد ضعیف را در فضاهای هسته بازتولید معرفی میکنیم. با استفاده از بسط تیلور تابع مجهول تکینگی معادله را رفع کرده و در ادامه با استفاده از پایه فضاهای هسته بازتولید معادله را حل میکنیم. در نهایت با چند مثال عددی کارایی روش مفروض را نشان میدهیم.

کلمات کلیدی : هسته منفرد ضعیف؛ معادلات انتگرال فردهلم؛ سری تیلور؛ فضای هسته بازتولید.