Iranian Journal of Numerical Analysis and Optimization
Vol. 12, No. 1, (2022), pp 201-227
DOI:10.22067/ijnao.2021.68926.1016
https://ijnao.um.ac.ir/

How to cite this article
Research Article

Using Mott polynomials operational matrices to optimize multi-dimensional fractional optimal control problems

S.A. Alavi*, A.R. Haghighi, A. Yari and F. Soltanian

Abstract

We offer a method for solving the fractional optimal control problems of multi-dimensional. We obtain a fractional derivative and multiplication operational matrix for Mott polynomials (M-polynomials). In the proposed method, the Caputo sense of the fractional derivative is applied on dynamical system. The main feature of this method is to reduce the problem into a system of algebraic equations in order to simplify it. We also show that by increasing the approximation points, the responses converge to the real answer. When the degree of fractional derivative approaches to 1, then the obtained solution approaches to the classical solution as well.

AMS subject classifications (2020): 49MXX, 34H05.

Keywords: Mott polynomials; Caputo derivative; fractional optimal control problems; operational matrix.

---

*Corresponding author
Received 16 February 2021; revised 24 November 2021; accepted 9 December 2021
Seyyed Ali Alavi
Department of Mathematics, Payame Noor University, Tehran, Iran.
mail: ali336alavi@gmail.com

Ahmad Reza Haghighi
Department of Mathematics, Technical and Vocational University, Tehran, Iran.
mail: ah.haghighi@gmail.com

Ayatollah Yari
Department of Mathematics, Payame Noor University, PO BOX 19395-3697, Tehran, Iran. mail:a_yary@yahoo.com

Fahimeh Soltanian
Department of Mathematics, Payame Noor University, PO BOX 19395-3697, Tehran, Iran. mail:f_soltanian@pnu.ac.ir

1 Introduction

Over the last years, many researchers have developed fractional calculus and its applications in physics, chemistry, engineering, and so on; see [2, 7, 8, 12]. A fractional dynamic system (FDs) is a system whose dynamics are expressed by fractional differential equations, and a fractional optimal control problem (FOCP) is an optimal control problem for an FDs [4, 26]. The theory of optimal control is an area in mathematics that has been developing for years, but the theory of fractional optimal control is a new area. Several authors studied this field [1, 3, 9, 22]. Some applications of FOCPs are given in [10, 19]. In most papers, one-dimensional FOCPs are considered where the problem is only with one state, one control variable, and one fractional differential equation [14, 15, 28]. In recent years, the use of different methods for solving optimal control problems has been considered by some researchers. A new numerical approach for solving FOCPs, including state and control inequality constraints using new biorthogonal multiwavelets, was made by Ashpazzadeh, Lakestani, and Yildirim [6]. In this paper, we consider FOCPs in the Caputo sense with multi-dimensional variables. In this method, the state and the control vectors do not necessarily have only one dimension.

An FOCP can be defined with respect to different definitions of fractional derivatives. Most important types of fractional derivatives are the Riemann–Liouville and the Caputo. In this paper, we consider the multi-dimensional FOCP as follows:
Determinate states $X(t) \in \mathbb{R}^n$ and controls $U(t) \in \mathbb{R}^k$, which

$$\text{Min } J(t, X(t), U(t)) = \int_0^1 f(t, X(t), U(t))dt, \tag{1}$$

subject to

$$D^\alpha x_i(t) = g_i(t, X(t), U(t)), \qquad 0 < \alpha \leq 1, 0 < t \leq 1, i = 1, \ldots, n, \tag{2}$$

and satisfy the initial condition

$$X(0) = X_0, \tag{3}$$

where

$$X(t) = [x_1(t), \ldots, x_n(t)]^T, \qquad U(t) = [u_1(t), \ldots, u_k(t)]^T, \tag{4}$$

$X_0 = [x_{0,1}(t), \ldots, x_{0,n}(t)]^T$, and also $f, g_i : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ are polynomial functions. The above problem reduces to a standard optimal control problem, when $\alpha = 1$.

The main purpose is to generalize the Mott operational matrix to fractional calculus. In the present work, we use Mott polynomials (MPs) for solving FOCPs. The method consists of expanding the solution by MPs with the unknown coefficients. The properties of MPs are used to evaluate the unknown coefficients and find an approximate solution for $X(t)$ and $U(t)$ in the problem. Also, illustrative examples are included to demonstrate the applicability of the new approach.

This article is organized as follows. Section 2 presents some of the preliminaries in fractional calculus. Section 3 describes the MPs and function approximation. We make an operational matrix for fractional integration, derivative, and multiplication by MPs in Section 4. In Section 5, we apply the MPs to solve multi-dimensional FOCPs. In Section 6, the convergence of the proposed method is discussed. In Section 7, the numerical examples are simulated to demonstrate the high performance of the proposed method. Finally, Section 8 concludes our work.

2 Some preliminaries in fractional calculus

This section provides some basic definitions of the fractional calculus.

Definition 1. We define from [11] $C_\mu = \{f(t) : f(t) > 0 \text{ for } t > 0, f(t) = t^p f_1(t) \text{ where } p > \mu, f_1 \in C[0,\infty)\}$ and $C_\mu^n = \{f(t) : f^{(n)}(t) \in C_\mu\}$, where $n \in N$ and $\mu \in \mathbb{R}$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq 1$, is defined as [11]

$$ {}_0I_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, & \alpha > 0, t > 0, \\ f(t), & \alpha = 0, \end{cases} \tag{5} $$

and for $n-1 < \alpha \leq n, n \in N, t > 0, f \in C_{-1}^n$, the fractional derivative of $f(t)$ in the Caputo sense is defined as

$$ \begin{aligned} {}_0^cD_t^\alpha f(t) &= I^{n-\alpha} D^n f(t) \\ &= \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \dfrac{d^n}{ds^n} f(s) ds, & n-1 < \alpha < n, n \in N, \\ f^{(n)}(t), & \alpha = n, n \in N, \end{cases} \end{aligned} \tag{6} $$

where D^n is the nth-order derivative.

Definition 3. For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$, we have [27]

$$ {}_0I_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}, \tag{7} $$

and for $n-1<\alpha\leq n, n\in N, t>0, f\in C_{\mu}^{n}, \mu\geq -1$, also, we have the following properties:

1. ${}_{0}^{c}D^{\alpha}I^{\alpha}f(t)=f(t)$,

2. $I^{\alpha}\,{}_{0}^{c}D^{\alpha}f(t)=f(t)-\sum_{j=0}^{n-1}f^{(t)}\left(0^{+}\right)\frac{x^{k}}{k!}$,

3. ${}_{0}^{c}D^{\beta}f(t)=I^{\alpha-\beta}D^{\alpha}f(t)$.

3 MPs

The MP $s_n(t)$ is defined by (see [21, 23, 25])

$$s_n(t)=(-1)^n\left(\frac{t}{2}\right)^n(n-1)!\sum_{k=0}^{h\left(\frac{n}{2}\right)}\binom{n}{k}(-1)^k\frac{t^{-2k}}{(n-2k-1)!},\tag{8}$$

where

$$h\left(\frac{n}{2}\right)=\begin{cases}\frac{n}{2} & \text{if } n \text{ is even},\\ \frac{n}{2}-\frac{1}{2} & \text{if } n \text{ is odd}.\end{cases}$$

From the above definition, it is obvious that $s_0(t)=1\neq 0$. For this reason $s_n(t)$ is a Sheffer set [29]. The first few MPs are

$$\begin{aligned}
s_0(t)&=1,\\
s_1(t)&=-\frac{1}{2}t,\\
s_2(t)&=\frac{1}{4}t^2,\\
s_3(t)&=\frac{3}{4}t-\frac{1}{8}t^3,\\
s_4(t)&=-\frac{3}{2}t^2+\frac{1}{16}t^4,\\
s_5(t)&=-\frac{15}{2}t+\frac{15}{8}t^3-\frac{1}{32}t^5.
\end{aligned}$$

Now we define the vector A_r, for $r=0,1,\ldots,n$. If r is odd, then for $i=0,1,\ldots,n$,

$$
\begin{aligned}
A_r &= b_{i_r}, && (9)\\
b_{i_r} &= \begin{cases} 0, & i = 2l, l \in N \cup \{0\}, \\ c_{i_r} & \text{otherwise}, \end{cases} && (10)\\
c_{i_r} &= \left(\frac{-1}{2}\right)^r (r-1)! \frac{\binom{r}{k_i}(-1)^{k_i}}{(r-2k_i-1)!}, \\
k_i &= \beta_j, \beta_j - 1, \dots, 0, \\
\beta_j &= \frac{r-1}{2} - j, \quad j = 0, 1, \dots, \frac{r-1}{2}. && (11)
\end{aligned}
$$

If r is even, then for $i = 0, 1, \dots, n$,

$$
\begin{aligned}
A_r &= b'_{i_r}, && (12)\\
b'_{i_r} &= \begin{cases} 0 & i = 2l+1, l \in N \cup \{0\}, \\ c'_{i_r} & \text{otherwise}, \end{cases} && (13)\\
c'_{i_r} &= \left(\frac{-1}{2}\right)^r (r-1)! \frac{\binom{r}{k_i}(-1)^{k'_i}}{(r-2k'_i-1)!}, \\
k'_i &= \beta'_j, \beta'_j - 1, \dots, 0, \\
\beta'_j &= \frac{r}{2} - j, \quad j = 0, 1, \dots, \frac{r}{2}. && (14)
\end{aligned}
$$

Hence, $S_r(t) = A_r T_r(t)$ for $(r = 0, 1, \dots, n)$, here $T_r(t) = [1, t, \dots, t^r]^T$. Then, we define an $(r+1) \times (r+1)$ lower triangular matrix A such that $A = [A_0, A_1, \dots, A_r]^T$ and $A_i (i = 0, 1, \dots, n)$ is a row vector of order r. As a result,

$$
\varphi_n(t) = A T_n(t), \tag{15}
$$

where

$$
\varphi_n(t) = \left[s_0(t), s_1(t), \dots, s_n(t)\right]^T. \tag{16}
$$

3.1 Function approximation

We recall here a theorem that was stated and proved in [20]. Suppose that $H = L^2[0,1]$ is a Hilbert space and that $\{s_0, s_1, \dots, s_n\}$ is the MPs of degree n on the interval $[0,1]$. We define $Y = \text{Span}\{s_0, s_1, \dots, s_n\}$. Let f be an arbitrary element in H. Since Y is a finite-dimensional subspace of the space H, the function f has the best unique approximation on Y like $f_n \in Y$, that is, there exists $f_n \in Y$ such that

$$\|f - f_n\|_2 \leq \|f - y\|_2, \qquad \text{for all } y \in Y, \tag{17}$$

where $\|f\|_2 = \sqrt{\langle f, f \rangle}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product. Since $f_n \in Y$, therefore f_n is a linear combination of the spanning basis of Y; that is, there exist $n+1$ coefficients

$$C = [c_0, c_1, \ldots, c_n] \in R \tag{18}$$

such that

$$f(t) \simeq f_n(t) = \sum_{j=0}^{n} c_j s_j(t) = C^T \varphi_n(t), \tag{19}$$

where

$$\|f - f_n\|_2 \to \min\,. \tag{20}$$

Then C can be obtained by

$$C = Q^{-1} \langle f(t), \varphi_n(t) \rangle, \tag{21}$$

where

$$Q = \langle \varphi_n(t), \varphi_n(t) \rangle = \int_0^1 \varphi_n(t) \varphi_n(t)^T dt. \tag{22}$$

Theorem 1. [20] Let X be an inner product space and let $M \neq \emptyset$ be a convex subset that is complete in the metric induced by the inner product. Then for every given $x \in X$, there exists unique $y \in M$ such that

$$\delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\| = \|x - y\|. \tag{23}$$

4 Mott operational matrices

4.1 Mott operational matrix of the fractional integration

In this section, we describe the MPs operational matrix of fractional integration of the vector φ_n. The operational matrix can be approximated as

$$I^{\alpha} \varphi_n(t) \simeq P \varphi_n(t), \tag{24}$$

where P is the $(n+1)\times(n+1)$ Riemann–Liouville fractional operational matrix of integration. We construct P as follows:

$$\begin{aligned} I^{\alpha}s_i(t) &= (-1)^i\left(\frac{1}{2}\right)^i(i-1)!\sum_{k=0}^{h\left(\frac{i}{2}\right)}\binom{i}{k}(-1)^k\frac{I^{\alpha}t^{i-2k}}{(i-2k-1)!} \\ &= (-1)^i\left(\frac{1}{2}\right)^i(i-1)!\sum_{k=0}^{h\left(\frac{i}{2}\right)}\frac{\binom{i}{k}(-1)^k}{(i-2k-1)!}t^{i-2k+\alpha}\frac{\Gamma(i-2k+1)}{\Gamma(i-2k+\alpha+1)}. \end{aligned} \tag{25}$$

Now we approximate $t^{i-2k+\alpha}$ by $n+1$ terms of the Mott basis

$$t^{i-2k+\alpha}\simeq\sum_{j=0}^{n}b_js_j(t), \tag{26}$$

where

$$\begin{aligned} b_j &= Q_j^{-1}\int_0^1 t^{i-2k+\alpha}s_j(t)dt \\ &= (-1)^j\left(\frac{1}{2}\right)^j(j-1)!\sum_{L=0}^{h\left(\frac{j}{2}\right)}\frac{\binom{j}{L}(-1)^L}{(j-2L-1)!}\times\frac{1}{i-2k+\alpha+j-2L+1}. \end{aligned} \tag{27}$$

Therefore we have

$$I^{\alpha}s_i(t)\simeq\sum_{j=0}^{n}B_{ij}s_j(t), \tag{28}$$

where

$$\begin{aligned} B_{ij} =&(-1)^{i+j}\left(\frac{1}{2}\right)^{i+j}(i-1)!(j-1)! \\ &\sum_{k=0}^{h\left(\frac{i}{2}\right)}\sum_{L=0}^{h\left(\frac{j}{2}\right)}\binom{j}{L}\binom{i}{k}(-1)^{k+L}\frac{1}{(i-2k-1)!(j-2L-1)!} \\ &\times\frac{1}{i+j-2L-2k+\alpha+1}, \end{aligned} \tag{29}$$

and

$$Q_j=\left\langle s_j(t),s_j(t)\right\rangle=\int_0^1 s_j(t)s_j(t)^T dt. \tag{30}$$

Finally, we obtain

$$P = \begin{bmatrix} B_{00} & \dots & B_{0n} \\ \vdots & \ddots & \vdots \\ B_{n0} & \dots & B_{nn} \end{bmatrix}, \tag{31}$$

where P is called the MPs operational matrix of fractional integration.

4.2 Mott operational matrix of the fractional derivative

In this section, we describe the MPs operational matrix of fractional derivative of the vector $\varphi_n(t)$. The operational matrix can be approximated as

$$ {}_0^cD_t^\alpha \varphi_n(t) \simeq T\varphi_n(t), \tag{32}$$

where T is the $(n+1)\times(n+1)$ operational matrix of the fractional derivative. We construct T as follows:

$$\begin{aligned} {}_0^cD_t^\alpha \varphi_n(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} \varphi_n(s)ds \\ &= \frac{1}{\Gamma(n-\alpha)} t^{n-\alpha-1} * \varphi_n^{(n)}(t), \end{aligned}$$

where $*$ denotes the convolution product and

$$\varphi_n^{(n)}(t) = D^n \varphi_n(t).$$

Therefore we have

$$\begin{aligned} D^\alpha \varphi_n(t) &= \frac{1}{\Gamma(n-\alpha)} t^{n-\alpha-1} * D^n \varphi_n(t) \\ &= \frac{D^n}{\Gamma(n-\alpha)} t^{n-\alpha-1} * \varphi_n(t) = \frac{D^n}{\Gamma(n-\alpha)} t^{n-\alpha-1} * AT_n(t) \\ &= \frac{AD^n}{\Gamma(n-\alpha)} t^{n-\alpha-1} * [1, t, \dots, t^n] \\ &= \frac{AD^n}{\Gamma(n-\alpha)} [t^{n-\alpha-1} * 1, t^{n-\alpha-1} * t, \dots, t^{n-\alpha-1} * t^n] \\ &= AD^n [D^\alpha 1, D^\alpha t, \dots, D^\alpha t^n] \\ &= AD^n \left[\frac{0!}{\Gamma(1-\alpha)} t^{-\alpha}, \frac{1!}{\Gamma(2-\alpha)} t^{1-\alpha}, \dots, \frac{n!}{\Gamma(n-\alpha+1)} t^{n-\alpha} \right] \\ &= AKD^n \bar{T}_n, \end{aligned}$$

where K is an $(n+1)\times(n+1)$ matrix where the entries are given by

$$K_{i,j} = \begin{cases} \dfrac{i!}{\Gamma(i+1-\alpha)}, & i=j, \\ 0, & i\neq j, \end{cases} \qquad i,j=0,1,\dots,n,$$

and

$$\bar{T}_n = [t^{-\alpha}, t^{1-\alpha}, \dots, t^{n-\alpha}]^T.$$

Now we approximate $t^{n-\alpha}$ by $n+1$ terms of the Mott basis

$$t^{n-\alpha} \simeq E^T \varphi_n(t),$$

where

$$\begin{aligned} E &= Q^{-1}\int_0^1 t^{n-\alpha}\varphi_n(t)dt = Q^{-1}\int_0^1 t^{n-\alpha} A T_n(t)dt \\ &= Q^{-1}A\int_0^1 t^{n-\alpha}[1,t,\dots,t^n]dt \\ &= Q^{-1}A\left[\frac{1}{n-\alpha+1}, \frac{1}{n-\alpha+2}, \dots, \frac{1}{n-\alpha+n}\right]^T, \end{aligned}$$

and

$$\begin{aligned} Q &= \langle \varphi_n, \varphi\rangle = \int_0^1 \varphi_n(t)\varphi_n(t)^T dt \\ &= \int_0^1 A T_n(t) T_n(t)^T A^T dt = A\int_0^1 T_n(t) T_n dt A^T = AHA^T, \end{aligned}$$

where H is the well-known Hilbert matrix

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \frac{1}{n+1} & \cdots & \frac{1}{2n+1} \end{bmatrix}.$$

Finally, we obtain

$${}_0^cD_t^\alpha \varphi_n(t) \simeq T\varphi_n(t),$$

where

$$T = AKD^nE. \tag{33}$$

Moreover, T is called the MPs operational matrix of fractional derivative.

4.3 The operational matrix of multiplication

The following property of the product of two Mott function vectors will be also used:

$$C^T \varphi_n(t)\varphi_n(t)^T \simeq \varphi_n(t)^T \tilde{C}^T, \tag{34}$$

where $\tilde{C}$ is the $(n+1)\times(n+1)$ multiplication operational matrix. To illustrate the calculation procedure, we let

$$C^T \varphi_n(t)\varphi_n(t)^T = \big[c_0(t), c_1(t), \dots, c_n(t)\big].$$

Now we approximate $C^T \varphi_n(t)\varphi_n(t)^T$ as follows:

$$c_i(t) \simeq \sum_{j=0}^{n} \tilde{c}_{ij} s_j(t) = \tilde{C}_i^T \varphi_n(t), \tag{35}$$

where

$$\tilde{C}_i = [\tilde{c}_{i0}, \tilde{c}_{i1}, \dots, \tilde{c}_{in}]^T. \tag{36}$$

Using (21), we obtain

$$c_i^k = \left\langle \sum_{j=0}^{n} \tilde{c}_{ij} s_j(t), s_k(t) \right\rangle = \sum_{j=0}^{n} \tilde{c}_{ij} d_{jk}(t), \qquad j,k = 0,\dots,n, \tag{37}$$

where $c_i^k = \big\langle c_i(t), s_k(t)\big\rangle$ and

$$d_{jk} = \big\langle s_j(t), s_k(t)\big\rangle. \tag{38}$$

Therefore

$$C_i^T = \tilde{C}_i^T D, \tag{39}$$

and

$$C_i = [c_i^0, c_i^1, \dots, c_i^n]^T, \tag{40}$$

where $D = [d_{jk}]$ is a matrix of order $(n+1)\times(n+1)$ given by (38). Also, $\tilde{C}_i^T$ in (39) is given by

$$\tilde{C}_i^T = C_i^T D^{-1}. \tag{41}$$

Hence, we get the operational matrix of multiplication as

$$\tilde{C}_0 = [\tilde{c}_{ij}]_{0\le i,j\le n}. \tag{42}$$

5 MPs for solving multi-dimensional FOCPs

Using Theorem 1, we can approximate the state functions $x_i(t)$ and the control functions $u_j(t)$ as

$$x_i(t) \approx C_i^T \varphi_n(t), \qquad i = 1, \dots, n, \tag{43}$$

$$u_j(t) \approx B_j^T \varphi_n(t), \qquad j = 1, \dots, k, \tag{44}$$

where $C_i, B_j \in R^{(m\times 1)\times 1}$. By using (32) and (43), we can write

$$D^\alpha x_i(t) \approx C_i^T D_\alpha \varphi_n(t), \qquad i = 1, \dots, n.$$

As a result, problem (1)–(4) reduces to

$$\text{Minimize} \int_0^1 f(t, C_1^T\varphi_n(t), \dots, C_n^T\varphi_n(t), B_1^T\varphi_n(t), \dots, B_k^T\varphi_n(t))dt,$$

subject to

$$C_i^T T\varphi_n(t) = g_i\big(t, C_1^T\varphi_n(t), \dots, C_n^T\varphi_n(t), B_1^T\varphi_n(t), \dots, B_k^T\varphi_n(t)\big),$$

with the initial conditions

$$C_i^T\varphi_n(0) = x_{0,i}, \qquad i = 1, \dots, n. \tag{45}$$

Since functions f, g_i are polynomials, we get the following approximations:

$$\int_0^1 f(t, X(t), U(t))dt \approx F(C_1, \dots, C_n, B_1, \dots, B_k), \tag{46}$$

$$g_i(t, X(t), U(t)) \approx G_i(C_1, \dots, C_n, B_1, \dots, B_k)\varphi_n(t), \qquad i = 1, \dots, n, \tag{47}$$

where $F, G_i : R^{(m+1)\times n} \times R^{(m+1)\times k} \to R^{1\times(m+1)}$. For each $i = 1, \dots, n$, we can generate algebraic equations from (47) as follows:

$$\tilde{G}_{i,j} = \int_0^1 (C_i^T T - G_i(C_1, \dots, C_n, B_1, \dots, B_k)\varphi_n(t) B_{jn}(t))dt = 0, \quad j = 0, \dots, n-1,$$

and from (45), we set $\tilde{G}_{i,n} = C_i^T \varphi_n(0) - x_{0,i}$.

Finally, the FOCP (1)–(4) has been reduced to a parameter optimization problem, which can be stated as follows:

Find C_i and B_j that

$$\text{Min} \quad F(C_1, \dots, C_n, B_1, \dots, B_k)$$

$$s.t. \quad \tilde{G}_{i,j}(C_1, \dots, C_n, B_1, \dots, B_k) = 0, \quad i = 1, \dots, n, j = 0, \dots, k.$$

We define the Lagrange function for the above problem as

$$L(C_1, \dots, C_n, B_1, \dots, B_k, \tilde{\lambda}_1, \dots, \tilde{\lambda}_n) = F(C_1, \dots, C_n, B_1, \dots, B_k) + \sum_{i=1}^{n} \sum_{j=0}^{k} \tilde{\lambda}_{i,j} \tilde{G}_{i,j}(C_1, \dots, C_n, B_1, \dots, B_k),$$

and consider the necessary conditions for the extremum and obtain the corresponding system of algebraic equations

$$\frac{\partial L}{\partial C_i} = 0, \quad \frac{\partial L}{\partial \tilde{\lambda}_i} = 0, \quad i = 1, \dots, n,$$
$$\frac{\partial L}{\partial B_j} = 0, \quad j = 1, \dots, k.$$

These equations can be solved for C_i, B_j, and $\tilde{\lambda}_i$ by Newton's iterative method. Then, we get the approximate value of the state functions $x_i(t)$ and the control functions $u_j(t)$ from (43) and (44), respectively.

6 Error estimation and convergence analysis

In the following theorem, the error estimation for the approximated functions will be expressed in terms of Gram determinant. For any given elements $x_1, x_2, \dots, x_n$ in a Hilbert space H, the Gram determinant of these elements is defined as follows [20]:

$$G(x_1,x_2,\ldots,x_n)=\begin{vmatrix}\langle x_1,x_1\rangle & \langle x_1,x_2\rangle & \ldots & \langle x_1,x_n\rangle\\ \langle x_2,x_1\rangle & \langle x_2,x_2\rangle & \ldots & \langle x_2,x_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle x_n,x_1\rangle & \langle x_n,x_2\rangle & \ldots & \langle x_n,x_n\rangle\end{vmatrix}. \tag{48}$$

Theorem 2. Suppose that H is a Hilbert space and that Y is a closed subspace of H such that $\dim Y < \infty$ and $y_1, y_2, \ldots, y_n$ is any basis for Y. Let x be an arbitrary element in H and let y_0 be the unique best approximation to x out of Y. Then from [20]

$$\|x-y_0\|_2^2=\frac{G(x,y_1,y_2,\ldots,y_n)}{G(y_1,y_2,\ldots,y_n)}, \tag{49}$$

where

$$G(x,y_1,y_2,\ldots,y_n)=\begin{vmatrix}\langle x,x\rangle & \langle x,y_1\rangle & \ldots & \langle x,y_n\rangle\\ \langle y_1,x\rangle & \langle y_1,y_1\rangle & \ldots & \langle y_1,y_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle y_n,x\rangle & \langle y_n,y_1\rangle & \ldots & \langle y_n,y_n\rangle\end{vmatrix}. \tag{50}$$

The following theorems illustrate that by increasing the number of MPs the error tends to zero.

Theorem 3. Let f be an arbitrary element in H. Then the function f has the best unique approximation on Y like $f_n \in Y$; that is,

$$\text{there exists } f_n \in Y \text{ s.t. for all } y \in Y : \|f-f_n\|_2 \le \|f-y\|_2, \tag{51}$$

where $\|f\|_2 = \sqrt{\langle f,f\rangle}$ and $\langle\cdot,\cdot\rangle$ denotes the inner product. Since $f_n \in Y$, therefore f_n is a linear combination of the spanning basis of Y; that is, there are $n+1$ coefficients

$$C=[c_0,c_1,\ldots,c_n]\in R, \tag{52}$$

such that

$$f(t)\simeq f_n(t)=\sum_{j=0}^{n}c_js_j(t)=C^T\varphi_n(t), \tag{53}$$

where

$$\|f-f_n\|_2\to\min. \tag{54}$$

Then C can be obtained by

$$C = Q^{-1}\langle f(t), \varphi_n(t)\rangle, \tag{55}$$

where

$$Q = \langle \varphi_n(t), \varphi_n(t)\rangle = \int_0^1 \varphi_n(t)\varphi_n(t)^T dt. \tag{56}$$

Theorem 4. Suppose that $f(t) \in L^2[0,1]$ is approximated by $f_n(t)$ as

$$f_n(t) = \sum_{i=0}^{n} c_i \beta_i(t) = C^T \varphi_n(t), \tag{57}$$

where C and $\varphi_n(t)$ are defined respectively in (55) and (16). Then

$$\lim_{m\to\infty} \|f(t) - f_n(t)\|_{L^2[0,1]} = 0. \tag{58}$$

The error vector eI^α of the operational matrix is given by

$$eI^\alpha = [eI_0^\alpha, eI_1^\alpha, \dots, eI_n^\alpha]^T = P\varphi_n(t) - I^\alpha \varphi_n(t). \tag{59}$$

From (49) and Theorem 2, we have

$$\left\| t^{i-2k+\alpha} - \sum_{j=0}^{n} b_j s_j \right\|_2 = \left(\frac{G(t^{i-2k+\alpha}, s_i(t), s_1(t), \dots, s_n(t))}{G(s_0(t), s_1(t), \dots, s_n(t))} \right). \tag{60}$$

Hence, according to (59) and (28), we have

$$\begin{aligned} \|eI_i^\alpha\|_2 &= \left| I^\alpha s_i(t) - \sum_{j=0}^{n} B_{ij} s_j(t) \right|_2 \\ &\leq \left[\left(-\frac{1}{2}\right)^{i+j} (i-1)!(j-1)! \right] \\ &\quad \times \sum_{k=0}^{h\left(\frac{i}{2}\right)} \sum_{L=0}^{h\left(\frac{j}{2}\right)} \binom{j}{L}\binom{i}{k} (-1)^{k+L} \frac{1}{(i-2k-1)!(j-2L-1)!} \\ &\quad \times \frac{1}{i+j-2L-2k+\alpha+1} \left| t^{i-2k+\alpha} - \sum_{j=0}^{n} b_j s_j \right|_2 \end{aligned}$$

$$
\begin{aligned}
&\leq \left[\left(-\frac{1}{2}\right)^{i+j}(i-1)!(j-1)!\right] \\
&\times \sum_{k=0}^{h\left(\frac{i}{2}\right)} \sum_{L=0}^{h\left(\frac{j}{2}\right)} \binom{j}{L}\binom{i}{k}(-1)^{k+L} \frac{1}{(i-2k-1)!(j-2L-1)!} \\
&\times \frac{1}{i+j-2L-2k+\alpha+1} \\
&\times \left(\frac{G(t^{i-2k+\alpha}, s_0(t), s_1(t), \ldots, s_n(t))}{G(s_0(t), s_1(t), \ldots, s_n(t))}\right)^{\frac{1}{2}}. \qquad (61)
\end{aligned}
$$

By considering Theorem 4 and using (61), one can conclude that by increasing the number of the Mott bases the vector eI^{α} tends to zero.

7 Numerical examples

To demonstrate the applicability of the numerical scheme, we apply the present method for the following illustrative examples.

Example 1. [16] Consider the following two-dimensional FOCP:

$$
\begin{aligned}
\text{Min} \quad & J[u,x] = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt, \\
\text{s.t.} \quad & D^{\alpha}x(t) = -x(t) + u(t), \\
& x(0) = 1.
\end{aligned}
$$

Our aim is to find the pair $(x(t), u(t))$, which minimizes the performance index J. The exact optimal solution of this problem for $\alpha = 1$ is as follows:

$$
\begin{aligned}
x(t) &= \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t), \\
u(t) &= (1+\sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2}+\beta)\sinh(\sqrt{2}t), \\
\beta &= -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})}.
\end{aligned}
$$

In Tables 1 and 2, the absolute errors of our proposed method and the method presented in [16] for $\alpha = 1$ are given. We see that the absolute errors of our approach are less than that of the method presented in [16]. Moreover, in Table 3, we show the comparison of the approximate optimal value of objective functional for $\alpha = 1$. It can be seen that the approximate value J tends to $J^* = 0.1929092980931791356$.

Table 1: Comparison of the absolute errors of approximate optimal state for Example 1

t	Present Method	Method in [16]
0.1	6.23e-9	2.11e-5
0.2	4.60e-10	9.71e-6
0.3	8.94e-9	4.08e-7
0.4	1.01e-9	5.76e-7
0.5	9.33e-9	5.66e-6
0.6	3.89e-10	9.25e-6
0.7	8.67e-9	8.35e-6
0.8	9.96e-10	4.34e-6
0.9	5.83e-9	2.59-6

Table 2: Comparison of the absolute errors of approximate optimal control for Example 1

t	Present Method	Method in [16]
0.1	1.1e-7	6.74e-6
0.2	1.57e-7	3.17e-6
0.3	3.49e-9	5.92e-7
0.4	1.41e-7	7.10e-7
0.5	3.84e-9	2.01e-6
0.6	1.38e-7	2.71e-6
0.7	9.80e-9	2.11e-6
0.8	1.50e-7	8.59e-7
0.9	1.08e-7	8.93-8

Table 3: Comparison of the approximate optimal value of J for Example 1

n	Present Method		Method in [16]	
	J	error of J	J[16]	error of J [16]
7	0.1929092980931	7.6e-15	0.192909308499	1.04e-8
8	0.1929092980931	7.6e-15	0.192909298458	3.64e-10
9	0.1929092980931	4.5e-18	0.192909298107	1.41e-11

Example 2. [13] Consider the following two-dimensional FOCP:

$$\begin{aligned}
\text{Min} \quad & J[u,x] = \frac{1}{2}\int_0^1 (x_1^2(t) + x_2^2 + u^2(t))dt, \\
s.t. \quad & \mathrm{D}^\alpha x_1(t) = -x_1(t) + x_2(t) + u(t), \\
& \mathrm{D}^\alpha x_2(t) = -2x_2(t), \\
& x_1(0) = x_2(0) = 1.
\end{aligned}$$

At $\alpha = 1$, the exact solutions are

$$x_1(t) = 0.018352e^{\sqrt{2}t} + 2.48165e^{-\sqrt{2}t} - \frac{3e^{-2t}}{2},$$
$$x_2(t) = e^{-2t},$$
$$u(t) = 0.044305e^{\sqrt{2}t} - 1.0279322e^{-\sqrt{2}t} + \frac{e^{-2t}}{2}$$

In Tables 4–6, the absolute errors when $\alpha = 1$ and $n = 3, 4, 6$ are demonstrated, and in Figure 1 (a-c), the results for $\alpha = 0.9$ and $n = 3, 4, 6$ are plotted. In Table 7, the comparison of our numerical results for the minimum values of J with different values of α at $n = 6, 7, 8$ with the results obtained in [24] is tabulated. In Figure 2 (a-c), we show that when α tends to 1, the approximate solutions tend to the exact solutions. In [13], a reasonable result was achieved with a large number of approximations ($n = 64, 128$), while in the present work, we achieve a satisfactory result with at most six elements of the Mott basis, which demonstrates the efficiency of the new method even when the number of approximations is not so great.

Table 4: Absolute errors of $x_1(t)$ for $\alpha = 1$ at various choices of n for Example 2

	n		
t	3	4	6
0.1	6.2546×10^{-4}	7.4441×10^{-4}	1.5002×10^{-5}
0.2	4.5464×10^{-3}	9.0972×10^{-4}	8.3446×10^{-6}
0.3	3.9041×10^{-3}	1.4174×10^{-5}	1.0492×10^{-5}
0.4	1.1462×10^{-3}	7.5444×10^{-4}	5.1504×10^{-6}
0.5	1.9425×10^{-3}	8.7140×10^{-4}	1.2149×10^{-5}
0.6	4.0841×10^{-3}	3.1893×10^{-4}	1.0835×10^{-6}
0.7	4.3836×10^{-3}	5.3794×10^{-4}	1.2403×10^{-5}
0.8	2.2303×10^{-3}	1.0500×10^{-3}	5.0792×10^{-6}
0.9	2.7809×10^{-3}	3.6221×10^{-4}	1.6075×10^{-5}

Table 5: Absolute errors of $x_2(t)$ for $\alpha = 1$ at various choices of n for Example 2

	n		
t	3	4	6
0.1	1.8817×10^{-4}	1.0437×10^{-3}	1.3953×10^{-5}
0.2	1.0761×10^{-2}	1.4093×10^{-3}	7.5730×10^{-6}
0.3	1.0648×10^{-2}	1.3573×10^{-4}	9.8425×10^{-6}
0.4	4.7251×10^{-3}	1.0583×10^{-3}	4.5687×10^{-6}
0.5	3.0146×10^{-3}	1.3381×10^{-3}	1.1291×10^{-5}
0.6	9.3023×10^{-3}	5.9271×10^{-4}	1.2293×10^{-6}
0.7	1.1462×10^{-2}	6.9627×10^{-4}	1.1414×10^{-5}
0.8	7.3019×10^{-3}	1.5621×10^{-3}	4.8958×10^{-6}
0.9	4.9714×10^{-3}	6.4096×10^{-4}	1.4835×10^{-5}

Table 6: Absolute errors of $u(t)$ for $\alpha = 1$ at various choices of n for Example 2

	n		
t	3	4	6
0.1	6.6070×10^{-4}	2.3079×10^{-4}	5.0375×10^{-6}
0.2	1.7968×10^{-4}	3.1360×10^{-4}	2.7248×10^{-6}
0.3	7.8105×10^{-4}	3.1989×10^{-5}	3.5467×10^{-6}
0.4	9.3073×10^{-4}	2.3164×10^{-4}	1.6400×10^{-6}
0.5	6.1527×10^{-4}	2.9416×10^{-4}	4.0622×10^{-6}
0.6	1.9916×10^{-5}	1.3128×10^{-4}	4.4423×10^{-7}
0.7	7.0047×10^{-4}	1.5236×10^{-4}	4.1084×10^{-6}
0.8	1.0445×10^{-3}	3.4459×10^{-4}	1.7637×10^{-6}
0.9	5.7713×10^{-4}	1.4011×10^{-4}	5.3528×10^{-6}

Table 7: Values of J when α approaches to 1, for Example 2

	PresentMethod			*Method of* [29]		
α	$n = 6$	$n = 7$	$n = 8$	$n = 6$	$n = 7$	$n = 8$
0.6	0.329 09	0.329 09	0.329 10			
0.7	0.351 62	0.351 62	0.351 62			
0.8	0.376 27	0.376 27	0.376 27	0.378 33	0.377 56	0.377 17
0.9	0.403 08	0.403 08	0.403 08	0.403 98	0.403 66	0.403 46
0.99	0.429 00	0.429 00	0.429 00	0.429 09	0.429 05	0.429 04
0.9999	0.431 95	0.431 95	0.431 95	0.431 96	0.431 96	0.431 96
1	0.431 98	0.431 98	0.431 98	0.431 99	0.431 98	0.431 98

Example 3. [1, 5, 22] Consider the following FOCP:

$$\begin{aligned} \text{Min} \quad & J[u,x] = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt, \\ s.t. \quad & \mathrm{D}^{\alpha} x(t) = t x(t) + u(t), \quad 0 \le \alpha \le 1, \\ & x(0) = 1. \end{aligned}$$

This problem has the exact solution $J = 0.4842676962287272$. In Figure 3 (a,b), the obtained results of the variables $x(t)$ and $u(t)$ are plotted for different values of α. In Figure 4 (a,b), the comparison between the exact solutions and the proposed method is plotted for $n = 8$ and $\alpha = 1$. In Table 8, the comparison of our numerical results for the minimum values of J for $\alpha = 1, 0.99$ at $n = 5$ with the results obtained in [18, 24] is tabulated. Obviously, our estimated results are in good agreement with them.

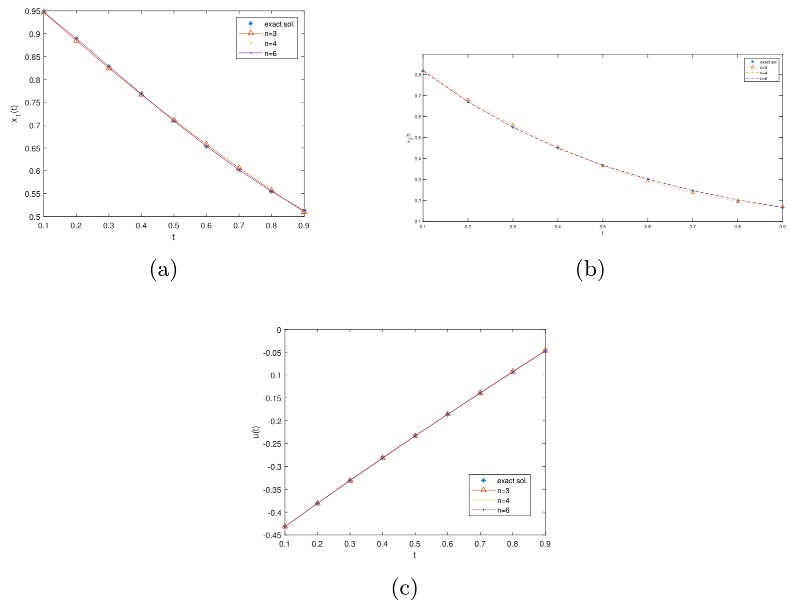


Figure 1: The behavior of the approximate solutions of Example 2 for $n = 3, 4, 6$ and $\alpha = 0.9$, with exact solution. $(a)x_1(t)$, $(b)x_2(t)$, $(c)u(t)$.

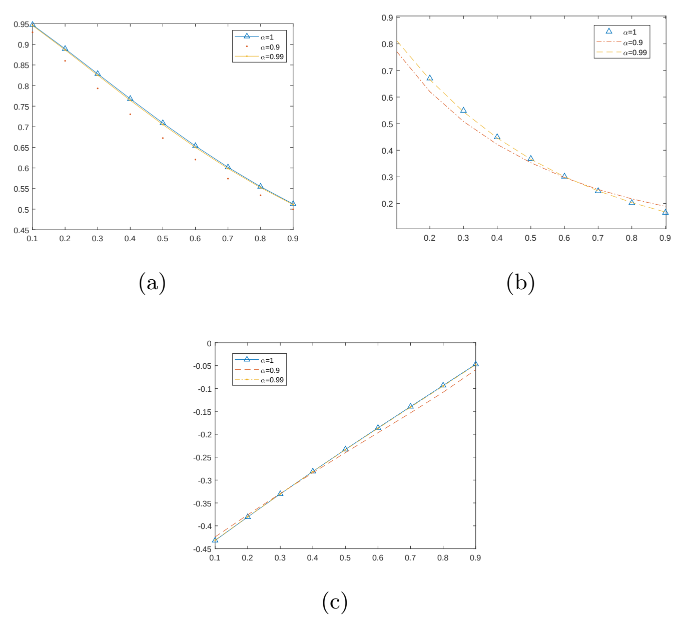
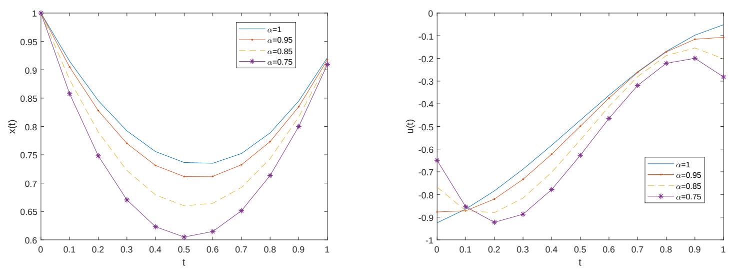


Figure 2: The behavior of the approximate solutions of Example 2 for $n = 6$ and $\alpha = 0.9, 0.99, 1$. $(a)x_1(t)$, $(b)x_2(t)$, $(c)u(t)$.

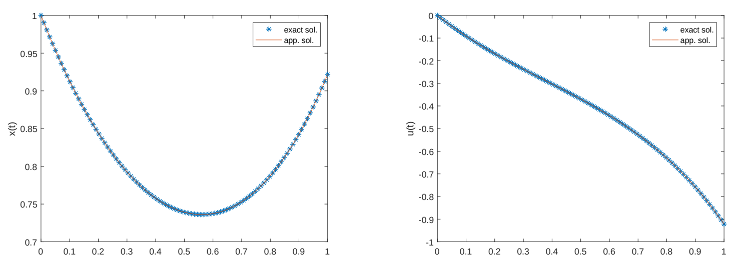
Table 8: The estimated values of J for different values of α and $n = 5$ for Example 3

α	Present Method	Method of [24]	Method of [18]
1	0.484267	0.484268	0.484268
0.99	0.483461	0.483468	0.483463
0.9	0.475874	0.476067	0.475883



(a) (b)

Figure 3: The behavior of the approximate solutions of Example 3 for $n = 8$ and $\alpha = 0.75, 0.85, 0.95, 1$, with exact solution. $(a)x(t)$, $(b)u(t)$.



(a) (b)

Figure 4: Comparison between the exact solutions and the proposed method of Example 3 for $n = 8$ and $\alpha = 1$. $(a)x(t)$, $(b)u(t)$.

Example 4. [30] In this example, the vibration of a spring-mass-damper system subjected to an external force is considered. In particular, the response to harmonic excitations, impulses, and step forcing functions is examined. In many environments, rotating machinery, motors, and so on cause periodic motions of structures to induce vibrations into other mechanical devices and structures nearby. On summing the forces, the equation for the forced vibration of the system in Figure 5 is obtained. It is common to approximate the driving forces $F(t)$

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t),$$

where m, c, and k are fixed numbers. Also $F(t)$ represents the control force derived from the action of an actuator force represented by $F(t) = bu(t)$, where b is a fixed number. The linear regulator problem has a specific application in the vibration suppression, and the performance index for this problem is defined as

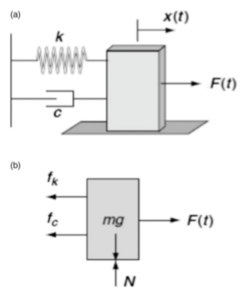


Figure 5: (a) Schematic of the forced mass–damper system assuming no friction on the surface and (b) free body diagram of the system of part (a) for Example 4

$$J = \frac{1}{2}\int_{t_0}^{t_f} (x^2(t) + au^2(t))dt,$$

where t_0 and t_f are initial and final times, respectively. We introduce the usual state variable notation

$$\begin{cases} x_1 = x, \\ x_2 = {}_0^cD_t^\alpha x(t). \end{cases}$$

Then, the equation of motion in a order can be written as

$$\begin{cases} {}_0^cD_t^\alpha x_1(t) = x_2, \\ {}_0^cD_t^\alpha x_2(t) = -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{b}{m}u. \end{cases}$$

By selecting $t_0 = 0$, $t_f = 1$, $c = 2$, and $m = a = b = k = 1$, the boundary conditions for this example are considered as

$$\begin{cases} x(0) = {}_0^cD_t^\alpha x(t)(0), \\ x(1) = {}_0^cD_t^\alpha x(t)(1), \end{cases}$$

or

$$\begin{cases} x_1(0) = x_2(0) = 1, \\ x_1(1) = x_2(1) = 0. \end{cases}$$

For $\alpha = 1$, we obtain the following exact solutions:

$$\begin{aligned} x_{1e}(t) =& [\frac{1393}{95}e^{-\frac{167}{152}t} + \frac{676}{219}e^{\frac{167}{152}t}](\sin \frac{76}{167}t) \\ &+ [\frac{1574}{423}e^{-\frac{167}{152}t} - \frac{1151}{423}e^{\frac{167}{152}t}](\cos \frac{76}{167}t), \\ x_{2e}(t) =& [\frac{2838}{613}e^{\frac{167}{152}t} - \frac{12872}{723}e^{-\frac{167}{152}t}](\sin \frac{76}{167}t) \\ &+ [\frac{1021}{395}e^{-\frac{167}{152}t} - \frac{626}{395}e^{\frac{167}{152}t}](\cos \frac{76}{167}t), \\ u_e(t) =& [\frac{16157}{890}e^{\frac{167}{152}t} - \frac{1134}{443}e^{-\frac{167}{152}t}](\sin \frac{76}{167}t) \\ &- [\frac{2547}{461}e^{\frac{167}{152}t} + \frac{2591}{1263}e^{-\frac{167}{152}t}](\cos \frac{76}{167}t), \end{aligned}$$

and $J_e = 13.00484741498823$, where $x_{1e}(t) = x(t)$. In Table 9, the results of J for different values of α and n are listed. It is seen that with the increase in the number of the Mott basis, the approximate value of J converges to the exact solution. Also Tables 10–12 demonstrate the approximation of $x_1(t)$, $x_2(t)$, and $u(t)$ for different values of n with $\alpha = 1$. Figure 6 (a-d) illustrates the behavior of state variables $x_1(t), x_2(t)$, control variable $u(t)$ and performance index J, respectively, for $n = 7$ and for different values of α with the exact solutions. Table 13 demonstrates the approximation of J, for $n = 8$ and different values of α. Also in Table 14, the absolute errors of $J, x_1(t), x_2(t)$, and $u(t)$ for Example 4 are calculated at various choices of n and $\alpha = 1$.

Table 9: Approximate solutions of J for different values of α and n for Example 4

	α		
n	1	0.99	0.9
3	**13.0**1904761904762	12.644 81	10.531 56
5	**13.0048478**586403	12.638 42	10.209 67
7	**13.004847414988**75	12.649 19	10.210 82
9	**13.0048474149882**2	12.654 24	10.220 66
11	**13.00484741498823**	12.656 99	10.227 68

H]

Table 10: Absolute errors of $x_1(t)$ for $\alpha = 1$ at various choices of n for Example 4

	n		
t	5	7	11
0	0	0	0
0.1	8.0624×10^{-6}	2.8936×10^{-9}	3.3240×10^{-13}
0.2	8.9073×10^{-6}	3.8441×10^{-9}	2.0139×10^{-13}
0.3	1.5140×10^{-6}	5.1600×10^{-9}	5.0016×10^{-13}
0.4	1.1564×10^{-5}	2.6349×10^{-9}	8.2645×10^{-13}
0.5	1.2533×10^{-5}	7.1014×10^{-9}	3.1086×10^{-15}
0.6	4.7572×10^{-6}	1.6704×10^{-9}	8.1934×10^{-13}
0.7	4.5973×10^{-6}	5.2971×10^{-9}	4.8572×10^{-13}
0.8	7.9685×10^{-6}	3.1233×10^{-9}	1.9929×10^{-13}
0.9	3.9680×10^{-6}	2.7298×10^{-9}	3.1830×10^{-13}
1	1.3322×10^{-15}	1.2212×10^{-15}	4.4409×10^{-16}

Table 11: Absolute errors of $x_2(t)$ for $\alpha = 1$ at various choices of n for Example 4

	n		
t	5	7	11
0	0	0	0
0.1	7.8816×10^{-5}	2.6708×10^{-8}	1.0362×10^{-11}
0.2	6.2887×10^{-5}	6.7790×10^{-8}	1.4781×10^{-11}
0.3	1.2314×10^{-4}	4.6618×10^{-8}	1.1712×10^{-11}
0.4	6.2434×10^{-5}	8.3958×10^{-8}	5.3897×10^{-12}
0.5	4.2027×10^{-5}	7.2681×10^{-9}	1.6518×10^{-11}
0.6	1.003×10^{-4}	8.4910×10^{-8}	5.3266×10^{-12}
0.7	7.2908×10^{-5}	3.2740×10^{-8}	1.1480×10^{-11}
0.8	8.3314×10^{-6}	6.6651×10^{-8}	1.4331×10^{-11}
0.9	5.8537×10^{-5}	1.7511×10^{-8}	9.9403×10^{-12}
1	7.7715×10^{-16}	1.2212×10^{-15}	7.7716×10^{-16}

Table 12: Absolute errors of $u(t)$ for $\alpha = 1$ at various choices of n for Example 4

	n		
t	5	7	11
0	3.9149×10^{-3}	4.0395×10^{-6}	1.2538×10^{-9}
0.1	9.5758×10^{-3}	1.5795×10^{-6}	2.1342×10^{-10}
0.2	1.3556×10^{-3}	6.0445×10^{-7}	2.8410×10^{-11}
0.3	1.7251×10^{-4}	1.1567×10^{-6}	2.5282×10^{-10}
0.4	8.5641×10^{-4}	2.3114×10^{-7}	2.8751×10^{-10}
0.5	1.0077×10^{-3}	1.1500×10^{-6}	3.2260×10^{-11}
0.6	3.5723×10^{-4}	3.5077×10^{-7}	3.0539×10^{-10}
0.7	5.0337×10^{-4}	1.0067×10^{-6}	2.0282×10^{-10}
0.8	8.2804×10^{-4}	6.34×10^{-7}	8.3564×10^{-11}
0.9	1.5161×10^{-4}	1.3112×10^{-6}	2.4373×10^{-10}
1	1.1831×10^{-3}	3.307×10^{-6}	1.1912×10^{-9}

Table 13: Approximation values of J for $n = 8$ and different values of α

n	$\alpha = 1$
1	13.004 847 414 988 234 117 130 714 182 4
0.99	12.676 065 068 414 313 965 482 218 517 6
0.9	10.416 927 546 051 989 730 195 819 221 4
0.8	9.004 530 163 410 743 299 622 066 577 65
0.7	8.363 433 733 210 117 712 725 797 350 86

Table 14: Absolute errors of J, x_1, x_2, u for $\alpha = 1$ and different values of n

	error			
n	error of J	error of x_1	error of x_2	error of u
11	7.6085×10^{-16}	4.846×10^{-13}	1.04×10^{-11}	2.792×10^{-10}
9	7.5908×10^{-15}	3.974×10^{-12}	7.744×10^{-11}	1.9×10^{-9}
8	3.8474×10^{-15}	2.729×10^{-10}	4.563×10^{-9}	9.6×10^{-8}
7	5.2381×10^{-12}	3.908×10^{-9}	5.603×10^{-8}	1.024×10^{-6}
5	4.4365×10^{-7}	7.474×10^{-6}	7.234×10^{-5}	9.419×10^{-4}

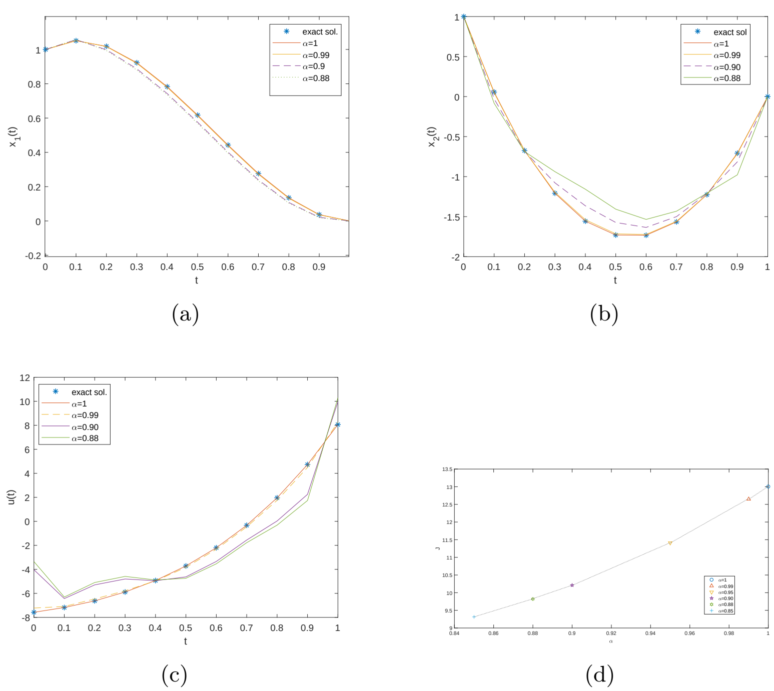


Figure 6: The behavior of the approximate solutions of Example 4 for $n = 7$ and $\alpha = 0.88, 0.90, 0.99, 1$, with exact solution. $(a)x_1(t)$, $(b)x_2(t)$, $(c)u(t)$, $(d)J$.

8 Conclusions

In this paper, a new numerical method has been derived to find the approximate solutions of the multi-dimensional FOCPs; this numerical method uses MPs. The Mott fractional integration matrix reduced the FOCP into an equivalent integral problem. By using the Mott fractional derivative and multiplication matrix, we can transform the equivalent functional integral equation problem into an algebraic system of equations, where this new problem is most easy to solve. Some examples are presented to demonstrate the validity and applicability of the new method. MATLAB (2018) is used for computations in this study.

References

1. Agrawal, O.P. *On a general formulation for the numerical solution of optimal control problems,* Int. J. Control 50 (1989) 627–638.

2. Agrawal O.P. *A general formulation and solution scheme for fractional optimal control problem*, Nonlinear Dynam, 38 (2004) 323–337.

3. Agrawal, O.P. and Baleanu, D. *A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems,* J. Vib. Control. 13 (2007) 1269–1281.

4. Ahmadi Darani, M. and Saadatmandi, A. *The operational matrix of fractional derivative of the fractional-order Chebyshev functions and its applications,* Comput. Methods Differ. Equ. 5(1) (2017), 67–87.

5. Akbarian, T. and Keyanpour, M. *A new approach to the numerical solution of fractional order optimal control problems,* Appl. Appl. Math. 8(2) (2013), 523–534.

6. Ashpazzadeh, E., Lakestani, M. and Yildirim, A. *Biorthogonal multiwavelets on the interval for solving multidimensional fractional optimal control problems with inequality constraint,* Optimal Control Appl. Methods 41(5) (2020), 1477–1494.

7. Atanacković, T.M., Pilipović, S., Stanković, B., and Zorica, D. *Fractional calculus with applications in mechanics. Wave propagation, impact and variational principles. Mechanical Engineering and Solid Mechanics Series,* ISTE, London; John Wiley & Sons, Inc., Hoboken, NJ, 2014.

8. Bagley, RL and Torvik PJ (1983) A theoretical basis for the application of fractional calculus to viscoelasticity, J. Rheol., 27, 201–210.

9. Baleanu, D., Defterli, O. and Agrawal, O.P. *A central difference numerical scheme for fractional optimal control problems,* J. Vib. Control. 15(4) (2009) 583–597.

10. Bohannan, G.W. *Analog fractional order controller in temperature and motor control applications,* J. Vib. Control. 14(9-10) (2008) 1487–1498.

11. Caputo, M. *Linear models of dissipation whose Q is almost frequency independent II,* Reprinted from Geophys. J. R. Astr. Soc. 13 (1967), no. 5, 529–539. Fract. Calc. Appl. Anal. 11 (2008), no. 1, 4–14.

12. Chow, T.S. *Fractional dynamics of interfaces between soft-nanoparticles and rough substrates,* Phys. Lett. A, 342, (2005) 148–155.

13. Defterli, O. *A numerical scheme for two-dimensional optimal control problems with memory effect,* Comput. Math. Appl. 59(5) (2010) 1630–1636.

14. Garrappa, R. *Numerical solution of fractional differential equations: A survey and a software tutorial,* https://www.mdpi.com/2227-7390/6/2/16

15. Ghomanjani, F. *A numerical technique for solving fractional optimal control problems and fractional Riccati differential equation,* J. Egyptian Math. Soc. 24(4) (2016), 638–643.

16. Habibli, M. and Noori Skandari, M.H. *Fractional Chebyshev pseudospectral method for fractional optimal control problems,* Optimal Control Appl. Methods 40(3) (2019), 558–572.

17. Hoda F. Ahmed *A numerical technique for solving multidimensional fractional optimal control problems,* Journal of Taibah University for Science, 12(5), (2018) 494–505.

18. Keshavarz, E., Ordokhani, Y. and Razzaghi, M. *A numerical solution for fractional optimal control problems via Bernoulli polynomials,* 22(18) (2016), 3889–3903.

19. Jesus, I.S. and Machado, J.T. *Fractional control of heat diffusion systems,* Nonlinear Dyn. 54 (2008) 263–282.

20. Kreyszig, E. *Introductory functional analysis with applications,* New York: John Wiley and sons. Inc, 1978.

21. Kruchinin, D.V. *Explicit formulas for some generalized Mott polynomials,* Advanced Studies in Contemporary Mathematics, 24 (3) (2014) 327–332.

22. Lotfi, A., Dehghan, M. and Yousefi, S.A. *A numerical technique for solving fractional optimal control problems,*Comput. Math. with Appl. 62 (2011) 1055–1067.

23. Maheswaran, A. and Elango, *Characterization of delta operator for Euler, Bernoulli of second kind and moot polynomials,* Int. J. Pure Appl. Math. 109 (2) 2016, (2016) 371–384.

24. Moghaddam, M.A., Yousef, E. and Lakestani, M. *Solving fractional optimal control problems using Genocchi polynomials,* Comput. Methods Differ. Equ. 9(1) (2021), 79–93.

25. Mott, N.F. *The polarization of electrons by double scattering*, Proc. R. Soc. Lond. A 135, (1932) 429–458.

26. Nemati. A., Yousefi, S.A. *A numerical method for solving fractional optimal control problems using Ritz method*, J. Comput. Nonlinear Dyn. 11(5) 051015-1 (2016) 7 pp.

27. Oldham, K.B. and Spanier, J. *The fractional calculus: Theory and application of differentiation and integration to arbitrary order,* Acad. Press, N. York and London, 1974.

28. Ross, B. *A brief history and exposition of the fundamental theory of fractional calculus, in fractional calculus and its Applications,* Lecture Notes in Mathematics, Vol.457, pp. 1–36, Springer, Berlin, 1975.

29. Weisstein, Eric W. “Mott Polynomial.” From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/MottPolynomial

30. Yari A *Numerical solution for fractional optimal control problems by Hermite polynomials,* J. Vib. Control. 25 (2021) 5–6.

31. Yousefi, S.A., Lotfi, A. and Dehghan, M. *The use of a Legendre multiwavelet collocation method for solving the fractional optimal control problems,* J. Vib. Control. 17(13)(2011) 2059–2065.

How to cite this article

Ali Alavi, S., Haghighi, A., Yari, A., Soltanian, F. Using MOTT polynomials operational matrices to optimize multi-dimensional fractional optimal control problems. *Iranian Journal of Numerical Analysis and Optimization*, 2022; 12(1): 201-227. doi: 10.22067/ijnao.2021.68926.1016