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Kudryashov method for exact solutions of isothermal magnetostatic atmospheres

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Abstract

The Kudryashov method to look for the exact solutions of the nonlinear differential equations is presented. The Kudryashov method is applied to search for the exact solutions of the Liouville equation and the Sinh-Poisson equation. The equations of magnetohydrostatic equilibria for a plasma in a gravitational field are investigated analytically. An investigation of a family of isothermal magnetostatic atmospheres with one ignorable coordinate corresponding to a uniform gravitational field in a plane geometry is carried out. The distributed current in the model J is directed along the x-axis where x is the horizontal ignorable coordinate. These equations transform to a single nonlinear elliptic equation for the magnetic vector potential u. This equation depends on an arbitrary function of u that must be specified.

Keywords: Kudryashov method; magnetostatic equilibria; nonlinear evolution equations; traveling waves.

1 Introduction

The equations of magnetostatic equilibria have been used extensively to model the solar magnetic structure [1, 4, 9, 11]. An investigation of a family of isothermal magnetostatic atmospheres with one ignorable coordinate corresponding to a uniform gravitational field in a plane geometry is carried out. The force balance consists of the force between $J \wedge B$ (B, magnetic field induction, J is the electric current density), the gravitational force, and gas pressure gradient force. However, in many models, the temperature distribution is specified a priori and direct reference to the energy equations is

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eliminated. In solar physics, the equations of magnetostatic have been used to model diverse phenomena, such as the slow evolution stage of solar flares, or the magnetostatic support of prominences [20]. The nonlinear equilibrium problem has been solved in several cases [3, 8, 17, 18]. In this paper, we obtain the exact analytical solutions for the Liouville and sinh-Poisson equations using the Kudryashov method. Because these two models will be special cases of magnetostatic atmospheres model. Also here there is force balance between different forces. The Kudryashov method was developed by Kudryashov on the basis of a procedure analogous to the first step of the test for the Painlev property [2, 7, 7, 9, 10]. The paper is organized as follows :
In Section 2, we describe the methodology of Kudryashov method for solving nonlinear evolution equations when the Riccati equation is used as the simplest equation. We describe the Basic equations in Section 3. We apply this methodology and obtain exact solutions of the Liouville and sinh-Poisson equations in Section 4. Finally, the concluding remarks are presented in Section 5.

2 Analysis of the Kudryashov method

We consider a partial differential equation and we assume that by means of an appropriate transformation this partial differential equation is transformed to a nonlinear ordinary differential equation in the form

$$P(u, u', u'', u''', ...) = 0. \tag{1}$$

Exact solution of this equation can be constructed as finite series

$$u(\xi) = \sum_{i=0}^{n} A_i (G(\xi))^i, \tag{2}$$

where $G(\xi)$ is a solution of some ordinary differential equation referred to as the simplest equation. The simplest equation has two properties:

1. the order of simplest equation should be less than the order of equation (1).
2. we should know the general solution of the simplest equation or at least exact analytical particular solution(s) of the simplest equation.

In this paper, we use the equation of Riccati, as the simplest equation. This equation is a well-known nonlinear ordinary differential equation which possesses the exact solution constructed by elementary function. In this paper for the Riccati equation

$$G'(\xi) = c\,G(\xi) + d\,G(\xi)^2, \tag{3}$$

we use the solution

$$G(\xi) = \frac{c\,\exp[c(\xi+\xi_0)]}{1-d\exp[c(\xi+\xi_0)]}; \qquad d<0, c>0, \tag{4}$$

and

$$G(\xi) = -\frac{c\,\exp[c(\xi+\xi_0)]}{1+d\exp[c(\xi+\xi_0)]}; \qquad d>0, c<0. \tag{5}$$

Here ξ_0 is a constant of integration. Now $u(\xi)$ can be determined explicitly by using the following three steps:

- Step (1). By considering the homogeneous balance between the highest nonlinear terms and the highest order derivatives of $u(\xi)$ in equation (1), the positive integer n in (2) is determined.
- Step (2). By substituting equation (2) with equation (3) into equation (1) and collecting all terms with the same powers of G together, the left hand side of equation (1) is converted into a polynomial. After setting each coefficient of this polynomial to zero, we obtain a set of algebraic equations in terms of $A_i\,(i=0,1,2,...,n)$, $c,\ d$.
- Step (3). Solving the system of algebraic equations and then substituting the results and the general solutions of (4) or (5) into (2) gives solutions of (1).

3 Basic equations

The relevant of magnetohydrostatic equations consisting of the equilibrium equation with force balance will be as:

$$J\wedge B-\rho\nabla\Phi-\nabla P=0, \tag{6}$$

which is coupled with Maxwells equations:

$$J=\frac{\nabla\wedge B}{\mu}, \tag{7}$$

$$\nabla\cdot B=0, \tag{8}$$

where P, ρ , μ and Φ are the gas pressure, the mass density, the magnetic permeability and the gravitational potential, respectively. It is assumed that the temperature is uniform in space and that the plasma is an ideal gas with equation of state $p=\rho R_0 T_0$, where R_0 is the gas constant and T_0 is the temperature. Then the magnetic field B can be written by the following:

$$B=\nabla u\wedge e_x+B_x\,e_x=(B_x,\frac{\partial u}{\partial z},\frac{-\partial u}{\partial y}). \tag{9}$$

The form of (9) for B ensures that $\nabla \cdot B = 0$, and there is no mono pole or defect structure. equation (6) requires the pressure and density be of the form [11]:

$$P(y,z) = P(u)\, e^{\frac{-z}{h}}, \quad \rho(y,z) = \frac{1}{(gh)} P(u)\, e^{\frac{-z}{h}}, \tag{10}$$

where $h = \frac{R_0\, T_0}{g}$ is the scale height. Substituting equations (7-10) in equation (6), we obtain

$$\nabla^2\, u + f(u)\, e^{\frac{-z}{h}} = 0, \tag{11}$$

where

$$f(u) = \mu\, \frac{dP}{du}. \tag{12}$$

Equation (12) gives

$$P(u) = P_0 + \frac{1}{\mu} \int f(u) du \tag{13}$$

Substituting equation (13) into equation (10), we obtain

$$P(y,z) = (P_0 + \frac{1}{\mu} \int f(u) du)\, e^{\frac{-z}{h}}, \tag{14}$$

$$\rho(y,z) = \frac{1}{gh} (P_0 + \frac{1}{\mu} \int f(u) du)\, e^{\frac{-z}{h}}, \tag{15}$$

where P_0 is constant. Taking transformation

$$x_1 + i\, x_2 = e^{\frac{-z}{l}}\, e^{\frac{iy}{l}} \tag{16}$$

equation (12) reduces to

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + l^2\, f(u)\, e^{(\frac{2}{l} - \frac{1}{h})z} = 0. \tag{17}$$

These equations have been given in Khater et al. (2000).

4 Application of the Kudryashov method

In this section, we will investigate the Kudryashov method for solving specific forms of $f(u)$.

4.1 Liouville equation

We first consider Liouville equation and the following equation will be special case of equation (17). Let us assume f(u) has the form (Dungey, 1953; Low, 1975):

$$f(u) = -\alpha^2 A_0 e^{-\frac{A}{A_0}}, \tag{18}$$

where A_0 and α^2 are constants. Hence

$$P(y,z) = (P_0 + \frac{\alpha^2 A_0^2}{2\mu} e^{\frac{-2A}{A_0}}) e^{\frac{-z}{h}}. \tag{19}$$

Inserting equation (18) into equation (17) we obtain

$$\nabla^2 A/A_0 = l^2 \alpha^2 e^{\frac{-2A}{A_0} + (\frac{2}{l} - \frac{1}{h})z}, \tag{20}$$

where $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. Let us set

$$\frac{A}{A_0} = \frac{z}{L} + w(y,z), \tag{21}$$

where L is a constant. Then equation (20) becomes

$$\nabla^2 w - l^2 \alpha^2 e^{-2w - (\frac{2}{L} + \frac{1}{h} - \frac{2}{l})z}. \tag{22}$$

Let us identify l by

$$\frac{2}{l} = \frac{2}{L} + \frac{1}{h}, \tag{23}$$

and inserting equation (23) into equation (22) we obtain a Liouville type

$$\phi_{xx} + \phi_{tt} - \alpha^2 l^2 e^{-2\phi} = 0. \tag{24}$$

In order to apply the Kudryashov method, we use the wave transformation $\xi = x - kt$ and change equation (24) into the form

$$(1 + k^2)\phi'' = \alpha^2 l^2 e^{-2\phi}, \tag{25}$$

we next use the transformation

$$v = e^{-2\phi}, \tag{26}$$

we obtain

$$(1 + k^2) v v'' - (1 + k^2) (v')^2 + 2\alpha^2 l^2 u^3 = 0, \tag{27}$$

with balancing according step (1) we get $n = 2$, therefore the solution of (27) can be expressed as follow:

$$v(\xi)=\sum_{i=0}^{2}A_i(G(\xi))^i. \tag{28}$$

Substituting equation (28) along with (3) into (27) and setting the coefficients of all powers of G to zero, we obtain a system of nonlinear algebraic equations for A_0, A_1, A_2 . Solving the resulting system with the help of mathematica, we have the following sets of solutions:

$$\begin{cases} A_0=0, \\ A_1=-\frac{cd(1+k^2)}{l^2\alpha^2}, \\ A_2=-\frac{d^2(1+k^2)}{l^2\alpha^2}, \end{cases} \tag{29}$$

where $\xi=x-kt$, λ,α,l are constants. Therefore, substituting (29) in (28) and general solution (3) according to (4), we obtain solution of (27) as follows:

$$v_1(\xi)=-\frac{c^2d(1+k^2)}{l^2\alpha^2}\frac{e^{(c(\xi+\xi_0))}}{(1-de^{(c(\xi+\xi_0))})^2}, \tag{30}$$

where $d<0$, $c>0$, $\xi=x-kt$. Using transformation

$$v=e^{-2\phi}, \tag{31}$$

we get solution of (24) as follows:

$$\phi_1(\xi)=-\frac{1}{2}\ln[-\frac{c^2d(1+k^2)}{l^2\alpha^2}\frac{e^{(c(\xi+\xi_0))}}{(1-de^{(c(\xi+\xi_0))})^2}]. \tag{32}$$

Now substituting (29) in (28) and general solution (3) according to (5), we obtain solution of (27) as follows:

$$v_2(\xi)=\frac{c^2d(1+k^2)}{l^2\alpha^2}\frac{e^{(c(\xi+\xi_0))}}{(1+de^{(c(\xi+\xi_0))})^2}, \tag{33}$$

where $d>0, c<0$, $\xi=x-kt$. Using transformation

$$v=e^{-2\phi}, \tag{34}$$

we get solution of (24) as follows:

$$\phi_2=-\frac{1}{2}\ln[\frac{c^2d(1+k^2)}{l^2\alpha^2}\frac{e^{(c(\xi+\xi_0))}}{(1+de^{(c(\xi+\xi_0))})^2}]. \tag{35}$$

4.2 Sinh-Poisson equation

In this section, we consider sinh-Poisson equation which plays an important role in the soliton model with BPS bound [4, 6]. Also, this equation will be special case of equation (17). If we assume

$$f(u) = -\frac{\beta^2}{4}(\frac{A_0}{h})\sinh(\phi). \tag{36}$$

The same as above we have

$$\phi_{xx} + \phi_{tt} = \beta^2 \sinh(\phi). \tag{37}$$

In order to apply the Kudryashov method, we use the wave transformation $\xi = x - kt$ and change equation (37) into the form

$$(1 + k^2)\phi'' = \beta^2 \sinh(\phi), \tag{38}$$

we next use the transformation

$$\begin{cases} v & = e^{\phi}, \\ \sinh(\phi) & = \frac{e^{\phi} - e^{-\phi}}{2}, \end{cases} \tag{39}$$

we obtain

$$2(1 + k^2)vv'' - 2(1 + k^2)\,(v')^2 - \beta^2(v^3 - v) = 0. \tag{40}$$

With balancing according to step (1) we get $n = 2$, therefore the solution of (40) can be expressed as follows:

$$v(\xi) = \sum_{i=0}^{2} A_i (G(\xi))^i. \tag{41}$$

Substituting equation (41) along with (3) into (40) and setting the coefficients of all powers of G to zero, we obtain a system of nonlinear algebraic equations for A_0, A_1, A_2 . Solving the resulting system with the help of mathematica, we have the following sets of solutions:

$$\begin{cases} A_0 = 1, \\ A_1 = \frac{4d}{c}, \\ A_2 = \frac{4d^2}{c^2}, \\ c = \pm\frac{\beta}{\sqrt{1+k^2}}, \end{cases} \tag{42}$$

where $\xi = x - kt$, λ,β are constants. Therefore, using Substituting (42) in (41) and general solution (3) according to (4), we obtain solution of (40) as follows:

$$v_1(\xi) = \frac{\left(1 + de^{c(\xi+\xi 0)}\right)^2}{\left(1 - de^{c(\xi+\xi 0)}\right)^2}; \qquad d < 0, \quad c > 0, \tag{43}$$

where $c = \frac{\beta}{\sqrt{1+k^2}}$ when $\beta > 0$ or $c = -\frac{\beta}{\sqrt{1+k^2}}$ when $\beta < 0$ and $\xi = x - kt$. Using transformation

$$v = e^{\phi}, \tag{44}$$

we get solution of (37) as follows:

$$\phi_1(\xi) = \ln[\frac{\left(1 + de^{c(\xi+\xi 0)}\right)^2}{\left(1 - de^{c(\xi+\xi 0)}\right)^2}], \tag{45}$$

where $c = \frac{\beta}{\sqrt{1+k^2}}$ when $\beta > 0$ or $c = -\frac{\beta}{\sqrt{1+k^2}}$ when $\beta < 0$. Now with Substituting (42) in (41) and general solution (3) according to (5), we obtain solution of (40) as follows:

$$v_2(\xi) = \frac{\left(1 - de^{c(\xi+\xi 0)}\right)^2}{\left(1 + de^{c(\xi+\xi 0)}\right)^2}; \qquad d > 0, \quad c < 0, \tag{46}$$

where $c = \frac{\beta}{\sqrt{1+k^2}}$ when $\beta < 0$ or $c = -\frac{\beta}{\sqrt{1+k^2}}$ when $\beta > 0$ and $\xi = x - kt$. Using transformation

$$v = e^{\phi}, \tag{47}$$

we get solution of (37) as follows:

$$\phi_2 = \ln[\frac{\left(1 - de^{c(\xi+\xi 0)}\right)^2}{\left(1 + de^{c(\xi+\xi 0)}\right)^2}], \tag{48}$$

where $c = \frac{\beta}{\sqrt{1+k^2}}$ when $\beta < 0$ or $c = -\frac{\beta}{\sqrt{1+k^2}}$ when $\beta > 0$.

5 Conclusion

This study shows that the Kudryashov method is quite efficient and practical and is well suited for use in finding exact solutions for the Liouville and Sinh-Poisson equations. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. In this paper, the Kudryashov method has been successfully used to obtain some exact travelling wave solutions for the Liouville and Sinh-Poisson equations.

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روش ساده ترین معادله برای جواب های دقیق اتمسفر مغناطیسی هم دما

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چکیده : روش کودریاشف برای جستجوی جواب های دقیق معادلات دیفرانسیل غیر خطی ارائه شده است. روش کودریاشف برای جستجو جوابهای دقیق معادله لیوویل و معادله سینویس هایپربولیک-پواسون به کار برده شده است. معادلات تعادل هیدرواستاتیک مغناطیس برای پلاسما در یک میدان گرانشی به صورت تحلیلی مورد بررسی قرار گرفته اند. بررسی یک خانواده از اتمسفر مغناطیسی هم دما با یک مختصات متناظر به یک میدان گرانشی یکنواخت در هندسه مسطحه انجام شده است. جریان توزیع شده در مدل J در امتداد محور x که در آن x مختصات افقی می باشد هدایت شده است. این معادلات تبدیل به یک معادله بیضی غیر خطی برای پتانسیل برداری مغناطیسی u می شوند. این معادله به یک تابع دلخواه u وابسته است که باید مشخص شود.

کلمات کلیدی : روش کودریاشف؛ تعادل مغناطیسی؛ معادلات تکامل غیر خطی؛ امواج تراولینگ.