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Research Article

Generalization of equitable efficiency in multiobjective optimization problems by the direct sum of matrices

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Abstract

We suggest an a priori method by introducing the concept of A_P-equitable efficiency. The preferences matrix A_P, which is based on the partition P of the index set of the objective functions, is given by the decision-maker. We state the certain conditions on the matrix A_P that guarantee the preference relation $\preceq_{eA_P}$ to satisfy the strict monotonicity and strict P-transfer principle axioms.

A problem most frequently encountered in multiobjective optimization is that the set of Pareto optimal solutions provided by the optimization process is a large set. Hence, the decision-making based on selecting a unique preferred solution becomes difficult. Considering models with A^r_P-equitable efficiency and A^∞_P-equitable efficiency can help the decision-maker for overcoming this difficulty, by shrinking the solution set.

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1 Introduction

A problem that sometimes occurs in classical multiobjective optimization is that the set of efficient solutions is a large set. By using a priori methods, we can generate finite sets of Pareto optimal solutions, which can help the decision-maker in the task of selecting the most appropriate solution. A priori methods are based on the preferences matrix, which evaluates how to combine the objective functions by the decision-maker to introduce a preference function. Note that in a priori methods, the preferences are expressed by the decision-maker before the solution process (e.g., setting goals or weights to the objective functions). The criticism about a priori methods is that it is very difficult for the decision-maker to beforehand define and accurately quantify his preferences; see [4].

The concept of equitable efficiency is a specific refinement of the Pareto efficiency. While the Pareto efficiency assumes that the criteria are uncomparable (not measured on a common scale), the equitability is based on the assumption that the criteria are comparable, impartial (anonymous), and that the Pigou–Dalton principle of transfer holds. The impartiality axiom makes the distribution of outcomes among the criteria more important than the assignment of outcomes to specific criteria. Therefore models are the equitable allocation of resources.

The equitable preference was first known as the generalized Lorenz dominance [8, 10]. Kostreva and Ogryczak [6] and Kostreva, Ogryczak, and Wierzbicki [7] are the first ones who introduced the concept of equitability into multiobjective programming. They analyzed solution properties and approaches to generating equitably efficient solutions. A complete preference structure of equitability is derived by Bataar and Wiecek [1]. Furthermore, the concept of equitability in multiobjective programming is generalized within a framework of convex cones by Mut and Wiecek [11]. They introduced the concept of A-equitable efficiency for solving the multiobjective optimization problems, where A is an arbitrary matrix with nonnegative entries, and they also showed that the preference relation $\preceq_{eA}$ satisfies the axioms of reflexivity, transitivity, and impartiality while the weak principle of transfer requires a condition on the matrix A. Because the preference relation $\preceq_{eA}$ does not satisfy the strict monotonicity and strict principle of transfer axioms in general, the set of A-equitably efficient solutions does not contain within the set of equitably efficient solutions and the set of Pareto optimal solutions for the same problem. Foroutannia and Merati [3] stated new conditions on the matrix A that guarantee to hold these axioms by the preference relation $\preceq_{eA}$.

Let the partition P of the index set of objective functions be given by the decision-maker according to the importance of the objective functions. The equitable rational preference relation is extended to P-equitable rational preference relations by Mahmodinejad and Foroutannia [9]. They showed that the concept of P-equitably efficient solutions is a specific refinement of Pareto optimality by adding the P-impartiality and P-transfer axioms. Moreover, they obtained the P-equitably efficient solutions by decomposing the original problem into a collection of smaller subproblems and then solved the subproblems by the concept of equitable efficiency.

The equitable optimization method is applied to problems such as portfolio, location, telecommunications, and resource allocation [12, 13, 14, 15, 16]. It should be noted that some authors have used the term "fair" rather than "equitable".

In this paper, we investigate a priori technique for attaining the decision-maker preferences by introducing the concept of A_P-equitable efficiency, where the preferences matrix A_P is based on the partition P of the index set of objective functions given by the decision-maker. The current study is an extension of some results obtained in [3, 9, 11].

The paper is organized as follows. Terminology and basic concepts are presented in Section 2. In Section 3, we introduce the concept of A_P-equitable efficiency and give some conditions that ensure that the preference relation $\preceq_{eA_P}$ is a P-equitable rational preference relation. In Section 4, the concept of A_P^r-equitably efficiency is examined to generate a subset of Pareto optimal solutions, for $r = 1, 2, \ldots$. In addition, a numerical example is provided to confirm the efficiency of this method. Finally, Section 5 concludes the paper.

2 Terminology and review of the equitable preference

Let $\mathbb{R}^m$ be the Euclidean vector space and let $y', y'' \in \mathbb{R}^m$. Then $y' \leqq y''$ means $y'_i \leq y''_i$ for $i = 1, \ldots, m$ and $y' < y''$ means $y'_i < y''_i$ for $i = 1, \ldots, m$, and also $y' \leq y''$ stands for $y' \leqq y''$ but $y' \neq y''$.

Consider a decision problem defined as an optimization problem with m objective functions. For simplification, we assume, without loss of generality, that the objective functions are to be minimized. The problem can be formulated as follows:

$$\begin{aligned} &\min \left(f_1(x), f_2(x), \ldots, f_m(x)\right), \\ &\text{subject to x} \in \text{X}, \end{aligned} \tag{1}$$

where x denotes the vector of decision variables in the feasible set X and $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ is the vector function that maps the feasible set X into the objective (criterion) space $\mathbb{R}^m$. We refer to the elements of the objective space as outcome vectors. An outcome vector y is attainable if it expresses outcomes of a feasible solution, that is, $y = f(x)$ for some $x \in X$. The set of all attainable outcome vectors is denoted by $Y = f(X)$.

In the single objective minimization problems, we compare the objective values at different feasible decisions to select the best decision. Decisions are ranked according to the objective values of those decisions, and any decision with the smallest objective value is called an optimal solution. Similarly, to make the multiobjective optimization model operational, one needs to assume some solution concepts specifying what it means to minimize multiobjective functions. The solution concepts are defined by the properties of the corresponding preference model. We assume that solution concepts depend only on the evaluation of the outcome vectors while not taking into account any other solution properties not represented within the outcome vectors. Thus, we can limit our considerations to the preferred model in the objective space Y.

In the rest of the section, some basic concepts and definitions of preference relations are reviewed from [3, 6, 9, 11]. Preferences are represented by a weak preference relation with the notation, $\preceq$, which allows us to compare pairs of outcome vectors y' and y'' in the objective space Y. We say $y' \preceq y''$ if and only if "y' is at least as good as y''" or "y' is weakly preferred to y''". In other words, $y' \preceq y''$ means that the decision-maker thinks that the outcome vector y' is at least as good as the outcome vector y''. From $\preceq$, we can derive two other important relations on Y.

Definition 1. Let $y', y'' \in \mathbb{R}^m$ and let $\preceq$ be a relation of weak preference defined on $\mathbb{R}^m \times \mathbb{R}^m$. The strict preference relation, $\prec$, is defined by

$$y' \prec y'' \Leftrightarrow (y' \preceq y'' \textit{and not } y'' \preceq y'), \tag{2}$$

and read y' is strictly preferred to y''. Also the indifference relation, $\simeq$, is defined by

$$y' \simeq y'' \Leftrightarrow (y' \preceq y'' \textit{and } y'' \preceq y'), \tag{3}$$

and read y' is indifferent to y''.

Definition 2. Preference relations satisfying the following axioms are called equitable rational preference relations:

1. Reflexivity: for all $y \in \mathbb{R}^m$, $y \preceq y$.
2. Transitivity: for all $y', y'', y''' \in \mathbb{R}^m$, $y' \preceq y''$ and $y'' \preceq y''' \Rightarrow y' \preceq y'''$.
3. Strict monotonicity: for all $y \in \mathbb{R}^m$, $y - \epsilon e_i \prec y$ for all $\epsilon > 0$, where e_i denotes the ith unit vector in $\mathbb{R}^m$, for all $i \in \{1, 2, \ldots, m\}$.
4. Impartial: for all $y \in \mathbb{R}^m$

   $$(y_1, y_2, \ldots, y_m) \simeq (y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)}),$$

   where τ stands for an arbitrary permutation of components of y.
5. Strict transfer principle: for all $y \in \mathbb{R}^m$ and for all $i, j \in \{1, 2, \ldots, m\}$

   $$y_i > y_j \Rightarrow y - \epsilon e_i + \epsilon e_j \prec y,$$

   where $0 < \epsilon < y_i - y_j$.

A preference relation with the axioms reflexivity, transitivity, and strict monotonicity is called rational preference relation. For $y', y'' \in Y$, we say that y' rationally dominates y'', and denote by $y' \prec_r y''$ if and only if $y' \prec y''$ for all rational preference relations $\preceq$. An outcome vector y is rationally nondominated if and only if there exist no other outcome vector y' such that y' rationally dominates y. Analogously, a feasible solution $x \in X$ is an efficient or Pareto optimal solution to the multiobjective problem (1) if and only if $y = f(x)$ is rationally nondominated. It has been shown in [6] that $y' \prec_r y''$ if and only if $y' \leq y''$. As a consequence, we can state that a feasible solution $x \in X$ is a Pareto optimal solution to the multiobjective problem (1) if and only if there exist no $x' \in X$ such that $f_i(x') \leq f_i(x)$ for $i = 1, 2, \ldots, m$, where at least one strict inequality holds.

The set of all Pareto optimal solutions $x \in X$ is denoted by X_E and called the efficient set. The set of all rationally nondominated points $y = f(x) \in Y$, where $x \in X_E$, is denoted by Y_N and called the nondominated set.

The equitable rational preference relations allow us to define the concept of equitably efficient solution.

Definition 3. Let $y', y'' \in Y$. We say that y' equitably dominates y'', and denote by $y' \prec_e y''$ if and only if $y' \prec y''$ for all equitable rational preference relations $\preceq$. An outcome vector y is equitably nondominated if and only if there exist no other outcome vector y' such that y' equitably dominates y. Analogously, a feasible solution x is called an equitably efficient solution of the multiobjective problem (1) if and only if $y = f(x)$ is equitably nondominated.

The set of all equitably efficient solutions $x \in X$ is denoted by X_{eE} and called the equitably efficient set. The set of all equitably nondominated points $y = f(x) \in Y$, where $x \in X_{eE}$, is denoted by Y_{eN} and called the equitably nondominated set.

Definition 4. Let $y \in \mathbb{R}^m$.

1. The function $\theta : \mathbb{R}^m \to \mathbb{R}^m$ is called an ordering map if and only if $\theta(y) = (\theta_1(y), \theta_2(y), \ldots, \theta_m(y))$, where $\theta_1(y) \geq \theta_2(y) \geq \cdots \geq \theta_m(y)$ in which $\theta_i(y) = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$, and τ is a permutation of the set $\{1, 2, \ldots, m\}$.

2. The function $\overline{\theta} : \mathbb{R}^m \to \mathbb{R}^m$ is called a cumulative ordering map if and only if $\overline{\theta}(y) = (\bar{\theta}_1(y), \bar{\theta}_2(y), \ldots, \bar{\theta}_m(y))$, where $\bar{\theta}_i(y) = \sum_{j=1}^{i} \theta_j(y)$ for $i = 1, 2, \ldots, m$ and the ordering map θ is given by part (1).

Note that $\overline{\theta}(y) = \Delta\theta(y)$, where

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix}.$$

is an $m \times m$ lower-triangular matrix and relates it to the equitable preference.

A relationship between the weak equitable preference relation $\preceq_e$ and the Pareto relation has been established in [6]. The following proposition shows finding nondominated points with respect to the weak equitable preference relation $\preceq_e$ can be done by means of Pareto preference.

Proposition 1. [6, Proposition 2.3] For any two vectors $y', y'' \in Y$, we have

$$y' \preceq_e y'' \Leftrightarrow \overline{\theta}(y') \leqq \overline{\theta}(y'') \Leftrightarrow \Delta\theta(y') \leqq \Delta\theta(y''),$$

where the ordering map θ and the cumulative ordering map $\overline{\theta}$ are given by Definition 2.

Now, we review the concept equitably with respect to any matrix A, which was introduced by Mut and Wiecek [11]. Assume that $A = (a_{ij})$ is an $m \times m$ matrix with real entries. Then the cumulative map $A(\theta) : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$A(\theta(y)) = \left(\sum_{j=1}^{m} a_{1j}\theta_j(y), \sum_{j=1}^{p} a_{2j}\theta_j(y), \dots, \sum_{j=1}^{m} a_{pj}\theta_j(y) \right).$$

Definition 5. Let $y', y'' \in Y$. We say that y' A-equitably dominates y'', and denote by $y' \prec_{eA} y''$ if and only if $A(\theta(y')) \leq A(\theta(y''))$. An outcome vector y is A-equitably nondominated if and only if there exist no other outcome vector y' such that y' A-equitably dominates y. Analogously, a feasible solution x is called an A-equitably efficient solution of the multiobjective problem (1) if and only if $y = f(x)$ is A-equitably nondominated.

The set of all A-equitably efficient solutions $x \in X$ is denoted by X_{eAE} and called the A-equitably efficient set. The set of all A-equitably nondominated points $y = f(x) \in Y$, where $x \in X_{eAE}$, is denoted by Y_{eAN} and called the A-equitably nondominated set.

Mut and Wiecek [11, Section 5] examined relationships between cone representations and the axioms of preference relation $\preceq_{eA}$. They showed that the relation $\preceq_{eA}$ satisfies the axioms of reflexivity, transitivity, and impartiality while the weak principle of transfer requires the following condition on the matrix A.

Weak transfer principle: For all $y \in \mathbb{R}^m$ and for all $i, j \in \{1, 2, \dots, m\}$

$$y_i > y_j \Rightarrow y - \epsilon e_i + \epsilon e_j \preceq y,$$

where $0 \leq \epsilon \leq y_i - y_j$.

Proposition 2. [11, Corollary 5.11] Let $A = [a_1, a_2, \dots, a_p]$, where a_i's are the columns of the matrix A, $i = 1, \dots, p$. The weak principle of transfer axiom for the generalized equitable preference $\preceq_{eA}$ is equivalent to the condition

$$a_1 \geqq a_2 \geqq \cdots \geqq a_p,$$

on the matrix A.

The preference relation $\preceq_{eA}$ does not satisfy the strict monotonicity and strict principle of transfer axioms in general. Therefore the set of A-equitably efficient solutions is not contained within the set of equitably efficient solutions and the set of Pareto optimal solutions for the same problem. Foroutannia and Merati extended the work done by Mut and Wiecek and stated new conditions on the matrix A that guarantee to satisfy these axioms by the preference relation $\preceq_{eA}$. They showed that the preference relation $\preceq_{eA}$ is an equitable rational preference relation if and only if

$$a_1 \geq a_2 \geq \cdots \geq a_m \geq 0,$$

where a_i is the ith column of the matrix A.

The concept of P-equitable rational preference relation has been introduced by Mahmodinejad and Foroutannia [9]. They studied some theoretical and practical aspects of the P-equitably efficient solutions and showed that the set of P-equitably efficient solutions is contained within the set of efficient solutions for the same problem.

Definition 6. Let $M = \{1, 2, \dots, m\}$ be the index set of objective functions $f = (f_1, f_2, \dots, f_m)$ and let n be a positive integer such that $n \leqslant m$. A collection $P = \{P_k \subseteq M : k = 1, 2, \dots, n\}$ is called a decomposition of M, and also it is said a partition of M if $\bigcup_{k=1}^{n} P_k = M$, and $P_i \cap P_j = \emptyset$ for all $i \neq j$, where $i, j \in \{1, 2, \dots, n\}$ and P_k is index set of objective functions in class k.

Definition 7. Rational preference relations satisfying the following axioms are called P-equitable rational preference relations:

1. P-impartiality: $y_{P_k} \simeq y_{\tau_{P_k}}$ for any permutation τ of components of y_{P_k}, $k = 1, \dots, n$.

2. Strict P-transfer principle:

$$y_i > y_j \Rightarrow y - \epsilon e_i + \epsilon e_j \prec y,$$

where $0 < \epsilon < y_i - y_j$ and $i, j \in P_k$ for $k = 1, \dots, n$.

When $n = 1$, that is, $P_1 = \{1, \dots, m\}$, each P-equitable rational preference relation becomes an equitable rational preference relation. For more details on the P-equitable rational preference relation, the reader may refer to [9].

Definition 8. Let $y', y'' \in Y$. We say that y' P-equitably dominates y'', and denote by $y' \prec_{Pe} y''$ if and only if $\bar{\theta}(y'_{P_k}) \leqq \bar{\theta}(y''_{P_k})$ for $k = 1, \dots, n$ and $\bar{\theta}(y'_{P_k}) \leq \bar{\theta}(y''_{P_k})$ for some $k \in \{1, \dots, n\}$. An outcome vector y is P-equitably nondominated if and only if there exist no other outcome vector y' such that y' P-equitably dominates y. Analogously, a feasible solution x is called an P-equitably efficient solution of the multiobjective problem (1) if and only if $y = f(x)$ is P-equitably nondominated.

The set of all P-equitably efficient solutions $x \in X$ is denoted by X_{PE} and called the P-equitably efficient set. The set of all P-equitably nondominated points $y = f(x) \in Y$, where $x \in X_{PE}$, is denoted by Y_{PN} and called the P-equitably nondominated set.

3 The concept of A_P-equitable efficiency

In this section, we suggest an a priori method that is based on the preferences matrix. The idea behind this is that the decision-maker classifies the objective functions in different classes and determines a partition $P = \{P_k \subseteq M : k = 1, 2, \dots, n\}$ of $\{1, 2, \dots, m\}$ according to the importance of objective functions. The decision-maker should give a preferences matrix A_k for objective functions in class P_k for $k = 1, 2, \dots, n$. We introduce the matrix $A_P = A_1 \oplus A_2 \oplus \cdots \oplus A_n$, which is the direct sum of the matrices $A_1, A_2, \dots, A_n$, that is,

$$A_P = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{bmatrix}.$$

The pairwise comparison matrix and its decompositions are one of the ways which the decision-maker can use to provide a preference matrix A_k for objective functions in the class P_k $(k = 1, 2, \dots, n)$. A pairwise comparison matrix is used to compute for relative priorities of objective functions. The entry (i, j) of a pairwise comparison matrix expresses the degree of the preference of the ith objective over the jth objective. For more details, the reader is referred to [5].

By the matrix A_P, the cumulative map $A_P(\theta) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as

$$A_P(\theta(y)) = (A_1(\theta(y_{P_1})), A_2(\theta(y_{P_2})), \dots, A_n(\theta(y_{P_n}))),$$

for $y \in Y$, where $y_{P_k} = (y_j)_{j \in P_k}$ for $k = 1, 2, \dots, n$. Note that A_k is a $|P_k| \times |P_k|$ matrix and $|P_k|$ is the cardinal of the set P_k.

Suppose that $y', y'' \in Y$ are two outcome vectors. Throughout this paper, the following notations is used:

$$A_P(\theta(y')) \leqq A_P(\theta(y'')) \Leftrightarrow A_k(\theta(y'_{P_k})) \leqq A_k(\theta(y''_{P_k})) \qquad (k = 1, 2, \dots, n),$$

$$A_P(\theta(y')) \leq A_P(\theta(y'')) \Leftrightarrow (A_P(\theta(y')) \leqq A_P(\theta(y''))\ \textit{and not}\ A_P(\theta(y'')) \leqq A_P(\theta(y'))),$$

and also

$$A_P(\theta(y')) = A_P(\theta(y'')) \Leftrightarrow (A_P(\theta(y')) \leqq A_P(\theta(y''))\ \textit{and}\ A_P(\theta(y'')) \leqq A_P(\theta(y'))).$$

The following definitions are necessary for the solution concepts of this paper.

Definition 9. Suppose that $y', y'' \in Y$ are two outcome vectors. We say that y' A_P-equitably dominates y'' if and only if $A_P(\theta(y')) \leq A_P(\theta(y''))$, and that is denoted by $y' \prec_{eA_P} y''$. Also we say that y is an A_P-equitably nondominated point if and only if there exit no y' such that $y' \prec_{eA_P} y$. A feasible solution $x \in X$ is an A_P-equitably efficient solution to the multiobjective problem (1) if and only if $y = f(x)$ is an A_P-equitably nondominated point.

The set of all A_P-equitably efficient solutions $x \in X$ is denoted by X_{eA_PE} and called the A_P-equitably efficient set. The set of all A_P-equitably nondominated points is denoted by Y_{eA_PN} and called the A_P-equitably nondominated set.

Note that the relation $\prec_{eA_P}$ becomes the equitable relation when $A_1 = \Delta$ and $P_1 = \{1, 2, \ldots, m\}$. Moreover, if A is an arbitrary matrix and $P_1 = \{1, 2, \ldots, m\}$, then Definition 5 holds. Also for $A_k = \Delta_{|P_k| \times |P_k|}$ $(k = 1, 2, \ldots, n)$, Definition 8 holds.

Similar to the relation of A_P-equitable dominance, we can define the relation of A_P-equitable indifference, $\simeq_{eA_P}$, and the relation of A_P-equitable weak dominance, $\preceq_{eA_P}$. We say that $y' \simeq_{eA_P} y''$ if and only if $A_P(\theta(y')) = A_P(\theta(y''))$, and also that $y' \preceq_{eA_P} y''$ if and only if $A_P(\theta(y')) \leqq A_P(\theta(y''))$.

It is clear that the preference relation $\preceq_{eA_P}$ satisfies the reflexivity, transitivity, and P-impartiality axioms. In continue, we express some conditions that guarantee the relation $\preceq_{eA_P}$ is a P-equitable rational preference relation. Throughout this section, we assume that $e_i^k \in \mathbb{R}^k$ is the unit vector with the ith component equal to one and the remaining ones equal to zero, where $k = 1, 2, \ldots$ and $i \in \{1, 2, \ldots, k\}$.

Theorem 1. The strict monotonicity axiom for the preference $\preceq_{eA_P}$ is equivalent to the condition

$$a_i^k \geq 0 \qquad (i = 1, 2, \ldots, |P_k|), \tag{4}$$

where a_i^k is the ith column of the matrix A_k and $k = 1, 2, \ldots, n$.

Proof. We first prove that if the matrix A_p satisfies condition (4), then the strict monotonicity axiom holds for the preference $\preceq_{eA_P}$. Let $y \in Y$, $i \in \{1, 2, \ldots, m\}$ and $y' = y - \epsilon e_i^m$, for $\epsilon > 0$. We show that $y' \prec_{eA_P} y$, this means that $A_j(\theta(y'_{P_j})) \leqq A_j(\theta(y_{P_j}))$ for $j = 1, 2, \ldots, n$ and $A_j(\theta(y'_{P_j})) \leq A_j(\theta(y_{P_j}))$, for some $j \in \{1, 2, \ldots, n\}$. There exists an index $k \in \{1, 2, \ldots, n\}$

such that $i \in P_k$. For $j \in \{1,2,\ldots,n\} - \{k\}$, we have $y'_{P_j} = y_{P_j}$, so $A_j(\theta(y'_{P_j})) = A_j(\theta(y_{P_j}))$. Since $y'_{P_k} = y_{P_k} - \epsilon e_i^{|P_k|}$, we have $y'_{P_k} \leq y_{P_k}$. Hence

$$\theta(y'_{P_k}) \leq \theta(y_{P_k}).$$

Because $a_j^k \geq 0$ for $j = 1,2,\ldots,|P_k|$, we obtain

$$\sum_{j=1}^{|P_k|} a_{ij}^k \theta_j(y'_{P_k}) \leqslant \sum_{j=1}^{|P_k|} a_{ij}^k \theta_j(y_{P_k}) \qquad (i = 1,2,\ldots,|P_k|),$$

and there is $i' \in \{1,2,\ldots,|P_k|\}$ such that

$$\sum_{j=1}^{|P_k|} a_{i'j}^k \theta_j(y'_{P_k}) < \sum_{j=1}^{|P_k|} a_{i'j}^k \theta_j(y_{P_k}).$$

So, $A_k(\theta(y'_{P_k})) \leq A_k(\theta(y_{P_k}))$ and the proof is complete.

Conversely, suppose that the strict monotonicity axiom holds for the preference $\preceq_{eA_P}$. For any $k \in \{1,2,\ldots,n\}$, we define the vector $y^j \in \mathbb{R}^m$ such that

$$y^j_{P_i} = \begin{cases} e_1^{|P_i|} + \cdots + e_j^{|P_i|} & for\ j \leqslant |P_i| \\ 0 & otherwise, \end{cases}$$

for $j = 1,2,\ldots,\max_{k=1,2,\ldots,n} |P_k|$ and $i = 1,2,\ldots,n$. Let $e \in \mathbb{R}^m$ be defined as $e_{P_k} = e_1^{|P_k|}$, for $k = 1,2,\ldots,n$. The strict monotonicity property implies that

$$y^j - e \prec_{eA_P} y^j, \qquad (j = 1,2,\ldots,\max_{k=1,2,\ldots,n} |P_k|),$$

which concludes that $a_j^k \geq 0$ for $j = 1,2,\ldots,|P_k|$ and $k = 1,2,\ldots,n$. Hence, the matrix A_P fulfills condition (4). $\square$

Remark 1. If $n = 1$ and $P_1 = \{1,2,\ldots,m\}$, then Theorem 3.1 in [3] holds.

To establish the strict P-transfer principle for the preference $\preceq_{eA_P}$, we need the following statement.

Proposition 3. [3, Proposition 3.1] Let $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ be two vectors in $\mathbb{R}^m$ such that

$$\sum_{j=1}^{i} x_j \leq \sum_{j=1}^{i} y_j \qquad (i = 1,2,\ldots,m),$$

where the strict inequality holds at least once. Also let $W = \left[w^1, w^2, \ldots, w^m\right]$ be a matrix $m \times m$ and let w^is be the columns of the matrix W for $i = 1, \ldots, m$. If

$$w^1 \geq w^2 \geq \cdots \geq w^m \geq 0, \tag{5}$$

then

$$\sum_{j=1}^{m} w_{ij} x_j \leq \sum_{j=1}^{m} w_{ij} y_j, \qquad (i = 1, 2, \ldots, m),$$

where the strict inequality holds at least once.

Corollary 1. Let x and y be vectors in $\mathbb{R}^m$. If $w_1 \geq w_2 \geq \cdots \geq w_m \geq 0$ and

$$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} y_i \qquad (n = 1, 2, \ldots, m),$$

then

$$\sum_{i=1}^{m} w_i x_i \leq \sum_{i=1}^{m} w_i y_i.$$

Proof. Let the matrix $W = (w_{ij})$ be defined by $w_{1j} = w_j$ for $j = 1, 2, \ldots, m$ and $w_{ij} = 0$ for $i = 2, \ldots, m$. Using Proposition 3, the proof is obvious. □

Theorem 2. The strict P-transfer principle for the preference $\preceq_{eA_P}$ is equivalent to the following condition:

$$a_1^k \geq a_2^k \geq \cdots \geq a_{|P_k|}^k, \tag{6}$$

where a_i^k's are the columns of the matrix A_k, for $i = 1, 2, \ldots, |P_k|$ and $k = 1, \ldots, n$.

Proof. Let $y \in Y$, $i, j \in P_k$, $y_i > y_j$, and $y' = y - \epsilon e_i^m + \epsilon e_j^m$, where $0 < \epsilon < y_i - y_j$. We show that $y' \prec_{eA_P} y$. This means that $A_l(\theta(y'_{P_l})) \leqq A_l(\theta(y_{P_l}))$ for $l = 1, 2, \ldots, n$ and $A_l(\theta(y'_{P_l})) \leq A_l(\theta(y_{P_l}))$, for some $l \in \{1, 2, \ldots, n\}$. If $l \in \{1, 2, \ldots, n\} - \{k\}$, then $y'_{P_l} = y_{P_l}$, so $A_l(\theta(y'_{P_l})) = A_l(\theta(y_{P_l}))$.

Let $\alpha \in R$ be such that

$$a_1^k + \alpha \geq a_2^k + \alpha \geq \cdots \geq a_{|P_k|}^k + \alpha \geq 0. \tag{7}$$

Since $y'_{P_k} = y_{P_k} - \epsilon e_i^{|P_k|} + \epsilon e_j^{|P_k|}$ and the equitable preference $\preceq_e$ satisfies the strict transfer principle, we have $\bar{\theta}(y'_{P_k}) \leq \bar{\theta}(y_{P_k})$. Hence, by Proposition 3 and (7), we have

$$\sum_{t=1}^{|P_k|} (a_{st}^k + \alpha)\theta_t(y'_{P_k}) \leqslant \sum_{t=1}^{|P_k|} (a_{st}^k + \alpha)\theta_t(y_{P_k}) \qquad (s = 1, 2, \ldots, |P_k|),$$

where the strict inequality holds at least s. On other hand, $\sum_{t=1}^{|P_k|}\theta_t(y'_{P_k}) = \sum_{t=1}^{|P_k|}\theta_t(y_{P_k})$ implies that

$$\sum_{t=1}^{|P_k|} a^k_{st}\theta_t(y'_{P_k}) \leqslant \sum_{t=1}^{|P_k|} a^k_{st}\theta_t(y_{P_k}) \qquad (s = 1, 2, \ldots, |P_k|),$$

where the strict inequality holds at least s. Hence, we have the desired result.

Conversely, suppose that the strict P-transfer axiom holds for the preference $\preceq_{eA_P}$. We define the vector $y^j \in \mathbb{R}^m$ such that

$$y^j_{P_i} = \begin{cases} 2e_1^{|P_i|} + \cdots + 2e_j^{|P_i|} & for\ j \leqslant |P_i| - 1 \\ 0 & otherwise, \end{cases}$$

for $j = 1, 2, \ldots, \max_{i=1,2,\ldots,n} |P_i|$ and $i = 1, 2, \ldots, n$.

Let $e^j \in \mathbb{R}^m$ be defined as $e^j_{P_i} = e_j^{|P_i|}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, \max_{i=1,2,\ldots,n} |P_i|$. The strict P-transfer property implies that

$$y^j - e^j + e^{j+1} \prec_{eA_P} y^j, \qquad (j = 1, 2, \ldots, \max_{i=1,2,\ldots,n} |P_i|),$$

which conclude the desired result. □

Remark 2. If $n = 1$ and $P_1 = \{1, 2, \ldots, m\}$, then Theorem 3.2 in [3] and Corollary 5.11 in [11] hold.

Theorems 1 and 2 imply that the preference relation $\preceq_{eA_P}$ is a P-equitable rational preference relation if and only if the matrix A_P fulfills conditions (4) and (6), that is,

$$a^k_1 \geq a^k_2 \geq \cdots \geq a^k_{|P_k|} \geq 0, \tag{8}$$

for $k = 1, \ldots, n$.

Theorem 3. Suppose that the matrix $A_P = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ satisfies condition (8). If $x \in X$ is an A_P-equitably efficient solution of multiobjective problem (1), then it is a P-equitably efficient solution of multiobjective problem (1). Moreover, $Y_{eA_PN} \subset Y_{PN}$.

Proof. Suppose that x is an A_P-equitably efficient solution to (1). If x is not a P-equitable efficient solution to (1), then a feasible solution x' must exist such that the outcome vectors $y = f(x)$ and $y' = f(x')$ satisfy $y' \prec_P y$, so $\bar{\theta}(y'_{P_k}) \leq \bar{\theta}(y_{P_k})$ for $k = 1, \ldots, n$. Using Proposition 3, we deduce that

$$\sum_{j=1}^{|P_k|} a^k_{ij}\theta_j(y'_{P_k}) \leqslant \sum_{j=1}^{|P_k|} a^k_{ij}\theta_j(y_{P_k}) \qquad (i = 1, 2, \ldots, |P_k|),$$

where the strict inequality holds at least once. Hence $y' \prec_{eA_P} y$, which contradicts the equitable A_P-efficiency of x. □

Remark 3. If $n = 1$ and $P_1 = \{1, 2, \ldots, m\}$, then Theorem 3.3 in [3] holds.

Since the P-equitably efficient set is contained within the efficient set, by applying Theorem 3, we can conclude $X_{eA_PE} \subset X_{PE} \subset X_E$, and hence $Y_{eA_PN} \subset Y_{PN} \subset Y_N$.

In general, the preference relation $\preceq_{eA_P}$ does not satisfy the strict monotonicity and the strict P-transfer axioms. Also condition (8) is necessary in Theorem 3. The truth of these statements is examined by the following example.

Example 1. Let

$$X = Y = \left\{(y_1, y_2) \colon y_1^2 + y_2^2 \leqslant 1 \text{ and } y_2 \geqslant y_1\right\}.$$

If $n = 1$, $A_P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, a $y = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$, and $\epsilon = 1/2$, then $y - \frac{1}{2}e_2 \not\preceq_{eA_P} y$ and $y - \frac{1}{2}e_2 + \frac{1}{2}e_1 \not\preceq_{eA_P} y$. Hence, the preference relation $\preceq_{eA_P}$ does not necessarily satisfy the strict monotonicity and the strict P-transfer axioms. Also, we have

$$Y_N = \left\{(y_1, y_2) \colon y_1^2 + y_2^2 = 1, -1 \leqslant y_1 \leqslant \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \leqslant y_2 \leqslant 0\right\},$$
$$Y_{eAN} = \left\{(y_1, y_2) \colon y_1^2 + y_2^2 = 1, -1 \leqslant y_1 \leqslant 0, 0 \leqslant y_2 \leqslant 1\right\}.$$

Moreover $Y_{PN} = \{(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$. Hence, $Y_{eA_PN} \nsubseteq Y_{PN}$ and $Y_{eA_PN} \nsubseteq Y_N$.

Note that Definition 9 permits one to express A_P-equitable efficiency for problem (1) in terms of the standard efficiency for the multiobjective problem with objectives $A_k(\theta(f_{Pk}(x)))$:

$$\min\{A_P(\theta(f(x))) : x \in X\}. \tag{9}$$

Theorem 4. A feasible solution $x \in X$ is an A_P-equitably efficient solution to the multiobjective problem (1) if and only if it is an efficient solution to the multiobjective problem (9).

Proof. The proof is trivial by Definition 9. □

Remark 4. If $n = 1$ and $P_1 = \{1, 2, \ldots, m\}$, then [11, Corollary 5.3] holds. Also, if $A_k = \Delta_{|P_k| \times |P_k|}$ for all $k = 1, 2, \ldots, n$, then [9, Theorem 3.2] holds.

4 The concept of A_P^∞-equitably efficiency

In this section, we investigate the inclusion relations among A_P^r-equitably efficient set, P-equitably efficient set, and efficient set. Then, we introduce the concept of A_P^{∞}-equitable efficient to generate a subset of efficient solutions, which aims to offer a limited number of representative solutions to the decision-maker.

Let $A_P = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and $B_P = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ be two $m \times m$ matrices. The combined cumulative map $(A_P \circ B_P)(\theta) : \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$(A_P \circ B_P)(\theta(y)) = A_P \left(\theta(B_P(\theta(y)))\right),$$

for $y \in Y$. If $y', y'' \in Y$, using the combined cumulative map, then we can say that y' $(A_P \circ B_P)$-equitably dominates y'' if and only if

$$A_P \left(\theta(B_P(\theta(y')))\right) \leq A_P \left(\theta(B_P(\theta(y'')))\right),$$

and that is denoted by $y' \prec_{e(A_P \circ B_P)} y''$. Also we say that y is an $(A_P \circ B_P)$-equitably nondominated point if and only if there exit no y' such that $y' \prec_{e(A_P \circ B_P)} y$. A feasible solution $x \in X$ is an $(A_P \circ B_P)$-equitably efficient solution to the multiobjective problem (1) if and only if $y = f(x)$ is an $(A_P \circ B_P)$-equitably nondominated point.

In order to make calculations easier, we present a condition on the matrix B_P whereby the vector $B_P(\theta(y))$ is decreasing for every outcome vector $y \in Y$.

Proposition 4. The condition

$$r_{ij}^{B_k} \geqslant r_{(i+1)j}^{B_k} \qquad (j = 1, 2, \ldots, |P_k|), \tag{10}$$

where $r_{ij}^{B_k} = \sum_{t=1}^{j} b_{it}^{k}$ for $i = 1, 2, \ldots, |P_k| - 1$ and $k = 1, 2, \ldots, n$, is equivalent to the statement that $B_k(\theta(y_{P_k}))$ is decreasing for all $y \in \mathbb{R}^m$.

Proof. Put $\theta_{|P_k|+1}(y_{P_k}) = 0$, by the Abel summation

$$\sum_{j=1}^{|P_k|} b_{ij}^{k} \theta_j(y_{P_k}) = \sum_{j=1}^{|P_k|} r_{ij}^{B_k} (\theta_j(y_{P_k}) - \theta_{j+1}(y_{P_k})),$$

we obtain the desired result. □

By the above proposition, we conclude that $\theta(B_P(\theta(y))) = B_P(\theta(y))$ and

$$(A_P \circ B_P)(\theta(y)) = A_P \left(\theta(B_P(\theta(y)))\right) = (A_P B_P)(\theta(y)),$$

where $A_P B_P$ is the product of the matrices A_P and B_p, and also

$$A_P B_P = A_1 B_1 \oplus A_2 B_2 \oplus \cdots \oplus A_n B_n. \tag{11}$$

It follows from what has been said above that the relation $\preceq_{e(A_P \circ B_P)}$ is equivalent to the relation $\preceq_{eA_P B_P}$, when the matrix B_P satisfies condition (10).

In continue, we study the relationship between A_P-equitably efficient solutions and $(A_P \circ B_P)$-equitably efficient solutions. To do this, we require the following statements.

Proposition 5. Let $A = (a_1, a_2, \dots, a_m)$ and $B = (b_1, b_2, \dots, b_m)$ be two $m \times m$ matrices, where a_j and b_j are the jth column of the matrices A and B, respectively. If $D = AB = (d_1, d_2, \dots, d_m)$, where d_j is the jth column of the matrix D, then the following statements hold:

(i) If $a_j \geq 0$ and $b_j \geq 0$ for all $j = 1, 2, \dots, m$, then $d_j \geq 0$ for all $j = 1, 2, \dots, m$.

(ii) If $a_j \geq 0$ for $j = 1, 2, \dots, m$ and $b_j \geq b_{j+1}$ for $j = 1, 2, \dots, m-1$, then $d_j \geq d_{j+1}$ for $j = 1, 2, \dots, m-1$.

(iii) If $r^A_{i,j} = \sum_{k=1}^{j} a_{ik}$ and $r^B_{i,j} = \sum_{k=1}^{j} b_{ik}$ are decreasing with respect to i for all $j = 1, 2, \dots, m$, and also if $r^A_{i,j} \geqslant 0$ for $i, j = 1, 2, \dots, m$, then $r^D_{i,j} = \sum_{k=1}^{j} d_{ik}$ is decreasing with respect to i for all $j = 1, 2, \dots, m$.

Proof. (i) We have

$$d_j = \left(\sum_{k=1}^{m} a_{ik} b_{kj} \right)_{i=1}^{m}.$$

The condition $b_j \geq 0$ implies that $b_{kj} \geqslant 0$ for all $k = 1, 2, \dots, m$ and $b_{k'j} > 0$ for some $k' \in \{1, 2, \dots, m\}$. Also, $a_{k'} \geq 0$ concludes that $a_{ik'} \geqslant 0$ for any $i = 1, 2, \dots, m$ and $a_{i'k'} > 0$ for some $i' \in \{1, 2, \dots, m\}$. Thus $a_{ik} b_{kj} \geqslant 0$ for any $i, k = 1, 2, \dots, m$ and $a_{i'k'} b_{k'j} > 0$, which means that $d_j \geq 0$.

(ii) The condition $b_j \geq b_{j+1}$ implies that $b_{kj} - b_{k(j+1)} \geqslant 0$ for all $k = 1, 2, \dots, m$ and $b_{k'j} - b_{k'(j+1)} > 0$ for some $k' \in \{1, 2, \dots, m\}$. Also, $a_{k'} \geq 0$ concludes that $a_{ik'} \geqslant 0$ for any $i = 1, 2, \dots, m$ and $a_{i'k'} > 0$ for some $i' \in \{1, 2, \dots, m\}$. Thus $a_{ik}(b_{kj} - b_{k(j+1)}) \geqslant 0$ for $i, k = 1, 2, \dots, m$ and $a_{i'k'}(b_{k'j} - b_{k'(j+1)}) > 0$, which means that

$$\sum_{k=1}^{m} a_{ik} b_{kj} \geqslant \sum_{k=1}^{m} a_{ik} b_{k(j+1)} \qquad (i = 1, 2, \dots, m),$$

and the strict inequality holds when $i = i'$. Hence $d_j \geq d_{j+1}$ for $j = 1, 2, \dots, m-1$.

(iii) Since $r^B_{i,j}$ is decreasing with respect to i for $j = 1, 2, \dots, m$, we obtain

$$\sum_{t=1}^{n} b_{it} \geqslant \sum_{t=1}^{n} b_{(i+1)t} \qquad (n = 1, 2, \dots, m).$$

For all $j \in \{1, 2, \dots, m\}$, we set $w_t = \sum_{k=1}^{j} a_{tk} = r^A_{tj}$. Using Corollary 1, we see that

$$\sum_{t=1}^{m}\sum_{k=1}^{j} a_{tk}b_{it} \geqslant \sum_{t=1}^{m}\sum_{k=1}^{j} a_{tk}b_{(i+1)t}.$$

Therefore

$$\sum_{k=1}^{j}\sum_{t=1}^{m} a_{tk}b_{it} \geqslant \sum_{k=1}^{j}\sum_{t=1}^{m} a_{tk}b_{(i+1)t},$$

and

$$\sum_{k=1}^{j} d_{ik} \geqslant \sum_{k=1}^{j} d_{(i+1)k},$$

for all $j = 1, 2, \ldots, m$. This completes the proof of part (*iii*). □

Theorem 5. Let $A_P = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and $B_P = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ be two $m \times m$ matrices. We have the following statements.

(i) If the matrix A_P satisfies condition (4) and the matrix B_P satisfies condition (8), then the matrix $A_P B_P$ fulfills condition (8). Thus, the preference relation $\preceq_{e(A_P B_P)}$ is a P-equitable rational preference relation. Moreover, if the matrix B_P satisfies condition (10), then the preference relation $\preceq_{e(A_P \circ B_P)}$ is a P-equitable rational preference relation.

(ii) If the matrices A_P and B_P satisfy condition (10) and also if the matrix A_P fulfills condition (4), then the matrix $A_P B_P$ satisfies condition (10).

Proof. By using relation (11) and Proposition 5, we obtain the desired results. □

Theorem 6. Let $A_P = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and $B_P = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ be two $m \times m$ matrices. Also let the matrix A_P satisfy condition (4) and the matrix B_P satisfy condition (8). If y' and y'' are two outcome vectors, then

$$y' \prec_{eB_P} y'' \Longrightarrow y' \prec_{e(A_P B_P)} y'',$$
$$y' \preceq_{eB_P} y'' \Longrightarrow y' \preceq_{e(A_P B_P)} y''.$$

Hence $Y_{e(A_P B_P)N} \subset Y_{eB_P N}$, which implies that $X_{e(A_P B_P)E} \subset X_{eB_P E}$. Moreover if the matrix B_P satisfies condition (10), then $Y_{e(A_P \circ B_P)N} \subset Y_{eB_P N}$ and $X_{e(A_P \circ B_P)E} \subset X_{eB_P E}$.

Proof. Let $y', y'' \in Y$ and $y' \prec_{eB_P} y''$. Then $B_k(\theta(y'_{P_k})) \leqq B_k(\theta(y''_{P_k}))$ for $k = 1, 2, \ldots, n$ and $B_{k'}(\theta(y'_{P_{k'}})) \leq B_{k'}(\theta(y''_{P_{k'}}))$ for some $k' \in \{1, 2, \ldots, n\}$. Hence

$$\sum_{j=1}^{|P_k|} b_{ij}^k \theta_j(y'_{P_k}) \leqslant \sum_{j=1}^{|P_k|} b_{ij}^k \theta_j(y''_{P_k}) \qquad (i = 1, 2, \ldots, |P_k| \text{ and } k = 1, 2, \ldots, n),$$

and there exists $i' \in \{1, 2, \ldots, |P_{k'}|\}$ such that

$$\sum_{j=1}^{|P_{k'}|} b_{i'j}^{k'} \theta_j(y'_{P_{k'}}) < \sum_{j=1}^{|P_{k'}|} b_{i'j}^{k'} \theta_j(y''_{P_{k'}}).$$

Now according to condition (4), we have $a_{ti}^k \geqslant 0$ for $i = 1, 2, \ldots, |P_k|$ and $k = 1, 2, \ldots, n$, and there exists $t' \in \{1, 2, \ldots, |P_{k'}|\}$ such that $a_{t'i'}^{k'} > 0$. This implies that

$$\sum_{j=1}^{|P_k|} (A_k B_k)_{tj} \theta_j(y'_{P_k}) \leqslant \sum_{j=1}^{|P_k|} (A_k B_k)_{tj} \theta_j(y''_{P_k})$$
$$(t = 1, 2, \ldots, |P_k| \text{ and } k = 1, 2, \ldots, n),$$

and the strict inequality holds when $k = k'$ and $t = t'$. Therefore $y' \prec_{e(A_P B_P)} y''$. Moreover, suppose that the matrix B_P fulfills condition (10). Since the preference relations $\prec_{e(A_P B_P)}$ and $\prec_{e(A_P \circ B_P)}$ are equivalent, the proof is complete. □

Let $A_P = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ be an $m \times m$ matrix and let $r = 1, 2, \ldots$. The cumulative map $A_P^r(\theta) : \mathbb{R}^m \to \mathbb{R}^m$ is defined as

$$A_P^r(\theta(y)) = (\underbrace{A_P \circ A_P \circ \cdots \circ A_P}_{r-times})(\theta(y)),$$

for $y \in Y$. If conditions (10) and (4) are satisfied by the matrix A_P, then

$$A_P^r(\theta(y)) = (\underbrace{A_P A_P \ldots A_P}_{r-times})(\theta(y)).$$

The following statement states the relationship among A_P^r-equitable efficient solutions, P-equitable efficient solutions, and efficient solutions to multiobjective problem (1).

Corollary 2. Suppose that the matrix A_P satisfies conditions (8) and (10). Then $Y_{eA_P^{r+1}N} \subset Y_{eA_P^r N} \subset Y_{PN} \subset Y_N$. Moreover, $X_{eA_P^{r+1}E} \subset X_{eA_P^r E} \subset X_{PE} \subset X_E$.

Proof. The first inclusion follows by replacing A_P^r instead of B_P, in Theorem 6. Also, by applying Theorem 3, we deduce the second inclusion. □

By using Corollary 2, we conclude the following statement for $P_1 = \{1, 2, \ldots, m\}$.

Corollary 3. Suppose that the matrix A satisfies conditions (8) and (10).Then $Y_{eA^{r+1}N} \subset Y_{eA^rN} \subset Y_{eN} \subset Y_N$. Moreover, $X_{eA^{r+1}E} \subset X_{eA^rE} \subset X_{eE} \subset X_E$.

Condition (8) in the above results are necessary. To investigate this fact, we give the following example.

Example 2. Let Y and A_P be defined as in Example 1. Although condition (10) holds, condition (8) does not hold, and we have

$$Y_{A^{4r}eN} = Y_N = \left\{(y_1, y_2) \colon y_1^2 + y_2^2 = 1, -1 \leqslant y_1 \leqslant \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \leqslant y_2 \leqslant 0\right\},$$
$$Y_{A^{4r+1}eN} = \left\{(y_1, y_2) \colon y_1^2 + y_2^2 = 1, -1 \leqslant y_1 \leqslant 0, 0 \leqslant y_2 \leqslant 1\right\},$$
$$Y_{eA^{4r+2}N} = \left\{(y_1, y_2) \colon y_1^2 + y_2^2 = 1, 0 \leqslant y_1 \leqslant \frac{1}{\sqrt{2}}, 0 \leqslant y_2 \leqslant 1\right\},$$
$$Y_{A^{4r+3}eN} = \left\{(y_1, -y_1) \colon y_1^2 + y_2^2 = 1, y_2 = y_1\right\},$$

for $r = 0, 1, 2, \ldots$. We observe that Corollary 2 does not hold.

According to Corollary 2, we offer an algorithm to compute the A_P^r-equitably efficient solutions to the multiobjective problem (1).

Algorithm 1

Input: Consider the feasible solution X and the objective functions f as in problem (1). Determine a partition $P = \{P_1, P_2, \ldots, P_n\}$ of $\{1, 2, \ldots, m\}$, a matrix A_P, and an integer $r \in \{1, 2, \ldots\}$, according to the decision-maker.
Step 1: Put $X_1 = X$ and $k = 1$.
Step 2: Solve the following multiobjective problem

$$\min\{A_P^k(\theta(f(x))) : x \in X_k\}. \tag{12}$$

Step 3: If $k = r$, stop. Otherwise, put $X_{k+1} = X_{eA_P^kE}$ and $k = k + 1$, go to Step 2.
Output: The set X_r is A_P^r-equitably efficient set.

In the first iteration of Algorithm 1, the A_P-equitably efficient solutions to the multiobjective problem (1) are computed. Then these solutions are gradually reduced in the next iterations. Finally, the A_P^r-equitably efficient solutions are obtained in the last iteration.

In the following example, we investigate Corollary 2 and Algorithm 1 and show that A_P^r-equitably efficient sets are reducing when r is increasing. For this purpose, a large number of random solutions are generated for the scalable test function. From this large set of solutions, efficient solutions, P-equitably efficient solutions, and A_P^r-equitably efficient solutions are calculated for $r = 1, 2, 3$.

Example 3. The test problem considered is the F1 (see [2])

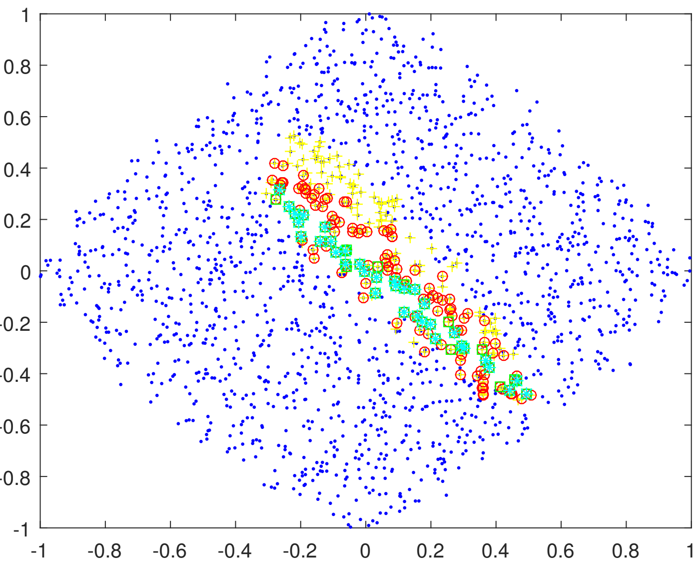

Figure 1: Efficient solutions, P-equitably efficient solutions, and A^r_P-equitably efficient solutions of the $F1$ problem (2 variables and 6 objectives) for $r = 1, 2, 3$.

$$\begin{aligned}
\min_{x\in R^2} \quad & y = \{f_1(x), f_2(x), f_3(x), f_4(x), f_5(x), f_6(x)\}\\
& f_1(x) = x_1^2 + (x_2+1)^2\\
& f_2(x) = (x_1-0.5)^2 + (x_2+0.5)^2\\
& f_3(x) = (x_1-1)^2 + x_2^2\\
& f_4(x) = (x_1+1)^2 + x_2^2\\
& f_5(x) = (x_1-0.5)^2 + (x_2-0.5)^2\\
& f_6(x) = x_1^2 + (x_2-1)^2\\
& x_1, x_2 \in [-1, 1].
\end{aligned}$$

In Figure 1 from 3000 random solutions, 1804 solutions (blue point) are efficient. Let $P_1 = \{1, 2, 3\}$, $P_2 = \{4, 5, 6\}$, and

$$A_1 = \begin{bmatrix} 1 & 0 & 0\\ 0.5 & 0.5 & 0\\ 0.4 & 0.4 & 0.2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 & 0\\ 0.5 & 0.4 & 0\\ 0.4 & 0.3 & 0.2 \end{bmatrix},$$

be given by the decision-maker. We obtain 230 P-equitably efficient solutions, 144 A_P-equitably efficient solutions, 44 A_P^2-equitably efficient solutions, and

34 A_P^3-equitably efficient solutions, which are shown by yellow plus sign, red circles, green square, and cyan star, respectively, in Figure 1.

We assume that the matrix A_P satisfies conditions (8) and (10). Using the results above, we can define infinite order dominance as follows:

$$\prec_{eA_P^\infty} = \bigcup_{r\in\mathbb{N}} \prec_{eA_P^r},$$

where $\mathbb{N} = \{1, 2, \ldots\}$. This means that,

$$y' \prec_{eA_P^\infty} y'' \Leftrightarrow y' \prec_{eA_P^r} y'' \quad (for\ some\ r \in \mathbb{N}).$$

Definition 10. The outcome vector y is A_P^∞-equitably nondominated if and only if there exist no other outcome vector y' such that $y' \prec_{eA_P^\infty} y$. Analogously, a feasible solution x is called an A_P^∞-equitably efficient solution to the multiobjective problem (1) if and only if $y = f(x)$ is A_P^∞-equitably nondominated.

Corollary 4. If the matrix A_P satisfies conditions (8) and (10), then $Y_{eA_P^\infty N} = \bigcap_{r\in\mathbb{N}} Y_{eA_P^r N}$ and $Y_{eA_P^\infty N} \subset Y_{PN} \subset Y_N$. Moreover, $X_{eA_P^\infty E} \subset X_{PE} \subset X_E$.

Proof. By applying Definition 10 and Corollary 2, the proof is trivial. □

Corollary 4 indicates that to reduce Pareto optimal solutions and P-equitably efficient solutions, we can use A_P^∞-equitably efficient solutions.

For $n = 1$ and $P_1 = \{1, 2, \ldots, m\}$, by applying Corollary 4, we conclude the following statement.

Corollary 5. Suppose that the matrix A satisfies conditions (8) and (10). Then $Y_{eA^\infty N} = \bigcap_{r\in\mathbb{N}} Y_{eA^r N}$ and $Y_{eA^\infty N} \subset Y_{eN} \subset Y_N$. Moreover, $X_{eA^\infty E} \subset X_{eE} \subset X_E$.

5 Conclusion

In this paper, we focused on a new concept of rational A_P-equitable efficiency for solving the multiobjective optimization problems, where the preferences matrix A_P is given by the decision-maker. This concept was obtained by rational preference relations on the certain cumulative vector $A_P(\theta(y))$ for $y \in Y$. We examined some conditions that ensure the preference relation $\preceq_{eA_P}$ is a P-equitable rational preference relation. Moreover, we expressed the concept of A_P^r-equitable efficiency to generate a subset of Pareto optimal solutions for $r = 1, 2, \ldots$. Also, we proved that the A_P^r-equitably efficient sets are decreasing with respect to r and that the intersection of these sets is the A_P^∞-equitably efficient set. Furthermore, an experiment was carried out on randomly generated solutions in order to better compare the efficient solutions, the P-equitably efficient solutions, and the A_P^r-equitably efficient solutions. This experiment indicated that the size of the A_P^r-equitably efficient sets is considerably smaller than the size of the efficient set.

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