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A numerical technique based on operational matrices for solving nonlinear integro-differential equations

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Abstract

This paper presents a computational method for solving two types of integro-differential equations, system of nonlinear high order Volterra-Fredholm integro-differential equation(VFIDEs) and nonlinear fractional order integro-differential equations. Our tools for this aims is operational matrices of integration and fractional integration. By this method the given problems reduce to solve a system of algebraic equations. Illustrative examples are included to demonstrate the efficiency and high accuracy of the method.

Keywords: Operational matrix of integration; Volterra-Fredholm; Nonlinear system of integro-differential equations; Fractional order; Legendre wavelet.

1 Introduction

Integro-differential equations frequently appear in all fields of sciences such as physics, chemistry and engineering problems [11, 20, 23, 24]. In last few decades fractional calculus and fractional differential equations have found application in several different disciplines, many important phenomena in electromagnetic, acoustics, viscolasticity, electrochemistry and material science are well described by differentiable and integro differentiable equation of fractional order[3, 22]. There are various numerical and analytical methods to solve such problems, for example, the homotopy perturbation method [4, 7, 8, 9], the Adomian decomposition method [5], fractional differential transform method [21] and Gronwald–Letnikov discretization method [6].

In recent years the approximation of orthogonal functions has been playing role in the solution of different kinds of mathematical and engineering problems such as identification, analysis and optimal control[15, 16, 18]. The main feature of this technique is to reduce the integro-differential equations

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to a nonlinear algebraic equation by introducing integration matrix of basis functions. In present article, we are concerned with the application of Legendre wavelet to the numerical solution of:
(I). Nonlinear fractional order integro-differential equations

$$D_{*t}^{\alpha}u(t) = f(t) + \int_0^x K(t, u(t), D_{*t}^{\alpha}u(t))dt, \qquad 0 \leq \alpha < 1. \tag{1}$$

(II). Nonlinear system of high order (VFIDe) of the form

$$\begin{aligned}\sum_{j=0}^{n} p_{ij}(x){u_l}^{(j)}(x) &= f_i(x) + \lambda_{i1}\int_0^x K_{i1}(x,t,u(t),u'(t),\ldots,u^{(n)}(t))dt \\ &+\lambda_{i2}\int_0^1 K_{i2}(x,t,u(t),u'(t),\ldots,u^{(n)}(t))dt, \quad i=1,\ldots,s,\end{aligned} \tag{2}$$

where $u^{(j)}(x) = \left(u_1^{(j)}, \ldots, u_s^{(j)}\right)$ for $j = 0, \ldots, n$ and initial conditions are

$$u_i^{(j)}(0) = a_j, \quad j = 0, 1 \ldots, n-1, \tag{3}$$

where $f(x)$,K, K_{i1} and K_{i2} are known functions assumed to be in $L^2(\mathbb{R})$ on the interval $0 \leq x, t \leq 1$, u(t) is unknown, K_{i1} and K_{i2} are nonlinear in $x, t, u(t), \ldots, u^{(n)}(t)$. This type of equations whose integrand contain high order derivatives arise in many fields such as theory of elasticity .

The article is organized as follows: in Section 2 we define the Legendre wavelets and operational matrix of integration. Section 3 is devoted to the solution of Eq. (1). In Section 4, we obtain an error bound for our method. Section 5, include our numerical findings and demonstrate the accuracy of the proposed scheme.

2 Preliminaries and notation

This section gives some necessary definition and mathematical preliminaries of the fractional calculus theory which are used further in this paper. The Riemann-Lioville fractional integration of order $\alpha > 0$ is defined as [14]

$$\begin{aligned}&I_t^{\alpha} f(t) = \tfrac{1}{\Gamma(\alpha)}\int_0^t \tfrac{f(\tau)}{(t-\tau)^{(1-\alpha)}}d\tau, \\ &I_t^0 f(t) = f(\tau),\end{aligned} \tag{4}$$

and its fractional derivative of order $\alpha > 0$ is normally used:

$$D_t^{\alpha} f(t) = (\frac{d}{dt})^n I_t^{n-\alpha} f(t) \;\; (n-1 < \alpha \leq n), \tag{5}$$

where n is an integer. For Riemann-Lioville definition, one has

$$I_t^{\alpha} t^{\upsilon} = \frac{\Gamma(\upsilon+1)}{\Gamma(\alpha+\upsilon+1)} t^{\upsilon+\alpha}. \tag{6}$$

The modified fractional differential operator D_{*t}^{α} proposed by Caputo is

$$D_{*t}^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) \ \ (n-1 < \alpha \leq n), \\ \\ \frac{d^n}{dt^n} f(t) \ \ \alpha = n \in \mathbb{N}, \end{cases} \tag{7}$$

where n is an integer. Caputos integral operator has an useful property:

$$I_t^{\alpha} D_{*t}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \qquad (n-1 < \alpha \leq n), \tag{8}$$

where n is an integer.

3 Properties of Legendre wavelet

3.1 Wavelets and Legendre wavelet

Wavelet constitute a family of functions constructed by a single function called the mother wavelet. When the dilation parameter a and translation parameter b vary continuously, we have the following family of continuous wavelet as [10]

$$\psi_{(a,b)} = |a|^{-1/2} \psi(\frac{t-b}{a}), \ \ a, b \in R, a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-m}, b = kb_0 a_0^{-m}, a_0 > 1, b_0 > 0$ and $m, k \in$ Z.We have the following family of discrete wavelets

$$\psi_{m.k}(t) = |a_0|^{m/2} \psi(a_0^m t - kb_0),$$

where $\psi_{m.k}(t)$ forms a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$, $b_0 = 1$, $\psi_{m.k}(t)$ forms an orthonormal basis.

The Legendre wavelets are defined on interval [0,1) see [16, 17].

$$\psi_{nm} = \begin{cases} \sqrt{m+\frac{1}{2}} 2^{k/2} L_m(2^k t - \hat{n}), for \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k}, \\ \\ 0 \quad otherwise, \end{cases}$$

where $m = 0, 1, \dots M-1$ and $n = 1, 2, 3, \dots, 2^{k-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthogonality. Here, $L_m(t)$ are the well-known Legendre polynomials of order m which are defined on the interval [- 1,1] and can be determined with the aid of the following recurrence formulae:

$$L_0(t) = 1, \quad L_1(t) = t,$$

$$L_{m+1}(t) = (\frac{2m+1}{m+1})tL_m(t) - (\frac{m}{m+1})L_{m-1}(t), \; m = 1, 2, 3, \ldots$$

3.2 Function approximation

Theorem. *A function $f(t)$ defined on [0,1) can be expanded as infinite sum of Legendre wavelets, and the series converges uniformly to the function $f(x)$, that is*

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \tag{9}$$

where, $c_{nm} = (f(t), \psi_{nm}(t))$, in which $(.,.)$ denote the inner product.
Proof. see[13].

If the infinite series in Eq. (9) is truncated, then it can be written as

$$f(t) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=1}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t), \tag{10}$$

where C and $\Psi(t)$ are $2^M \times 1$ matrices given by

$$C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M}]^T, \tag{11}$$

$$\Psi(t) = [\psi_{10}, \psi_{11}, \ldots, \psi_{1M-1}, \psi_{20}, \ldots, \psi_{2M-1}, \ldots, \psi_{2^{k-1}0}, \ldots, \psi_{2^{k-1}M}]^T. \tag{12}$$

Now we want to find an upper bound to the estimate error . Suppose that $f(x)$ is a $(m+1)-$times differentiable function on $\Omega = [0, 1)$. An error function between $f(x)$ and its Legendre-wavelet approximation $f_{nm}(x)$ is defined on every subinterval $\Omega_n = [\frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k}]$ as

$$e_{nm}(x) = f(x) - f_{nm}(x) = f(x) - c_{nm}\psi_{nm}(x). \tag{13}$$

Then we can write

$$\| e_{nm}(x)\|^2 = \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} |f(x) - c_{nm}\psi_{nm}(x)|^2. \tag{14}$$

Since $\psi_{nm}(x)$ is a polynomial of degree m,we can use the error bound for interpolation of degree m on Ω_n that is

$$|f(x)-p_n(x)| \leq \frac{h^{(m+1)}}{4(m+1)} \max_{\xi\in[\frac{\hat{n}-1}{2^k},\frac{\hat{n}+1}{2^k}]} \left|f^{(m+1)}(\xi)\right|, \tag{15}$$

where $h=\frac{1}{2^k m}$.By Eq. (14)and Eq. (15)

$$\begin{aligned} \| e_{nm}(x)\|^2 &\leq \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}} \left| \frac{h^{(m+1)}}{4(m+1)} \max_{\xi\in[\frac{\hat{n}-1}{2^k},\frac{\hat{n}+1}{2^k}]} \left|f^{(m+1)}(\xi)\right| \right|^2 \\ &\leq \frac{1}{2^k} \left| \frac{h^{(m+1)}}{4(m+1)} \max_{\xi\in[\frac{\hat{n}-1}{2^k},\frac{\hat{n}+1}{2^k}]} \left|f^{(m+1)}(\xi)\right| \right|^2 \end{aligned}. \tag{16}$$

According to above equation we find an error bound for each subinterval as

$$\| e_{nm}(x)\| \leq \frac{1}{2^{k/2}} \frac{h^{(m+1)}}{4(m+1)} \max_{\xi\in[\frac{\hat{n}-1}{2^k},\frac{\hat{n}+1}{2^k}]} \left|f^{(m+1)}(\xi)\right|. \tag{17}$$

Then for error on Ω we get

$$\| e(x)\| \leq \frac{1}{2^{k/2}} \frac{h^{(m+1)}}{4(m+1)} \max_{\xi\in[0,1]} \left|f^{(m+1)}(\xi)\right|. \tag{18}$$

3.3 The Legendre wavelets operational matrix of integration

The integration of the Vector defined in Eq.(12) can be obtained as

$$\int_0^t \Psi(t')dt' = P\Psi(t), \tag{19}$$

where P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix for integration [18]

$$P = \frac{1}{2^k}\begin{bmatrix} L & H & H & H & \cdots & H \\ 0 & L & H & H & \cdots & H \\ 0 & 0 & L & H & \cdots & H \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & L & H \\ 0 & 0 & 0 & 0 & \cdots & L \end{bmatrix}.$$

H and L are $M \times M$ matrices given by :

$$H = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$L = \begin{bmatrix} 1 & \frac{1}{3^{1/2}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{3^{1/2}}{3} & 0 & \frac{3^{1/2}}{3\times5^{1/2}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{5^{1/2}}{5\times3^{1/2}} & 0 & \frac{5^{1/2}}{5\times7^{1/2}} & & 0 & 0 & 0 \\ 0 & 0 & -\frac{7^{1/2}}{7\times5^{1/2}} & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\frac{(2M-3)^{1/2}}{(2M-3)(2M-5)^{1/2}} & 0 & \frac{(2M-3)^{1/2}}{(2M-3)(2M-1)^{1/2}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{(2M-1)^{1/2}}{(2M-1)(2M-3)^{1/2}} & 0 \end{bmatrix}.$$

3.4 Operational matrix of fractional integration

We defined a m-set of Block Pulse function (BPF)as:

$$b_i(t) = \begin{cases} 1, & i/m \leq t < (i+1)/m \\ 0, & otherwise, \end{cases} \tag{20}$$

where $i = 0, 1, 2, \ldots (m-1)$.

The function $b_i(t)$ are disjoint and orthogonal. That is

$$b_i(t) b_j(t) = \begin{cases} 0, & i \neq j \\ b_i(t), & i = l. \end{cases} \tag{21}$$

The Legendre wavelet may be expanded into m-terms of block pulse function (BPF) as

$$\Psi_m(t) = \Phi_{m\times m} B_m(t), \tag{22}$$

where

$$B_m(t) \triangleq [\, b_0(t)\ b_1(t)\ \ldots\ b_i(t)\ \ldots\ b_{(m-1)}(t)\,]^T. \tag{23}$$

The Block Pulse operational matrix of the fractional integration give in [12] F^α as following:

$$(I_t^\alpha B_m)(t) \approx F^\alpha B_m(t), \tag{24}$$

where

$$F^{\alpha} = \frac{1}{m^{\alpha}} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{(m-1)} \\ 0 & 1 & \xi_1 & \cdots & \xi_{(m-2)} \\ 0 & 0 & 1 & \cdots & \xi_{(m-3)} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$. The Legendre wavelet operational matrix of of fractional integration is defined in [19] as

$$P^{\alpha}_{m\times m} = \Phi_{m\times m} F^{\alpha} \Phi^{-1}_{m\times m}, \tag{25}$$

so the fractional integration of vector in Eq. (12) is defined as

$$(I^{\alpha}_t \Psi)(t) \approx P^{\alpha} \Psi_m(t). \tag{26}$$

4 Application to nonlinear system of VFIDEs

Here, before presenting our method, we prove the next lemma. By this lemma we can approximate the high order derivative of a function by Legendre wavelet.

Lemma. *Suppose that $u(x) = C^T\Psi(x)$ where C and $\Psi(x)$ are defined in Eq. (11) and Eq. (12), then*

$$u^{(k)}(x) = (C^T P^{-k} - \sum_{i=0}^{k-1} u_0^{(i)} E^T P^{i-k})\Psi(x), \tag{27}$$

where P is operational matrix of integration, $u^{(i)}(0) = u_0^{(i)}$ and E is defined as $E^T\Psi(t) = 1$.

Proof. suppose that $f(x) = u^k(x)$ and we approximate $u(x)$ and $f(x)$ by Legender wavelet as

$$\begin{cases} u(x) = C^T\Psi(x), \\ f(x) = F^T\Psi(x), \end{cases} \tag{28}$$

by integrating $f(t)$ on $[0,t]$

$$\begin{aligned} &\textstyle\int_0^t \int_0^t \cdots \int_0^t f(t') \underbrace{dt' \ldots dt'}_{k-times} = \int_0^t \int_0^t \cdots \int_0^t F^T P \Psi(t') \underbrace{dt' \ldots dt'}_{(k-1)times} \\ &\quad = \textstyle\int_0^t \int_0^t \cdots \int_0^t F^T P^2 \Psi(t') \underbrace{dt' \ldots dt'}_{(k-2)times} \\ &\quad \vdots \\ &\quad = F^T P^k \Psi(t), \end{aligned} \tag{29}$$

Since $f(x) = u^k(x)$, then

$$\int_0^t \int_0^t \cdots \int_0^t f(t') \underbrace{dt' \cdots dt'}_{k-times} = \int_0^t \int_0^t \cdots \int_0^t (u^{(k-1)}(t) - u^{(k-1)}(0)) \underbrace{d(t') \cdots dt'}_{(k-1)-times}$$
$$= \int_0^t \int_0^t \cdots \int_0^t u^{(k-1)}(t) \underbrace{dt' \cdots dt'}_{(k-1)-times} u^{(k-1)}(0) \int_0^t \int_0^t \cdots \int_0^t \underbrace{dt' \cdots dt'}_{(k-1)-times}$$
$$\vdots$$
$$= u(t) - u^{(0)}(0) - u^{(1)}(0) - \cdots - u^{(k-1)}(0) \int_0^t \int_0^t \cdots \int_0^t \underbrace{dt' \cdots dt'}_{(k-1)-times}, \tag{30}$$

by $u^{(i)}(0) = u_0^{(i)}$ we get

$$u_0^{(i)} \int_0^t \int_0^t \cdots \int_0^t \underbrace{dt' \cdots dt'}_{k-times} = u_0^{(i)} E^T P^i \Psi(t), \tag{31}$$

Eq. (29)-(31) result

$$F^T P^k \Psi(t) = C\Psi(t) - u_0^{(0)} E^T \Psi(t) - u_0^{(1)} E^T P \Psi(t) - \cdots - u_0^{(k-1)} E^T P^{k-1} \Psi(t)$$
$$= C^T \Psi(t) - \sum_{i=0}^{k-1} u_0^{(i)} E^T P^i \Psi(t), \tag{32}$$

Since the basis functions are linear independent, we omit $\Psi(t)$ from both sides of Eq. (32), then this equation can be written as

$$F^T P^k = C^T - \sum_{i=0}^{k-1} u_0^{(i)} E^T P^i, \tag{33}$$

and then

$$F^T = C^T P^{-k} - \sum_{i=0}^{k-1} u_0^{(i)} E^T P^{i-k}, \tag{34}$$

according to Eq. (28)

$$u^{(k)}(x) = (C^T P^{-k} - \sum_{i=0}^{k-1} u_0^{(i)} E^T P^{i-k}) \Psi(x). \tag{35}$$

This ends the proof of lemma. □

To solve Eq. (2) by Legendre wavelets, we assume that each $u_\ell(x)$ has the expansion as

$$u_\ell(x) = C_\ell^T \Psi(x), \qquad \ell = 1, \ldots, s, \tag{36}$$

by Eq. (35) the derivative expansion is given by

$$y_\ell^{(k)}(x) = (C_\ell^T P^{-k} - \sum_{i=0}^{k-1} y_{\ell 0}^{(i)} E^T P^{i-k})\Psi(x), \qquad \ell = 1,\dots,s, \tag{37}$$

substituting Eq. (36) and Eq. (37) in Eq. (2) results for $\ell = 1,\dots,s$

$$\begin{aligned} p_{\ell 0} C_\ell^T \Psi(x) + \sum_{i=1}^{n} p_{\ell i}(x)(C_\ell^T P^{-i} - \sum_{m=0}^{i-1} u_{\ell 0}^{(m)} E^T P^{m-i})\Psi(x) = f_\ell(x) \\ +\lambda_{\ell 1} \int_0^1 K_{\ell 1}(x,t,C_1^T\Psi(t),\dots,(C^T P^{-n} - \sum_{i=0}^{n-1} u_0^{(i)} E^T P^{i-n})\Psi(x))dt \\ +\lambda_{\ell 2} \int_0^x K_{\ell 2}(x,t,C_1^T\Psi(t),\dots,(C^T P^{-n} - \sum_{i=0}^{n-1} u_0^{(i)} E^T P^{i-n})\Psi(x))dt. \end{aligned} \tag{38}$$

by suitable collocation points, the zeros of Chebyshve polynomials [16]

$$x_i = \cos(\frac{(2i-1)\pi}{2^k M}), \quad i = 1,\dots,2^{k-1}M, \tag{39}$$

we collocate the Eq. (38). In order to use the Gaussian integration formula for Eq. (38), we transfer the t-intervals $[0, x_i]$ and [0,1] into ζ_1 and ζ_2 intervals [-1,1] by

$$\zeta_1 = \frac{2}{x_i} t - 1, \qquad \zeta_2 = 2t - 1. \tag{40}$$

Let

$$\left\{ \begin{aligned} H_{\ell 1}(x_j,t) = K_{\ell 1}(x_j,t,C_1^T\Psi(t),\dots,(C^T P^{-n} - \sum_{i=0}^{n-1} y_0^{(i)} E^T P^{i-n})\Psi(x)), \\ H_{\ell 2}(x_j,t) = K_{\ell 2}(x_j,t,C_1^T\Psi(t),\dots,(C^T P^{-n} - \sum_{i=0}^{n-1} u_0^{(i)} E^T P^{i-n})\Psi(x)), \end{aligned} \right. \quad \ell = 1,\dots,s. \tag{41}$$

We rewrite Eq. (38) as

$$\begin{aligned} p_{\ell 0} C_\ell^T \Psi(x_j) + \sum_{i=1}^{n} p_{\ell i}(x_j)(C_\ell^T P^{-i} - \sum_{m=0}^{i-1} u_{\ell 0}^{(m)} E^T P^{m-i})\Psi(x_j) = f_\ell(x_j) \\ +\lambda_{l1} \tfrac{x_j}{2} \int_{-1}^1 H_{\ell 1}(x_j, \tfrac{x_j}{2}(\zeta_1+1))d\zeta_1 \\ +\tfrac{\lambda_{\ell 2}}{2} \int_{-1}^1 H_{\ell 2}(x_j, \tfrac{1}{2}(\zeta_2+1))d\zeta_2, \qquad \ell = 1,\dots,s, \end{aligned} \tag{42}$$

and with the Gaussian integration

$$\begin{aligned} p_{\ell 0} C_\ell^T \Psi(x_j) + \sum_{i=1}^{n} p_{\ell i}(x_j)(C_\ell^T P^{-i} - \sum_{m=0}^{i-1} u_{\ell 0}^{(m)} E^T P^{m-i})\Psi(x_j) \approx f_\ell(x_j) \\ +\lambda_{\ell 1} \tfrac{x_j}{2} \sum_{h=1}^{s_1} \omega_{1h} H_{\ell 1}(x_j, \tfrac{x_j}{2}(\zeta_{1h}+1)) \\ +\tfrac{\lambda_{\ell 2}}{2} \sum_{h=1}^{s_2} \omega_{2h} H_{\ell 2}(x_j, \tfrac{1}{2}(\zeta_{2h}+1)), \qquad \ell = 1,\dots,s, \end{aligned} \tag{43}$$

where ζ_{1h} and ζ_{2h} are s_1 and s_2 zeros of Legendre polynomials L_{s_1+1} and L_{s_2+1} respectively, and ω_{1h}, ω_{2l} are the corresponding weights. If we assume that

$$\begin{cases} A_\ell(x) = p_{\ell 0} C_\ell^T \Psi(x_j) + \sum\limits_{i=1}^{n} p_{\ell i}(x_j)(C_\ell^T P^{-i} - \sum\limits_{m=0}^{i-1}) u_{\ell 0}^{(m)} E^T P^{m-i}) \Psi(x_j) \\ \quad -\lambda_{\ell 1} \frac{x_j}{2} \sum\limits_{h=1}^{s_1} \omega_{1h} H_{\ell 1}(x_j, \frac{x_j}{2}(\zeta_{1h}+1)) - \frac{\lambda_{\ell 2}}{2} \sum\limits_{h=1}^{s_2} \omega_{2h} H_{\ell 2}(x_j, \frac{1}{2}(\zeta_{2h}+1)), \\ B_\ell(x) = f_\ell(x), \qquad \ell = 1, \ldots, s, \end{cases} \tag{44}$$

Then our problem has the next matrix representation form

$$\begin{pmatrix} A_1(x_1) \\ \vdots \\ A_1(x_{2^{k-1}M}) \\ ----- \\ \vdots \\ ----- \\ A_s(x_1) \\ \vdots \\ A_s(x_{2^{k-1}M}) \end{pmatrix} = \begin{pmatrix} B_1(x_1) \\ \vdots \\ B_1(x_{2^{k-1}M}) \\ ----- \\ \vdots \\ ----- \\ B_s(x_1) \\ \vdots \\ B_s(x_{2^{k-1}M}) \end{pmatrix}$$

This $2^{k-1}Ms \times 2^{k-1}Ms$ nonlinear system of equations which can be solved using Newton iterative method for the elements of C.

5 Application to nonlinear fractional order integro-differential equations

In this section we want to apply the operational matrix of fractional integration to fractional order integro-differential equation. Assume that we approximate $D_{*t}^{\alpha} u(x)$ by Legndre wavelet as

$$D_{*t}^{\alpha} u(x) = K^T \Psi(x), \tag{45}$$

then Eq. (8)and Eq. (26) result

$$u(x) = K^T P_{m\times m}^{\alpha} \Psi(x) + u(0). \tag{46}$$

By Eq. (45)and Eq. (46) we rewrite Eq. (1) as

$$K^T \Psi(x) = f(x) + \int_0^x k(t, K^T P_{m\times m}^{\alpha} \Psi(t) + u(0), K^T \Psi(t)) dt \tag{47}$$

Assume that

$$H(t) = k(t, K^T P^{\alpha}_{m\times m}\Psi(t) + u(0), K^T\Psi(t)), \tag{48}$$

and like the last chapter, we collocated this equation by Eq. (39)in $2^{k-1}M$ points and then use the Gaussian integration. Finally, we can write Eq. (39) as

$$K^T\Psi(x_i) = f(x_i) + \sum_{h=1}^{s_1} \frac{x_i}{2}\omega_{1h} H(\frac{x_i}{2}(\zeta_h + 1)) \quad i = 1, \dots, 2^{k-1}M \tag{49}$$

which is the $2^{k-1}M \times 2^{k-1}M$ nonlinear of system equation which can be solved using Newton iterative method for the elements of C.

6 Numerical examples

In this section we consider some examples which show that operational matrices are powerful and demonstrate the accuracy of our method.

Example 5.1. Consider the nonlinear system of integro-differential equation

$$\begin{cases} 3xu_1(x) + u_1''(x) = 5x^3 + 2u_2'(x) - \int_0^x (u_2'(t) + u_1(t)u_3''(t))dt, + \int_0^1 xu_1'(t)u_2'(t)dt, \\ 2u_2'(x) + u_2''(x) = -4x^2 - xu_1(x) + \int_0^x (txu_2'(t)u_1''(t) + u_3'(t))dt + \int_0^1 x^2u_3(t) + u_2'(t)u_1''(t)dt \\ x/3y_3(x) + u_3''(x) = 2 - \frac{4}{3}x^3 + u_1''^2(x) - 2u_1^2(x) + \int_0^x (x^2u_2(t) + u'^2(t) + t^3u_3''(t))dt + \int_0^1 x^2u_1'(t)dt \\ u_1(0) = u_1'(0) = 0, u_2(0) = 0, u_2'(0) = 1, u_3(0) = u_3(0) = 0, \end{cases} \tag{50}$$

which has the exact solution $u_1(x) = x^2$, $u_2(x) = x$ and $u_3(x) = 3x^2$. Figure.1 show the absolute error when we apply our method for $M = 3$ and $k = 1$.

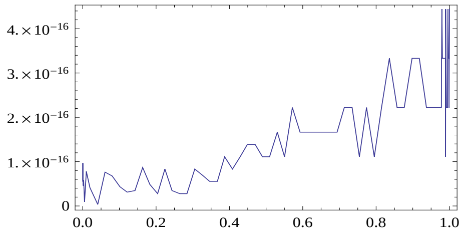
It is clear form figures that our approximate solution is in good agreement with exact one.

Example 5.2. As a second example, consider the nonlinear system given in [2, 1]

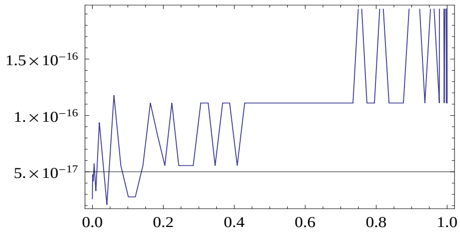
$$\begin{cases} u_1'(x) = 1 - \frac{1}{2}u_2'(x) + \int_0^x ((x-t)u_2(t) + u_1(t)u_2(t))dt, \\ u_2'(x) = 2x + \int_0^x ((x-t)u_1(t) - u_2^2(t) + u_1^2(t))dt, \\ u_1(0) = 0, u_2(0) = 1, \end{cases} \tag{51}$$

which has the exact solution $u_1(x) = sinh(x)$ and $u_2(x) = cosh(x)$ for $M = 6$ and $k = 1$.

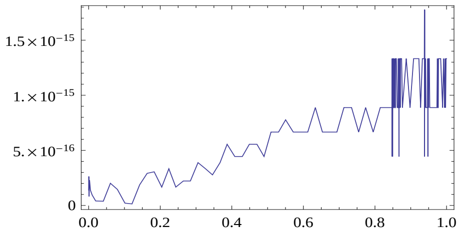
Results for Example 5.2 are reported in Table 1 for $u_1(x_i)$ and $u_2(x_i)$.



(a). Error For $u_1(x)$



(b). Error For $u_2(x)$



(c). Error For $u_3(x)$

Figure 1: Absolute error for Example 5.1 for M=3 And k=1

Example 5.3. Consider the nonlinear fractional order integro differential equation given in[7]

$$D_{*t}^{\alpha}u(t) = 1 + \int_0^x u(t)D_{*t}^{\alpha}u(t)dt \qquad 0 \le x < 10 \le \alpha < 1 \tag{52}$$

The exact solution of this problem for $\alpha = 1$ is $\sqrt{2}Tan(\frac{\sqrt{2}}{2}t)$ we solve this equation for $m = 20$ and different α numerical results are shown in Figure 2.

Example 5.4. Finally Consider the nonlinear fractional order integro differential equations in[7]

$$D_{*t}^{\alpha}u(t) = -1 + \int_0^x u^2(t)dt 0 \le x < 1 \quad 0 \le \alpha < 1 \tag{53}$$

subject to the initial conditions $y(0) = 0$. Table 2 shows the numerical results for $\alpha = 0.8, 0.9, 1$ when $m = 20$. From Table 2 we can see that the approximate solutions obtained by our method are in good agreement with the exact solution for $\alpha = 1$, and with the approximate solutions for $\alpha = 0.8, 0.9$ in [7].

Table 1: Numerical result of Example 5.2

	Error for $M=6, k=1$		Error for Method of [2] $N=5$		Error for Method of [1] $N=6$	
x_i	$u_2(x)$	$u_1(x)$	$u_2(x)$	$u_1(x)$	$u_2(x)$	$u_1(x)$
0	1.70×10^{-6}	7.66×10^{-7}	0	0	0	0
0.1	6.82×10^{-7}	3.42×10^{-7}	1.3×10^{-8}	1×10^{-8}	1.41×10^{-9}	1.41×10^{-9}
0.2	2.85×10^{-7}	8.94×10^{-8}	7.98×10^{-7}	1.33×10^{-7}	9.15×10^{-8}	9.15×10^{-8}
0.3	5.17×10^{-7}	2.50×10^{-7}	9.06×10^{-6}	2.17×10^{-6}	1.06×10^{-6}	1.06×10^{-6}
0.4	1.17×10^{-7}	1.22×10^{-8}	5.06×10^{-5}	1.53×10^{-5}	6.03×10^{-6}	6.03×10^{-6}
0.5	5.43×10^{-7}	2.40×10^{-7}	1.90×10^{-4}	6.64×10^{-5}	2.34×10^{-5}	2.34×10^{-5}
0.6	1.75×10^{-7}	1.08×10^{-7}	5.05×10^{-4}	2.12×10^{-4}	7.08×10^{-5}	7.08×10^{-5}
0.7	4.65×10^{-7}	2.43×10^{-7}	1.36×10^{-3}	5.27×10^{-4}	1.81×10^{-5}	1.81×10^{-5}
0.8	2.68×10^{-7}	4.02×10^{-7}	2.87×10^{-3}	1.05×10^{-3}	4.10×10^{-4}	4.10×10^{-4}
0.9	7.65×10^{-7}	5.10×10^{-7}	5.34×10^{-3}	1.66×10^{-3}	8.45×10^{-4}	8.45×10^{-4}
1	2.85×10^{-6}	1.44×10^{-6}	8.71×10^{-3}	1.17×10^{-3}	1.62×10^{-3}	1.62×10^{-3}

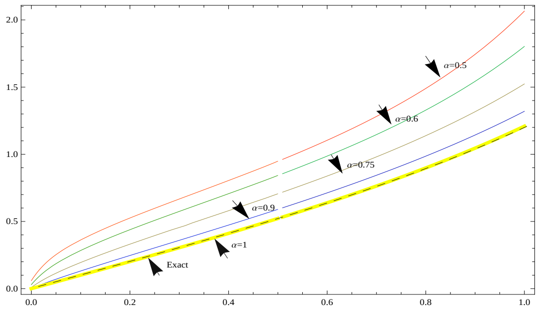


Figure 2: Numerical result for Example 5.3 for different α and m=20

7 Conclusion

Most nonlinear integro-differential equation with nonlinear differential part are usually difficult to solve analytically. In many cases it is required to obtain the approximate solution. We have shown that the properties of operational of matrix of integration and operational matrix of fractional integration together with Legendre wavelet can reduce the system of nonlinear integro-differential equation and nonlinear fractional order integro differential equation to a system of algebraic equations. The advantage of this method is that it can solve high and fractional order integro-differential equation easier and more time efficient. Also we found an error bound. Although we solved our problem by Legender wavelet, other orthogonal basis also can be used. Illustrative examples show the high accuracy of the method in compar-

Table 2: Numerical result of Example 5.4

x_i	Exact solution $\alpha = 1$	Our method			Method of [7]		
		$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$
0	0	0	-0.00013	-0.00046	0	0	0
0.0625	-0.06250	-0.06249	-0.08574	-0.11683	-0.06250	-0.08574	-0.11682
0.125	-0.12498	-0.124977	-0.15995	-0.20327	-0.12498	-0.15997	-0.20328
0.1875	-0.18740	-0.18749	-0.23023	-0.28080	-0.18740	-0.23024	-0.28082
0.2500	-0.24968	-0.24966	-0.29788	-0.35269	-0.24968	-0.29790	-0.35272
0.3125	-0.31171	-0.31172	-0.36339	-0.42026	-0.31171	-0.36342	-0.42039
0.3750	-0.37336	-0.37333	-0.42695	-0.48409	-0.37336	-0.42689	-0.48413
0.4375	-0.43446	-0.43443	-0.48858	-0.54446	-0.43446	-0.48861	-0.54451
0.5000	-0.49482	-0.49478	-0.54818	-0.60140	-0.49482	-0.54824	-0.60150
0.5625	-0.55423	-0.55418	-0.60565	-0.65501	-0.55423	-0.60571	-0.65510
0.6250	-0.61243	-0.61237	-0.66078	-0.70511	-0.61243	-0.66086	-0.70521
0.6875	-0.66917	-0.66910	-0.71337	-0.75162	-0.66917	-0.71345	-0.75172
0.7500	-0.72415	-0.72418	-0.76318	-0.79440	-0.72415	-0.76327	-0.79451
0.8125	-0.77710	-0.77710	-0.80997	-0.83330	-0.77710	-0.81006	-0.83341
0.8750	-0.82767	-0.82771	-0.85348	-0.86820	-0.82767	-0.85395	-0.86831
0.9375	-0.87557	-0.87564	-0.89349	-0.89896	-0.87557	-0.89361	-0.89908

ison with other methods. This procedure can also be used for solving other functional equations such as ordinary and partial differential equations.

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