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Research Article

A robust method for optimal control problems governed by system of Fredholm integral equations in mechanics

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Abstract

This essay considers an optimal control problem (**OCP**) governed by a system of Fredholm integral equations (**FIE**). In this paper, collocation approach with utilizing Lagrange polynomials is introduced to transform the OCP into a nonlinear programming problem (**NLP**). An efficient optimization method in Mathematica software is utilized to solve NLP. The convergence analysis is discussed, which show the theoretical structure behind the propounded technique under some assumptions. In this essay, computational outcomes are given to demonstrate the adaptability, forthrightness, and relationship of the calculations manufactured. A practical real-world problem involving hanging chain in classical mechanic is also dissolved utilizing the approach proposed.

AMS subject classifications (2020): 49M25, 90C30.

Keywords: Collocation method; Lagrange polynomials; Optimal control; System of Fredholm integral equation; Convergence analysis.

1 Introduction

The mathematical formulations of most physical and mechanical engineering models lead to FIE. FIEs occur in different real world models like the mass dispensation of polymers in polymeric melt, linear forward modeling, signal processes etc [7, 26]. The OCPs inclusive integral equations are a different class of OCPs that have a substantial role in population dynamics, economy, heat conduction, continuum mechanics, electrical power centers, water

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resource management, and mass transport [13, 19]. In particular, optimal control of systems managed by FIE is significant in usages like the OCP regarding the Ornstein-Uhlenbeck process which arises from statistical communication theory [18]. In recent years, many methods for solving OCPs of FIE, whether direct or indirect, have been proposed. One can read about this subject in references [2, 1, 8, 10, 16, 20, 24].

Problem$\mathcal{B}$: Determine the optimal control vector u^* and the corresponding optimal state vector x^* for a category of OCPs governed by a system of nonlinear FIEs described as follows:

$$\min\ J(x,u) = \int_0^T F(t, x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t))dt, \tag{1}$$

subject to

$$x_j(t) = y_j(t) + \int_0^T K_j(t, s, x_1(s), \cdots, x_n(s), u_1(s), \cdots, u_m(s))ds, \quad j = 1, \cdots, n, \tag{2}$$

where $x = [x_1, x_2, \cdots, x_n]^T \in \Re^n$, $u = [u_1, u_2, \cdots, u_m]^T \in \Re^m$, $F \in C([0,T], \Re^n, \Re^m)$ and $K_j \in C([0,T], [0,T], \Re^n, \Re^m)$.

Many researchers have been solved and carefully examined non-linear system of FIEs by utilizing orthogonal basis functions. For example, see the references [3, 4, 21, 22]. The orthogonal nature causes the solution of the preliminary problem to transmute into an algebraic system of equations. More ever, by utilizing an operational matrix of integral or derivative of these functions, the computational convolution of the problem is substantially diminished.

For dissolving the mentioned problem, the Lagrange polynomials and Gauss–-Legendre (**GL**) quadrature rule are used. A robust direct approach is established to transform the problem $\mathcal{B}$ into a NLP. This method is based on the expansion of control and state functions with Lagrange polynomials together with the GL integration method. We have utilized Wolfram Mathematica 12 for obtaining the solution of the NLP. Convergence analysis and related theorems are discussed in details. The reliability of the approach is investigated numerically by dissolving some illustrative examples. The OCP, which modeled the problem of classical mechanics for power lines are also described and solved by propounded approach.

The remaining structure of our article is given as follows: In Section 2, by using approximation of control and state variables, we obtain a NLP corresponding to problem $\mathcal{B}$ and also discuss about the method of solving the resulted NLP. Convergence analysis is given in Section 3. In Section 4, we demonstrate the efficiency of the propounded method to several examples and show the computational results of the method. Finally, in Section 5, concluding comments and future extensions are summarized.

2 Proposed method

Consider the problem $\mathcal{B}$ in equations (1) and (2). In this section, by utilizing the properties of Lagrange polynomials and GL integration, the considered OCP is changed to a NLP, then the function NMinimze in Wolfram Mathematica software is used to solve the resulted NLP.

2.1 Discretization of problem $\mathcal{B}$

Firstly, we approximate the control function $u(t)$ and corresponding state function $x(t)$ over the interval $[0,T]$ by the following linear combination of the Lagrange polynomials

$$x_j(t) \approx x_{j\,N}(t) = \sum_{i=0}^{N} x_{j\,i} Q_i(t) = X_j{}^T Q(t), \qquad j = 1, \cdots, n, \tag{3}$$

and

$$u_k(t) \approx u_{k\,N}(t) = \sum_{i=0}^{N} u_{k\,i} Q_i(t) = U_k{}^T Q(t), \qquad k = 1, \cdots, m, \tag{4}$$

where $Q(t) = [Q_0(t), Q_1(t), \cdots, Q_N(t)]^T$, $X_j = [x_{j0}, x_{j1}, x_{j2}, \cdots, x_{jN}]^T$ and $U_k = [u_{k0}, u_{k1}, u_{k2}, \cdots, u_{kN}]^T$, in which x_{ji} and u_{ki} for $i = 0, 1, \cdots, N$, $j = 1, \cdots, n$ and $k = 1, \cdots, m$ are the unknown coefficients and, $Q_i(t)$ are the Lagrange polynomials which are defined as follows:

$$Q_i(t) = \prod_{\substack{0 \le s \le N \\ s \ne i}} \frac{t - t_s}{t_i - t_s} = \frac{w(t)}{(t - t_i) w'(t_i)}, \tag{5}$$

where $w(t) = \prod_{s=0}^{N} (t - t_s)$ and t_i, for $i = 0, 1, 2, \cdots, N$, are the zeros of Legendre polynomials [11] transformed to the interval $[0,T]$. The transfer is done according to the interval in which the problem is defined. By discritizing the FIE given in Eq (2) by using (3) and (4) and collocation nodes t_q ,we obtain for $j = 1, \cdots, n$ and $q = 0, \cdots, N$

$$x_{jN}(t_q) = y(t_q) + \int_0^T K_j(t_q, s, x_{1N}(s), \cdots, x_{nN}(s), u_{1N}(s), \cdots, u_{mN}(s)) ds. \tag{6}$$

For using GL quadrature formula, a simple linear transmutation must be produced with the following form

$$s = \frac{T}{2}(\mu + 1), \tag{7}$$

Then, the dynamical system (2) is converted to

$$x_{jN}(t_q) = y(t_q) + \frac{T}{2}\int_{-1}^{1} K_j(t_q, \frac{T}{2}(\mu+1), x_{1N}(\frac{T}{2}(\mu+1)), \cdots, x_{nN}(\frac{T}{2}(\mu+1)),$$
$$u_{1N}(\frac{T}{2}(\mu+1)), \cdots, u_{mN}(\frac{T}{2}(\mu+1)))d\mu. \tag{8}$$

We apply the GL quadrature for approximating the integral given in Eq (8)

$$x_j(t_q) = y(t_q) + \frac{T}{2}\sum_{l=0}^{N} w_l K_j(t_q, \frac{T}{2}(\mu_l+1), x_{1N}(\frac{T}{2}(\mu_l+1)), \cdots, x_{nN}(\frac{T}{2}(\mu_l+1)),$$
$$u_{1N}(\frac{T}{2}(\mu_l+1)), \cdots, u_{mN}(\frac{T}{2}(\mu_l+1))), \tag{9}$$

where w_l is defined as follows:

$$w_l = \frac{2}{(1-(\mu_l)^2)(P'_{N+1}(\mu_l))},$$

in which P_{N+1} is the Legendre polynomial of order $N+1$ [11]. By using GL quadrature and equations (3) and (4), we can approximate the cost functional in equation (1) as follows:

$$J(x,u) = \int_0^T F\left(t, x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)\right) dt \approx \frac{T}{2} \tag{10}$$

$$\int_{-1}^{1} F\left(\frac{T}{2}(\mu+1), x_{1N}\left(\frac{T}{2}(\mu+1)\right), \cdots, x_{nN}\left(\frac{T}{2}(\mu+1)\right), u_{1N}\left(\frac{T}{2}(\mu+1)\right)\right.$$
$$\left., \cdots, u_{mN}\left(\frac{T}{2}(\mu+1)\right)\right) d\mu \approx \frac{T}{2}\sum_{l=0}^{N} w_l F\left(\frac{T}{2}(\mu_l+1), x_{1N}\left(\frac{T}{2}(\mu_l+1)\right),\right.$$
$$\left., \cdots, x_{nN}\left(\frac{T}{2}(\mu_l+1)\right), u_{1N}\left(\frac{T}{2}(\mu_l+1)\right), \cdots, u_{mN}\left(\frac{T}{2}(\mu_l+1)\right)\right).$$

Eventually, the OCP governed by a system of FIE in problem $\mathcal{B}$ has been reduced to a NLP by (10) as objective functional and (9) as constraints. We call this NLP as problem $\mathcal{B}_{\mathcal{N}}$.

2.2 Optimization algorithm

In this paper, for solving NLP, which we call as problem $\mathcal{B}_{\mathcal{N}}$, given in equations (9) and (10), the **Nminimize** function in Wolfram Mathematica has been utilized. **Nminimize** find global optimum numerically. This function implement several algorithms for solving optimization problems according to the kind of optimization problem. Now by solving this NLP and determining the vectors X_j and U_k for $j = 1, \cdots, n$ and $k = 1, \cdots, m$, we can find a numerical solution of problem $\mathcal{B}$ given in equations (1) and (2).

3 Convergence analysis

We suppose that the finite set, $\mathcal{P}_N$, consists of all Lagrange polynomials from degree less than or equal to N. The set of $\mathcal{P}_N$ is linearly independent and a basis of the vector space for polynomials of degree at most equal to N. If $\mathcal{F}(t)$ is a function in $L^2[0,T]$, $\mathcal{F}(t)$ has a best estimation $\mathcal{F}_0 \in \mathcal{P}_N$, so we possess [17]

$$\forall\ g \in \mathcal{L}^2[0,T]: \quad ||\mathcal{F} - \mathcal{F}_0||_2 \leq ||\mathcal{F} - g||_2. \tag{11}$$

Suppose $\mathcal{S}_n \in \mathcal{P}_N$, then there is coefficients s_l, $l = 0, 1, \cdots, N$, whereas

$$\mathcal{S}_n = \sum_{i=0}^{N} s_l \mathcal{Q}_l(t) = \mathcal{S}^T \mathcal{Q}(t), \tag{12}$$

where $S = [s_0, s_1, \cdots, s_N]$, and $\mathcal{Q}(t) = [Q_0(t), Q_1(t), \cdots, Q_N(t)]$.

Theorem 1. Let $\mathcal{F} \in L^2[0,1]$ be estimated by Lagrange polynomials $\{\mathcal{Q}_i\}_{i=0}^N$ that is, $\mathcal{F}_N := \sum_{i=0}^{N} C_i \mathcal{Q}_i(t)$. If e_N is absolutely error of $\mathcal{F}_N$, then $||\mathcal{F} - \mathcal{F}_N||_2 \leq \frac{L}{(N+2)!}$ and $\lim_{N\to\infty} e_N = 0$ and by increasing N; the errors quickly tends to zero. L is a constant.

Proof. The proof is given in [23]. □

Theorem 1 indicates the accuracy estimation of Lagrange polynomials.

Definition 1. The vectors $x(t) \in \Re^n$ and $u(t) \in \Re^m$ are called admissible, if they satisfy in equation (2). The set of admissible pairs $\{(x,u)\}$, for $j = 1, \cdots, n$, is defined as follows:

$$\begin{aligned}\psi = \{(x(t), u(t)) \Big| x_j(t) = y_j(t) + \int_0^T K_j(t, s, x_1(s), \cdots, \\ x_n(s), u_1(s), \cdots, u_m(s)) ds\}.\end{aligned} \tag{13}$$

Define U as the admissible control functions. Let

$$x_N = (x_{1N}, x_{2N}, \cdots, x_{nN}), \qquad u_N = (u_{1N}, u_{2N}, \cdots, u_{mN}), \tag{14}$$

then we have

$$x_{jN}(t) = X_j^T \mathcal{Q}(t), \quad j = 1, \cdots, n; \qquad u_{iN}(t) = U_i^T \mathcal{Q}(t), \quad i = 1, \cdots, m, \quad (15)$$

and

$$\psi_N = \Big\{ (x_N(t), u_N(t)) \Big| x_{jN}(t) = y(t) + \int_0^T K_j(t, s, x_{1N}(s), \cdots, x_{nN}(s), u_{1N}(s), \cdots, u_{mN}(s)) ds \Big\}. \quad (16)$$

The proof of the following theorem is given by utilizing of the same argument in [25], but some revisions are given according to the problem of our paper.

Theorem 2. Assume $J_i^* = \inf_{\psi_i} J$ and $J^* = \inf_{\psi} J$ and J^* is finite and unique, then the following relation is hold.

$$J_1^* \geq J_2^* \geq \cdots J_r^* \geq \cdots \geq J^* = inf_{\psi} J(x, t). \quad (17)$$

Proof. From the definition of ψ_i, the following relation is hold

$$\psi_1 \subset \psi_2 \subset \psi_3 \cdots \subset \psi_N \subset \cdots \subset \psi. \quad (18)$$

According to (18), $\{J_i^*\}$ is a non-increasing sequence that is bounded from below, so it is convergent to J^*. □

Remark 1. The computational distance can be transmuted from $[0, T]$ to any ideal interval via an affine alteration.

Lemma 1. Suppose that t_i and w_i for $1 \leq i \leq N$ are GL nodes and weights respectively, Let $f(t)$ be Rimann integrable. Then [9]

$$\int_{-1}^{1} f(t) dt = \lim_{N \to \infty} \sum_{i=1}^{N} f(t_i) w_i. \quad (19)$$

Theorem 3. Assume that $x^* \in \Re^n$ is exact state solution of equation (2) with $u^* \in \Re^m$ as exact control functions, let x_N^* and u_N^* be numerical solutions of

$$x_{jN}(t_q) = y(t_q) + \int_0^T K(t_q, s, x_{1N}(s) \cdots, x_{nN}(s), u_{1N}, \cdots, u_{mN})) ds. \quad (20)$$

Also assume that the function $K(t, s, x_1(s), \cdots, x_n(s), u_1(s), \cdots, u_m(s))$ satisfies Lipschitz condition with respect to variables x and u as follows:

$$\begin{aligned} &||K(t, s, x_1(s), \cdots, x_n(s), u_1(s), \cdots, u_m(s)) \\ &\quad - K(t, s, x'_1(s), \cdots, x'_n(s), u'_1(s), \cdots, u'_m(s))||_2 \\ &\qquad \leq L_K \{ ||x_1(s) - x'_1(s)||_2 + \cdots + ||x_n(s) - x'_n(s)||_2 + ||u_1(s) - u'_1(s)||_2 \end{aligned}$$

$$+\cdots+||u_n(s)-{u'}_m(s)||_2\}, \tag{21}$$

where L_K is Lipschitz constant. Then we have:

$$||x^*(t_p)-x^*_N(t_p)||_2 \leq \left(\frac{\mathcal{A}}{(N+1)!}\right), \tag{22}$$

where A is a constant which is non-dependent to N.

Proof. We have for $j=1,\cdots,n$,

$$\begin{aligned}
||x^*_j(t)-x^*_{jN}(t)|| \leq{}& || \int_0^T K\left(t,s,x^*_1(s),\cdots,x^*_n(t),u^*_1(s),\cdots,u^*_m(s)\right)ds \\
&- \int_0^T K\left(t,s,x^*_{1N}(s),\cdots,x^*_{nN}(s),u^*_{1N}(s),\cdots,u^*_{mN}(s)\right)ds||_2 \\
\leq{}& \int_0^T ||K\left(t,s,x^*_1(s),\cdots,x^*_n(t),\ u^*_1(s),\cdots,u^*_m(s)\right)ds \\
&-K\left(t,s,x^*_{1N}(s),\ \cdots,x^*_{nN}(s),u^*_{1N}(s),\cdots,u^*_{mN}(s)\right)ds||_2 \\
\leq{}& L_K\int_0^T ||x^*_1(s)-x^*_{1N}(s)||_2+\cdots+||x^*_n(s)-x^*_{nN}(s)||_2 \\
&+||u^*_1(s)-u^*_{1N}(s)||_2+\cdots+, ||u^*_m(s)-u^*_{mN}(s)||_2 ds \\
={}& L_K\sum_{i=1}^n\int_0^T ||x^*_i(s)-x^*_{iN}(s)||_2 ds \\
&+L_K\sum_{i=1}^m\int_0^T ||u^*_i(s)-u^*_{iN}(s)||_2 ds
\end{aligned} \tag{23}$$

$$\leq L_K T\sum_{i=1}^n ||x^*_i(s)-x^*_{iN}(s)||_2 + L_K T\sum_{i=1}^m ||u^*_i(s)-u^*_{iN}(s)||_2 \tag{24}$$

$$\begin{aligned}
&\leq 2L_K T\left(\frac{\sum_{i=1}^n L_i}{(N+1)!}\right)+2L_K T\left(\frac{\sum_{i=1}^m L'_i}{(N+1)!}\right) \\
&\leq \frac{2L_K T}{(N+1)!}\left(\sum_{i=1}^n L_i+\sum_{i=1}^m L'_i\right)\leq \frac{\mathcal{A}}{(N+1)!}.
\end{aligned} \tag{25}$$

In (25) $\mathcal{A} = 2L_K T\left(\sum_{i=1}^n L_i+\sum_{i=1}^m L'_i\right)$, so by increasing N for $j = 1,2,\cdots,n$, $||x^*_j(t)-x^*_{jN}(t)||$ goes to zero. □

4 Numerical experiments

In this section, the proficiency of the proposed technique is shown by applying to some given numerical examples of OCP for systems governed by FIE.

Example 1. Observe the following OCP

$$\min \ J(x(t), u(t)) = \int_0^1 (x(t) - 0.8182 - 2.7273t^2)^2 + (u(t) - t^2)^2 dt, \quad (26)$$

subject to

$$x(t) = t^2 + \int_0^1 (t^2 + u(s))x(s)ds, \quad (27)$$

with the exact solutions, $u^*(t) = t^2$ and $x^*(t) = 0.8182 + 2.7273t^2$.

By exerting the propounding method, the approximate solutions are approximately $x^*(t) = 0.8182 - 8.78094 * 10^{-15}t + 2.7273t^2$ and $u^*(t) = 1.47439 * 10^-6 + 1.t^2$. The optimal value of J^* for $N = 2$ is $1.20133 * 10^{-11}$.

Example 2. Investigate the following minimization problem

$$J(x_1(t), x_2(t), u(t)) = \int_{-1}^1 (x_1(t) - t^2 + 1)^2 + (x_2(t) - 1 + t^2)^2 + (u(t) - \sin(t))^2 dt, \quad (28)$$

subject to FIE system

$$x_1(t) = t^2 - 1 + \int_{-1}^1 t^3 u(s) \left(x_1(s) + x_2(s)\right) ds,$$
$$x_2(t) = 1 - t^2 + \int_{-1}^1 (s - u(s))(x_1(s) + x_2(s))ds.$$

The precise optimal solutions of this OCP are $x_1^*(t) = t^2 - 1$, $x_2^*(t) = 1 - t^2$ and $u^*(t) = \sin(t)$.

After dissolving with the propounded approach, we obtain the following result

$$x_1{}^*(t) = -1 + 5.55112 \times 10^{-17}t + t^2,$$
$$x_2{}^*(t) = 1 + 5.55112 \times 10^{-17} - 1t^2$$
$$u^*(t) = -1.50469 \times 10^{-24} + 0.902957t - 1.11022 \times 10^{-16}t^2,$$

which approximates the exact functions with good approximation. The Maclaurin series of sine function is $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$. As we can see in the approximate control function $u^*(t) = -1.50469 \times 10^{-24} + 0.902957t - 1.11022 \times 10^{-16}t^2$, the constant sentence and the coefficient of sentence includes x^2 is approximately equal to zero. The optimal value of objective function for different value of N is given in Table 1. The numerical results of this example with sinc wavelet for $k = 5$ and $N = 10, \quad 20$ given in [16] are reported in Table 1, which is comparable with our method. The value of the objective function for $N = 2$ is more accurate than its value for $N = 3$, because for $N = 2$, the quadratic functions of the state are approximated with better accuracy. The plot of approximate function $u^*(t)$ for $N = 3$ is given in Figure 1. The plots of the absolute error of state functions x_1 and x_2 for $N = 3$ are demonstrated in Figures 2 and 3.

Table 1: Optimal value of J^* for Example 2

N	2	3
J^*	8.55969×10^{-33}	1.07179×10^{-31}
N [16]	10	20
J^*	6.88×10^{-12}	6.88×10^{-12}

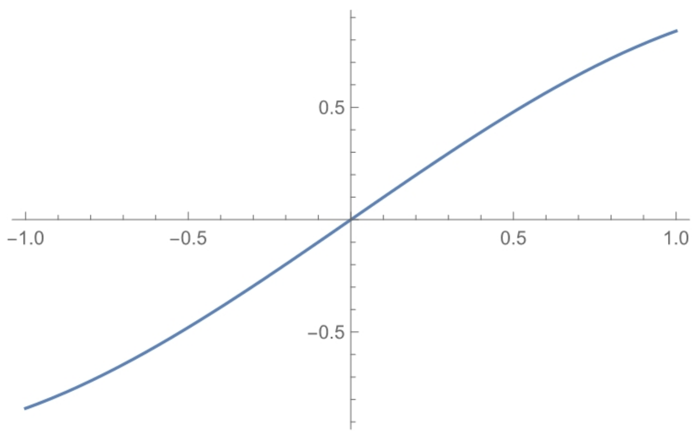


Figure 1: The approximate function of u^* in Example 2

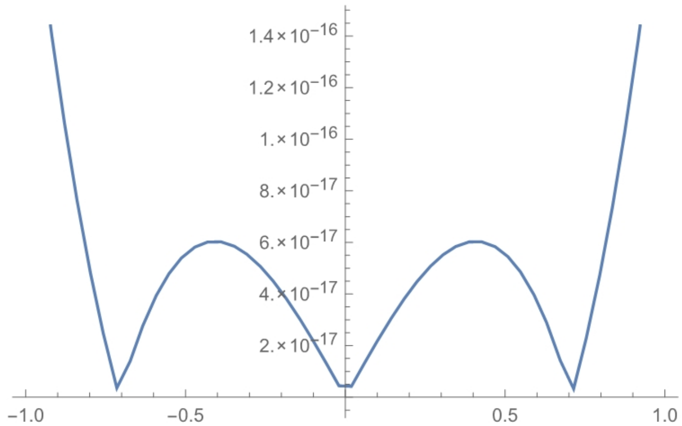


Figure 2: The absolute error of x_1^* in Example 2

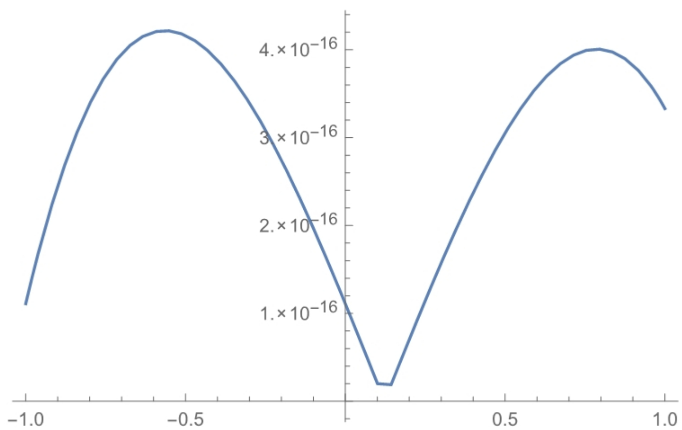


Figure 3: The absolute error of x_2^* in Example 2

Example 3. Consider the following OCP governed by FIE systems

$$\min \int_0^1 (x_1(t)-t)+\left(x_2(t)-t^2\right)+\left(u(t)-e^{-t}\right)$$

subject to

$$x_1(t)=t-\frac{e^{-t}}{4}+\int_0^1 u(t)x_1(s)x_2(s)ds,$$

$$x_2(t)=t^2-\frac{e^{-t}}{5}+\int_0^1 u(t){x_1}^2(s)x_2(s)ds$$

The exact solution of this OCP is ${x_1}^*(t)=t$, ${x_2}^*(t)=t^2$ and $u^*(t)=e^{-t}$.

The approximate and exact optimal control and state functions for $N = 2$ are shown in Figure 4, 5 and 6, respectively. The optimal values of J^* for different values of N are given in the Table 2.

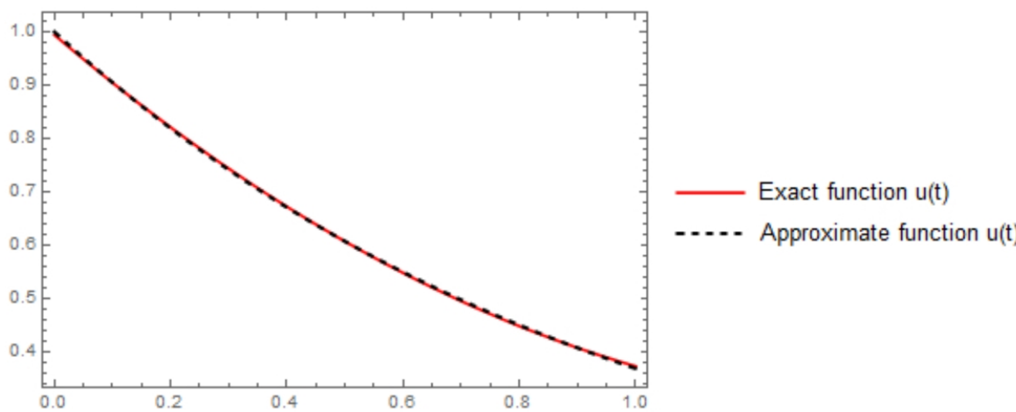


Figure 4: The exact and approximate optimal control for $N = 2$ in Example 3

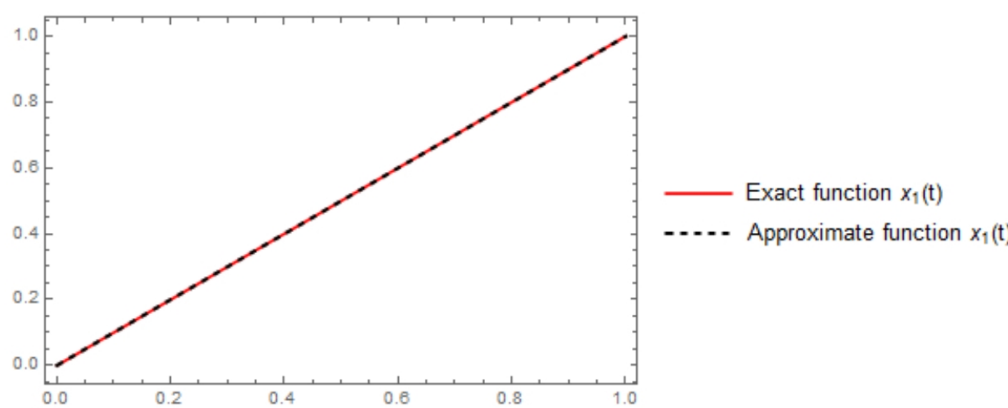


Figure 5: The exact and approximate optimal state $x_1(t)$ for $N = 2$ in Example 3

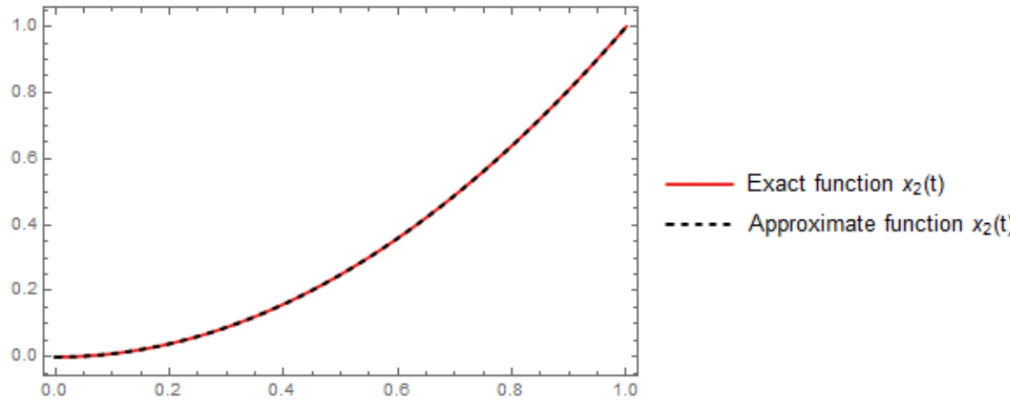


Figure 6: The exact and approximate optimal state $x_2(t)$ for $N = 2$ in Example 3

Table 2: Optimal value of J^* for Example 3

N	2	3	4
J^*	1.10430×10^{-7}	5.40995×10^{-10}	1.11407×10^{-12}

Example 4 (Hanging chain)**.** In this example, we investigate a problem from classical mechanics, which appears in power lines. If a flexible chain is hung from two points, it forms a U shape that is called a catenary curve. The resulting shape of the chain depends on the mass distribution along the chain. Consider the following FIE from [14],

$$x(s) = g \int_0^L G(s,t)\phi(t)dt, \tag{29}$$

Here, $x(s)$ and $\phi(s)$ are demonstrated the displacement of the chain and the mass density of the chain respectively. g is the gravitational constant and

$$G(s,t) = \begin{cases} \frac{s(L-t)}{T_0 L} & 0 \leq s \leq t \\ \frac{t(L-s)}{T_0 L} & t \leq s \leq L \end{cases} \tag{30}$$

In (30), T_0 is the constant tension. In this example, we are looking for the mass density distribution as the control function $\phi(t)$ that leads to a prescribed shape $x(t)$ as its corresponding state. Therefore, the OCP problem is given as the minimization of the cost functional

$$J(x,u) = \int_0^L \left(\frac{g(tL^2 - t^3)}{6T_0} - x(t) \right)^2 + (\phi(x) - x)^2 \, dx \tag{31}$$

subject to equation (29). The optimal control and state solutions are $\phi^*(x) = x$ and $x^*(t) = \frac{g(tL^2-t^3)}{6T_0}$. In this example, we consider the following parameters as $L = 1$, $g = 9.8$, $T0 = 1$. By applying the proposed algorithm for $N = 3$, the graphs of the exact and approximate controls and trajectories are given in figures 7 and 8. The optimal value of J^* is 3.59596×10^{-4}. The graph of error for control and state function is given in figures 9a and 9b.

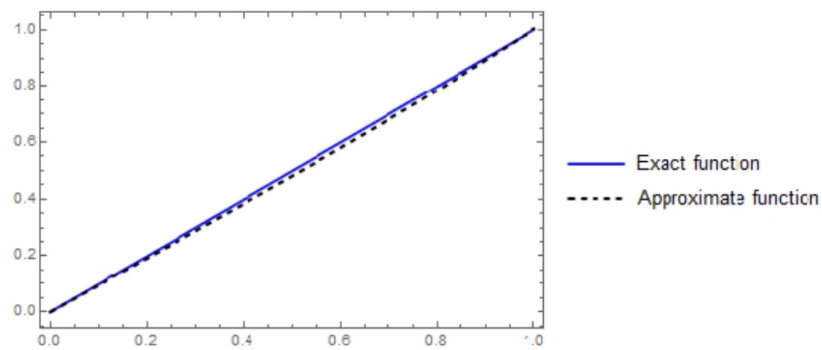


Figure 7: The exact and approximate optimal control in Example 4

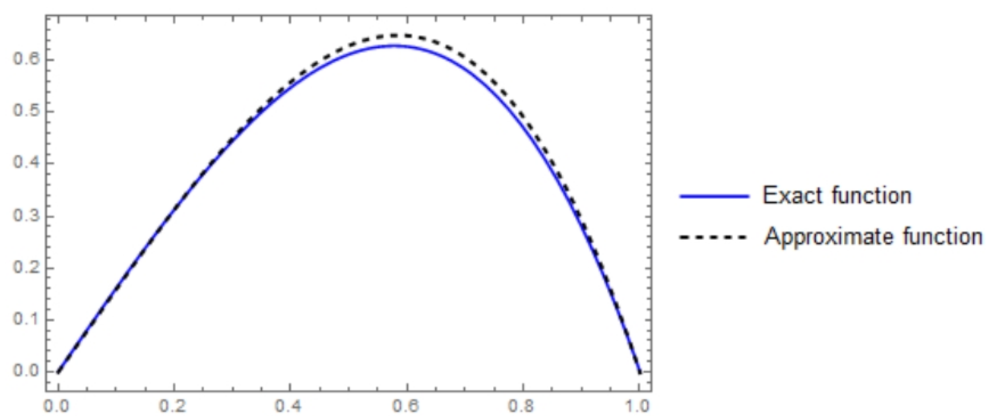
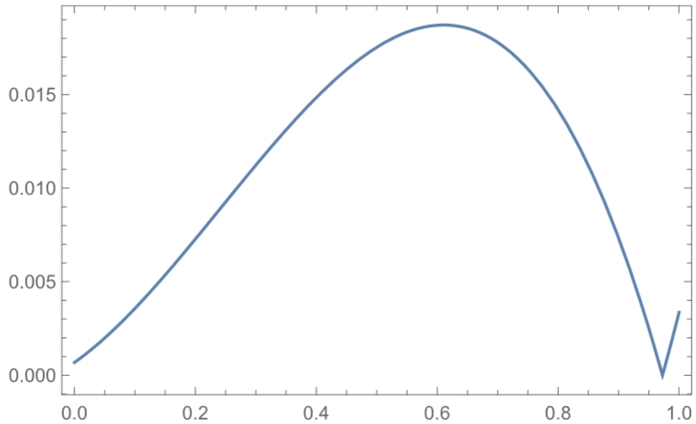
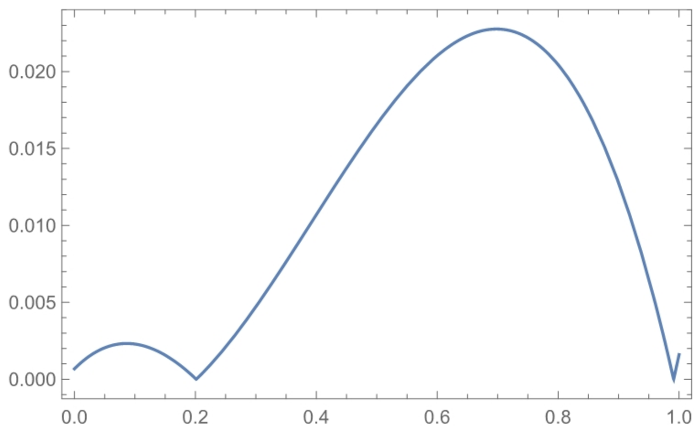


Figure 8: The exact and approximate optimal state in Example 4



(a) The error of state function in Example 4



(b) The errors of control function in Example 4

Figure 9

Example 5. Find the optimal control u^* and corresponding optimal state x^* that minimize objective function

$$\min \ J = \int_0^1 (x_1(t) - \sin(t))^2 + \left(x_2(t) - e^{-3t}\right)^2 + (u(t) - t)^2 \, dt$$

subject to

$$x_1(t) = h_1(t) - \int_0^1 \left(-t(u(s))^2 x_1(s) + tu(s)x_2(s)\right) ds,$$
$$x_2(t) = h_2(t) - \int_0^1 (t(u(s)+1)x_1(s) + t^2 u(s)x_2(s))ds.$$

where $h_1(t) = -\sin(5t) - \frac{23\cos(5)}{125}t + \frac{2\sin(5)}{15}t + \frac{107}{1125}t - \frac{4e^{-3}}{9}t$ and $h_2(t) = e^{-3t} + \frac{2cos(5)}{5}t - \frac{sin(5)}{25}t - \frac{t}{5} - \frac{4e^{-3}}{9}t^2 + \frac{t^2}{9}$. The exact solutions are $x_1^*(t) = \sin(t)$, $x_2^*(t) = e^{-3t}$ and $u^*(t) = t$. Figures 10, 11 and 12 are the graph of exact and approximate solutions for $N = 3$ and $N = 5$, respectively.

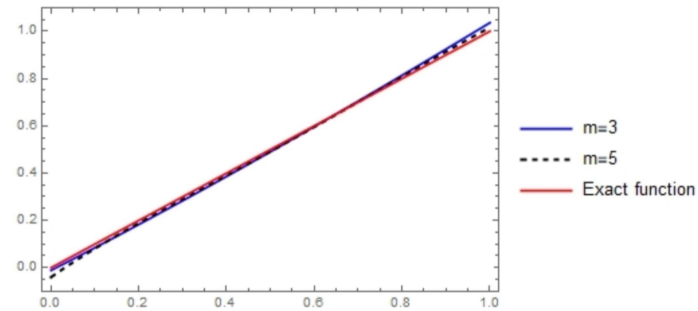


Figure 10: The precise and proximate optimal control in Example 5

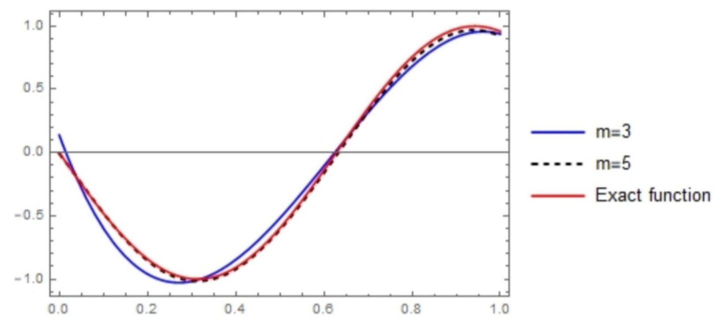


Figure 11: The accurate and proximate optimal state x_1 in Example 5

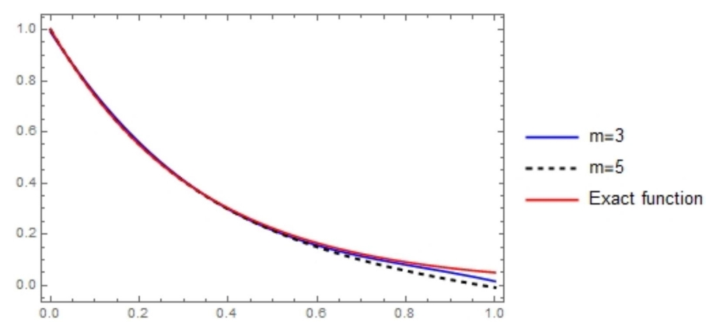


Figure 12: The accurate and proximate optimal state x_2 in Example 5

Example 6. Find the optimal solutions that minimize objective function

$$\min J = \int_0^1 \left(x_1(t) - (t + e^t)\right)^2 + \left(x_2(t) - e^t\right)^2 + \left(x_3(t) - (1 + \cos(t))\right)^2 + \left(u(t) - e^t\right)^2 dt$$

subject to

$$x_1(t) + \int_0^1 3su(t)x_2(s)ds = t + 4e^t,$$

$$x_2(t) - \int_0^1 6te^s x_1(s) - t^2 x_3(s)ds = e^t - 3t(e^2 + 1) + t^2(\sin(1) + 1)$$

$$x_3(t) - \int_0^1 (4(x_3(s) - 1) + x_2(s)ds = \cos(t) - (e + 4\sin(1)) + 2$$

The exact solutions of this example are $x_1^*(t) = t + e^t$, $x_2^*(t) = e^t$, $x_3^*(t) = 1 + \cos(t)$ and $u^*(t) = e^t$. The absolute errors of u^*, x_1^*, x_2^* and x_3^* for $N = 3$, $N = 4$ and $N = 5$ are represented in Tables 3 and 4.

Table 3: Absolute errors of control and state functions in Example 6

	N	$t = 0$	$t = 0.25$	$t = 0.5$	$t = 0.75$	$t = 1$
E_{x_1}	3	7.51115×10^{-4}	2.45764×10^{-4}	4.67421×10^{-4}	2.56842×10^{-4}	9.41652×10^{-4}
	4	3.99592×10^{-5}	7.55614×10^{-6}	5.88693×10^{-7}	9.47055×10^{-6}	486114×10^{-5}
	5	1.83038×10^{-6}	9.19475×10^{-7}	8.9541×10^{-7}	9.96471×10^{-7}	2.13043×10^{-6}
E_{x_2}	3	4.91524×10^{-4}	2.60616×10^{-4}	5.32509×10^{-4}	3.01636×10^{-4}	5.72133×10^{-4}
	4	2.49372×10^{-5}	1.28883×10^{-5}	1.78345×10^{-7}	1.4376×10^{-5}	2.98331×10^{-5}
	5	1.05009×10^{-6}	1.08341×10^{-6}	1.13318×10^{-6}	1.15422×10^{-6}	1.19212×10^{-6}
E_{x_3}	3	3.36834×10^{-4}	1.04755×10^{-4}	3.23772×10^{-4}	9.70093×10^{-4}	3.05991×10^{-4}
	4	6.42153×10^{-6}	3.47200×10^{-6}	9.92872×10^{-6}	4.26175×10^{-6}	8.97689×10^{-6}
	5	6.20439×10^{-7}	6.10935×10^{-7}	5.82297×10^{-7}	5.87852×10^{-7}	5.74393×10^{-7}
E_u	3	4.34033×10^{-4}	2.24118×10^{-4}	6.13891×10^{-4}	2.32091×10^{-4}	5.65502×10^{-4}
	4	1.93484×10^{-5}	1.40343×10^{-5}	7.62504×10^{-7}	1.68747×10^{-5}	2.46499×10^{-5}
	5	8.13269×10^{-7}	1.17233×10^{-6}	1.16398×10^{-6}	1.27319×10^{-6}	9.68081×10^{-7}

Table 4: Absolute errors of objective functions in Example 6

N	3	4	5
J^*	1.01824×10^{-7}	2.36092×10^{-10}	4.34198×10^{-13}

5 Conclusion

The FIEs have rich physical and Engineering backgrounds. These equations got superior fondness across plenty of disciplines and broadly utilized in dynamical machinery with chaotic or quasi-chaotic treatment. In this article, a robust and efficient numerical approach based on Lagrange polynomials and collocation approach is applied to obtain the solutions of OCP governed by FIE system. At last, the prescribed problem is converted to a NLP. The resulted NLP is solved by optimization algorithm. The preciseness of Lagrange collocation approach can be readily concluded from the ameliorated results of our presented approach. The propounded approach can be extended for the approximation of solutions in OCPs involving FIE with fractional [5, 6, 12, 27] and singular [15] kernels, however some modifications will be needed in the method and convergence analysis.

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