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Research Article

Legendre wavelet method combined with the Gauss quadrature rule for numerical solution of fractional integro-differential equations

M. Riahi Beni

Abstract

In this paper, we use a novel technique to solve the nonlinear fractional Volterra-Fredholm integro-differential equations (FVFIDEs). To this end, the Legendre wavelets are used in conjunction with the quadrature rule for converting the problem into a linear or nonlinear system of algebraic equations, which can be easily solved by applying mathematical programming techniques. Only a small number of Legendre wavelets are needed to obtain a satisfactory result. Better accuracies are also achieved within the method by increasing the number of polynomials. Furthermore, the existence and uniqueness of the solution are proved by preparing some theorems and lemmas. Also, error estimation and convergence analyses are given for the considered problem and the method. Moreover, some examples are presented and their results are compared to the results of Chebyshev wavelet, Nyström, and Newton–Kantorovitch methods to show the capability and validity of this scheme.

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1 Introduction

Fractional calculus has been applied in physics in recent years, although it has a long history in mathematics. Most of the physical phenomenon can be mod-

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Mohsen Riahi Beni
Department of Mathematics, Higher Education Complex of Saravan, Saravan, Iran. e-mail: m.riahi@saravan.ac.ir

eled by using fractional calculus [34, 10]. Applications of fractional differential equations and fractional integral equations create a wide area of research for many researchers [3, 4, 12, 11]. It has been applied to model the nonlinear oscillation of earthquakes, fluid-dynamic traffic, frequency-dependent damping behavior of many viscoelastic materials, continuum and statistical mechanics, solid mechanics, economics, signal processing, and control theory. To better analyze these systems, it is required to obtain the solution of these equations. However, mostly, finding the analytical solution of these equations cannot be possible, so considering more accurate numerical solutions can be helpful. In the literature, there are various techniques for solving fractional ordinary differential equations (FODEs), fractional partial differential equations (FPDEs), fractional integro-differential equations (FIDEs), and dynamic systems with fractional derivatives, such as analytical and semi-analytical methods (homotopy analysis method, Adomian's decomposition method, etc.) [1, 13, 14, 15, 17, 39, 35] and numerical methods (finite difference schemes, collocation method, Tau method, etc.) [6, 36, 16, 19, 30, 5, 22].

We can find some other famous methods for the numerical solutions of these kinds of equations. For example, spline collocation [29], analytical Lie group approach [28], fractional differential transform [25], least-squares [20], rationalized Haar function [27], exp-function method [8], and many others. In [32], a novel Legendre wavelet Petrov–Galerkin method was presented for fractional Volterra integro-differential equations. The Chebyshev wavelet method [23] has been used to nonlinear fractional Volterra-Fredholm integro-differential equations (FVFIDEs) with mixed boundary conditions. Approximate solutions of Volterra-Fredholm integro-differential equations of fractional order based on the Sinc-collocation method have been discussed in [2]. In [24], the Nyström and Newton–Kantorovitch methods were described for solving FVFIDEs with mixed boundary conditions. Systems of integro-differential equations [37] have been solved by using the spectral second kind Chebyshev wavelets scheme. Also, nonlinear Volterra integro-differential equations of fractional order have been solved numerically using the Euler wavelet method [38]. The reliable modification of the Laplace Adomian decomposition method [9] has been applied to solve nonlinear Volterra-Fredholm integral equations. Haar wavelet bases have been employed for solving the integro-differential equation [7]. In [18], hybrid Bernstein block-pulse functions have been used for solving systems of fractional integro-differential equations.

Orthogonal polynomials and functions apply to different problems because of their appropriate attributes. These functions and methods such as collocation, Galerkin, and Tau, are applied to reduce the solutions of different problems to the solutions of a system of algebraic equations. In this work, the Legendre wavelets are implemented to obtain an approximate solution for nonlinear FVFIDEs. We consider the fractional FVFIDEs in a Banach space as follows:

$$\left({}_0^C D_x^\alpha y\right)(x) + Vy(x) + Fy(x) = g(x), \;\; m-1 < \alpha \leq m, \;\; m \in \mathbb{N}, \qquad (1)$$

where ${}_0^C D_x^\alpha$ is in the Caputo sense fractional derivative, $g(x)$ is a given continuous function, $x \in J = [a,b]$, $y(x)$ is unknown function, and

$$Vy(x) = \int_a^x k_1(x,\xi) N_1(y(\xi))\, d\xi, \;\; Fy(x) = \int_a^b k_2(x,\xi) N_2(y(\xi))\, d\xi,$$

in which $k_i : J \times J \to \mathbb{R}$, $i = 1,2$, are continuous functions and $N_i : J \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are Lipschitz nonlinear continuous functions. Equation (1) is subject to the following mixed boundary conditions:

$$\sum_{j=1}^{m} \left[\lambda_{ij} y^{(j-1)}(a) + \eta_{ij} y^{(j-1)}(b)\right] = \gamma_i, \;\; i = 1, 2, \ldots, m. \qquad (2)$$

The main characteristic of this technique is that it reduces these problems to a system of algebraic equations. This approach is based on converting the FVFIDEs into mixed Volterra-Fredholm integral equations through integration, approximating various terms involved in the equation by truncated Legendre wavelet series and using the operational matrices to eliminate the integral, derivation, and product operations.

The rest of the paper is organized as follows: In section 2, some essential mathematical preliminaries and definitions of fractional calculus are introduced. In section 3, the properties of Legendre wavelets and function approximations are discussed. In the next section, the Gauss quadrature Legendre wavelet method (GQLWM) is constructed for solving FVFIDEs. In section 5, we study the convergence and error analysis of our algorithm. The existence and uniqueness of the solution are investigated in section 6. In section 7, the proposed method is applied on two examples to demonstrate the efficiency and accuracy of the present method. At last, a brief conclusion is given in section 8.

2 Basic definitions and notations of the fractional calculus

In this section, some definitions and properties of the fractional calculus, which will be used, are presented. For more details, see [21, 26, 31, 33].

Definition 1. The Gamma function is intrinsically tied with fractional calculus. The definition of the gamma function is given by

$$\Gamma(\alpha) = \int_0^\infty e^{-\xi} \xi^{\alpha-1} d\xi, \;\; \mathrm{Re}(\alpha) > 0.$$

Definition 2. A real function $g(x)$, $x > 0$, is said to be in the space C_θ, $\theta \in \mathbb{R}$ if there exists a real number $p > \theta$ such that $g(x) = x^p g_1(x)$, where $g_1(x) \in C\,[0,\infty)$ and it is said to be in the space C_θ^k, $k \in \mathbb{N}\bigcup\{0\}$ if and only if $g^{(k)} \in C_\theta$.

Definition 3. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $g \in C_\theta$, $\theta \geq -1$, is defined as

$$ {}_0I_x^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} g(\xi) d\xi. $$

Some properties of the Riemann-Liouville fractional integral are as follows:

1. ${}_0I_x^0 g(x) = g(x)$,
2. $\left({}_0I_x^\alpha {}_0I_x^\beta g\right)(x) = \left({}_0I_x^\beta {}_0I_x^\alpha g\right)(x) = \left({}_0I_x^{\alpha+\beta} g\right)(x)$,
3. ${}_0I_x^\alpha (x-\mu)^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)}(x-\mu)^{\alpha+\lambda}, \quad \alpha \geq 0,\ \lambda > -1.$

Similar to the integer-order integration, the Riemann-Liouville fractional integral is a linear operator, which means that

$$ {}_0I_x^\alpha \left(\sum_{i=1}^m c_i f_i(x) \right) = \left(\sum_{i=1}^m c_i \right) {}_0I_x^\alpha f_i(x), $$

where $\{c_i\}_{i=1}^m$ are constants.

Definition 4. The fractional derivative of $g(x)$ in the Caputo sense is defined as

$$ {}_0^C D_x^\alpha g(x) = {}_0I_x^{n-\alpha} g^{(n)}(x), $$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $x > 0$, and $g \in C_{-1}^n$.

The relation between the Riemann-Liouville operator and Caputo operator is given by the following lemma.

Lemma 1. If $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, then ${}_0^C D_x^\alpha {}_0I_x^{n-\alpha} g(x) = g(x)$ and

$$ {}_0I_x^{n-\alpha} {}_0^C D_x^\alpha g(x) = g(x) - \sum_{k=0}^{n-1} g^{(k)}(0^+) \frac{x^k}{k!}, \ \ x > 0. $$

3 Legendre wavelets and functions approximation

3.1 Properties of Legendre wavelets

Wavelets are a family of functions constructed from the dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as

$$\Psi_{a,b}(x) = |a|^{-\frac{1}{2}} \Psi\left(\frac{x}{a} - \frac{b}{a}\right), \;\; a, b \in \mathbb{R}, \;\; a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 0$, $b_0 > 0$ and n and k are positive integers, then the following family of discrete wavelets will be obtained

$$\psi_{k,n}(x) = |a_0|^{\frac{k}{2}} \psi(a_0^k x - nb_0),$$

where $\psi_{k,n}(x)$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, with $a_0 = 2$ and $b_0 = 1$, $\psi_{k,n}(x)$ form an orthonormal basis.
Legendre wavelets $\psi_{n,m}(x) = \psi(k, \hat{n}, m, x)$ have four arguments; $\hat{n} = 2n - 1$ for $n = 1, 2, \ldots, 2^{k-1}$, $k \in \mathbb{Z}^+$, m as the order of Legendre polynomials and x, which is the normalized time. They are defined on the interval $[0, 1)$ by

$$\psi_{n,m}(x) = \psi(k, \hat{n}, m, x) = \begin{cases} 2^{\frac{k-1}{2}} (2m+1)^{\frac{1}{2}} L_m^*(x), & \frac{\hat{n}-1}{2^k} \leq x < \frac{\hat{n}+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where $m = 0, 1, \ldots, M - 1$, $n = 1, 2, \ldots, 2^{k-1}$, the dilation and translation parameters are $a = 2^{-k}$ and $b = \hat{n}2^{-k}$, respectively. Moreover, $L_m^*(x) = L_m(2^k x - \hat{n})$ are shifted Legendre polynomials on interval $[0, 1)$, which are orthogonal with respect to the weight function $w(x) = 1$. Also, L_m's can be determined by the following recurrence formulas:

$$\begin{aligned} &L_0(x) = 1, \;\; L_1(x) = x, \\ &L_{m+1}(x) = \left(\frac{2m+1}{m+1}\right) x L_m(x) - \left(\frac{m}{m+1}\right) L_{m-1}(x), \;\; m = 1, 2, \ldots. \end{aligned}$$

3.2 Function approximation by Legendre wavelets

A function $y(x)$, which is square integrable in $[0, 1)$, may be expressed in terms of the Legendre wavelet as

$$y(x) = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N} \bigcup \{0\}} y_{n,m} \psi_{n,m}(x), \tag{4}$$

where $y_{n,m} = \langle y(x), \psi_{n,m}(x) \rangle$ and $< \cdot, \cdot >$ denotes the inner product. If the infinite series in (4) is truncated, then one obtains

$$y(x) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} y_{n,m}\psi_{n,m}(x) = Y^T\Psi(x), \tag{5}$$

where the coefficient vector Y and the Legendre wavelet function vector $\Psi(x)$ are given by

$$Y = [y_{1,0}, y_{1,1}, \dots, y_{1,M-1}, y_{2,0}, y_{2,1}, \dots, y_{2,M-1}, \dots, y_{2^{k-1},0}, y_{2^{k-1},1}, \dots, y_{2^{k-1},M-1}]^T, \tag{6}$$

$$\Psi(x) = [\psi_{1,0}(x), \psi_{1,1}(x), \dots, \psi_{1,M-1}(x), \dots, \psi_{2^{k-1},0}(x), \psi_{2^{k-1},1}(x), \dots, \psi_{2^{k-1},M-1}(x)]^T. \tag{7}$$

Similarly, we can approximate the functions $k_i(x,\xi) \in L^2([0,1]\times[0,1])$ as

$$k_i(x,\xi) = \Psi(x)^T K_i \Psi(\xi), \tag{8}$$

where K_i, $i = 1, 2$ are $2^{k-1}M \times 2^{k-1}M$ matrices.

4 GQLWM for solving FVFIDEs

Let us consider the nonlinear FVFIDE with mixed boundary conditions given by (1)–(2). To approximate the functions $y(x)$ and $g(x)$ by $y(x) = Y^T\Psi(x)$ and $g(x) = G^T\Psi(x)$, assume that

$$N_1(y(\xi)) = u(\xi), \;\; N_2(y(\xi)) = v(\xi), \tag{9}$$

where $u(\xi)$ and $v(\xi)$ are as follows:

$$u(\xi) = U^T\Psi(\xi), \;\; v(\xi) = V^T\Psi(\xi), \tag{10}$$

in which U^T and V^T are defined similarly as in (6). Applying the operator ${}_0I_x^{\alpha}$ on both sides of (1), results in

$$\begin{aligned} y(x) - \sum_{k=0}^{n-1} y^{(k)}(0^+)\frac{x^k}{k!} =& \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1} g(\tau)d\tau \\ & - \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\int_0^{\tau} k_1(\tau,\xi)u(\xi)d\tau d\xi \\ & - \frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\int_0^{1} k_2(\tau,\xi)v(\xi)d\tau d\xi. \end{aligned}$$

Replacing the exact solution $y(x)$ by $Y^T\Psi(x)$ and using their approximations by (5), (8)–(10), one gets

$$
\begin{aligned}
Y^T\Psi(x)-&\sum_{k=0}^{n-1} Y^T\Psi^{(k)}(0^+)\frac{x^k}{k!}\\
&=\frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}G^T\Psi(\tau)d\tau\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\int_0^\tau \Psi(\tau)^T K_1\Psi(\xi)U^T\Psi(\xi)d\tau d\xi\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_0^x (x-\tau)^{\alpha-1}\int_0^1 \Psi(\tau)^T K_2\Psi(\xi)V^T\Psi(\xi)d\tau d\xi. \qquad (11)
\end{aligned}
$$

Collocating (11) in $M2^{k-1}$ Legendre wavelet-Gauss collocation points x_j, $j = 1, 2, \ldots, M2^{k-1}$, on interval $[0,1]$, we arrive at the following system:

$$
\begin{aligned}
Y^T\Psi(x_j)-&\sum_{k=0}^{n-1} Y^T\Psi^{(k)}(0^+)\frac{x_j^k}{k!}\\
&=\frac{1}{\Gamma(\alpha)}\int_0^{x_j} (x_j-\tau)^{\alpha-1}G^T\Psi(\tau)d\tau\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_0^{x_j} (x_j-\tau)^{\alpha-1}\int_0^\tau \Psi(\tau)^T K_1\Psi(\xi)U^T\Psi(\xi)d\tau d\xi\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_0^{x_j} (x_j-\tau)^{\alpha-1}\int_0^1 \Psi(\tau)^T K_2\Psi(\xi)V^T\Psi(\xi)d\tau d\xi. \qquad (12)
\end{aligned}
$$

In a similar way, the mixed boundary conditions (2) are approximated as follows:

$$
\sum_{j=1}^{m}\left[\lambda_{ij}Y^T\Psi^{(j-1)}(0)+\eta_{ij}Y^T\Psi^{(j-1)}(1)\right]=\gamma_i,\quad i=1,2,\ldots,m. \qquad (13)
$$

Now, we use the quadrature rule to approximate the integral involved in this equation, which has zero error integration for polynomial integrands of degree less than or equal to $1 + M2^k$ with $M2^{k-1}$ Legendre-Gauss nodes. To do this, $M2^{k-1}$ intervals $[0, x_j]$ are transferred to a fixed interval $[-1, 1]$ through the transformations $\tau = (x_j/2\ )(s + 1)$. Applying the Gaussian quadrature, system (12) is converted to

$$
\begin{aligned}
Y^T\Psi(x_j)-&\sum_{k=0}^{n-1} Y^T\Psi^{(k)}(0^+)\frac{x_j^k}{k!}\\
&=\frac{1}{\Gamma(\alpha)}\int_0^{x_j} (x_j-\tau)^{\alpha-1}G^T\Psi(\tau)d\tau\\
&\quad-\frac{1}{\Gamma(\alpha)}\frac{x_j}{2}\sum_{l=1}^{M2^{k-1}} w_l\Big(\frac{x_j}{2}(1-s_l)\Big)^{\alpha-1}\int_0^\sigma [\Psi(\sigma)]^T K_1\Psi(\xi)U^T\Psi(\xi)d\xi\\
&\quad-\frac{1}{\Gamma(\alpha)}\frac{x_j}{2}\sum_{l=1}^{M2^{k-1}} w_l\Big(\frac{x_j}{2}(1-s_l)\Big)^{\alpha-1}\int_0^1 [\Psi(\sigma)]^T K_2\Psi(\xi)V^T\Psi(\xi)d\xi, \qquad (14)
\end{aligned}
$$

where $\sigma = \frac{x_j}{2}(s_l+1))$ and w_l for $l = 1, 2, \dots, M2^{k-1}$ are the weight functions on $[-1, 1]$. Taking into account (9)–(10) and using collocation points x_j, $j = 1, 2, \dots, M2^{k-1}$, one obtains

$$N_1(Y^T\Psi(x_j)) = U^T\Psi(x_j), \ \ N_2(Y^T\Psi(x_j)) = V^T\Psi(x_j). \tag{15}$$

Combining (13)–(15), the main problem is reduced to a system of nonlinear algebraic equations. By solving this system, the unknown values Y^T will be obtained, and by relation (5), the approximate solution of system (1) will be determined.

5 Convergence and error analysis

In this section, we will obtain an estimation for the error bound of our numerical method. Also, its convergence analysis will be discussed. For this purpose, assume that $L_m^*(x) = L_m(2x-1)$ and that $\Lambda(x) = [L_0^*(x), L_1^*(x), \dots, L_N^*(x)]^T$. So, the function $y(x) \in L^2[0, 1]$ can be expressed in terms of the Legendre polynomials basis $\Lambda(x)$ as

$$y(x) = \sum_{n=0}^{N} l_n L_n^*(x) = L^T\Lambda(x),$$

where $L = [l_0, l_1, \dots, l_N]^T$.

Lemma 2. Suppose that $y \in C^{m+1}[0, 1]$ and that $S = \operatorname{span}\{L_0^*, L_1^*, \dots, L_N^*\}$ is a vector space. If $L^T\Lambda(x)$ is the best approximation of $y(x)$ out of S, then the error bound of presented method is as follows:

$$\left\| y - L^T\Lambda \right\|_2 \le \frac{\sqrt{2\chi^{2m+3}}M_{m+1}}{(m+1)!\sqrt{2m+3}},$$

in which $M_{m+1} = \max\left\{\left|f^{m+1}(x)\right| : \ 0 \le x \le 1\right\}$ and $\chi = \max\{1-\xi, \xi\}$.

Proof. The Taylor polynomials, implies that

$$Ty(x) = y_0(\xi) + (x-\xi)y'(\xi) + \frac{(x-\xi)^2}{2!}y''(\xi) + \cdots + \frac{(x-\xi)^m}{m!}y^{(m)}(\xi),$$

where $\xi \in (0, 1)$. Therefore, there exists $\lambda \in (0, 1)$ such that

$$|y(x) - Ty(x)| \le \left| \frac{(x-\xi)^{m+1}}{(m+1)!} y^{(m+1)}(\lambda) \right|.$$

Since $L^T\Lambda(x)$ is the best approximation of $y(x)$, one obtains

$$\begin{aligned}\left\|y-L^T\Lambda\right\|_2^2 &\le \|y-Ty\|_2^2=\int_0^1 |y(x)-Ty(x)|^2dx\\ &\le \int_0^1 \left|\frac{(x-\xi)^{m+1}}{(m+1)!}y^{(m+1)}(\lambda)\right|^2 dx\\ &\le \frac{M_{m+1}^2}{[(m+1)!]^2}\int_0^1 (x-\xi)^{2m+2}dx \le \frac{2\chi^{2m+3}M_{m+1}^2}{(2m+3)[(m+1)!]^2},\end{aligned}$$

and the proof is completed. □

Theorem 1. Assume that $y \in C^{m+1}[0,1]$ and that its approximation by the Legendre wavelets is $\tilde{y}(x)=Y^T\Psi(x)$. Then, its mean error bound is as follows:

$$\|y-\tilde{y}\|_2 \le \frac{2^{(m+1)(1-k)}\sqrt{2}M_{m+1}}{(m+1)!\sqrt{2m+3}}.$$

Proof. Dividing $[0,1]$ to 2^{k-1} subintervals $I_{k,n}=[(n-1)/2^{k-1}\ ,n/2^{k-1}\ ]$, $n=1,2,\ldots,2^{k-1}$ with the restriction that $\tilde{y}$ is a polynomial of degree less than $m+1$ and also using Lemma 2, we get

$$\begin{aligned}\|y-\tilde{y}\|_2^2 &\le \int_0^1 |y(x)-\tilde{y}(x)|^2dx\\ &=\sum_{n=1}^{2^{k-1}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} |y(x)-\tilde{y}(x)|^2dx\\ &\le \sum_{n=1}^{2^{k-1}}\left[\frac{\sqrt{2}\bar{M}_n 2^{\frac{(1-k)(2m+3)}{2}}}{(m+1)!\sqrt{2m+3}}\right]^2=\frac{2^{(1-k)(2m+3)+1}}{[(m+1)!]^2(2m+3)}\sum_{n=1}^{2^{k-1}}\bar{M}_n^2\\ &\le \frac{2^{(1-k)(2m+2)+1}M_{m+1}^2}{[(m+1)!]^2(2m+3)},\end{aligned}$$

where $\bar{M}_n=\max\left\{\left|y^{(m+1)}(x)\right|,\ \ x\in I_{k,n}\right\}$, which completes the proof. □

Lemma 3. Suppose that for $y(x)\in[0,1)$, there exists $\beta_y\in\mathbb{R}$ such that $|y(x)|\le\beta_y$. Then, the sum of Legendre coefficients of $y(x)$ defined in (5) is absolutely convergent if $|y_{n,m}|\le\sqrt{2^{1-k}}\beta_y$.

Proof. The function $y(x)\in L^2[0,1]$ can be expressed as the Legendre wavelet basis as defined in (5). For $m\ge 0$, we have

$$\begin{aligned}|y_{n,m}| &= |<y,\psi_{n,m}>| = \left|\int_0^1 y(x)\psi_{n,m}(x)dx\right| \le \int_0^1 |y(x)||\psi_{n,m}(x)|dx\\ &\le \beta_y\int_0^1 |\psi_{n,m}(x)|dx = \beta_y\int_{I_{n,k}} |\psi_{n,m}(x)|dx\\ &= \beta_y 2^{\frac{k-1}{2}}(2m+1)^{\frac{1}{2}}\int_{I_{n,k}} \left|L_m(2^kx-2n+1)\right| dx.\end{aligned}$$

Setting $s=2^kx-2n+1$, one obtains

$$|y_{n,m}| \le \beta_y 2^{\frac{-k-1}{2}}(2m+1)^{\frac{1}{2}}\int_{-1}^1 |L_m(s)|ds.$$

The Hölder's inequality implies that

$$|y_{n,m}| \le \beta_y 2^{\frac{-k-1}{2}}(2m+1)^{\frac{1}{2}}\frac{2}{\sqrt{2m+1}} = \sqrt{2^{1-k}}\beta_y.$$

This means that the series $\sum\limits_{n=1}^{2^{k-1}}\sum\limits_{m=0}^{M-1} y_{n,m}$ is absolutely convergent as $k\to\infty$. □

Theorem 2. The series solution $\tilde{y}(x)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} y_{n,m}\psi_{n,m}(x)$ defined by (5) is convergent with respect to L^2-norm on $[0,1]$ if the sum of absolute values of the Legendre coefficients $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}|y_{nm}|$ for continuous function $y(x)$ forms a convergent series.

Proof. Let $L^2(\mathbb{R})$ be a Hilbert space and let $\psi_{n,m}$ defined in (3) form an orthogonal basis with $\psi_{n,m}(x)=\Delta_j(x)$ and $\delta_j=<\tilde{y}(x),\Delta_j(x)>$. Define a sequence of partial sums $\{S_n\}$ as

$$S_n(x)=\sum_{j=0}^{n}\delta_j\Delta_j(x).$$

Now, for every $\epsilon>0$, there exists $N_\epsilon>0$ such that for every $n>m>N_\epsilon$,

$$\begin{aligned}\|S_n(x)-S_m(x)\|_2^2 &= \int_0^1\left|\sum_{l=m+1}^{n}\delta_l\Delta_l(x)\right|^2 dx \le \sum_{l=m+1}^{n}|\delta_l|^2\int_0^1|\Delta_l(x)|^2dx\\ &= \sum_{l=m+1}^{n}|\delta_l|^2.\end{aligned}$$

Using Lemma 3, $\sum_{l=0}^{\infty}|\delta_l|^2$ is absolutely convergent. Hence, according to the Cauchy criterion, we have

$$\sum_{l=m+1}^{n} |\delta_l|^2 < \epsilon^2.$$

So, $\|S_n(x) - S_m(x)\|_2 \leq \sqrt{\epsilon^2} = \epsilon$. Thus, the sequence of partial sum of the series is a Cauchy sequence. Therefore, it convergent with respect to L^2-norm and this completes the proof. □

6 Existence and uniqueness

In order to study the uniqueness of the solution, we consider the FVFIDEs (1) in an operator form as

$$\left({}_0^C D_x^\alpha y\right)(x) = g(x) - K_1 N_1 y - K_2 N_2 y, \tag{16}$$

where

$$K_1 N_1 y = \int_a^x k_1(x,\xi) N_1(y(\xi))\, d\xi, \quad K_2 N_2 y = \int_a^b k_2(x,\xi) N_2(y(\xi))\, d\xi.$$

Applying the operator ${}_0I_x^\alpha$ on both sides of (16), one obtains

$$y(x) = f(x) + {}_0I_x^\alpha \left[g(x) - K_1 N_1 y - K_2 N_2 y\right],$$

where $f(x) = \sum_{k=0}^{n-1} y^{(k)}(0^+) x^k/k!$. This equation can be written as $\mathcal{T} y = y$, where $\mathcal{T}$ is as follows:

$$\mathcal{T} y(x) = f(x) + {}_0I_x^\alpha \left[g(x) - K_1 N_1 y - K_2 N_2 y\right].$$

Let $(C\,[0,1], ||\infty||)$ be a Banach space of all continuous functions with the norm $||h||_\infty = \max_x |h(x)|$. Also, the operators N_1 and N_2 satisfy the Lipschitz condition on $[0,1]$ with Lipschitz constants J_1 and J_2 as

$$|N_1\tilde{y}(x) - N_1 y(x)| \leq J_1 \left|\tilde{y}(x) - y(x)\right|, \quad |N_2\tilde{y}(x) - N_2 y(x)| \leq J_2 \left|\tilde{y}(x) - y(x)\right|.$$

In the following theorem with these assumptions, we show that the FVFIDEs (1) have a unique solution.

Theorem 3. Suppose that the nonlinear operators N_1 and N_2 satisfy the following relation

$$J_1 \|N_1\|_\infty + J_2 \|N_2\|_\infty < \Gamma(\alpha + 1).$$

Then the FVFIDE (1) has a unique solution.

Proof. Let $\mathcal{T} : C\,[0,1] \to C\,[0,1]$ be defined as

$$\mathcal{T}y(x) = f(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} \left[g(\xi) - K_1 N_1 y(\xi) - K_2 N_2 y(\xi)\right] d\xi.$$

Suppose that $\tilde{y}, y \in C\,[0,1]$. Then for every positive x, we have

$$\begin{aligned}
&\mathcal{T}\tilde{y}(x) - \mathcal{T}y(x) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} \left[K_1 N_1 y(\xi) - K_1 N_1 \tilde{y}(\xi) + K_2 N_2 y(\xi) - K_2 N_2 \tilde{y}(\xi)\right] d\xi.
\end{aligned}$$

Therefore, one obtain

$$\begin{aligned}
&\left|\mathcal{T}\tilde{y}(x) - \mathcal{T}y(x)\right| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^x |x-\xi|^{\alpha-1} \left[|K_1|\,|N_1 y(\xi) - N_1 \tilde{y}(\xi)| + |K_2|\,|N_2 y(\xi) - N_2 \tilde{y}(\xi)|\right] d\xi \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^x |x-\xi|^{\alpha-1} \left[|K_1|\, J_1 \,|\tilde{y}(\xi) - y(\xi)| + |K_2|\, J_2 \,|\tilde{y}(\xi) - y(\xi)|\right] d\xi \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^x |x-\xi|^{\alpha-1} \left[\|K_1\|_\infty J_1 + \|K_2\|_\infty J_2\right] \|\tilde{y}(\xi) - y(\xi)\|_\infty \, d\xi \\
&\le \frac{|x|^\alpha}{\Gamma(\alpha+1)} \left(\|K_1\|_\infty J_1 + \|K_2\|_\infty J_2\right) \|\tilde{y}(\xi) - y(\xi)\|_\infty \\
&\le \frac{1}{\Gamma(\alpha+1)} \left(\|K_1\|_\infty J_1 + \|K_2\|_\infty J_2\right) \|\tilde{y}(\xi) - y(\xi)\|_\infty .
\end{aligned}$$

Hence,

$$\|\mathcal{T}\tilde{y}(x) - \mathcal{T}y(x)\|_\infty \le L \|\tilde{y}(\xi) - y(\xi)\|_\infty,$$

where $L = \left[1/\Gamma(\alpha+1)\right] \left(\|K_1\|_\infty J_1 + \|K_2\|_\infty J_2\right)$. Since $L < 1$, by the contraction mapping theorem, this problem has a unique solution in $C\,[0,1]$. $\square$

7 Numerical examples

In this section, to demonstrate the capability and accuracy of our method, we apply it to some examples and compare the quality of the computed solutions with some other well-known methods. In order to demonstrate the error, we will calculate the maximum absolute error (MAE) by the following formula:

$$MAE = \max_{0 \le x < 1} \left\{ |y(x) - \tilde{y}(x)| \right\}.$$

Also for comparison, in Example 2, the root-mean-square error (RMSE) is calculated. This error is as follows:

$$\text{RMSE} = \sqrt{\int_0^1 \left[y(x) - \tilde{y}(x)\right]^2 dx}.$$

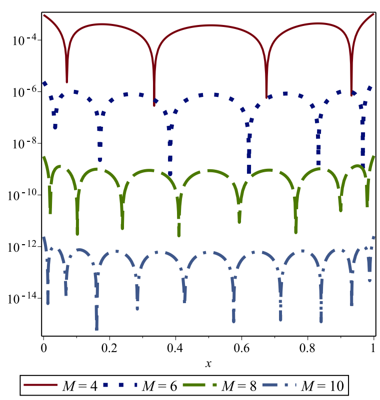
Example 1. Consider the following nonlinear FVFIDEs:

$$\left({}_{0}^{C}D_{x}^{\frac{\sqrt{7}}{2}}y\right)(x)-\int_{0}^{x}\frac{1+2\xi}{1+y(\xi)}d\xi-\int_{0}^{1}(1+2\xi)y(\xi)d\xi=g(x),$$

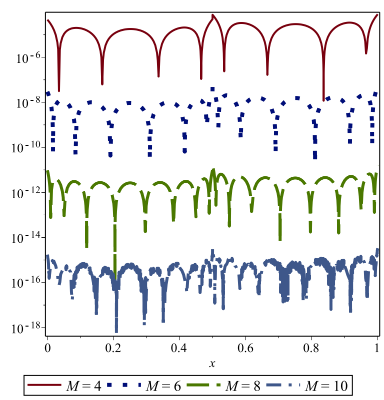
subject to the boundary conditions $y(0)=0$ and $y(1)=2$ in which

$$g(x)=\frac{2x^{2-\frac{\sqrt{7}}{2}}}{\Gamma\left(3-\frac{\sqrt{7}}{2}\right)}-\ln(x^2+x+1)+1-e^2.$$

The exact solution of this problem is $y(x)=x^2+x$. Numerical results for Example 1 are indicated in Table 1, which show that the results of our method are better than the results obtained in [24, 23] by the Chebyshev wavelet method (CWM), Nyström and Newton–Kantorovitch methods, respectively. Table 2 analyzes the exact solution y and the approximate solutions of this algorithm by CWM and GQLWM. One can see that the present method has excellent accuracy with respect to CWM. Table 3 shows the maximum absolute error between the exact and approximate solutions for various choices of M and k. The outcomes reveal that the results of GQLWM by using only a small number of bases are very promising and superior to CWM and Nyström and Newton–Kantorovitch methods. Furthermore, we compare the absolute error functions for different values of k and M in Figure 1. These results confirm that with increasing the amounts of k and M, the error will be decreased.



(a) $k=1$



(b) $k=2$

Figure 1: Error history of the presented method for various values for M and k of Example 1.

Table 1: Comparison between our method, CWM [23] and method in [24] for Example 1.

x		0.1	0.3	0.5	0.7	0.9
Error [24]	$k=3,\ n=160$	$7.81E-06$	$1.55E-05$	$1.65E-05$	$1.27E-05$	$5.07E-06$
	$k=3,\ n=320$	$1.92E-06$	$3.84E-06$	$4.10E-06$	$3.15E-06$	$1.25E-06$
	$k=3,\ n=640$	$4.65E-07$	$9.31E-07$	$9.92E-07$	$7.65E-07$	$3.04E-07$
Error [23] for $k=1,\ M=19$		$3.47E-12$	$6.08E-12$	$6.21E-12$	$4.68E-12$	$1.84E-12$
GQLWM	$k=2,\ M=4$	$2.93-06$	$1.01E-05$	$1.99E-06$	$1.08E-05$	$4.27E-06$
	$k=3,\ M=5$	$8.17E-09$	$5.11E-10$	$5.05E-11$	$6.78E-10$	$9.48E-09$
	$k=4,\ M=6$	$1.69E-23$	$4.22E-20$	$9.91E-21$	$7.06E-20$	$4.45E-23$
	$k=1,\ M=10$	$1.01E-12$	$4.67E-13$	$1.00E-12$	$4.71E-13$	$1.02E-12$
	$k=1,\ M=19$	$1.16E-28$	$2.11E-30$	$8.86E-29$	$3.35E-30$	$1.30E-28$

Table 2: The numerical solution of Example 1.

x	y(exact)	y(CWM)	y(GQLWM)
0.0	0.00	0.00000000000000000001	0.00000000000000000000
0.1	0.11	0.11000000000347509219	0.10999999999999999999
0.2	0.24	0.24000000000518991700	0.23999999999999999999
0.3	0.39	0.39000000000608906284	0.38999999999999999999
0.4	0.56	0.56000000000639391977	0.55999999999999999999
0.5	0.75	0.75000000000621724965	0.74999999999999999999
0.6	0.96	0.96000000000562994250	0.95999999999999999999
0.7	1.19	1.19000000000468149700	1.18999999999999999999
0.8	1.44	1.44000000000340880150	1.43999999999999999999
0.9	1.77	1.71000000000184060200	1.70999999999999999999
1	2.00	1.99999999999999999990	2.00000000000000000000

Table 3: Maximum absolute error between the exact and approximate solutions for various choices of M and k for Example 1.

M		4	7	10	13	16	19
MAE	$k=1$	$4.41E-04$	$4.06E-08$	$1.03E-12$	$4.02E-16$	$3.01E-23$	$1.57E-28$
	$k=2$	$7.68E-05$	$9.41E-10$	$3.05E-15$	$3.96E-21$	$2.55E-27$	$1.61E-34$
	$k=3$	$5.46E-06$	$8.23E-12$	$3.39E-18$	$5.49E-25$	$4.42E-32$	$2.02E-39$

Example 2. Consider the following nonlinear fractional integro-differential equation:

$$\left({}_0^C D_x^\alpha y\right)(x) = g(x) + \int_0^1 x\xi[y(\xi)]^2 d\xi, \;\; y(0)=0,$$

where $0<\alpha\leq 1$ and $g(x)$ is as follows:

Case 1. [23] $g(x) = \frac{4}{3}\Gamma\left(\frac{3}{4}\right)^{-1} x^{\frac{3}{4}} - \frac{1}{4}x$ with $\alpha=\frac{1}{4}$ and exact the solution is $y(x)=x$.

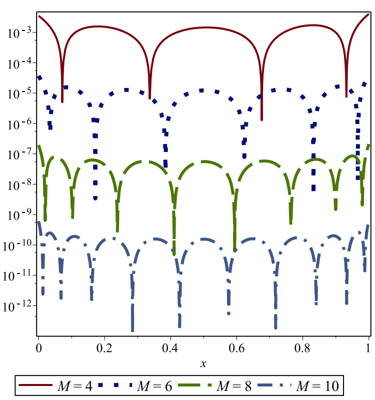
Case 2. [23] $g(x) = \frac{64}{15}\Gamma\left(\frac{3}{4}\right)\frac{\sqrt{2}}{\pi}x^{\frac{9}{4}} - \frac{1}{8}x$ with $\alpha = \frac{3}{4}$ and the exact solution is $y(x) = x^3$.

Case 3. [40] $g(x) = 1 - \frac{1}{4}x$ with $\alpha = 1$ and the exact solution is $y(x) = x$.

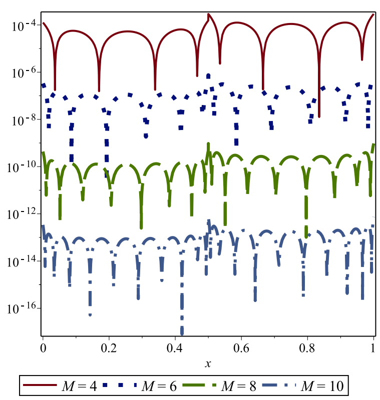
We apply our method to this example with various values of M and k, and the results are presented in Tables 4 and 5. The maximum absolute error of $y(x)$ by GQLWM for $k = 1$ with different values of M has been compared to the error of CWM [23] for cases 1 and 2. These results demonstrate the validity and capability of GQLWM with respect to CWM. The graphs of absolute error functions for different values of M and k with $\alpha = 1/4$, $\alpha = 1$, and $\alpha = 3/4$ are given in Figures 2, 3, and 4, respectively. These figures reveal that the error will be decreased when M is increased. Table 5 compares the RMSE of GQLWM with the results in [40] for different values of k and M when $\alpha = 1$. This table reveals that, for a certain value of k, as M increases, the accuracy increases. Also, for a certain value of M, as k increases, the accuracy increases as well. Therefore, GQLWM for solving this problem is very effective and more accurate with respect to CWM and the method in [40].

Table 5: RMSE for some k and M of GQLWM and the second kind Chebyshev wavelet method [40] for Example 2 (case 3).

	Method in [40]	GQLWM			
M	2	2	6	10	14
$k=3$	$2.970E-07$	$2.348E-09$	$4.236E-11$	$1.003E-12$	$1.320E-13$
$k=4$	$1.861E-08$	$7.391E-10$	$9.863E-13$	$4.002E-15$	$7.5691E-17$
$k=5$	$1.164E-09$	$8.541E-12$	$8.786E-16$	$9.251E-18$	$2.015E-20$



(a) $k = 1$

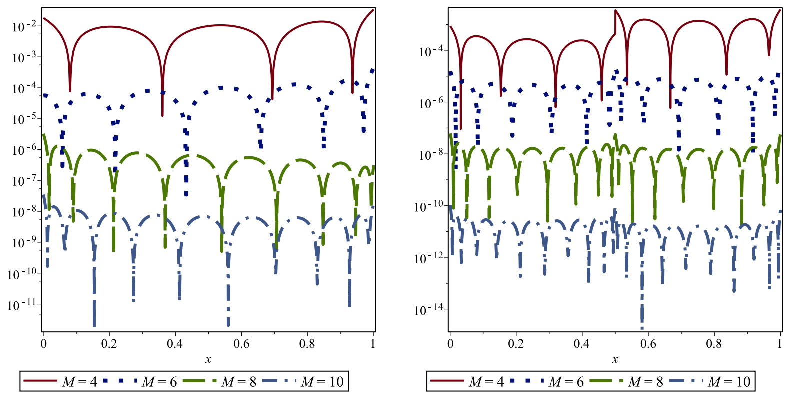


(b) $k = 2$

Figure 2: Error history of the presented method for various values for M and k of Example 2 with $\alpha = \frac{1}{4}$.

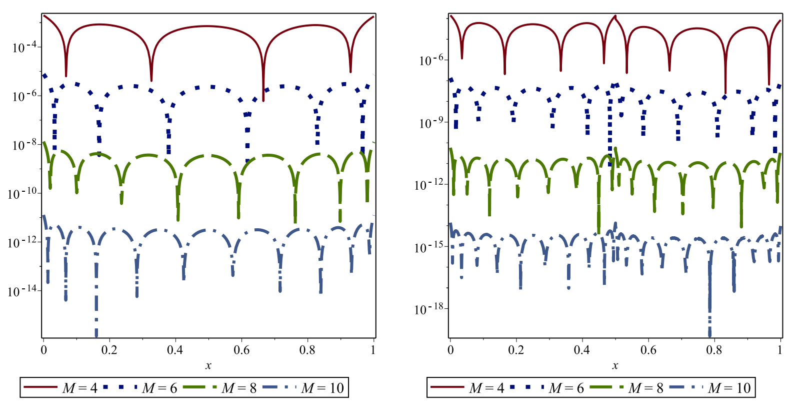
Table 4: Comparison between our method and CWM [23] for Example 2 (cases 1 and 2) when $k = 1$.

α	$\alpha = \frac{3}{4}$				$\alpha = \frac{1}{4}$			
	Error[23]	GQLWM			Error[23]	GQLWM		
x	$M = 19$	$M = 13$	$M = 16$	$M = 19$	$M = 19$	$M = 13$	$M = 16$	$M = 19$
0.0	$1.68E-20$	0	0	0	$7.92E-21$	0	0	0
0.1	$9.06E-13$	$1.75E-13$	$1.85E-16$	$1.27E-20$	$6.57E-18$	$6.53E-16$	$4.00E-19$	$2.51E-24$
0.2	$3.99E-12$	$4.05E-12$	$7.53E-17$	$3.92E-23$	$3.62E-18$	$5.89E-15$	$1.63E-19$	$2.91E-25$
0.3	$1.03E-11$	$3.76E-12$	$1.47E-16$	$1.49E-20$	$2.65E-18$	$5.44E-15$	$3.80E-19$	$6.18E-24$
0.4	$2.10E-11$	$2.21E-12$	$1.53E-16$	$1.06E-20$	$2.11E-18$	$3.41E-15$	$4.19E-19$	$4.08E-24$
0.5	$3.73E-11$	$8.35E-14$	$1.48E-16$	$1.83E-22$	$1.85E-18$	$4.07E-16$	$4.25E-19$	$2.29E-25$
0.6	$6.05E-11$	$2.38E-12$	$1.42E-16$	$1.04E-20$	$1.65E-18$	$2.61E-15$	$4.28E-19$	$4.33E-24$
0.7	$9.16E-11$	$3.92E-12$	$1.23E-16$	$1.51E-20$	$1.49E-18$	$4.63E-15$	$3.94E-19$	$5.92E-24$
0.8	$1.31E-10$	$4.23E-12$	$1.48E-16$	$3.71E-22$	$1.40E-18$	7	$1.81E-19$	$1.23E-25$
0.9	$1.82E-10$	$3.11E-13$	$1.67E-17$	$1.32E-20$	$1.07E-18$	$4.98E-15$	$5.10E-19$	$4.92E-24$
1	$2.45E-10$	0	0	0	$5.42E-17$	0	0	0



(a) $k = 1$ (b) $k = 2$

Figure 3: Error history of the presented method for various values for M and k of Example 2 with $\alpha = \frac{3}{4}$.



(a) $k = 1$ (b) $k = 2$

Figure 4: Error history of the presented method for various values for M and k of Example 2 with $\alpha = 1$.

8 Conclusion

In this paper, the GQLWM, which is an efficient technique for the numerical solution of nonlinear FVFIDEs, has been proposed. Using this method, the system of nonlinear FVFIDEs was reduced to a system of algebraic equations. The numerical solution of the resulted system was approximated by the Gauss quadrature formula with respect to the Legendre weight function. By solving this nonlinear system, the numerical solution was obtained. Moreover, the convergence, error analysis, existence, and uniqueness of the proposed method were discussed. It was evident that, for a certain value of k, as M increases, the accuracy of the GQLWM is increased. Also, for a certain value of M, as k increases, the accuracy is increased, as well. The method was applied to two examples, and the obtained results were compared to some other well-known methods. This comparison showed that the GQLWM is a suitable and powerful technique for solving the nonlinear FVFIDEs. In the end, we note that the method can be easily extended and applied to multi-dimensional integral equations or systems of FVFIDEs easily with some modifications. We also believe that it shall not be difficult to extend this approach to nonlinear equations of general form, which will be the subject of future researches.

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