Iranian Journal of Numerical Analysis and Optimization
Vol. 13, No. 3, 2023, pp 500–531
https://doi.org/10.22067/ijnao.2022.73126.1209
https://ijnao.um.ac.ir/

How to cite this article
Research Article

Cubic hat-functions approximation for linear and nonlinear fractional integral-differential equations with weakly singular kernels

H. Ebrahimi and J. Biazar*

Abstract

In the current study, a new numerical algorithm is presented to solve a class of nonlinear fractional integral-differential equations with weakly singular kernels. Cubic hat functions (CHFs) and their properties are introduced for the first time. A new fractional-order operational matrix of integration via CHFs is presented. Utilizing the operational matrices of CHFs, the main problem is transformed into a number of trivariate polynomial equations. Error analysis and the convergence of the proposed method are evaluated, and the convergence rate is addressed. Ultimately, three examples are provided to illustrate the precision and capabilities of this algorithm. The numerical results are presented in some tables and figures.

AMS subject classifications (2020): Primary 26A33, Secondary 47N20, 33Exx, 65D15.

Keywords: Fractional integral-differential equations; Numerical algorithms; Weakly singular kernels; Cubic hat functions; Fractional operational matrix.

---

*Corresponding author

Received 25 December 2022; revised 31 March 2023; accepted 12 April 2023

Hamed Ebrahimi
Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran.
e-mail: hamed_ebrahimi@phd.guilan.ac.ir

Jafar Biazar
Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran.
e-mail: biazar@guilan.ac.ir

1 Introduction

The mathematical modeling of many phenomena in various branches of science leads to nonlinear integral-differential equations. Fractional calculus is applied extensively by many scientists in the mathematical modeling and control of numerous dynamic systems [30, 31]. This class of equations arises in the field of signal processing [21], waves and brain modeling [18, 26], radiative equilibrium [17], and so on. Commonly, it is impractical to obtain an analytical solution to integral-differential equations. As a result, the improvement of some numerical methods and the introduction of new high-accuracy numerical algorithms is very important to obtain approximate solutions. So various numerical methods have been developed to solve these types of equations by many researchers. Some of the prominent methods are modified differential transform [15, 20], Adomian decomposition, Homotopy analysis [9, 12], Galerkin [10, 28], collocation [16, 25], product integration [1], Euler wavelets [7], haar wavelets [4], Legendre wavelets [29], Chebyshev wavelets [13], Hermite cubic splines [27], Hat functions [11], Taylor series [8], and so forth. The nonlinear fractional integral-differential equation with a weakly singular kernel appears in the following form:

$$
\begin{aligned}
&{}_0^C D_t^{\alpha} u(t) = g(t) + p(t)u(t) + \lambda \int_0^t (t-s)^{-\beta} u^m(s)ds, \quad \alpha > 0, \quad 0 < \beta < 1, \\
&\quad u^{(i)}(0) = u_0^{(i)}, \quad i = 0, 1, \dots, \lceil \alpha \rceil - 1, \quad m \in \mathbb{N}, \quad t \in I(t),
\end{aligned} \tag{1}
$$

where $u(t)$ is an unknown function to be determined, λ is an appropriate parameter, $g(t)$ and $p(t)$ are known continuous functions on $I(T) := [0, T]$, and ${}_0^C D_t^{\alpha}$ is the Caputo fractional differential operator of order α. Some numerical methods convert such an integral-differential equation into a system of algebraic equations that can be easily solved.

Wang and Zhu [32] applied the second kind of Chebyshev wavelets method to give approximate solutions for the fractional integral-differential equations with a weakly singular kernel. Nemati and Lima [22] applied a numerical method based on modified hat functions (MHFs) for solving the problem (1). Xie et al. [33] used the Haar wavelets to solve a coupled system of fractional-order integral-differential equations. Riahi Beni [29] proposed a novel technique for nonlinear fractional Volterra–Fredholm integro-differential equations. Also, a numerical solution for a fractional integro-differential equation via a method based on the Gegenbauer wavelets was suggested by Özaltun, Konuralp, and Gümgüm [23]. In this paper, we introduce a high-precision numerical algorithm for the problem (1) in terms of Cubic hat-functions (CHFs).

The present work discusses some of the properties of Riemann–Liouville integral operators to solve the nonlinear fractional integral-differential equations. Applying the operational matrix method, the principal problem will be reduced to solving several nonlinear trivariate polynomial equations. In Section 2, some basic definitions and characteristics of fractional calculus are presented. Section 3 is devoted to introducing the operational matrix of CHFs basis. The fourth section studies the absolute error of approximation of a function by a truncated series of CHFs. The fifth section presents a numerical method for Problem (1). The convergence analysis of the proposed scheme is discussed in Section 6. To show the validity and accuracy of the utilized approach, three numerical examples are provided in Section 7, and the paper ends in Section 8, with a conclusion and discussion.

2 Basic concepts and definitions

In this section, some definitions and properties, which have been used in this manuscript, are explained. In this research, the Riemann–Liouville integral operator of the αth order (I_t^{α}) and the Caputo fractional differential operator of order α (${}_0^cD_t^{\alpha}$) will be used. They are well addressed in [24].

Definition 1. Suppose that $\alpha \in R$, $n-1<\alpha\leq n$, $n\in\mathbb{N}$, and let $u(t)$ be a continuous function defined on $[0,1]$. The Caputo fractional derivative of order $\alpha>0$ is defined as follows:

$$
{}_0^CD_t^{\alpha}u(t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{(n-\alpha-1)}\frac{d^n}{d\tau}u(\tau)d\tau, & n-1<\alpha<n,\\ u^{(n)}(t), & \alpha=n,\end{cases} \tag{2}
$$

wherein

$$
\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt.
$$

Definition 2. Assume that $\alpha>0$ and that $u(t)$ is a continuous function defined on the closed interval $[0,1]$. The Riemann–Liouville integral operator of order α is defined as follows:

$$
I_t^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}u(\tau)d\tau. \tag{3}
$$

The Riemann–Liouville integral operator and the Caputo fractional derivative operator satisfy the following properties [24]:

$$
\begin{aligned}
&I_t^{\alpha}(I_t^{\beta}u(t))=I_t^{\beta}(I_t^{\alpha}u(t))=I_t^{\alpha+\beta}u(t),\quad \alpha,\beta>0,\\
&I_t^{\alpha}({}_0^CD_t^{\alpha}u(t))=u(t)-\sum_{i=0}^{n-1}u^{(i)}(0)\frac{t^i}{i!},\quad n-1<\alpha\leq n,\quad t>0.
\end{aligned} \tag{4}
$$

2.1 Definition of CHFs

First, let us state a history of Hat functions and then some definitions and properties of CHFs. In 2011, Babolian and Mordad [6] described generalized Hat functions to solve systems of Fredholm and Volterra integral equations. In 2016, Mirzaee and Hadadiyan [19] introduced MHFs to solve Volterra–Fredholm integral equations. In this paper, we improve the hat functions method and use the method for solving linear and nonlinear fractional integral-differential equations with weakly singular kernels. CHFs are defined on the closed interval $[0, T]$ and have a hat-like shape. The interval is divided into n subintervals, with equal lengths h, where $h = \frac{T}{n}$, and $n = 3K$, $K \in \mathbb{N}$.
CHFs are defined as follows:

$$\phi_0(t) = \begin{cases} -\frac{1}{6h^3}(t-h)(t-2h)(t-3h), & 0 \le t \le 3h, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

For $i = 3\nu - 2$, $\nu = 1, 2, \dots, n/3$,

$$\phi_i(t) = \begin{cases} \frac{1}{2h^3}(t-(i-1)h)(t-(i+1)h)(t-(i+2)h), & (i-1)h \le t \le (i+2)h, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

For $i = 3\nu - 1$, $\nu = 1, 2, \dots, n/3$,

$$\phi_i(t) = \begin{cases} -\frac{1}{2h^3}(t-(i-2)h)(t-(i-1)h)(t-(i+1)h), & (i-2)h \le t \le (i+1)h, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

When $i = 3\nu$, $\nu = 1, 2, \dots, (n-3)/3$,

$$\phi_i(t) = \begin{cases} \frac{1}{6h^3}(t-(i-3)h)(t-(i-2)h)(t-(i-1)h), & (i-3)h \le t \le ih, \\ -\frac{1}{6h^3}(t-(i+1)h)(t-(i+2)h)(t-(i+3)h), & ih \le t \le (i+3)h, \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

and

$$\phi_n(t) = \begin{cases} \frac{1}{6h^3}(t-(n-3)h)(t-(n-2)h)(t-(n-1)h), & (n-3)h \le t \le nh, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

A function $u(t)$ can be expressed in terms of CHFs as follows:

$$u(t) \approx u_n(t) = \sum_{i=0}^{n} a_i \phi_i(t) = A^{\mathrm{T}} \Phi(t) = \Phi(t)^T A, \tag{10}$$

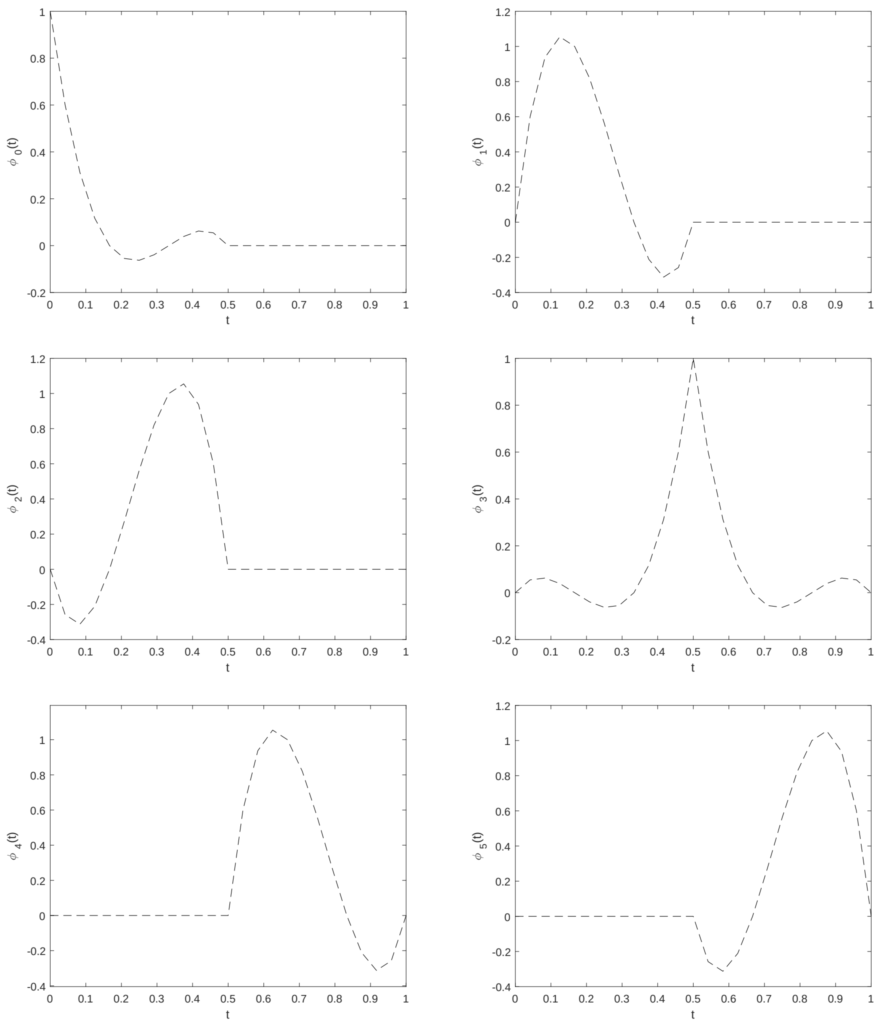
so that

$$\Phi(t) = [\phi_0(t), \phi_1(t), \dots, \phi_n(t)]^T, \tag{11}$$

and

$$A = [a_0, a_1, \ldots, a_n]^T, \tag{12}$$

wherein $a_i = u(ih)$, $i = 0, \ldots, n$, are unknown coefficients of the CHFs. Figure 1 shows the CHFs plotted on the interval $[0, 1]$ for $n = 6$ using MATLAB package.



2.1.1 Properties of CHFs

Using the CHFs definition, the following properties can be obtained:

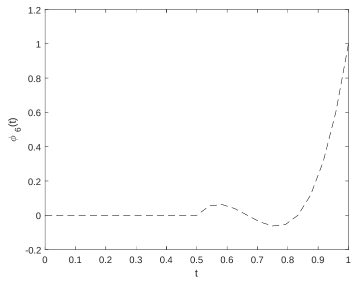


Figure 1: Plots of the CHFs, up to $n = 6, T = 1$

$$\sum_{i=0}^{n} \phi_i(t) = 1, \quad \phi_i(jh) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{13}$$

Multiplying both sides of this summation to $\phi_j(t)$ attains

$$\left(\sum_{i=0}^{n} \phi_j(t)\phi_i(t) \right) = \phi_j(t). \tag{14}$$

Thus, for $t = jh$, we have

$$\begin{aligned} &\sum_{i=0}^{n} \phi_j(jh)\phi_i(jh) = \phi_j(jh), \\ &[(\phi_j(jh)\phi_0(jh)) + \cdots + (\phi_j(jh)\phi_j(jh)) + \cdots + (\phi_j(jh)\phi_n(jh))] = \psi_j(jh), \\ &[(\phi_j(jh) \times 0) + \cdots + (\phi_j(jh) \times \phi_j(jh)) + \cdots + (\phi_j(jh) \times 0)] = \phi_j(jh). \end{aligned} \tag{15}$$

As a result,

$$\phi_j(jh)\phi_j(jh) = \phi_j(jh). \tag{16}$$

Taking these properties, one has

$$\phi_i(jh)\phi_j(jh) \approx \begin{cases} \phi_i(jh), & j = i, \\ 0, & j \neq i. \end{cases} \tag{17}$$

Then, from the relations (17) and (11), it can be concluded that

$$\Phi(t)\Phi^T(t) \simeq \operatorname{diag}\left[\phi_0(t), \phi_1(t), \ldots, \phi_{n-1}(t), \phi_n(t)\right]^T = \operatorname{diag}\left(\Phi(t)\right). \tag{18}$$

2.1.2 Nonlinear approximation of CHFs

Using (18) and (10), $u^m(t)$, $m = 1, 2, \ldots$, can be calculated as follows:

$$u^2(t) \simeq A^T\Phi(t)\Phi^T(t)A = A^T \operatorname{diag}(\Phi(t))A = A^T \operatorname{diag}(A)\Phi(t)$$

$$
\begin{aligned}
&= A_2^T\Phi(t), A_2 = [a_0^2, a_1^2, \dots, a_n^2]^T,\\
u^3(t) &\simeq u^2(t)u(t) = A_2^T\Phi(t)\Phi^T(t)A = A_2^T \operatorname{diag}(\Phi(t))A = A_2^T \operatorname{diag}(A)\Phi(t)\\
&= A_3^T\Phi(t), A_3 = [a_0^3, a_1^3, \dots, a_n^3]^T,\\
&\vdots\\
u^m(t) &\simeq \sum_{i=0}^{n} {a_i}^m \phi_i(t) = A_m^T\Phi(t), \quad A_m = [a_0^m, a_1^m, \dots, a_n^m]^T. \qquad (19)
\end{aligned}
$$

3 Operational matrices of CHFs

In this part of the study, we achieve the fractional-order integral operational matrix using CHFs.

3.1 Fractional order operational matrix of integration

Let us state the following theorem.

Theorem 1. Let $\Phi(t)$ be given by (11) and let $\alpha > 0$. Then

$$I_t^\alpha \Phi(t) \simeq Q^\alpha \Phi(t), \qquad (20)$$

where Q^α is called the $(n+1)\times(n+1)$ operational matrix of fractional integration of order α and is defined as follows:

$$
Q^{(\alpha)} = \frac{h^\alpha}{6\Gamma(\alpha+4)}
\begin{pmatrix}
0 & \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 & \rho_6 & \cdots & \rho_{n-2} & \rho_{n-1} & \rho_n\\
0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \cdots & \sigma_{n-2} & \sigma_{n-1} & \sigma_n\\
0 & \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 & \kappa_5 & \kappa_6 & \cdots & \kappa_{n-2} & \kappa_{n-1} & \kappa_n\\
0 & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \cdots & \mu_{n-2} & \mu_{n-1} & \mu_n\\
0 & 0 & 0 & 0 & \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_{n-5} & \sigma_{n-4} & \sigma_{n-3}\\
0 & 0 & 0 & 0 & \kappa_1 & \kappa_2 & \kappa_3 & \cdots & \kappa_{n-5} & \kappa_{n-4} & \kappa_{n-3}\\
0 & 0 & 0 & 0 & \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n-5} & \mu_{n-4} & \mu_{n-3}\\
0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \sigma_1 & \sigma_2 & \sigma_3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \kappa_1 & \kappa_2 & \kappa_3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \mu_1 & \mu_2 & \mu_3
\end{pmatrix}, \qquad (21)
$$

wherein

$$
\begin{aligned}
\rho_k &= 6k^\alpha(\alpha+3)(\alpha+2)(\alpha+1) - 11k^{\alpha+1}(\alpha+3)(\alpha+2)\\
&\quad + 12k^{\alpha+2}(\alpha+3) - 6k^{\alpha+3}, \quad k = 1, 2, 3,
\end{aligned}
$$

$$
\begin{aligned}
\rho_k &= 6k^{\alpha}(\alpha+3)(\alpha+2)(\alpha+1) - \left(11k^{\alpha+1} - 2(k-3)^{\alpha+1}\right)(\alpha+3)(\alpha+2) \\
&\quad + 6\left(2k^{\alpha+2} + (k-3)^{\alpha+2}\right)(\alpha+3) \\
&\quad - 6\left(k^{\alpha+3} - (k-3)^{\alpha+3}\right), \qquad k = 4,\dots,n, \\
\sigma_k &= 3\left(6(k)^{\alpha+1}(\alpha+3)(\alpha+2) - 10(k)^{\alpha+2}(\alpha+3) + 6(k)^{\alpha+3}\right), \quad k = 1,2,3, \\
\sigma_k &= 9\left(2(k)^{\alpha+1} - (k-3)^{\alpha+1}\right)(\alpha+3)(\alpha+2) \\
&\quad - 6\left(5(k)^{\alpha+2} + 4(k-3)^{\alpha+2}\right)(\alpha+3) \\
&\quad + 18\left((k)^{\alpha+3} - (k-3)^{\alpha+3}\right), \quad k = 4,\dots,n, \\
\kappa_k &= -3\left(3(k)^{\alpha+1}(\alpha+3)(\alpha+2) - 8(k)^{\alpha+2}(\alpha+3) + 6(k)^{\alpha+3}\right), \quad k = 1,2,3, \\
\kappa_k &= -9\left((k)^{\alpha+1} - 2(k-3)^{\alpha+1}\right)(\alpha+3)(\alpha+2) \\
&\quad + 6\left(4(k)^{\alpha+2} + 5(k-3)^{\alpha+2}\right)(\alpha+3) \\
&\quad - 18\left((k)^{\alpha+3} - (k-3)^{\alpha+3}\right), \qquad k = 4,\dots,n, \\
\mu_k &= 2(k)^{\alpha+1}(\alpha+3)(\alpha+2) - 6(k)^{\alpha+2}(\alpha+3) + 6(k)^{\alpha+3}, \qquad k = 1,2,3, \\
\mu_k &= 2\left((k)^{\alpha+1} - 11(k-3)^{\alpha+1}\right)(\alpha+3)(\alpha+2) - 6(k)^{\alpha+2}(\alpha+3) \\
&\quad + 6\left((k)^{\alpha+3} - 2(k-3)^{\alpha+3}\right), \qquad k = 4,5,6, \\
\mu_k &= 2\left((k)^{\alpha+1} - 11(k-3)^{\alpha+1} + (k-6)^{\alpha+1}\right)(\alpha+3)(\alpha+2) \\
&\quad - 6\left((k)^{\alpha+2} - (k-6)^{\alpha+2}\right)(\alpha+3) \\
&\quad + 6\left((k)^{\alpha+3} - 2(k-3)^{\alpha+3} + (k-6)^{\alpha+3}\right), \quad k = 7,\dots,n. \qquad (22)
\end{aligned}
$$

Proof. First, for $\phi_i(t)$, $i = 0,\dots,n$, we have the definition of the Riemann–Liouville integral operator as follows:

$$
I_t^{\alpha}\phi_i(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\phi_i(\tau)d\tau. \qquad (23)
$$

We expand $I_t^{\alpha}\phi_i(t)$, in terms of the cubic hat basis functions as follows:

$$
I_t^{\alpha}\phi_i(t) \simeq \sum_{j=0}^{n}\gamma_{ij}\phi_j(t), \quad i = 0,\dots,n, \qquad (24)
$$

where the values of $I_t^{\alpha}\phi_i(t)$ at jth node point, (jh), represent the coefficients γ_{ij}. Thus, we have

$$\gamma_{ij} = \frac{1}{\Gamma(\alpha)} \int_0^{jh} (jh-\tau)^{\alpha-1} \phi_i(\tau) d\tau, \quad i,j = 0,1,\dots,n. \tag{25}$$

Using equations (5)–(9), we calculate the integral (25). For $i = 0$, by substituting (5) in (25), we introduce the coefficient as follows:

$$\gamma_{0j} = -\frac{h^\alpha}{6\Gamma(\alpha+4)} \begin{cases} \begin{pmatrix} 6j^{\alpha+3} - 12j^{\alpha+2}(\alpha+3) \\ +11j^{\alpha+1}(\alpha+3)(\alpha+2) \\ -6j^{\alpha}(\alpha+3)(\alpha+2)(\alpha+1), \end{pmatrix}, & j \le 2, \\ \\ \begin{pmatrix} 6\left(j^{\alpha+3} - (j-3)^{\alpha+3}\right) \\ -6\left(2j^{\alpha+2} + (j-3)^{\alpha+2}\right)(\alpha+3) \\ +\left(11j^{\alpha+1} - 2(j-3)^{\alpha+1}\right)(\alpha+3)(\alpha+2) \\ -6j^{\alpha}(\alpha+3)(\alpha+2)(\alpha+1), \end{pmatrix}, & j \ge 3. \end{cases} \tag{26}$$

For $i = 3\nu - 2$, $\nu = 1,2,\dots,n/3$, we obtain

$$\gamma_{ij} = \frac{h^\alpha}{2\Gamma(\alpha+4)} \begin{cases} 0, & j \le i-1, \\ \begin{pmatrix} 6(j-i+1)^{\alpha+3} \\ -10(j-i+1)^{\alpha+2}(\alpha+3) \\ +6(j-i+1)^{\alpha+1}(\alpha+3)(\alpha+2) \end{pmatrix}, & i \le j \le i+2, \\ \begin{pmatrix} 6\left((j-i+1)^{\alpha+3} - (j-i-2)^{\alpha+3}\right) \\ -2\left(5(j-i+1)^{\alpha+2}\right. \\ \left. +4(j-i-2)^{\alpha+2}\right)(\alpha+3) \\ +3\left(2(j-i+1)^{\alpha+1}\right. \\ \left. -(j-i-1)^{\alpha+1}\right)(\alpha+3)(\alpha+2) \end{pmatrix}, & j \ge i+3. \end{cases} \tag{27}$$

For $i = 3\nu - 1$, $\nu = 1,2,\dots,n/3$, replacing (7) into Eq. (25) yields

$$\gamma_{ij} = -\frac{h^\alpha}{2\Gamma(\alpha+4)} \begin{cases} 0, & j \le i-2, \\ \begin{pmatrix} 6(j-i+2)^{\alpha+3} \\ -8(j-i+2)^{\alpha+2}(\alpha+3) \\ +3(j-i+2)^{\alpha+1}(\alpha+3)(\alpha+2) \end{pmatrix}, & i-1 \le j \le i+1, \\ \begin{pmatrix} 6\left((j-i+2)^{\alpha+3} - (j-i-1)^{\alpha+3}\right) \\ -2\left(4(j-i+2)^{\alpha+2} \right. \\ \left. +5(j-i-1)^{\alpha+2}\right)(\alpha+3) \\ +3\left((j-i+1)^{\alpha+1} \right. \\ \left. -2(j-i-1)^{\alpha+1}\right)(\alpha+3)(\alpha+2) \end{pmatrix}, & j \ge i+2. \end{cases} \tag{28}$$

Now, we attain (25) for $i = 3\nu$, $\nu = 1, 2, \ldots, n/3$,

$$\gamma_{ij} = \frac{h^\alpha}{6\Gamma(\alpha+4)} \begin{cases} 0, & j \le i-3, \\ \begin{pmatrix} 6(j-i+3)^{\alpha+3} \\ -6(j-i+3)^{\alpha+2}(\alpha+3) \\ +2(j-i+3)^{\alpha+1}(\alpha+3)(\alpha+2) \end{pmatrix}, & i-2 \le j \le i, \\ \begin{pmatrix} 6\left((j-i+3)^{\alpha+3} - 2(j-i)^{\alpha+3}\right) \\ -6(j-i+3)^{\alpha+2}(\alpha+3) \\ +2\left((j-i+3)^{\alpha+1} \right. \\ \left. -11(j-i)^{\alpha+1}\right)(\alpha+3)(\alpha+2) \end{pmatrix}, & i+1 \le j \le i+3, \\ \begin{pmatrix} 6\left((j-i+3)^{\alpha+3} \right. \\ \left. -2(j-i)^{\alpha+3} + (j-i-3)^{\alpha+3}\right) \\ -6\left((j-i+3)^{\alpha+2} \right. \\ \left. -(j-i-3)^{\alpha+2}\right)(\alpha+3) \\ +2\begin{pmatrix} (j-i+3)^{\alpha+1} \\ -11(j-i)^{\alpha+1} \\ +(j-i-3)^{\alpha+1} \end{pmatrix}(\alpha+3)(\alpha+2) \end{pmatrix}, & j \ge i+3, \end{cases} \tag{29}$$

Consider $3\nu - 2 = i$ in (27), $3\nu - 1 = i$ in (28), and $3\nu = i$ in (29). Then apply $3\nu + k = j$ to all (26)–(29), $\nu = 1, \ldots, n/3$ and $k = 1, \ldots, n$. Some simple manipulations completes the proof. □

As a result of using (10) and (20), we can approximate the integral of a nonlinear function as follows:

$$I_t^{\alpha}u^m(t) \simeq I_t^{\alpha}\left(\sum_{i=0}^{n} a_i^m \phi_i(t)\right) \simeq I_t^{\alpha}\left(A_m^T\Phi(t)\right) \simeq A_m^T Q^{\alpha}\Phi(t), \quad ,m = 1,2,\ldots. \tag{30}$$

For instance, when $\alpha = 1$ and $n = 3$, using the operational matrix (21), we get

$$\begin{cases} \text{Examples} & \text{Composite trapezoidal rule} \quad \text{CHFs} \\ & \text{solutions with } n=6 \quad \text{solutions} \\ \text{Example 1}: \int_0^1 \sin s \, \cos^3 s \; ds & 0.2251 \quad 0.2327 \\ \text{Example 2}: \int_0^{\frac{2}{3}} 2^x x^3 ds & 0.0742 \quad 0.0730 \\ \text{Example 3}: \int_0^1 x^{\frac{1}{2}} \ln(x+1)\, ds & 0.3055 \quad 0.3053 \end{cases} \tag{31}$$

4 Error analysis

In this section, our analysis shows that when using CHFs to approximate a function, the order of accuracy is $O(h^4)$. Let us approximate a function $u(t)$, as (10), where

$$u_n(t) = \sum_{i=0}^{n} u(ih)\phi_i(t), \quad n = 3K, \quad K \in \mathbb{N}. \tag{32}$$

In the first step, for $t \in (jh,\, (j+1)h)$, $j = 0,\, 3,\, 6, \ldots, n-3$, using (5)–(9) and doing some computation, we obtain

$$\begin{aligned} u_n(t) &= \sum_{i=0}^{n} u(jh)\phi_i(t) \\ &= \phi_j(t)u\left(jh\right) + \phi_{j+1}(t)u\left((j+1)h\right) \\ &\quad + \phi_{j+2}(t)u\left((j+2)h\right) + \phi_{j+3}(t)u\left((j+3)h\right) \\ &= u(jh)\left(\frac{\left(t-(j+1)h\right)\left(t-(j+2)h\right)\left(t-(j+3)h\right)}{-6h^3}\right) \\ &\quad + u(jh+h)\left(\frac{\left(t-jh\right)\left(t-(j+2)h\right)\left(t-(j+3)h\right)}{2h^3}\right) \\ &\quad + u(jh+2h)\left(\frac{\left(t-jh\right)\left(t-(j+1)h\right)\left(t-(j+3)h\right)}{-2h^3}\right) \\ &\quad + u(jh+3h)\left(\frac{\left(t-jh\right)\left(t-(j+1)h\right)\left(t-(j+2)h\right)}{6h^3}\right). \end{aligned}$$

Then by simplifying the current relationship, we have

$$
\begin{aligned}
u_n(t) = u(jh) &\left(\frac{(t-jh)^3 - 6h(t-jh)^2 + 11h^2(t-jh) - 6h^3}{-6h^3} \right) \\
&+ u(jh+h) \left(\frac{(t-jh)^3 - 5h(t-jh)^2 + 6h^2(t-jh)}{2h^3} \right) \\
&+ u(jh+2h) \left(\frac{(t-jh)^3 - 4h(t-jh)^2 + 3h^2(t-jh)}{-2h^3} \right) \\
&+ u(jh+3h) \left(\frac{(t-jh)^3 - 3h(t-jh)^2 + 2h^2(t-jh)}{6h^3} \right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
u_n(t) = u(jh)& \\
&+ (t-jh) \left(\frac{-11u(jh) + 18u(jh+h) - 9u(jh+2h) + 2u(jh+3h)}{6h} \right) \\
&+ \frac{(t-jh)^2}{2} \left(\frac{2u(jh) - 5u(jh+h) + 4u(jh+2h) - u(jh+3h)}{h^2} \right) \\
&+ \frac{(t-jh)^3}{6} \left(\frac{-u(jh) + 3u(jh+h) - 3u(jh+2h) + u(jh+3h)}{h^3} \right).
\end{aligned}
$$

It is known that the kth, $k = 1, 2, 3$, order derivative of $u(t)$ about the point (jh) is as follows:

$$
\begin{aligned}
u'(jh) &= \frac{-11u(jh) + 18u(jh+h) - 9u(jh+2h) + 2u(jh+3h)}{6h} + O(h^4), \\
u''(jh) &= \frac{2u(jh) - 5u(jh+h) + 4u(jh+2h) - u(jh+3h)}{h^2} + O(h^4), \\
u'''(jh) &= \frac{-u(jh) + 3u(jh+h) - 3u(jh+2h) + u(jh+3h)}{h^3} + O(h^4).
\end{aligned}
$$

So, assuming $h \to 0$ results in

$$
u_n(t) = u(jh) + (t-jh)u'(jh) + \frac{(t-jh)^2}{2} u''(jh) + \frac{(t-jh)^3}{6} u'''(jh) + O(h^4). \tag{33}
$$

Expanding $u(t)$ in the Taylor's series, about the point $t = jh$, we have

$$
u(t) = \sum_{k=0}^{3} \frac{(t-jh)^k}{k!} u^{(k)}(jh) + O(t-jh)^4. \tag{34}
$$

According to (33) and (34), we can obtain the error between the exact and approximate values of $u(t)$ as follows:

$$u(t) - u_n(t) = O(t - jh)^4. \tag{35}$$

Thus, for $t \to jh$, $j = 0, 3, 6, \ldots, n-3$ and $h \to 0$, since $(t - jh) < h$, from (35), we attain

$$|u(t) - u_n(t)| = O(h^4). \tag{36}$$

In the second step, for $j = 1, 4, 7, \ldots, n-2$ and $jh < t < (j+1)h$, we get

$$\begin{aligned}
u_n(t) &= \sum_{i=0}^{n} u(jh)\phi_i(t) \\
&= \phi_{j-1}(t)u\left((j-1)h\right) + \phi_j(t)u\left(jh\right) \\
&\quad + \phi_{j+1}(t)u\left((j+1)h\right) + \phi_{j+2}(t)u\left((j+2)h\right) \\
&= u(jh - h)\left(\frac{(t - jh)\left(t - (j+1)h\right)\left(t - (j+2)h\right)}{-6h^3}\right) \\
&\quad + u(jh)\left(\frac{\left(t - (j-1)h\right)\left(t - (j+1)h\right)\left(t - (j+2)h\right)}{2h^3}\right) \\
&\quad + u(jh + h)\left(\frac{\left(t - (j-1)h\right)(t - jh)\left(t - (j+2)h\right)}{-2h^3}\right) \\
&\quad + u(jh + 2h)\left(\frac{\left(t - (j-1)h\right)(t - jh)\left(t - (j+1)h\right)}{6h^3}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
u_n(t) = u(jh)& \\
&+ (t - jh)\left(\frac{-2u(jh-h) - 3u(jh) + 6u(jh+h) - u(jh+2h)}{6h}\right) \\
&+ \frac{(t - jh)^2}{2}\left(\frac{u(jh-h) - 2u(jh) + u(jh+h)}{h^2}\right) \\
&+ \frac{(t - jh)^3}{6}\left(\frac{-u(jh-h) + 3u(jh) - 3u(jh+h) + u(jh+2h)}{h^3}\right).
\end{aligned} \tag{37}$$

As a reminder, the derivatives of $u(t)$ about the point jh are as follows:

$$\begin{aligned}
u'(jh) &= \frac{-2u(jh-h) - 3u(jh) + 6u(jh+h) - u(jh+2h)}{6h} + O(h^4), \\
u''(jh) &= \frac{u(jh-h) - 2u(jh) + u(jh+h)}{h^2} + O(h^4), \\
u'''(jh) &= \frac{-u(jh-h) + 3u(jh) - 3u(jh+h) + u(jh+2h)}{h^3} + O(h^4).
\end{aligned} \tag{38}$$

As $h \to 0$, from (37)–(38), we get

$$u_n(t) = u(jh) + (t - jh)u'(jh) + \frac{(t-jh)^2}{2}u''(jh) + \frac{(t-jh)^3}{6}u'''(jh) + O(h^4). \tag{39}$$

Considering (39), (34), and $h \to 0$ for $j = 1, 4, 7, \ldots, n-2$, one has

$$|u(t) - u_n(t)| = O(h^4). \tag{40}$$

In the final step, for $j = 2, 5, 8, \ldots, n-1$ and $t \in (jh, (j+1)h)$, we attain

$$\begin{aligned}
u_n(t) &= \sum_{i=0}^{n} u(jh)\phi_i(t) \\
&= \phi_{j-2}(t)u\left((j-2)h\right) + \phi_{j-1}(t)u\left((j-1)h\right) \\
&\quad + \phi_j(t)u\left(jh\right) + \phi_{j+1}(t)u\left((j+1)h\right) \\
&= u(jh-2h)\left(\frac{(t-(j-1)h)\,(t-jh)\,(t-(j+1)h)}{-6h^3}\right) \\
&\quad + u(jh-h)\left(\frac{(t-(j-2)h)\,(t-jh)\,(t-(j+1)h)}{2h^3}\right) \\
&\quad + u(jh)\left(\frac{(t-(j-2)h)\,(t-(j-1)h)\,(t-(j+1)h)}{-2h^3}\right) \\
&\quad + u(jh+h)\left(\frac{(t-(j-2)h)\,(t-(j-1)h)\,(t-jh)}{6h^3}\right).
\end{aligned}$$

As a result,

$$\begin{aligned}
u_n(t) = u(jh) &+ (t-jh)\left(\frac{u(jh-2h) - 6u(jh-h) + 3u(jh) + 2u(jh+h)}{6h}\right) \\
&+ \frac{(t-jh)^2}{2}\left(\frac{u(jh-h) - 2u(jh) + u(jh+h)}{h^2}\right) \\
&+ \frac{(t-jh)^3}{6}\left(\frac{-u(jh-2h) + 3u(jh-h) - 3u(jh) + u(jh+h)}{h^3}\right).
\end{aligned} \tag{41}$$

On the other hand, according to the derivatives of $u(t)$ about the point jh, (41) can be written as follows:

$$u_n(t) = u(jh) + (t - jh)u'(jh) + \frac{(t-jh)^2}{2}u''(jh) + \frac{(t-jh)^3}{6}u'''(jh) + O(h^4). \tag{42}$$

Thus, assuming (42), (34), and $h \to 0$, for $j = 2, 5, 8, \ldots, n-1$, one has

$$|u(t) - u_n(t)| = O(h^4). \tag{43}$$

Finally, for $t \in (jh,(j+1)h)$, $j = 0,1,2,\dots,n$, and $h \to 0$, using (36), (40), and (43), we get

$$|u(t) - u_n(t)| = O(h^4). \tag{44}$$

5 Numerical algorithm

In this section, a numerical algorithm is offered to solve problem (1). Consider the following nonlinear fractional integral-differential equation with weakly singular kernel :

$$\begin{gathered} {}_0^C D_t^{\alpha} u(t) = g(t) + p(t)u(t) + \lambda \int_0^t (t-s)^{-\beta} u^m(s)ds, \\ \alpha > 0, \quad 0 < \beta < 1, \quad m \in \mathbb{N}, \quad t \in I(t). \end{gathered} \tag{45}$$

First, putting $-\beta = \omega - 1$, $0 < \omega < 1$ in the third term on the right of this equation, we get

$$\int_0^t (t-s)^{-\beta} u^m(s)ds = \Gamma(\omega) \left(\frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} u^m(s)ds \right), \quad 0 < \omega < 1. \tag{46}$$

By the definition of Riemann–Liouville fractional integral operator, [24], the current relationship can be rewritten as follows:

$$\int_0^t (t-s)^{-\beta} u^m(s)ds = \Gamma(\omega) I_t^{\omega} \left(u^m(t) \right). \tag{47}$$

Now, by applying (47), the Riemann–Liouville integral operator of order α on the both sides of (45), one gets

$$\begin{aligned} &I_t^{\alpha} \left({}_0^C D_t^{\alpha} u(t) \right) = I_t^{\alpha} \left(g(t) \right) + I_t^{\alpha} \left(p(t)u(t) \right) + \lambda \Gamma(\omega) I_t^{\alpha} I_t^{\omega} \left(u^m(t) \right), \\ &u(t) = z(t) + I_t^{\alpha} \left(g(t) \right) + I_t^{\alpha} \left(p(t)u(t) \right) + \lambda \Gamma(\omega) I_t^{\alpha+\omega} \left(u^m(t) \right), \end{aligned} \tag{48}$$

where

$$z(t) = \sum_{i=0}^{\lceil \alpha \rceil - 1} u^{(i)}(0) \frac{t^i}{i!}, \quad \alpha > 0.$$

Now, by approximating the functions in (48) by CHFs (10) and (19), we attain

$$u(t) \simeq \sum_{i=0}^{n} a_i \phi_i(t) = A^T \Phi(t), \quad A_m = [a_0, a_1, \dots, a_n]^T. \tag{49}$$

$$u^m(t) \simeq \sum_{i=0}^{n} {a_i}^m \phi_i(t) = A_m^T \Phi(t), \quad A_m = [a_0^m, a_1^m, \dots, a_n^m]^T. \tag{50}$$

$$z(t) \simeq \sum_{i=0}^{n} z(ih)\phi_i(t) = z^T\Psi(t), \quad Z = [z(0), z(h), \ldots, z(nh)]^T, \tag{51}$$

$$g(t) \simeq \sum_{i=0}^{n} g(ih)\phi_i(t) = G^T\Psi(t), \quad G = [g(0), g(h), \ldots, g(nh)]^T, \tag{52}$$

and

$$p(t) \simeq \sum_{i=0}^{n} p(ih)\phi_i(t) = p^T\Psi(t), \quad P = [p(0), p(h), \ldots, p(nh)]^T, \tag{53}$$

wherein n is an integer multiple of 3. Utilizing (20)–(22), and (18) and the substitution of (49)–(53) in (48) become

$$A^T\Phi(t) = Z^T\Phi(t) + I_t^\alpha(G^T\Phi(t)) + I_t^\alpha(P^T\Phi(t)\Phi(t)^T\mathrm{A}) + \lambda\Gamma(\omega)I_t^{\alpha+\omega}(\mathrm{A}_m^T\Phi(t)),$$
$$A^T\Phi(t) - Z^T\Phi(t) - I_t^\alpha(G^T\Phi(t)) - I_t^\alpha(P^T diag\,(\Phi(t))\,\mathrm{A}) - \lambda\Gamma(\omega)I_t^{\alpha+\omega}(\mathrm{A}_m^T\Phi(t)) = 0,$$
$$A^T\Phi(t) - Z^T\Phi(t) - I_t^\alpha(G^T\Phi(t)) - I_t^\alpha(P^T diag\,(A)\,\Phi(t)) - \lambda\Gamma(\omega)I_t^{\alpha+\omega}(\mathrm{A}_m^T\Phi(t)) = 0,$$
$$A^T\Phi(t) - Z^T\Phi(t) - G^TQ^{(\alpha)}\Phi(t) \; - P^T diag\,(A)\,Q^{(\alpha)}\Phi(t) - \lambda\Gamma(\omega)A_m^TQ^{(\alpha+\omega)}\Phi(t) = 0.$$

Thus

$$A^T - Z^T - G^TQ^{(\alpha)} \; - P^T\,\mathrm{diag}\,(A)\,Q^{(\alpha)} - \lambda\Gamma(\omega)A_m^TQ^{(\alpha+\omega)} = 0,$$
$$\alpha > 0, \quad 0 < \omega < 1, \quad \omega = 1 - \beta. \tag{54}$$

This system has the dimension $(n+1) \times (n+1)$.
Suppose that

$$Q^{(\alpha)} = [\gamma_{ij}]\,, \quad Q^\omega = [\theta_{ij}]\,, \quad i,j = 0,1,2,\ldots,n. \tag{55}$$

Then, from the operational matrix (23), one gets

$$\gamma_{i0} = \theta_{i0} = 0, \quad i = 0,1,2,\ldots,n,$$
$$\gamma_{ij} = \theta_{ij} = 0, \quad j = 1,3,\ldots,n-1, \quad i = j+3, j+4, \ldots, n,$$
$$\gamma_{ij} = \theta_{ij} = 0, \quad j = 2,4,\ldots,n, \quad i = j+2, j+3, \ldots, n.$$

Using (54), the unknown coefficients can be determined. We start to find the first unknown coefficient as follows:

$$a_0 = z(0). \tag{56}$$

In the next step, we get

$$
\textbf{system 1:}\begin{cases}
eq_1 : [a_1] - [z(1h)] - \left[\sum_{i=0}^{3} g(ih)\gamma_{i1}\right] \\
\quad - \left[\sum_{i=0}^{3} p(ih)\gamma_{i1}a_i\right] - \left[\lambda\Gamma(\omega)\sum_{i=0}^{3}\theta_{i1}{a_i}^m\right] = 0, \\
eq_2 : [a_2] - [z(2h)] - \left[\sum_{i=0}^{3} g(ih)\gamma_{i2}\right] \\
\quad - \left[\sum_{i=0}^{3} p(ih)\gamma_{i2}a_i\right] - \left[\lambda\Gamma(\omega)\sum_{i=0}^{3}\theta_{i2}{a_i}^m\right] = 0, \\
eq_3 : [a_3] - [z(3h)] - \left[\sum_{i=0}^{3} g(ih)\gamma_{i3}\right] \\
\quad - \left[\sum_{i=0}^{3} p(ih)\gamma_{i3}a_i\right] - \left[\lambda\Gamma(\omega)\sum_{i=0}^{3}\theta_{i3}{a_i}^m\right] = 0.
\end{cases}
$$

Solving the first system allows us to calculate a_1, a_2, and a_3, then we solve the following system:

$$
\textbf{system 2:}\begin{cases}
eq_4 : [a_4] - [z(4h)] - \left[\sum_{i=0}^{6} g(ih)\gamma_{i4}\right] \\
\quad - \left[\sum_{i=0}^{6} p(ih)\gamma_{i4}a_i\right] - \left[\lambda\Gamma(\omega)\sum_{i=0}^{6}\theta_{i4}{a_i}^m\right] = 0, \\
eq_5 : [a_5] - [z(5h)] - \left[\sum_{i=0}^{6} g(ih)\gamma_{i5}\right] \\
\quad - \left[\sum_{i=0}^{6} p(ih)\gamma_{i5}a_i\right] - \left[\lambda\Gamma(\omega)\sum_{i=0}^{6}\theta_{i5}{a_i}^m\right] = 0, \\
eq_6 : [a_6] - [z(6h)] - \left[\sum_{i=0}^{6} g(ih)\gamma_{i6}\right] \\
\quad - \left[\sum_{i=0}^{6} p(ih)\gamma_{i6}a_i\right] - \left[\lambda\Gamma(\omega)\sum_{i=0}^{6}\theta_{i6}{a_i}^m\right] = 0.
\end{cases}
$$

By solving **system 2**, the values of the unknown parameters a_4, a_5, and a_6 are calculated. Then we can get the values of a_7, a_8, and a_9 using **system 3**:

$$
\textbf{system 3:}\begin{cases}
eq_7 : [a_7] - [z(7h)] - \left[\sum_{i=0}^{9} g(ih)\gamma_{i7}\right] \\
\quad \left[\sum_{i=0}^{9} p(ih)\gamma_{i7}a_i\right] - \left[\lambda\Gamma(\omega)\sum_{i=0}^{9}\theta_{i7}{a_i}^m\right] = 0, \\
eq_8 : [a_8] - [z(8h)] - \left[\sum_{i=0}^{9} g(ih)\gamma_{i8}\right] \\
\quad - \left[\sum_{i=0}^{9} p(ih)\gamma_{i8}a_i\right] - \left[\lambda\Gamma(\omega)\sum_{i=0}^{9}\theta_{i8}{a_i}^m\right] = 0, \\
eq_9 : [a_9] - [z(9h)] - \left[\sum_{i=0}^{9} g(ih)\gamma_{i9}\right] \\
\quad - \left[\sum_{i=0}^{9} p(ih)\gamma_{i9}a_i\right] - \left[\lambda\Gamma(\omega)\sum_{i=0}^{9}\theta_{i9}{a_i}^m\right] = 0.
\end{cases}
$$

The process can be continued up to the following form:

$$\textbf{system n/3:}\begin{cases} eq_{n-2}:[a_{n-2}]-[z((n-2)h)]-\left[\sum_{i=0}^{n}g(ih)\gamma_{i(n-2)}\right] \\ \quad -\left[\sum_{i=0}^{n}p(ih)\gamma_{i(n-2)}a_i\right]-\left[\lambda\Gamma(\omega)\sum_{i=0}^{n}\theta_{i(n-2)}{a_i}^m\right]=0, \\ eq_{n-1}:[a_{n-1}]-[z((n-1)h)]-\left[\sum_{i=0}^{n}g(ih)\gamma_{i(n-1)}\right] \\ \quad -\left[\sum_{i=0}^{n}p(ih)\gamma_{i(n-1)}a_i\right]-\left[\lambda\Gamma(\omega)\sum_{i=0}^{n}\theta_{i(n-1)}{a_i}^m\right]=0, \\ eq_{n}:[a_{n}]-[z(nh)]-\left[\sum_{i=0}^{n}g(ih)\gamma_{in}\right] \\ \quad -\left[\sum_{i=0}^{n}p(ih)\gamma_{in}a_i\right]-\left[\lambda\Gamma(\omega)\sum_{i=0}^{n}\theta_{in}{a_i}^m\right]=0. \end{cases}$$

As a result, the values of a_{n-2}, a_{n-1}, and a_n are derived using system $n/3$. Therefore, we can obtain an approximate solution via (10). To solve the nonlinear equations, see [34]. The computations were handled by MATLAB package. The following theorem outlines the proposed method.

Theorem 2. Consider the principal problem (1). To obtain a numerical solution to (1) using CHFs, the following iterative algorithm is offered:

Proof. See the scheme proposed in this section, (56)–(57). □

6 Convergence analysis

In this section, we will verify the convergence of the numerical proposed scheme.

Theorem 3. Let $u_n(t)$ be the numerical solution of (1) obtained by the proposed method in Section (5). Moreover, $u(t)$ is an exact solution and $E_n(t)$ is the residual error for numerical solution. Also, suppose that M and K are positive constants. Then, $E_n(t)$ tends to zero, as $n \to \infty$, where

$$M=\sup_{t,\tau\in[0,T]}\left|\Gamma^{-1}(\alpha)(t-\tau)^{\alpha-1}p(\tau)\right|,$$
$$K=\sup_{t,\tau\in[0,T]}\left|\lambda mL\Gamma^{-1}(\alpha+\omega)(t-\tau)^{\alpha+\omega-1}\right|.$$

Proof. Applying the Riemann–Liouville integral operator of order α and (48), it is appropriate to rewrite (1) in the integral form

$$u(t)=z(t)+I_t^{\alpha}\left(g(t)\right)+I_t^{\alpha}\left(p(t)u(t)\right)+\lambda\Gamma(\omega)I_t^{\alpha+\omega}\left(u^m(t)\right), \tag{57}$$

where

Algorithm 1 An algorithm for approximation using CHsF

Step 1: Inputs, n (integer multiple of 3), $\alpha, \beta, \lambda, T, g(t), p(t)$, $u^{(i)}(0), \quad i = 0, 1, \dots, \lceil \alpha \rceil - 1$.

Step 2: Set $\omega = 1 - \beta$, $h = T/n$, and $t_i = ih$, $i = 0, \dots, n$.

Step 3: $z(t) = \sum_{i=0}^{n-1} u^{(i)}(0) \frac{t^i}{i!}$.

Step 4: Compute the elements of $Q^{(\alpha)} = [\gamma_{ij}]$ and $Q^{(\alpha+\omega)} = [\theta_{ij}]$, $i, j = 0, \dots, n$.

Step 5: Set and solve recursive trivariable system v, $v = 1 : n/3$.

$a_0 = z(0)$,

for $v = 1 : n/3$

Solution of the vth system determines the unknown parameter.

$$\text{system}\, v : \begin{cases} \left[a_{(3v-2)}\right] - [z((3v-2)h)] - \left[\sum_{k=0}^{3v} g(ih)\gamma_{i(3v-2)}\right] \\ - \left[\sum_{k=0}^{3v} p(ih)\gamma_{i(3v-2)} a_i\right] \\ - \left[\Gamma(\omega) \sum_{i=0}^{3v} \theta_{i(3v-2)} {a_i}^m\right] = 0, \\ \left[a_{(3v-1)}\right] - [z((3v-1)h)] - \left[\sum_{k=0}^{3v} g(ih)\gamma_{i(3v-1)}\right] \\ - \left[\sum_{i=0}^{3v} p(ih)\gamma_{i(3v-1)} a_i\right] \\ - \left[\Gamma(\omega) \sum_{i=0}^{3v} \theta_{i(3v-1)} {a_i}^m\right] = 0, \\ \left[a_{(3v)}\right] - [z((3v)h)] - \left[\sum_{i=0}^{3v} g(ih)\gamma_{i(3v)}\right] \\ - \left[\sum_{i=0}^{3v} p(ih)\gamma_{i(3v)} a_i\right] \\ - \left[\Gamma(\omega) \sum_{i=0}^{3v} \theta_{i(3v)} {a_i}^m\right] = 0, \end{cases}$$

end.

Step 6: Calculate fully $a_i, \quad i = 0, 1, \dots, n$.

Step 7: Define CHFs: $(\phi_i(t), \quad i = 0, 1, \dots, n)$.

Step 8: Determine the approximate solutions: $u_n(t) = \sum_{i=0}^{n} a_i \phi_i(t)$.

$$z(t) = \sum_{i=0}^{\lceil \alpha \rceil - 1} u^{(i)}(0) \frac{t^i}{i!}, \quad \alpha > 0, \quad \omega = 1 - \beta, \quad 0 < \beta < 1, \quad t \in I(t).$$

Thus, $u_n(t)$ satisfies the following equation:

$$u_n(t) = z(t) + I_t^{\alpha}\left(g(t)\right) + I_t^{\alpha}\left(p(t)u_n(t)\right) + \lambda\Gamma(\omega) I_t^{\alpha+\omega}\left(u_n^m(t)\right). \tag{58}$$

If the residual function $E_n(t)$ is not zero, then we can obtain it by using the following relation:

$$E_n(t) = e_n[u](t) - J_n^{\alpha}[u](t) - V_n^{\alpha+\omega}[u^m](t), \tag{59}$$

where

$$e_n[u](t) = u(t) - u_n(t), \tag{60}$$

$$J_n^{\alpha}[u](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} p(\tau)(u(\tau) - u_n(\tau))d\tau, \tag{61}$$

and

$$V_n^{\alpha+\omega}[u^m](t) = \frac{\lambda}{\Gamma(\alpha+\omega)}\int_0^t (t-\tau)^{\alpha+\omega-1}(u^m(\tau) - u_n^m(\tau))d\tau. \tag{62}$$

Then, we get

$$|E_n(t)| \leq |e_n[u](t)| + |J_n^{\alpha}[u](t)| + \left|V_n^{\alpha+\omega}[u^m](t)\right|. \tag{63}$$

For $t \in (ih,\ (i+3)h), i = 0,\ 3, 6, \ldots, n-3$, according to (44), the approximation of the absolute error using CHFs yields

$$|u(t) - u_n(t)| = O(h^4). \tag{64}$$

By using (60), we have

$$|e_n[u](t)| = O(h^4), \tag{65}$$

when $h \to 0$, $|e_n[u](t)| \to 0$. Then, by using (61) and (64), we attain

$$\begin{aligned} |J_n^{\alpha}[u](t)| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^t (t-\tau)^{\alpha-1} p(\tau)(u(\tau) - u_n(\tau))d\tau\right| \\ &\leq \frac{|p(\tau)|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left|u(\tau) - u_n(\tau)\right| d\tau \\ &\leq \frac{|p(\tau)|}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left|u(\tau) - u_n(\tau)\right| d\tau \\ &\leq MO(h^4), \end{aligned} \tag{66}$$

wherein

$$M = \sup_{t,\tau\in[0,T]}\left|\Gamma^{-1}(\alpha)(t-\tau)^{\alpha-1}p(\tau)\right| \text{ and when} h \to 0, \quad \left|J_n^{\beta-\alpha}[u](t)\right| \to 0.$$

In addition, the following inequality holds [11]:

$$|u^m(t) - u_n^m(t)| \leq mL\,|u(t) - u_n(t)|\,, \tag{67}$$

where $L = \left|(\max(u(t), u_n(t)))^{m-1}\right|$. As well, from (62), (64), and (67), we have

$$\begin{aligned}
\left|V_n^{\alpha+\omega}[u^m](t)\right| &= \frac{1}{\Gamma(\alpha+\omega)}\left|\lambda\int_0^t (t-\tau)^{\alpha+\omega-1}(u^m(\tau) - u_n^m(\tau))d\tau\right| \\
&\leq \frac{|\lambda|}{\Gamma(\alpha+\omega)}\int_0^t (t-\tau)^{\alpha+\omega-1}\left|(u^m(\tau) - u_n^m(\tau))\right| d\tau \\
&\leq \frac{mL\,|\lambda|}{\Gamma(\alpha+\omega)}\int_0^t (t-\tau)^{\alpha+\omega-1}\left|(u(\tau) - u_n(\tau))\right| d\tau \\
&\leq KO(h^4),
\end{aligned} \tag{68}$$

wherein

$K = \sup_{t,\tau\in[0,T]}\left|\lambda mL\Gamma^{-1}(\alpha+\omega)(t-\tau)^{\alpha+\omega-1}\right|$ and as $h \to 0$, $\left|V_n^{\alpha+\omega}[u^m](t)\right| \to 0$.

Then, from relations (65), (66), (68), and (63), it is obvious that $|E_n(t)|$ tends to zero, as $h \to 0$, or $n \to \infty$. □

7 Numerical examples

In this section, the theoretical results of the previous sections are used for solving linear and nonlinear fractional integral-differential equations with the weakly singular kernel, that is, the initial condition equation (1). For assessing the accuracy of the scheme, let us define the maximum absolute error (L_∞-norm error) as

$$\|\xi_n\|_\infty = \sup_{[t_i=ih]_{i=0}^n}\{|u(t_i) - u_n(t_i)|\}\,. \tag{69}$$

Using this definition, the order of convergence, with respect to this norm, is introduced as follows:

$$\text{Rate} = \log_2\left(\frac{\|\xi_n\|_\infty}{\|\xi_{2n}\|_\infty}\right), \tag{70}$$

For some problems, there are no exact solutions, so the L_2-norm error is calculated by the following formula:

$$\widetilde{\|\xi_n\|_2} = \left(\sum_{i=0}^n (u_n(t_i) - u_{2n}(t_i))^2\right)^{\frac{1}{2}}, \quad t_i = ih, \quad i = 0,\dots,n, \tag{71}$$

where $u_n(t),\ n = T/h$ is the approximate solution defined as (49). In addition, the results of different values of α are compared with each other and with MHFs method, [22].

Example 1. Consider a nonlinear integral-differential equation with weakly singular kernel [22]:

$$
\begin{aligned}
{}_{\circ}^{C}D_t^{\alpha}u(t) &= g(t) + p(t)u(t) + \int_0^t (t-s)^{-\beta}u^2(s)ds, \quad t \in [0,1],\\
g(t) &= 3t^2 - \left(\frac{\sqrt{\pi}\Gamma(7)}{\Gamma\left(\frac{15}{6}\right)}\right)t^{\frac{13}{2}}, \quad p(t) = 0, \quad u(0) = 0.
\end{aligned}
$$

For $\alpha = 1$ and $\beta = \frac{1}{2}$, the exact solution is $u(t) = t^3$. Approximate numerical results using different values of n are shown in Tables 1–3 and Figures 2–4. Table 1 shows the approximate and exact solutions to the problem at some points. Also, the L_∞-norm errors and convergence orders obtained by the current method are compared with the MHFs method [22] and the methods presented in [14, 5] in Table 2. Table 3 shows the comparison of the result of the l_2-norm error $\widetilde{\|\xi_n\|_2}$ obtained by the proposed method. It is clear from Table 2 that the results of the present method for less or similar n are better than the results obtained in [22]. Figure 2 indicates the behavior of absolute errors for Example 1. Also, Figure 3 shows the logarithm of the L_∞-norm errors. As can be seen from the plot, as n increases, the error decreases. Also, the comparison of the results obtained for different values of alpha with the exact solutions of the equation is plotted in Figure 4.

Table 1: Numerical results of Example 1

Points s	Exact solutions $u(s)$	Approximate solutions $n = 12$	Approximate solutions $n = 24$	Approximate solutions $n = 48$
0	0.00000000000	0.00000000000	0.00000000000	0.00000000000
$1/12$	0.00057870370	0.00057871890	0.00057870463	0.00057870368
$1/6$	0.00462962963	0.00462979683	0.00462962554	0.00462962982
$1/4$	0.01562500000	0.01562510533	0.01562500074	0.01562499998
$1/3$	0.03703703704	0.03703629382	0.03703707090	0.03703703603
$5/12$	0.07233796296	0.07233970584	0.07233791123	0.07233796502
$1/2$	0.12500000000	0.12500014128	0.12499999725	0.12499999984
$7/12$	0.19849537037	0.19849178055	0.19849551759	0.19849536600
$2/3$	0.29629629630	0.29630236028	0.29629611403	0.29629630318
$3/4$	0.42187500000	0.42187499884	0.42187498758	0.42187499953
$5/6$	0.57870370370	0.57869407976	0.57870407228	0.57870369262
$11/12$	0.77025462963	0.77026833822	0.77025420450	0.77025464509
1	1.00000000000	0.99999965644	0.99999997045	0.99999999902

Table 2: Comparison of the L_∞-norm error and convergence order for Example 1

MHFs method [22]			CHFs method		
n	$\|\xi_n\|_\infty$	Rate of convergence	n	$\|\xi_n\|_\infty$	Rate of convergence
4	$4.81704E-03$	3.93	3	$5.64350E-03$	4.22
8	$3.15569E-04$	3.49	6	$3.03330E-04$	4.47
16	$2.81379E-05$	3.88	12	$1.37086E-05$	4.71
32	$1.90987E-06$	3.93	24	$5.24867E-07$	4.85
64	$1.25460E-07$	3.95	48	$1.82605E-08$	4.92
128	$8.09506E-09$	3.97	96	$6.04389E-10$	4.96
256	$5.16785E-10$	–	192	$1.950051E-11$	–

	Method presented in [14]	Method presented in [5]
n	$\|\xi_n\|_\infty$	$\|\xi_n\|_\infty$
4	$3.52E-09$	$3.5E-04$
8	$1.40E-14$	$1.11022E-16$
16	$1.51E-14$	$1.11022E-16$

Table 3: Numerical results of the L_2-norm error functions $\widetilde{\|\xi_n\|_2}$ for Example 1

n	3	6	12	24	48	96	192
$\widetilde{\|\xi_n\|_2}$	$7.9E-03$	$3.5E-04$	$1.9E-05$	$9.3E-07$	$4.4E-08$	$2.0E-09$	$9.6E-11$

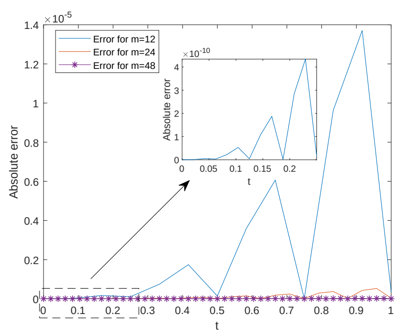


Figure 2: Absolute errors of Example 1, for $n = 12, 24, 48$

Example 2. Consider the following nonlinear fractional integral-differential equation with weakly singular kernel [22] :

$$\begin{aligned} {}^{C}_{\circ}D_t^{\alpha} u(t) &= g(t) + p(t)u(t) + \int_0^t (t-s)^{-\beta} u^2(s)ds, \quad t \in [0,1], \\ g(t) &= \frac{3\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{11}{6}\right)} t^{\frac{5}{6}} - t^{\frac{5}{2}} - \frac{32}{35} t^{\frac{7}{2}}, \\ p(t) &= t, \quad u(0) = 0. \end{aligned}$$

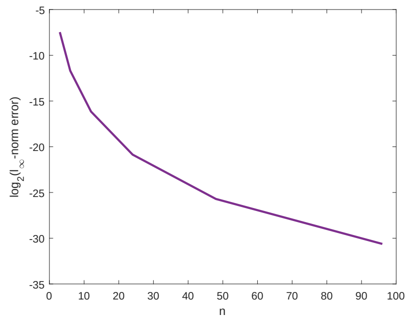


Figure 3: Logarithm of the L_∞-norm error in Example 1

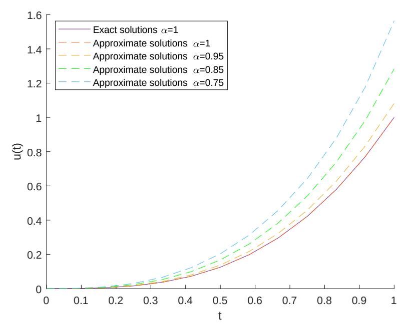


Figure 4: Exact and approximate solutions of Example 1, for $n = 12$

For $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{2}$, the analytic solution is $u(t) = t^{\frac{3}{2}}$. Approximate numerical results using different values of n are shown in Tables 4–6, and Figures 5–7. Table 4 shows the approximate and exact solutions to the problem at some points. Table 5 indicates the L_∞-norm errors and convergence orders for various values of n. Table 6 shows the L_2-norm errors at some values of n. As can be compared in Table 5, this new method provides a higher order of convergence compared to the other method. Figure 5 indicates the absolute errors of Example 2, at $n = 12, 24, 48$. Figure 6 shows that the logarithm of the L_∞-norm error decreases as n increases. A comparison between the changes in the fractional orders of the equation is shown in Figure 7.

Example 3. Consider the following linear fractional integral-differential equation with weakly singular kernel [22]:

$$
{}_{\circ}^{C}D_t^{\alpha}u(t) = g(t) + p(t)u(t) + \int_0^t (t-s)^{-\beta}u(s)ds, \quad t \in [0,1],
$$

$$
g(t) = \frac{6t^{\frac{8}{3}}}{\Gamma\left(\frac{11}{3}\right)} + \left(\frac{32}{35} - \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{17}{6}\right)}\right)t^{\frac{11}{6}} + \Gamma\left(\frac{7}{3}\right)t,
$$

Table 4: Numerical results of Example 2

Points s	Exact solutions $u(s)$	Approximate solutions $n = 12$	Approximate solutions $n = 24$	Approximate solutions $n = 48$
0	0.00000000	0.00000000	0.00000000	0.00000000
$1/12$	0.02405626	0.02340582	0.02394753	0.02400902
$1/6$	0.06804138	0.06771773	0.06790284	0.06800459
$1/4$	0.12500000	0.12449685	0.12487612	0.12496621
$1/3$	0.19245009	0.19199415	0.19232923	0.19241677
$5/12$	0.26895718	0.26849729	0.26883160	0.26892254
$1/2$	0.35355339	0.35305089	0.35341692	0.35351566
$7/12$	0.44552819	0.44495789	0.44537319	0.44548528
$2/3$	0.54433105	0.54365866	0.54414702	0.54428012
$3/4$	0.64951905	0.64868490	0.64929131	0.64945599
$5/6$	0.76072577	0.75964752	0.76043179	0.76064433
$11/12$	0.87764152	0.87619255	0.87724539	0.87753181
1	1.00000000	0.99796079	0.99944320	0.99984577

Table 5: Comparison of the L_∞-norm error and convergence order for Example 2

MHFs method [22]			CHFs method		
n	$\|\xi_n\|_\infty$	Rate of convergence	n	$\|\xi_n\|_\infty$	Rate of convergence
4	$1.39991E-02$	2.08	3	$4.05382E-02$	2.38
8	$3.30803E-03$	1.90	6	$7.80949E-03$	1.94
16	$8.84780E-04$	1.84	12	$2.03921E-03$	1.87
32	$2.46509E-04$	1.83	24	$5.56795E-04$	1.85
64	$6.92324E-05$	1.51	48	$1.54232E-04$	1.84
128	$2.42498E-05$	1.50	96	$4.30102E-05$	1.84
256	$8.57216E-06$	–	192	$1.20325E-05$	–

Table 6: Numerical results of the L_2-norm error functions $\widetilde{\|\xi_n\|_2}$ for Example 2

n	3	6	12	24	48	96	192
$\widetilde{\|\xi_n\|_2}$	$3.5E-02$	$7.2E-03$	$2.4E-03$	$8.5E-04$	$3.3E-04$	$1.3E-04$	$5.0E-05$

$$p(t) = -\frac{32}{35}t^{\frac{1}{2}}, \quad u(0) = 0.$$

For $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$, the analytic solution is $u(t) = t^3 + t^{\frac{4}{3}}$. Tables 7–9 and Figures 8–10 show approximate numerical results using different values of n. Table 7 indicates the approximate and exact solutions to the problem at some grid points. Table 7 shows the advantage of the proposed method compared to the MHF method by presenting the order of convergence and the maximum norm error. Figure 8 shows the behavior of absolute errors for Example 3. Figure 9 shows the logarithm of the L_∞-norm errors. In

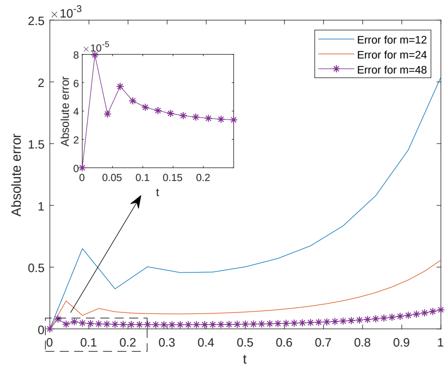


Figure 5: Absolute errors of Example 2, for $n = 12, 24, 48$

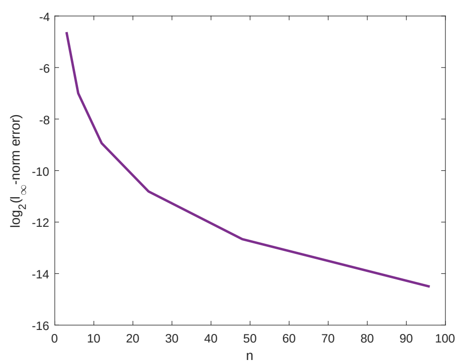


Figure 6: Logarithm of the L_∞-norm error in Example 2

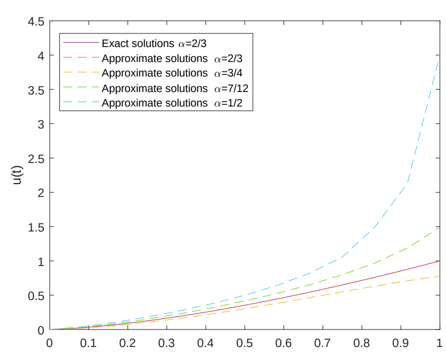


Figure 7: Exact and approximate solutions of Example 2,for $n = 12$

addition, a comparison of the results for different values of α with the exact solution of the equation is shown in Figure 10.

Table 7: Numerical results of Example 3

Points s	Exact solutions $u(s)$	Approximate solutions $n = 12$	Approximate solutions $n = 24$	Approximate solutions $n = 48$
0	0.00000000	0.00000000	0.00000000	0.00000000
$1/12$	0.03697789	0.03708156	0.03699820	0.03698362
$1/6$	0.09634983	0.09644877	0.09637681	0.09635574
$1/4$	0.17311513	0.17323059	0.17314347	0.17312128
$1/3$	0.26815746	0.26828582	0.26818799	0.26816396
$5/12$	0.38354663	0.38368969	0.38357860	0.38355352
$1/2$	0.52185026	0.52199562	0.52188416	0.52185755
$7/12$	0.68589934	0.68605454	0.68593554	0.68590705
$2/3$	0.87868327	0.87885234	0.87872136	0.87869145
$3/4$	1.10329522	1.10346969	1.10333552	1.10330386
$5/6$	1.36290039	1.36308529	1.36294318	1.36290952
$11/12$	1.66071634	1.66091514	1.66076132	1.66072598
1	2.00000000	2.00020619	2.00004743	2.00001015

Table 8: Comparison of the L_∞-norm error and convergence order for Example 3

MHFs method [22]			CHFs method		
n	$\Vert\xi_n\Vert_\infty$	Rate of convergence	n	$\Vert\xi_n\Vert_\infty$	Rate of convergence
4	$1.14967E-03$	1.86	3	$1.68472E-03$	1.20
8	$3.15569E-04$	1.83	6	$7.35848E-04$	1.84
16	$8.90424E-05$	2.20	12	$2.06192E-04$	2.12
32	$1.93665E-05$	2.30	24	$4.74299E-05$	2.22
64	$3.94225E-06$	2.09	48	$1.01541E-05$	2.27
128	$9.26153E-07$	2.13	96	$2.10201E-06$	2.30
256	$2.12088E-07$	–	192	$4.27476E-07$	–

Table 9: Numerical results of the L_2-norm error $\widetilde{\Vert\xi_n\Vert_2}$ for Example 3

n	3	6	12	24	48	96	192
$\widetilde{\Vert\xi_n\Vert_2}$	$1.2E-03$	$1.0E-03$	$4.2E-04$	$1.4E-04$	$4.2E-05$	$1.2E-05$	$3.6E-06$

8 Conclusion

In this paper, we proposed a numerical scheme for solving a class of nonlinear fractional integral-differential equations with weakly singular kernels

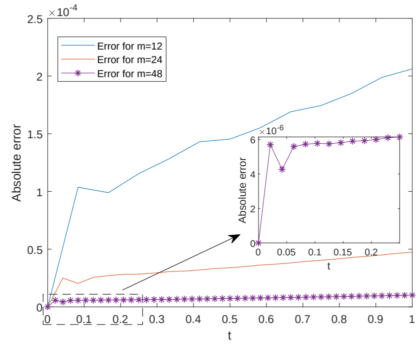


Figure 8: Absolute errors of Example 3, for $n = 12, 24, 48$

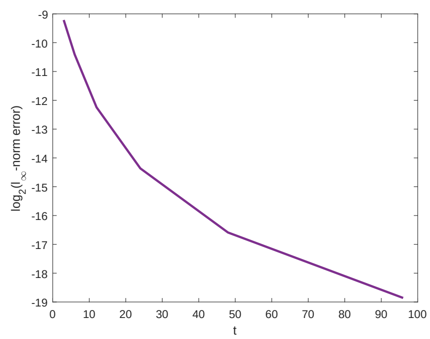


Figure 9: Logarithm of the L_∞-norm error in Example 3

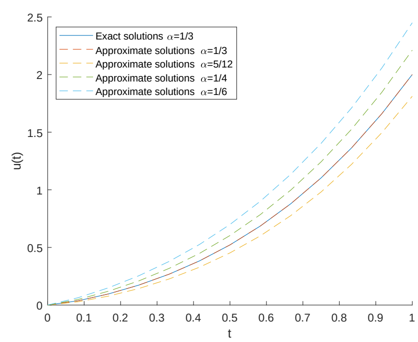


Figure 10: Exact and approximate solutions of Example 3, for $n = 12$

based on CHFs. CHF and the corresponding operational matrix were introduced. The proposed method transforms the original problem into an iterative algorithm, including polynomial equations with three unknown coefficients, using the fractional-order operational matrix of integration. An analysis of the method's absolute errors and convergence was conducted. In order to validate the accuracy and effectiveness of this new method, three numerical examples were presented. In Example 1, the absolute error is lower at the nodal points near the beginning of the interval, as shown in Figure 2. Table 2 shows that the new method offers more accurate solutions at the same lengths h than the MHFs approaches. In Examples 2 and 3, the error clearly increases as the time variable approaches one; see Figures 5 and 8, respectively. A study of the results shows that, generally, as n increases, the accuracy of the approximate solution increases, and the absolute error decreases. One of the advantages of this proposed algorithm is that instead of solving a system of $(n+1)\times(n+1)$ equations, it only needs to solve $n/3$ systems of three-variable nonlinear equations. In addition, the order of convergence for the cubic hat functions is $O(h^4)$, while the order of convergence for the generalized hat functions method [6] and the MHFs method [22] are $O(h^2)$ and $O(h^3)$, respectively. Finally, the proposed method (CHFs) can be used for a large number of similar problems, and we will continue to work on developing this method.

Acknowledgements

We want to thank the anonymous referees for the careful reading of our article and for giving constructive feedback that can significantly help to improve the quality of the article.

References

[1] Abdelaty, A.M., Roshdy, M., Said, L.A. and Radwan, A. G. *Numerical simulations and FPGA implementations of fractional-order systems based on product integration rules*, IEEE Access. 8 (2020), 102093–102105.

[2] Ahmed, S.A., Elzaki, T.M. and Hassan, A.A. *Solution of integral differential equations by new double integral transform (Laplace–Sumudu transform)*, Abstr. Appl. Anal. (2020), Art. ID 4725150, 7.

[3] Al-Ahmad, S., Sulaiman, I.M. and Mamat, M. *An efficient modification of differential transform method for solving integral and integro-differential equations*, Aust. J. Math. Anal. Appl. 17(2) (2020), Art. 5, 15.

[4] Amin, R., Shah, K., Asif, M., Khan, I. and Ullah, F. *An efficient algorithm for numerical solution of fractional integro-differential equations via Haar wavelet*, J. Comput. Appl. Math. 381(113028) (2021), 17.

[5] Arsalan Sajjadi, S., Saberi Najafi, H. and Aminikhah, H. *A numerical study on the non-smooth solutions of the nonlinear weakly singular fractional Volterra integro-differential equations*, Math. Methods Appl. Sci. 46(4) (2023), 4070–4084.

[6] Babolian, E. and Mordad, M. *A numerical method for solving systems of linear and nonlinear integral equations of the second kind by hat basis functions*, Comput. Math. Appl. 62(1) (2011), 187–198.

[7] Behera, S. and Ray, S. Saha. *An efficient numerical method based on Euler wavelets for solving fractional order pantograph Volterra delay-integro-differential equations*, J. Comput. Appl. Math. 406(113825) (2022), 23.

[8] Behera, S. and Ray, S. Saha. *On a wavelet-based numerical method for linear and nonlinear fractional Volterra integro-differential equations with weakly singular kernels*, Comput. Appl. Math. 41(5) (2022), 1–32.

[9] Biazar, J. *Solution of systems of integral–differential equations by Adomian decomposition method*, Appl. Math. Comput. 168(2) (2005), 1232–1238.

[10] Biazar, J. and Ebrahimi, H. *Orthonormal Bernstein polynomials for Volterra integral equations of the second kind*, Int. J. Appl. Math. Res. 9(1) (2019), 9–20.

[11] Biazar, J. and Ebrahimi, H. *A numerical algorithm for a class of nonlinear fractional Volterra integral equations via modified hat functions*, J. Integral Equ. Appl. 34(3) (2022), 295–316.

[12] Biazar, J. and Montazeri, R. *Optimal homotopy asymptotic and multistage optimal homotopy asymptotic methods for solving system of volterra integral equations of the second kind*, J. Appl. Math. 2019 (2019), Art. ID 3037273, 17.

[13] Derakhshan, M. *A numerical scheme based on the Chebyshev functions to find approximate solutions of the coupled nonlinear sine-Gordon equations with fractional variable orders*, Abstr. Appl. Anal. 2021 (2021), Art. ID 8830727, 20.

[14] Du, H., Chen, Z. and Yang, T. *A stable least residue method in reproducing kernel space for solving a nonlinear fractional integro-differential equation with a weakly singular kernel*, Appl. Numer. Math. 157 (2020), 210–222.

[15] Hashemi Mehne, S.H. *Differential transform method: A comprehensive review and analysis*, Iranian Journal of Numerical Analysis and Optimization 12(3) (Special Issue), (2022), 629–657.

[16] Kukreja, V. K. *An improvised collocation algorithm with specific end conditions for solving modified Burgers equation*, Numer. Methods Partial Differ. Equ. 37(1) (2021), 874–896.

[17] Kythe, P.K. and Puri, P. *Computational methods for linear integral equations*, Birkhäuser Boston, Inc., Boston, MA, 2002.

[18] Leitman, M.J. *An integro-differential equation for plane waves propagating into a random fluid: asymptotic behavior*, SIAM J. Math. Anal. 12(4) (1981), 560–571.

[19] Mirzaee, F. and Hadadiyan, E. *Numerical solution of Volterra–Fredholm integral equations via modification of hat functions*, Appl. Math. Comput. 280 (2016), 110–123.

[20] Moosavi Noori, S.R. and Taghizadeh, N. *Modified differential transform method for solving linear and nonlinear pantograph type of differential and Volterra integro-differential equations with proportional delays*, Adv. Differ. Equ. 2020(1) (2020), 1–25.

[21] Ndiaye, A. and Mansal, F. *Existence and uniqueness results of Volterra-Fredholm integro-differential equations via Caputo fractional derivative*, J. Math. (2021), Art. ID 5623388, 8.

[22] Nemati, S. and Lima, P M. *Numerical solution of nonlinear fractional integro-differential equations with weakly singular kernels via a modification of hat functions*, Appl. Math. Comput. 327 (2018), 79–92.

[23] Özaltun, G., Konuralp, A. and Gümgüm, S. *Gegenbauer wavelet solutions of fractional integro-differential equations*, J. Comput. Appl. Math. 420(114830) (2023), 11.

[24] Podlubny, I. *Fractional differential equations*, Math. Sci. Eng. 198 (1999), Academic Press.

[25] Qiao L. and Xu, D. *A fast ADI orthogonal spline collocation method with graded meshes for the two-dimensional fractional integro-differential equation*, Adv. Comput. Math. 47(5) (2021), 1–22.

[26] Quentin, R., King, J.R., Sallard, E., Fishman, N., Thompson, R., Buch E.R. and Cohen, L.G. *Differential brain mechanisms of selection and maintenance of information during working memory* J. Neurosci. 39(19) (2019), 3728–3740.

[27] Rabbath, C.A. and Corriveau, D. *A comparison of piecewise cubic Hermite interpolating polynomials, cubic splines and piecewise linear functions for the approximation of projectile aerodynamics*, Defence Technology 15(5) (2019), 741–757.

[28] Rakshit G. and Rane, A.S. *Asymptotic expansion of iterated Galerkin solution of Fredholm integral equations of the second kind with Green's kernel*, J. Integral Equ. Appl. 32(4) (2020), 495–507.

[29] Riahi Beni, M. *Legendre wavelet method combined with the Gauss quadrature rule for numerical solution of fractional integro-differential equations*, Iranian Journal of Numerical Analysis and Optimization 12(1) (2022), 229–249.

[30] Sabatier, J., Aoun, M., Oustaloup, A., Gregoire, G., Ragot, F. and Roy, P. *Fractional system identification for lead acid battery state of charge estimation*, Signal Process 86(10) (2006), 2645–2657.

[31] Vinagre, B.M. ,Monje, C.A. ,Calderón A.J. and Suárez, J.I. *Fractional PID controllers for industry application. A brief introduction*, J. Vib. Control, 13(9-10) (2007), 1419–1429.

[32] Wang, Y. and Zhu, Li. *SCW method for solving the fractional integro-differential equations with a weakly singular kernel*, Appl. Math. Comput. 275 (2022), 72–80.

[33] Xie, J., Wang, T., Ren, Z., Zhang J. and Quan, L. *Haar wavelet method for approximating the solution of a coupled system of fractional-order integral–differential equations*, Math. Comput. Simul 163 (2019), 80–89.

[34] Yang, Z. *Gröbner Bases for Solving Multivariate Polynomial Equations*, Computing Equilibria and Fixed Points: The Solution of Nonlinear Inequalities (1999) 265–288.

How to cite this article

Ebrahimi, H. and Biazar, J., Cubic hat-functions approximation for linear and nonlinear fractional integral-differential equations with weakly singular kernels. *Iran. j. numer. anal. optim.*, 2023; 13(3): 500-531. https://doi.org/10.22067/ijnao.2022.73126.1209