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Research Article

Exponentially fitted tension spline method for singularly perturbed differential difference equations

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Abstract

In this article, singularly perturbed differential difference equations having delay and advance in the reaction terms are considered. The highest-order derivative term of the equation is multiplied by a perturbation parameter ε taking arbitrary values in the interval $(0, 1]$. For the small value of ε, the solution of the equation exhibits a boundary layer on the left or right side of the domain depending on the sign of the convective term. The terms with the shifts are approximated by using the Taylor series approximation. The resulting singularly perturbed boundary value problem is solved using an exponentially fitted tension spline method. The stability and uniform convergence of the scheme are discussed and proved. Numerical examples are considered for validating the theoretical analysis of the scheme. The developed scheme gives an accurate result with linear order uniform convergence.

AMS subject classifications (2020): 65L11; 65L03; 65L70.

Keywords: Differential difference; Exponentially fitted; Singularly perturbed problem; Tension spline; Uniform convergence.

1 Introduction

A large number of mathematical models have appeared in different areas of science and engineering that take into account not just the present state of a physical system but also its past history. These models are described by

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certain classes of functional differential equations often called delay differential equations or differential difference equations (DDEs). The class of DDEs with characteristics of delay/advance and singularly perturbed behavior is known as singularly perturbed differential difference equations (SPDDEs). The DDEs with delay or advance term play an important role in modeling many real life phenomena in bioscience, control theory, economics, and engineering [5]. Some applications are the mathematical modeling of population dynamics and epidemiology [19], physiological kinetics [4], blood cell production [25], and so on.

In SPDDEs model process, the evaluation not only depends on the current state of the system but also includes the past history. A number of model problems in science and engineering take the forms of SPDDEs [32]; we list a few of them: neuron variability model in computational neuroscience, optimal control theory problems, and model describing the motion of sunflower.

For the perturbation parameter, when ε tends to zero, the smoothness of the solution of the singularly perturbed problems deteriorates and it forms boundary layer; see [6, 27]. In the case where ε is very small, standard numerical methods such as FDM, FEM, and collocation method lead to oscillations in the computed solutions. To handle the oscillation, a large number of mesh points are required, which is not practical; see [32].

The solution methods of SPDDEs have received great attention in recent years because of their wide applications. It is of theoretical and practical interest to consider numerical methods for such problems [40]. Adilaxmi et al. [1, 2] proposed the exponentially fitted nonstandard FDM and integration method using a nonpolynomial interpolating function. In articles [21, 22, 23, 24], Lange and Miura developed asymptotic methods for solving a class of SPDDEs. The authors extend the matched asymptotic method initially developed for solving BVPs to obtain an approximate solution for SPDDEs. In articles [9, 11, 12, 13], Kadalbajoo and Sharma developed ε-uniform numerical methods using fitted mesh techniques. Swamy, Phaneendra, and Reddy [35] used the exponentially fitted Galerkin method for treating the problem. The authors in [36, 37] developed a fourth-order FDM with an exponential fitting factor. Melesse, Tiruneh, and Deresecite [26] used the initial value technique to treat the problem. They showed the applicability of the scheme by considering different examples. Ranjan and Prasad [31] used the modified fitted FDM for solving the problem. Sirisha, Phaneendra, and Reddy [34] developed a finite difference scheme using the procedure of domain decomposition. Kumar and Sharma [20] applied the B-spline collocation method to approximate the solution of the SPDDEs. Mohapatra and Natesan [28] applied the fitted mesh FDM using the equidistributed grid technique. In [40], Woldaregay and Duressa developed the exponentially fitted FDM with Richardson extrapolation techniques.

Different authors in [3, 7, 8, 14, 15, 16, 17, 30] have applied the tension spline method for treating singularly perturbed reaction diffusion or convection diffusion problems. To the best of the authors' knowledge, the exponentially fitted tension spline method has not been developed for treating SPDDEs. Developing uniformly convergent schemes is an active research area [33]. This motivates us to develop an accurate and uniformly convergent scheme using the exponentially fitted tension spline method.

Notations: N denotes the number of mesh interval in the discretization, C denotes a positive constant independent of ε and N, and the norm $\|\cdot\|$ denotes the supremum norm.

2 Continuous problem

Consider a class of SPDDE of the form

$$-\varepsilon u''(x)+a(x)u'(x)+\alpha(x)u(x-\delta)+\omega(x)u(x)+\beta(x)u(x+\eta) = f(x), \quad x \in \Omega, \tag{1}$$

with the interval conditions

$$\begin{aligned} u(x) =&\phi(x), \quad x \in [-\delta, 0], \\ u(x) =&\psi(x), \quad x \in [1, 1+\eta], \end{aligned} \tag{2}$$

where $\Omega = (0,1)$, $\varepsilon \in (0,1]$ is the singular perturbation parameter, and δ and η are delay and advance parameters satisfying $\delta, \eta < \varepsilon$. The functions $a(x), \alpha(x), \omega(x), \beta(x), f(x), \phi(x)$, and $\psi(x)$ are assumed to be sufficiently smooth and bounded for the existence of unique solution. The coefficient functions $\alpha(x), \omega(x)$, and $\beta(x)$ are assumed to satisfy

$$\alpha(x) + \omega(x) + \beta(x) \geq \alpha + \omega + \beta =: \theta > 0, \quad \text{for all } x \in \bar{\Omega},$$

where the constants α, ω, and β are lower bounds of $\alpha(x), \omega(x)$, and $\beta(x)$, respectively.

In the case when $\delta, \eta = 0$, equations (1)–(2) reduce to a singularly perturbed boundary value problem, in which for small ε, it exhibits boundary layer. The layer is maintained for $\delta, \eta \neq 0$ but sufficiently small.

2.1 Properties of the continuous solution

In the case when $\delta, \eta < \varepsilon$, using Taylor's series approximation for the terms with deviating, the argument is appropriate [38]. Using the Taylor series approximation, we approximate

$$\begin{aligned} u(x-\delta) \approx& u(x) - \delta u'(x) + (\delta^2/2)u''(x) + O(\delta^3), \\ u(x+\eta) \approx& u(x) + \eta u'(x) + (\eta^2/2)u''(x) + O(\eta^3). \end{aligned} \tag{3}$$

Replacing (3) in (1) gives

$$-c_\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \quad x \in \Omega, \tag{4}$$

with the boundary conditions

$$u(0) = \phi(0), \quad u(1) = \psi(1), \tag{5}$$

where $c_\varepsilon = \varepsilon^2 - (\delta^2/2)\alpha - (\eta^2/2)\beta$, $p(x) = a(x) - \delta\alpha(x) + \eta\beta(x)$, and $q(x) = \alpha(x) + \beta(x) + \omega(x)$. For small values of δ and η, (1)–(2) and (4)–(5) are asymptotically equivalent, since the difference between these two equations is $O(\delta^3, \eta^3)$. The differential operator L is denoted for the differential equation in (4) and defined as

$$Lu(x) = -c_\varepsilon u''(x) + p(x)u'(x) + q(x)u(x).$$

The problem in (4)–(5) exhibits the regular boundary layer of thickness $O(c_\varepsilon)$, and the position of the boundary layer depends on the conditions: If $p(x) < 0$, then the left boundary layer exists, and if $p(x) > 0$, then the right boundary layer exists. In the case when $p(x), x \in \Omega$, the change sign interior layer will exist [10].

The problem obtained by setting $c_\varepsilon = 0$ in (4)–(5) is called the reduced problem and given as

$$\begin{aligned} p(x)u_0'(x) + q(x)u_0(x) = f(x), &\quad \text{for all } x \in \Omega, \\ u_0(0) = \phi(0), \quad u_0(1) \neq \psi(1). & \end{aligned} \tag{6}$$

For the right boundary layer case, it does not satisfy the right boundary condition, and for the left boundary layer case, it is given as

$$\begin{aligned} p(x)u_0'(x) + q(x)u_0(x) = f(x), &\quad \text{for all } x \in \Omega, \\ u_0(0) \neq \phi(0), \quad u_0(1) = \psi(1). & \end{aligned} \tag{7}$$

For small values of c_ε, the solution $u(x)$ of (4)–(5) is very close to the solution $u_0(x)$ of (6) or (7).

Lemma 1 (The maximum principle [40])**.** For sufficiently smooth function z on Ω, satisfying $z(0) \geq 0$, $z(1) \geq 0$, and $Lz(x) \geq 0$, for all $x \in \Omega$, implies that $z(x) \geq 0$, for all $x \in \bar{\Omega}$.

Lemma 2 (Stability estimate)**.** The solution $u(x)$ of the continuous equation (4)–(5) satisfies the bounded

$$|u(x)| \leq \theta^{-1}\|f\| + \max\{|\phi(0)|, |\psi(1)|\}. \tag{8}$$

Proof. It is proved by the construction of barrier function and using the maximum principle. Let us define barrier functions $\vartheta^{\pm}(x)$ as $\vartheta^{\pm}(x) = \theta^{-1}\|f\| + \max\{\phi(0), \psi(1)\} \pm u(x)$. On the boundary points, we obtain

$$\vartheta^{\pm}(0) = \theta^{-1}\|f\| + \max\{\phi(0), \psi(1)\} \pm u(0) \geq 0,$$
$$\vartheta^{\pm}(1) = \theta^{-1}\|f\| + \max\{\phi(0), \psi(1)\} \pm u(1) \geq 0.$$

On the differential operator, we have

$$\begin{aligned} L\vartheta^{\pm}(x) =& - c_\varepsilon \vartheta''_{\pm}(x) + p(x)\vartheta'_{\pm}(x) + q(x)\vartheta_{\pm}(x) \\ =& - c_\varepsilon(0 \pm u''(x)) + p(x)\big(0 \pm u'(x)\big) + q(x)\big(\theta^{-1}\|f\| + \max\{\phi(0), \psi(1)\} \\ & \pm u(x)\big) \\ =& q(x)\big(\theta^{-1}\|f\| + \max\{\phi(0), \psi(1)\}\big) \pm f(x) \\ \geq& 0, \quad \text{since } q(x) \geq \theta > 0. \end{aligned}$$

By using the hypothesis of the maximum principle, we obtain $\vartheta^{\pm}(x) \geq 0$, for all $x \in \bar{\Omega}$, which implies the required bound. □

In the next lemma, we obtain a bound for the derivatives of solution.

Lemma 3. Derivatives of the solutions of the problem in (4)–(5) satisfy the bound

$$|u^{(k)}(x)| \leq C\big(1 + c_\varepsilon^{-k} \exp\big(\frac{-p^* x}{c_\varepsilon}\big)\big), \quad x \in \Omega, \quad 0 \leq k \leq 4,$$

for the left boundary layer problem and

$$|u^{(k)}(x)| \leq C\big(1 + c_\varepsilon^{-k} \exp\big(\frac{-p^*(1-x)}{c_\varepsilon}\big)\big), \quad x \in \Omega, \quad 0 \leq k \leq 4,$$

for the right boundary layer problem.

Proof. See [6]. □

3 Numerical scheme formulation

In this article, an exponentially fitted tension spline method is proposed for solving equations (1)–(2). The exponential fitting factor is used to hinder the influence of the perturbation parameter in the boundary layer region. The theory of the asymptotic method is used for developing the exponential fitting factor. We consider and treat the left and right boundary layer problems separately.

3.1 Exponentially fitted tension spline method

Let $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$ be a uniform partition for $[0, 1]$ such that $x_i = ih,\ i = 0, 1, 2, \ldots, N$. A function $S(x, \tau) = S(x)$ is a class of $C^2(\bar{\Omega})$, which interpolates $u(x)$ at the mesh points x_i depending on the parameter τ, reduces to cubic spline in $\bar{\Omega}$ as $\tau \to 0$, and is termed as a parametric cubic spline function [17, 3]. In $[x_i, x_{i+1}]$, the spline function $S(x)$ satisfies the differential equation

$$S''(x) - \tau S(x) = [S''(x_i) - \tau S(x_i)]\frac{x_{i+1} - x}{h} + [S''(x_{i+1}) - \tau S(x_{i+1})]\frac{x - x_i}{h}, \tag{9}$$

where $S(x_i) = u(x_i)$ and $\tau > 0$ is termed as a cubic spline in compression. Solving the linear second-order differential equation in (9) and determining the arbitrary constants from the interpolation conditions $S(x_{i+1}) = u(x_{i+1}), S(x_i) = u(x_i)$, we get

$$\begin{aligned} S(x) =& \frac{h^2}{\lambda^2 \sinh\lambda}\big[M_{i+1}\sinh(\frac{\lambda(x - x_i)}{h}) + M_i \sinh(\frac{\lambda(x_{i+1} - x)}{h})\big] \\ &- \frac{h^2}{\lambda^2}\big[(M_{i+1} - \frac{\lambda^2}{h^2}u(x_{i+1}))(\frac{x - x_i}{h}) + (M_i - \frac{\lambda^2}{h^2}u(x_i))(\frac{x_{i+1} - x}{h})\big], \end{aligned} \tag{10}$$

where $\lambda = h\tau^{1/2}$ and $M_j = u''(x_j)$ for $j = i \pm 1, i$.

Now, differentiating (10) and letting $x \to x_i$, on the interval $[x_i, x_{i+1}]$, we obtain

$$S'(x_i^+) =\frac{u(x_{i+1}) - u(x_i)}{h} - \frac{h}{\lambda^2}\big[M_{i+1}(1 - \frac{\lambda}{\sinh\lambda}) + M_i(\lambda\coth\lambda - 1)\big], \tag{11}$$

and on the interval $[x_{i-1}, x_i]$, we obtain

$$S'(x_i^-) =\frac{u(x_i) - u(x_{i-1})}{h} + \frac{h}{\lambda^2}\big[M_i(\lambda\coth\lambda - 1) + M_{i-1}(1 - \frac{\lambda}{\sinh\lambda})\big]. \tag{12}$$

Equating the left- and right-hand derivatives at x_i gives

$$\begin{aligned} &\frac{u(x_i) - u(x_{i-1})}{h} + \frac{h}{\lambda^2}\big[M_i(\lambda\coth\lambda - 1) + M_{i-1}(1 - \frac{\lambda}{\sinh\lambda})\big] \\ &= \frac{u(x_{i+1}) - u(x_i)}{h} - \frac{h}{\lambda^2}\big[M_{i+1}(1 - \frac{\lambda}{\sinh\lambda}) + M_i(\lambda\coth\lambda - 1)\big], \\ &\qquad\qquad\qquad\qquad i = 1, 2, \ldots, N - 1. \end{aligned} \tag{13}$$

Rearranging, we obtain the tridiagonal system

$$\lambda_1 M_{i-1}+2\lambda_2 M_i+\lambda_1 M_{i+1} = \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2}, \quad i = 1, 2, \dots, N-1, \tag{14}$$

where $\lambda_1 = \frac{1}{\lambda^2}(\frac{\lambda}{\sinh\lambda} - 1)$ and $\lambda_2 = \frac{1}{\lambda^2}(1 - \lambda\coth\lambda)$.
The condition of continuity given in (14) ensures the continuity of the first-order derivatives of the spline $S(x)$ at interior nodes.
Now, substituting $-c_\varepsilon M_j = f(x_j) - p(x_j)u'(x_j) - q(x_j)u(x_j)$ for $j = i-1, i$ and $i+1$, we obtain

$$\begin{aligned} L^h u(x_i) \equiv &- \frac{c_\varepsilon}{h^2}[u(x_{i-1}) - 2u(x_i) + u(x_{i+1})] + \lambda_1[p(x_{i-1})u'(x_{i-1}) \\ &+ q(x_{i-1})u(x_{i-1})] + 2\lambda_2[p(x_i)u'(x_i) + q(x_i)u(x_i)] \\ &+ \lambda_1[p(x_{i+1})u'(x_{i+1}) + q(x_{i+1})u(x_{i+1})] \\ =&\lambda_1 f(x_{i-1}) + 2\lambda_2 f(x_i) + \lambda_1 f(x_{i+1}) + T_1(h), \quad i = 1, 2, \dots, N-1, \end{aligned} \tag{15}$$

where $T_1(h)$ is the truncation error in the above discretization, one can see the detail in [3], and it is given by

$$T_1(h) = \frac{h^4}{3}(-2\lambda_1 + \lambda_2)p(x_i)u'''(\zeta_i) + \frac{h^4}{12}(1 - 12\lambda_1)p(x_i)c_\varepsilon u^{(4)}(\zeta_i) + O(h^6), \tag{16}$$

for any choice of λ_1 and λ_2 whose sum is $1/2$, except $\lambda_1 = 1/12$ and $\lambda_2 = 5/12$. For the choice $\lambda_1 = 1/12, \lambda_2 = 5/12$, we have

$$T_1(h) = \frac{c_\varepsilon h^6}{240} u^{(6)}(\zeta_i), \quad \zeta_i \in [x_{i-1}, x_{i+1}]. \tag{17}$$

Next, we use the left-shifted, central, and right-shifted finite difference approximation as

$$\begin{aligned} u'(x_{i-1}) =& \frac{-3u(x_{i-1}) + 4u(x_i) - u(x_{i+1})}{2h} + O(h), \\ u'(x_i) =& \frac{u(x_{i+1}) - u(x_{i-1})}{2h} + O(h^2), \quad \text{and} \\ u'(x_{i+1}) =& \frac{3u(x_{i+1}) - 4u(x_i) + u(x_{i-1})}{2h} + O(h). \end{aligned} \tag{18}$$

Substituting (18) in (15) leads to

$$\begin{aligned} Lu_i \equiv &- \frac{c_\varepsilon}{h^2}[u_{i-1} - 2u_i + u_{i+1}] + \lambda_1[p(x_{i-1})(\frac{-3u_{i-1} + 4u_i - u_{i+1}}{2h}) + q(x_{i-1})u_{i-1}] \\ &+ 2\lambda_2[p(x_i)(\frac{u_{i+1} - u_{i-1}}{2h}) + q(x_i)u_i] + \lambda_1[p(x_{i+1})(\frac{3u_{i+1} - 4u_i + u_{i-1}}{2h}) \\ &+ q(x_{i+1})u_{i+1}] = \lambda_1 f(x_{i-1}) + 2\lambda_2 f(x_i) + \lambda_1 f(x_{i+1}) + T_2(h), \\ &\qquad\qquad i = 1, 2, \dots, N-1, \end{aligned} \tag{19}$$

where u_i denotes the approximation of $u(x_i)$ in the above discretization and $T_2(h) = O(h) + T_1(h)$.

3.1.1 Case I: Left boundary layer problem

In this case, the boundary layer occurs on the left side of the domain. From the theory of singular perturbations, the zeros-order asymptotic solution of (4)–(5) is given as [29]

$$u(x) = u_0(x)+\frac{p(0)}{p(x)}(\phi(0)-u_0(0))\exp\big(-\int_0^x \big(\frac{p(x)}{c_\varepsilon(x)}-\frac{q(x)}{p(x)}\big)dx\big)+O(c_\varepsilon). \quad (20)$$

Using the Taylor series about $x = 0$ for $p(x)$ and $q(x)$ and simplifying give

$$u(x) = u_0(x) + (\phi(0) - u_0(0))\exp(-p(0)x) + O(c_\varepsilon), \quad (21)$$

where u_0 is the solution of the reduced problem. The domain $[0, 1]$ is discretized into N equal number of subintervals, each of length h. Let $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$ be the points such that $x_i = ih,\ i = 0, 1, 2, \ldots, N$.

Considering h small enough, the discretized form of (21) becomes

$$u(ih) \simeq u_i = u_0(ih) + (\phi(0) - u_0(0))\exp(-p(0)(i\rho)), \quad (22)$$

where $\rho = h/c_\varepsilon$ and $h = 1/N$. Similarly, we write

$$\begin{aligned} u_{i+1} &= u_0((i + 1)h) + (\phi(0) - u_0(0))\exp(-p(0)((i + 1)\rho)),\\ u_{i-1} &= u_0((i - 1)h) + (\phi(0) - u_0(0))\exp(-p(0)((i - 1)\rho)). \end{aligned} \quad (23)$$

In order to handle the influence of the perturbation parameter, the exponentially fitting factor σ_1 is multiplied on the term containing c_ε as

$$\begin{aligned} L_L^h u_i \equiv &-\frac{c_\varepsilon\sigma_1}{h^2}[u_{i-1} - 2u_i + u_{i+1}] + \lambda_1[p(x_{i-1})(\frac{-3u_{i-1} + 4u_i - u_{i+1}}{2h})\\ &+ q(x_{i-1})u_{i-1}] + 2\lambda_2[p(x_i)(\frac{u_{i+1} - u_{i-1}}{2h}) + q(x_i)u_i]\\ &+ \lambda_1[p(x_{i+1})(\frac{3u_{i+1} - 4u_i + u_{i-1}}{2h}) + q(x_{i+1})u_{i+1}]\\ =&\lambda_1 f(x_{i-1}) + 2\lambda_2 f(x_i) + \lambda_1 f(x_{i+1}) + T_3(h), \quad i = 1, 2, \ldots, N - 1. \end{aligned} \quad (24)$$

Multiplying both sides of (24) by h, denoting $c_\varepsilon/h = \rho$, and taking the limit as $h \to 0$, we obtain

$$- \lim_{h\to 0} \frac{\sigma_1}{\rho}[u_{i-1} - 2u_i + u_{i+1}] + \lambda_1 \lim_{h\to 0}[p((i-1)h)(-3u_{i-1} + 4u_i - u_{i+1})]$$
$$+ 2\lambda_2 \lim_{h\to 0}[p(ih)(u_{i+1} - u_{i-1})] + \lambda_1 \lim_{h\to 0}[p((i+1)h)(3u_{i+1} - 4u_i + u_{i-1})] = 0. \quad (25)$$

The expression in (25) simplifies to

$$\lim_{h\to 0} \frac{c_\varepsilon \sigma_1}{\rho}[u_{i-1} - 2u_i + u_{i+1}] = \lim_{h\to 0}(\lambda_1 + \lambda_2)[p(0)(u_{i+1} - u_{i-1})]. \quad (26)$$

Using the results in (22) and (23), we obtain

$$\begin{aligned}\lim_{h\to 0}[u_{i+1} - 2u_i + u_{i-1}] =& (\phi(0) - u_0(0)) \exp(-p(0)i\rho) \\ & \times [\exp(-p(0)\rho) - 2 + \exp(p(0)\rho)], \\ \lim_{h\to 0}[u_{i+1} - u_{i-1}] =& (\phi(0) - u_0(0)) \exp(-p(0)i\rho) \\ & \times [\exp(-p(0)\rho) - \exp(p(0)\rho)].\end{aligned} \quad (27)$$

Using the result in (27) into (26), we obtain the exponential fitting factor as

$$\sigma_1 = p(0)\rho(\lambda_1 + \lambda_2) \coth(p(0)\frac{\rho}{2}). \quad (28)$$

Hence, the required finite difference scheme becomes

$$\begin{aligned}L_L^h u_i \equiv& - \frac{c_\varepsilon \sigma_1}{h^2}[u_{i-1} - 2u_i + u_{i+1}] + \lambda_1[p(x_{i-1})(\frac{-3u_{i-1} + 4u_i - u_{i+1}}{2h}) \\ & + q(x_{i-1})u_{i-1}] + 2\lambda_2[p(x_i)(\frac{u_{i+1} - u_{i-1}}{2h}) + q(x_i)u_i] \\ & + \lambda_1[p(x_{i+1})(\frac{3u_{i+1} - 4u_i + u_{i-1}}{2h}) + q(x_{i+1})u_{i+1}] \\ =& \lambda_1 f(x_{i-1}) + 2\lambda_2 f(x_i) + \lambda_1 f(x_{i+1}) + T_3(h), \quad i = 1, 2, \ldots, N-1,\end{aligned} \quad (29)$$

with the boundary conditions $u_0 = \phi(0)$ and $u_N = \psi(1)$. The bound of truncation error $T_3(h)$ is derived in Theorem 1.

3.1.2 Case II: Right boundary layer problem

In this case, the boundary layer is on the right side of the domain. From the theory of singular perturbations, the zeros-order asymptotic solution of (4)–(5) is given as [29]

$$u(x) = u_0(x) + \frac{p(1)}{p(x)}(\psi(1) - u_0(1)) \exp\big(-\int_x^1 (\frac{p(x)}{c_\varepsilon} - \frac{q(x)}{p(x)})dx\big) + O(c_\varepsilon). \quad (30)$$

Using the Taylor series about $x = 1$ for $p(x)$ and $q(x)$ and simplifying, we obtain

$$u(x) = u_0(x) + (\psi(1) - u_0(1)) \exp\big(-p(1)(1-x)\big) + O(c_\varepsilon), \tag{31}$$

where u_0 is the solution of the reduced problem. The domain $[0, 1]$ is discretized into N equal number of subintervals, each of length h. Let $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$ be the points such that $x_i = ih,\ i = 0, 1, 2, \dots, N$.

Considering h is small enough, the discretized form of (21) becomes

$$u(ih) \simeq u_i = u_0(ih) + (\psi(1) - u_0(1)) \exp(-p(1)(1/c_\varepsilon - i\rho)), \tag{32}$$

where $\rho = h/c_\varepsilon$ and $h = 1/N$. Similarly, we write

$$\begin{aligned} u_{i+1} =& u_0((i+1)h) + (\psi(1) - u_0(1)) \exp(-p(1)(1/c_\varepsilon - (i+1)\rho)),\\ u_{i-1} =& u_0((i-1)h) + (\psi(1) - u_0(1)) \exp(-p(1)(1/c_\varepsilon - (i-1)\rho)). \end{aligned} \tag{33}$$

Using the similar procedure as the left boundary layer case, we obtain the exponential fitting factor as

$$\sigma_2 = p(1)\rho(\lambda_1 + \lambda_2) \coth(\frac{p(1)\rho}{2}). \tag{34}$$

Hence, the required finite difference scheme becomes

$$\begin{aligned} L_R^h u_i \equiv & -\frac{c_\varepsilon \sigma_2}{h^2}[u_{i-1} - 2u_i + u_{i+1}] + \lambda_1[p(x_{i-1})(\frac{-3u_{i-1} + 4u_i - u_{i+1}}{2h})\\ & + q(x_{i-1})u_{i-1}] + 2\lambda_2[p(x_i)(\frac{u_{i+1} - u_{i-1}}{2h}) + q(x_i)u_i]\\ & + \lambda_1[p(x_{i+1})(\frac{3u_{i+1} - 4u_i + u_{i-1}}{2h}) + q(x_{i+1})u_{i+1}]\\ =& \lambda_1 f(x_{i-1}) + 2\lambda_2 f(x_i) + \lambda_1 f(x_{i+1}) + T_3(h), \quad i = 1, 2, \dots, N-1, \end{aligned} \tag{35}$$

with the boundary conditions $u_0 = \phi(0)$ and $u_N = \psi(1)$.

3.2 Stability and uniform convergence

In this section, we discuss the uniform stability and convergence for the right boundary layer problems. Similarly, one can do it for the left boundary layer case. First, we prove the discrete comparison principle for the scheme in (35) for the existence of the unique discrete solution.

We observe that the nonzero entries of the coefficient matrix of $L_R^h u_i$ are given by

$$
\begin{aligned}
a_{i,i-1} =& -\frac{c_\varepsilon\sigma_2}{h^2} - \lambda_1\big(\frac{3p(x_{i-1})}{2h} - \frac{p(x_{i+1})}{2h} + q(x_{i-1})\big) - \lambda_2\frac{p(x_i)}{h},\\
a_{i,i} =& \frac{2c_\varepsilon\sigma_2}{h^2} - \lambda_1(\frac{2p(x_{i-1})}{h} - \frac{2p(x_{i+1})}{h}) + q(x_i),\\
a_{i,i+1} =& -\frac{c_\varepsilon\sigma_2}{h^2} - \lambda_1\big(\frac{p(x_{i-1})}{2h} + \frac{3p(x_{i+1})}{2h} + q(x_{i+1})\big) + \lambda_2\frac{p(x_i)}{h}.
\end{aligned} \tag{36}
$$

For each $i = 1, 2, \ldots, N-1$ and for arbitrary values of h and c_ε, we have $a_{i,i-1} < 0$, $a_{i,i+1} < 0$, and $a_{i,i} > 0$.

Lemma 4 (Discrete comparison principle)**.** Assume that, for a mesh function u_i, there exists a comparison function v_i such that $L^h u_i \le L^h v_i,\ i = 1, 2, \ldots, N-1$ and if $u_0 \le v_0$ and $u_N \le v_N$, then $u_i \le v_i,\ i = 0, 1, 2, \ldots, N$.

Proof. The matrix associated with operator L^h_R is of size $(N+1)\times(N+1)$ and where for $i = 1$ and $i = N-1$, the terms involving u_0 and u_N have been moved to the right-hand side. It is easy to see that the matrix of coefficients is diagonally dominant and has nonpositive off-diagonal entries. Hence, the matrix is an irreducible M matrix. See the details of proof in [18]. □

Lemma 5 (Discrete uniform stability estimate)**.** The solution of the discrete scheme in (29) satisfies the bound

$$
|u_i| \le \theta^{-1}\|L^h_R u_i\| + \max\{|u_0|, |u_N|\}. \tag{37}
$$

Proof. Let $r = \theta^{-1}\|L^h_R u_i\| + \max\{u_0, u_N\}$, and define the barrier function $\vartheta^\pm_i$ by $\vartheta^\pm_i = r \pm u_i$. On the boundary points, we obtain

$$
\begin{aligned}
\vartheta^\pm_0 &= r \pm u_0 = \theta^{-1}\|L^h_R u_i\| + \max\{u_0, u_N\} \pm \phi(0) \ge 0,\\
\vartheta^\pm_N &= r \pm u_N = \theta^{-1}\|L^h_R u_i\| + \max\{u_0, u_N\} \pm \psi(1) \ge 0.
\end{aligned}
$$

On the discretized spatial domain $x_i,\ \ 0 < i < N$, we obtain

$$
\begin{aligned}
L^h_R\theta^\pm_i \equiv& -\frac{c_\varepsilon\sigma_1}{h^2}[(r \pm u_{i-1}) - 2(r \pm u_i) + (r \pm u_{i+1})]\\
&+ \lambda_1[p(x_{i-1})(\frac{-3(r \pm u_{i-1}) + 4(r \pm u_i) - (r \pm u_{i+1})}{2h})\\
&+ q(x_{i-1})(r \pm u_{i-1})] + 2\lambda_2[p(x_i)(\frac{(r \pm u_{i+1}) - (r \pm u_{i-1})}{2h})\\
&+ q(x_i)(r \pm u_i)] + \lambda_1[p(x_{i+1})(\frac{(r \pm u_{i+1}) - 4(r \pm u_i) + 3(r \pm u_{i-1})}{2h})\\
&+ q(x_{i+1})(r \pm u_{i+1})]\\
=&[\lambda_1 q(x_{i-1}) + 2\lambda_2 q(x_i) + \lambda_1 q(x_{i+1})]\big(\theta^{-1}\|L^h_R u_i\| + \max\{u_0, u_N\}\big)\\
&\pm [\lambda_1 f(x_{i-1}) + 2\lambda_2 f(x_i) + \lambda_1 f(x_{i+1})] \ge 0, \quad \text{since } q_i \ge \theta > 0.
\end{aligned}
$$

From Lemma 4, we obtain $\vartheta^\pm_i \ge 0$, for all $x_i \in \bar{\Omega}^N$. Hence, the required bound is obtained. □

Now for $z>0$, C_1 and C_2 are constants, and we have

$$C_1\frac{z^2}{z+1}\le z\coth(z)-1\le C_2\frac{z^2}{z+1},\quad \text{and}\quad c_\varepsilon\frac{(h/c_\varepsilon)^2}{h/c_\varepsilon+1}=\frac{h^2}{h+c_\varepsilon},\tag{38}$$

giving that

$$\left|c_\varepsilon\left[p(1)\rho(\lambda_1+\lambda_2)\coth(\frac{p(1)\rho}{2})-1\right]D^+D^-u(x_i)\right|\le\frac{Ch^2}{h+c_\varepsilon}\left\|u''(x_i)\right\|,\tag{39}$$

since $\lambda_1+\lambda_2\le 1/2$. We obtain the bound

$$\begin{aligned}\left|c_\varepsilon[u''(x_i)-\sigma D^+D^-u(x_i)]\right| =&\Big|c_\varepsilon\left[p(1)\rho(\lambda_1+\lambda_2)\coth(\frac{p(1)\rho}{2})-1\right]D^+D^-u(x_i)\\ &+c_\varepsilon\big(u''(x_i)-D^+D^-u(x_i)\big)\Big|\\ \le&\frac{Ch^2}{h+c_\varepsilon}\left\|u''(x_i)\right\|+Cc_\varepsilon h^2\left\|u^{(4)}(x_i)\right\|.\end{aligned}\tag{40}$$

Now, let us denote the right-shifted, central, and left-shifted finite differences, respectively, as

$$\begin{aligned}D^Ru(x_i)=&\frac{u_{i-1}-4u_i+3u_{i+1}}{2h},\quad D^0u(x_i)=\frac{u_{i+1}-u_{i-1}}{2h},\quad\text{and}\\ D^Lu(x_i)=&\frac{-3u_{i-1}+4u_i-u_{i+1}}{2h}.\end{aligned}$$

Using Taylor's series approximation, we obtain the bound

$$\begin{aligned}|u'(x_{i-1})-D^Lu(x_{i-1})|\le& Ch\left\|u''(\zeta)\right\|,\\ |u'(x_i)-D^0u(x_i)|\le& Ch^2\left\|u'''(\zeta)\right\|,\quad\text{and}\\ |u'(x_{i+1})-D^Ru(x_{i-1})|\le& Ch\left\|u''(\zeta)\right\|,\end{aligned}\tag{41}$$

where $\|u''(\zeta)\|=\max_{x_0\le x_i\le x_N}|u''(x_i)|$ and $\|u'''(\zeta)\|=\max_{x_0\le x_i\le x_N}|u'''(x_i)|$.

The next theorem gives the truncation error bound for the proposed scheme.

Theorem 1. Let $u(x_i)$ and u_i be the solution of (4)–(5) and (29), respectively. Then, the following error estimate holds:

$$\left|Lu(x_i)-L_R^hu_i\right|\le Ch\big(1+c_\varepsilon^{-3}\exp\big(-\frac{p^*(1-x_i)}{c_\varepsilon}\big)\big).\tag{42}$$

Proof. Consider the truncation error bound in the above discretization

$$\begin{aligned} \left|Lu(x_i) - L_R^h u_i\right| \leq & \left|Lu(x_i) - L^h u(x_i)\right| + \left|L^h u(x_i) - L_R^h u_i\right| \\ \leq & \left\|T_1(h)\right\| + \left|c_\varepsilon u''(x_i) - c_\varepsilon \sigma D^+ D^- u(x_i)\right| \\ & + |u'_{i-1} - D^L u(x_i)| + |u'_i - D^0 u(x_i)| \\ & + |u'_{i+1} - D^R u(x_i)|. \end{aligned} \tag{43}$$

Using the bounds in (40), (41), and (43) and using

$$T_1(h) \leq Ch^4(-2\lambda_1 + \lambda_2)\left\|u'''(\zeta_i)\right\| + Cc_\varepsilon h^4(1 - 12\lambda_1)\left\|u^{(4)}(\zeta_i)\right\|,$$

we obtain

$$\begin{aligned} \left|Lu(x_i) - L_R^h u_i\right| \leq & \frac{Ch^2}{h + c_\varepsilon}\|u''(x_i)\| + c_\varepsilon Ch^2\|u^{(4)}(x_i)\| + \lambda_1 Ch\|u''(\zeta)\| \\ & + \lambda_2 Ch^2\|u'''(\zeta)\| + Ch^4(-2\lambda_1 + \lambda_2)\left\|u'''(\zeta_i)\right\| \\ & + Cc_\varepsilon h^4(1 - 12\lambda_1)\left\|u^{(4)}(\zeta_i)\right\|. \end{aligned}$$

Using the bounds for the derivatives of the solution in Lemma 3 gives

$$\begin{aligned} \left|Lu(x_i) - L_R^h u_i\right| \leq & \frac{Ch^2}{h + c_\varepsilon}\left(1 + c_\varepsilon^{-2} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right) + Ch^2\left[c_\varepsilon\left(1 + c_\varepsilon^{-4} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right)\right. \\ & + Ch\lambda_1\left(1 + c_\varepsilon^{-2} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right) + Ch^2\lambda_2\left(1 + c_\varepsilon^{-3} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right) \\ & + Ch^4(-2\lambda_1 + \lambda_2)\left(1 + c_\varepsilon^{-3} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right) \\ & + Cc_\varepsilon h^4(1 - 12\lambda_1)\left(1 + c_\varepsilon^{-4} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right) \\ \leq & \frac{Ch^2}{h + c_\varepsilon}\left(1 + c_\varepsilon^{-2} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right) + Ch^2\left(c_\varepsilon + c_\varepsilon^{-3} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right) \\ & + Ch\lambda_1\left(1 + c_\varepsilon^{-2} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right) + Ch^2\lambda_2\left(1 + c_\varepsilon^{-3} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right) \\ & + Ch^4(-2\lambda_1 + \lambda_2)\left(1 + c_\varepsilon^{-3} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right) \\ & + Ch^4(1 - 12\lambda_1)\left(c_\varepsilon + c_\varepsilon^{-3} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right) \\ \leq & Ch\left(1 + c_\varepsilon^{-3} e^{\left(\frac{-p^*(1-x_i)}{c_\varepsilon}\right)}\right), \quad \text{since} \quad c_\varepsilon^{-3} \geq c_\varepsilon^{-2}. \end{aligned}$$

□

Lemma 6. For a fixed number of mesh numbers N and for $c_\varepsilon \to 0$, it holds

$$\lim_{c_\varepsilon \to 0} \max_{1 \leq i \leq N-1} \frac{\exp\left(\frac{-\alpha(x_i)}{c_\varepsilon}\right)}{c_\varepsilon^m} = 0, \quad \lim_{c_\varepsilon \to 0} \max_{1 \leq i \leq N-1} \frac{\exp\left(\frac{-\alpha(1-x_i)}{c_\varepsilon}\right)}{c_\varepsilon^m} = 0, \tag{44}$$

for $m = 1, 2, 3, \ldots$, where $x_i = ih, h = 1/N,$ for all $i = 1, 2, \ldots, N-1$.

Proof. See [39]. □

Theorem 2. Let $u(x_i)$ and u_i be the solution of (4)–(5) and (29), respectively. Then it satisfies the error bound

$$\sup_{0<c_\varepsilon \ll 1} \|u(x_i) - u_i\| \le Ch. \tag{45}$$

Proof. Using the result in Lemma 6 into Theorem 1 and using the result in Lemma 5, we obtain the required bound. □

4 Numerical results and discussion

In this section, we consider examples to illustrate the theoretical analysis of the proposed scheme.

Example 1. Consider the problem from [28]

$$-\varepsilon u''(x) + (1 + e^{(x^2)})u'(x) + xe^x u(x-\delta) + x^2 u(x) + (1 - e^{-x})u(x+\eta) = -1$$

with interval conditions $u(x) = 1,\ -\delta \le x \le 0$, and $u(x) = -1,\ 1 \le x \le 1+\eta$.

Example 2. Consider the problem from [28]

$$-\varepsilon u''(x) + u'(x) - 2u(x-\delta) + 5u(x) - u(x+\eta) = 0$$

with interval conditions $u(x) = 1,\ -\delta \le x \le 0$ and $u(x) = -1,\ 1 \le x \le 1+\eta$. The exact solution is given as

$$u(x) = \frac{(1 + e^{m_2})e^{m_1 x} - (1 + e^{m_1})e^{m_2 x}}{e^{m_2} - e^{m_1}},$$

where

$$m_1 = \frac{-(-1 - 2\delta + \eta) + \sqrt{(-1 - 2\delta + \eta)^2 - 4(\varepsilon + \delta^2 + \eta^2/2)}}{2(\varepsilon + \delta^2 + \eta^2/2)},$$
$$m_2 = \frac{-(-1 - 2\delta + \eta) - \sqrt{(-1 - 2\delta + \eta)^2 - 4(\varepsilon + \delta^2 + \eta^2/2)}}{2(\varepsilon + \delta^2 + \eta^2/2)}.$$

Example 3. Consider the problem from [28]

$$-\varepsilon u''(x) - u'(x) + 2u(x-\delta) + 5u(x) - u(x+\eta) = 0$$

with interval conditions $u(x) = 1,\ -\delta \le x \le 0$ and $u(x) = 0,\ 1 \le x \le 1+\eta$. The exact solution is given as

$$u(x) = \frac{e^{m_1 x + m_2} - e^{m_1 + m_2 x}}{e^{m_2} - e^{m_1}},$$

where

$$m_1 = \frac{-(1+2\delta+\eta) + \sqrt{(1+2\delta+\eta)^2 - 4(\varepsilon - \delta^2 + \eta^2/2)}}{2(\varepsilon - \delta^2 + \eta^2/2)},$$
$$m_2 = \frac{-(1+2\delta+\eta) - \sqrt{(1+2\delta+\eta)^2 - 4(\varepsilon - \delta^2 + \eta^2/2)}}{2(\varepsilon - \delta^2 + \eta^2/2)}.$$

Example 4. Consider the problem

$$-\varepsilon u''(x) - (1+\exp(-x^2))u'(x) - xu(x-\delta) - x^2u(x) - (1.5-\exp(-x))u(x+\eta) = 1$$

with interval conditions $u(x) = 1,\ -\delta \leq x \leq 0$ and $u(x) = 1,\ 1 \leq x \leq 1+\eta$.

Since, the exact solution of the variable coefficient problems is not known, we applied the double mesh technique to calculate the maximum absolute error.

Let U_i^N denote the computed solution of the problem on N number of mesh points and let U_i^{2N} denote the computed solution on a double number of mesh points $2N$ by including the mid-points $x_{i+1/2} = \frac{x_{i+1}+x_i}{2}$ into the mesh points. The maximum absolute error is given by

$$E^N_{\varepsilon,\delta,\eta} = \max_i |U_i^N - u(x_i)|, \text{ or } E^N_{\varepsilon,\delta,\eta} = \max_i |U_i^N - U_i^{2N}|,$$

and the ε-uniform error is calculated using $E^N = \max_{\varepsilon,\delta,\eta} \left|E^N_{\varepsilon,\delta,\eta}\right|$. The rate of convergence of the scheme is calculated using $r^N_{\varepsilon,\delta,\eta} = \log_2\left(E^N_{\varepsilon,\delta,\eta}/E^{2N}_{\varepsilon,\delta,\eta}\right)$, and the ε-uniform rate of convergence is calculated using $r^N = \log_2\left(E^N/E^{2N}\right)$.

Table 1: Example 1, maximum absolute error of the scheme for $\lambda_1 = 1/12, \lambda_2 = 5/12$.

$\varepsilon \downarrow$	$N = 2^5$	2^6	2^7	2^8	2^9	2^{10}
2^0	9.4299e-04	5.6749e-04	3.0865e-04	1.6056e-04	8.1847e-05	4.1316e-05
2^{-2}	1.4597e-03	6.2724e-04	2.9779e-04	1.4502e-04	7.1636e-05	3.5613e-05
2^{-4}	1.5547e-03	8.9197e-04	4.0302e-04	1.5441e-04	7.2554e-05	3.5885e-05
2^{-6}	2.6082e-03	1.0350e-03	4.1282e-04	2.3469e-04	1.0215e-04	3.9059e-05
2^{-8}	3.0394e-03	1.5279e-03	6.7558e-04	2.6400e-04	1.0473e-04	5.9369e-05
2^{-10}	3.0383e-03	1.5588e-03	7.8897e-04	3.8892e-04	1.7038e-04	6.6326e-05
2^{-12}	3.0377e-03	1.5585e-03	7.8910e-04	3.9701e-04	1.9905e-04	9.7666e-05
2^{-14}	3.0376e-03	1.5584e-03	7.8907e-04	3.9699e-04	1.9911e-04	9.9709e-05
2^{-16}	3.0376e-03	1.5584e-03	7.8906e-04	3.9699e-04	1.9911e-04	9.9708e-05
2^{-18}	3.0376e-03	1.5584e-03	7.8906e-04	3.9699e-04	1.9911e-04	9.9708e-05
2^{-20}	3.0376e-03	1.5584e-03	7.8906e-04	3.9699e-04	1.9911e-04	9.9708e-05
E^N	3.0376e-03	1.5584e-03	7.8906e-04	3.9699e-04	1.9911e-04	9.9708e-05
r^N	0.9629	0.9819	0.9910	0.9955	0.9978	-

Four examples with their solution exhibiting a boundary layer are considered. Examples 1 and 2 have the boundary layer on the right side of the domain and Examples 3 and 4 have it on the left side of the domain. For the detail, one can observe in Figure 1, the layer formation of the solutions for different values of ε. In Figure 2, the influence of the delay parameter on the solution profile is given by considering different values of the delay

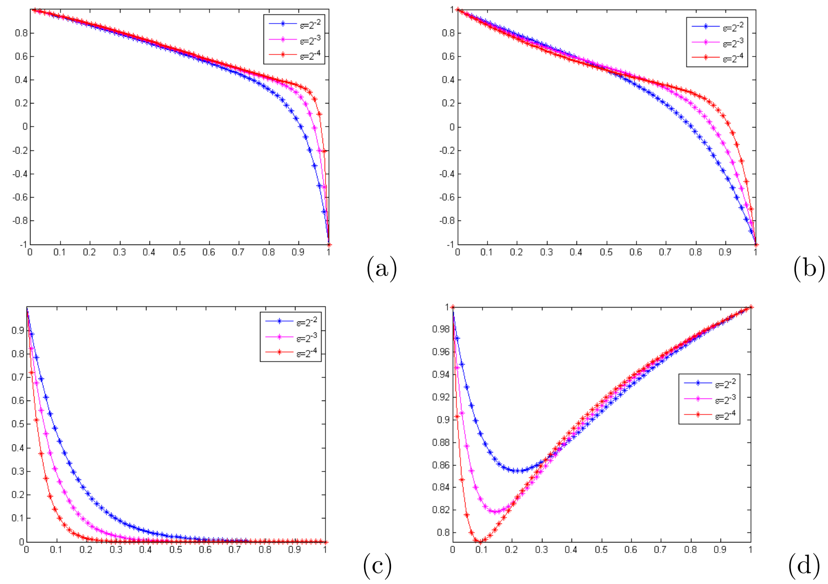

Figure 1: Boundary layer formation for different values of ε and $\delta = 0.6\varepsilon, \eta = 0.5\varepsilon$, (a) Example 1, (b) Example 2, (c) Example 3 and (d) Example 4.

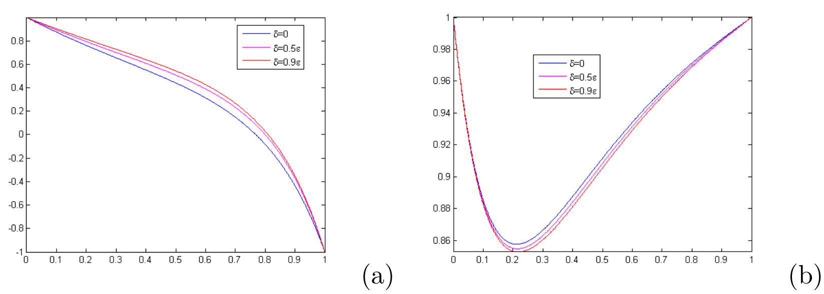

Figure 2: Solution profile for different values of delay parameter for $\varepsilon = 2^{-2}$, (a) Example 2, (b) Example 4.

parameter for $\varepsilon = 2^{-2}$. The maximum absolute error of the proposed scheme for $\lambda_1 = 1/12$ and $\lambda_2 = 5/12$ is given in Tables 1–4, for different values of the perturbation parameter ε. One observes that as $\varepsilon \to 0$ in each column, the maximum absolute error becomes stable and uniform. This indicates that the proposed scheme is uniformly convergent. In the last two rows of these tables, the uniform error and uniform rate of the convergence of the scheme are given. In Table 5, the rate of convergence of the scheme is given for different values of ε ranging from 2^{-12} to 2^{-20}. It is observed that the scheme gives a linear order uniform convergence. In Tables 6–8, the comparison of the proposed scheme with the result in [28] is given. As we observe, the uni-

Table 2: Example 2, maximum absolute error of the scheme for $\lambda_1 = 1/12, \lambda_2 = 5/12$.

$\varepsilon \downarrow$	N = 2^5	2^6	2^7	2^8	2^9	2^{10}
2^0	7.4098e-04	3.7006e-04	1.8494e-04	9.2446e-05	4.6218e-05	2.3108e-05
2^{-2}	2.1452e-03	1.0603e-03	5.2699e-04	2.6277e-04	1.3121e-04	6.5561e-05
2^{-4}	3.3473e-03	1.5615e-03	7.5382e-04	3.7142e-04	1.8438e-04	9.1872e-05
2^{-6}	4.4315e-03	2.1468e-03	9.3461e-04	4.3381e-04	2.0955e-04	1.0317e-04
2^{-8}	6.2230e-03	2.7096e-03	1.1361e-03	5.5377e-04	2.4090e-04	1.1169e-04
2^{-10}	6.2230e-03	3.2715e-03	1.6132e-03	6.9087e-04	2.8575e-04	1.3957e-04
2^{-12}	6.3987e-03	3.2747e-03	1.6565e-03	8.3275e-04	4.0708e-04	1.7359e-04
2^{-14}	6.4018e-03	3.2751e-03	1.6567e-03	8.3326e-04	4.1787e-04	2.0915e-04
2^{-16}	6.4025e-03	3.2752e-03	1.6568e-03	8.3329e-04	4.1788e-04	2.0915e-04
2^{-18}	6.4025e-03	3.2752e-03	1.6568e-03	8.3329e-04	4.1788e-04	2.0915e-04
2^{-20}	6.4025e-03	3.2752e-03	1.6568e-03	8.3329e-04	4.1788e-04	2.0915e-04
E^N	6.4025e-03	3.2752e-03	1.6568e-03	8.3329e-04	4.1788e-04	2.0915e-04
r^N	0.9671	0.9832	0.9915	0.9957	0.9986	-

Table 3: Example 3, maximum absolute error of the scheme for $\lambda_1 = 1/12, \lambda_2 = 5/12$.

$\varepsilon \downarrow$	N = 2^5	2^6	2^7	2^8	2^9	2^{10}
2^0	9.7173e-04	4.7967e-04	2.3842e-04	1.1886e-04	5.9346e-05	2.9652e-05
2^{-2}	2.2582e-03	1.0918e-03	5.3781e-04	2.6712e-04	1.3311e-04	6.6445e-05
2^{-4}	4.1684e-03	1.9446e-03	9.3525e-04	4.6062e-04	2.2884e-04	1.1409e-04
2^{-6}	9.0197e-03	3.4155e-03	1.3580e-03	3.1052e-04	2.0955e-04	1.5427e-04
2^{-8}	1.3302e-02	6.2951e-03	2.4966e-03	9.1533e-04	3.7245e-04	1.7575e-04
2^{-10}	1.3600e-02	7.2675e-03	3.6950e-03	1.6616e-03	6.4185e-04	2.3339e-04
2^{-12}	1.3615e-02	7.2783e-03	3.7679e-03	1.9170e-03	9.4973e-04	4.2126e-04
2^{-14}	1.3618e-02	7.2804e-03	3.7690e-03	1.9182e-03	9.6771e-04	4.8587e-04
2^{-16}	1.3619e-02	7.2809e-03	3.7690e-03	1.9183e-03	9.6778e-04	4.8607e-04
2^{-18}	1.3619e-02	7.2809e-03	3.7690e-03	1.9183e-03	9.6778e-04	4.8607e-04
2^{-20}	1.3619e-02	7.2809e-03	3.7690e-03	1.9183e-03	9.6778e-04	4.8607e-04
E^N	1.3619e-02	7.2809e-03	3.7690e-03	1.9183e-03	9.6778e-04	4.8607e-04
r^N	0.9034	0.9499	0.9744	0.9871	0.9935	-

form error and uniform rate of convergence of the proposed scheme is better than that of in [28].

5 Conclusion

This article dealt with the numerical treatment of SPDDEs having shifts on the reaction terms. The solution of the considered problem exhibited the boundary layer on the left or right side of the domain as $\varepsilon \to 0$. The terms involving the shift were approximated using the Taylor series approximation. The exponentially fitted tension spline method was used for treating the resulting singularly perturbed boundary value problem. The first derivative terms were approximated using left-shifted, central, and right-shifted finite

Table 4: Example 4, maximum absolute error of the scheme for $\lambda_1 = 1/12, \lambda_2 = 5/12$.

$\varepsilon \downarrow$	N = 2^5	2^6	2^7	2^8	2^9	2^{10}
2^0	1.9784e-04	9.7513e-05	4.8442e-05	2.4139e-05	1.2049e-05	6.0194e-06
2^{-2}	3.7410e-04	2.0860e-04	1.0982e-04	5.6313e-05	2.8509e-05	1.4342e-05
2^{-4}	1.4674e-04	1.6633e-04	1.2390e-04	7.2396e-05	3.8825e-05	2.0070e-05
2^{-6}	1.4706e-03	3.3509e-04	4.4397e-05	4.6273e-05	3.5377e-05	2.0784e-05
2^{-8}	2.0794e-03	1.0391e-03	3.9571e-04	8.8140e-05	1.1611e-05	1.2172e-05
2^{-10}	2.0806e-03	1.0780e-03	5.4814e-04	2.6668e-04	1.0074e-04	2.2328e-05
2^{-12}	2.0806e-03	1.0780e-03	5.4844e-04	2.7658e-04	1.3881e-04	6.7101e-05
2^{-14}	2.0806e-03	1.0780e-03	5.4844e-04	2.7658e-04	1.3888e-04	6.9589e-05
2^{-16}	2.0806e-03	1.0780e-03	5.4844e-04	2.7658e-04	1.3888e-04	6.9589e-05
2^{-18}	2.0806e-03	1.0780e-03	5.4844e-04	2.7658e-04	1.3888e-04	6.9589e-05
2^{-20}	2.0806e-03	1.0780e-03	5.4844e-04	2.7658e-04	1.3888e-04	6.9589e-05
E^N	2.0806e-03	1.0780e-03	5.4844e-04	2.7658e-04	1.3888e-04	6.9589e-05
r^N	0.9486	0.9750	0.9876	0.9939	0.9969	-

Table 5: Example 1, ($r^N_{\varepsilon,\delta,\eta}$) of the scheme for $\lambda_1 = 1/12, \lambda_2 = 5/12$.

$\varepsilon \downarrow$	N = 2^5	2^6	2^7	2^8	2^9
	Example 1				
2^{-12}	0.9628	0.9819	0.9910	0.9960	1.0272
2^{-14}	0.9629	0.9819	0.9910	0.9955	0.9978
2^{-16}	0.9629	0.9819	0.9910	0.9955	0.9978
2^{-18}	0.9629	0.9819	0.9910	0.9955	0.9978
2^{-20}	0.9629	0.9819	0.9910	0.9955	0.9978
	Example 4				
2^{-12}	0.9486	0.9750	0.9876	0.9946	1.0487
2^{-14}	0.9486	0.9750	0.9876	0.9939	0.9969
2^{-16}	0.9486	0.9750	0.9876	0.9939	0.9969
2^{-18}	0.9486	0.9750	0.9876	0.9939	0.9969
2^{-20}	0.9486	0.9750	0.9876	0.9939	0.9969

Table 6: Comparison of uniform error and uniform rate of convergence of Example 1.

$\varepsilon \downarrow$	N = 2^5	2^6	2^7	2^8	2^9	2^{10}
	Propose	Scheme				
E^N	3.0376e-03	1.5584e-03	7.8906e-04	3.9699e-04	1.9911e-04	9.9708e-05
r^N	0.9629	0.9819	0.9910	0.9955	0.9978	-
	Result	in [28]				
E^N	8.9743e-02	4.6893e-02	2.4148e-02	1.5602e-02	7.5110e-03	3.5319e-03
r^N	0.9364	0.9575	0.6302	1.0547	1.0886	-

difference approximation. The stability of the scheme was investigated using the comparison principle and solution bound. The uniform convergence of the scheme was proved, and it gave the first-order uniform convergent. The performance of the scheme was compared with some published articles and it gave an accurate result.

Table 7: Comparison of uniform error and uniform rate of convergence of Example 2.

$\varepsilon \downarrow$	N = 2^5	2^6	2^7	2^8	2^9	2^{10}
	Propose	Scheme				
E^N	6.4025e-03	3.2752e-03	1.6568e-03	8.3329e-04	4.1788e-04	2.0915e-04
r^N	0.9671	0.9832	0.9915	0.9957	0.9986	-
	Result	in [28]				
E^N	1.0273e-01	6.1537e-02	3.8643e-02	2.2077e-02	1.2395e-02	7.0772e-03
r^N	0.7393	0.6712	0.8074	0.8328	0.8085	-

Table 8: Comparison of uniform error and uniform rate of convergence of Example 3.

$\varepsilon \downarrow$	N = 2^5	2^6	2^7	2^8	2^9	2^{10}
	Propose	Scheme				
E^N	1.3619e-02	7.2809e-03	3.7690e-03	1.9183e-03	9.6778e-04	4.8607e-04
r^N	0.9034	0.9499	0.9744	0.9871	0.9935	-
	Result	in [28]				
E^N	9.6126e-02	5.7165e-02	3.3247e-02	1.8984e-02	1.0685e-02	5.9444e-03
r^N	0.7498	0.7819	0.8084	0.8293	0.8459	-

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