An operational approach for solving fractional pantograph differential equation

H. Ebrahimi* and K. Sadri

Abstract

The aim of the current paper is to construct the shifted fractional-order Jacobi functions (SFJFs) based on the Jacobi polynomials to numerically solve the fractional-order pantograph differential equations. To achieve this purpose, first the operational matrices of integration, product, and pantograph, related to the fractional-order basis, are derived (operational matrix of integration is derived in Riemann–Liouville fractional sense). Then, these matrices are utilized to reduce the main problem to a set of algebraic equations. Finally, the reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments. Also, some theorems are presented on existence of solution of the problem under study and convergence of our method.

Keywords: Fractional pantograph differential equation; Fractional-order Jacobi functions; Operational matrices; Caputo derivative; Riemann–Liouville integral.

1 Introduction

In recent decades, fractional calculus and fractional differential equations have attracted the interest of many mathematicians and researchers. Fractional differential equations appear in various fields of sciences and engineering such as visco-elastic materials [1], propagation of spherical flames [16], electromagnetism [7], fluid mechanics [13], and continuum and statistical

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mechanics [15]. The fractional nature of this class of equations makes their solving very difficult. For this reason, many researchers and authors have tried to generalize the existing methods to simply apply and numerically or analytically solve the fractional-order functional equations. These methods include Spectral method [5, 6], Tau method [2], Muntz–Legendre Tau Method [17], Collocation method [27], and many others. In this regard, valuable works have been presented, for example ,Dehghan and et al. applied the Jacobi polynomials to solve fractional variational problems [4]. In [8], the authors utilized new fractional-order Legendre functions to solve weakly singular Volterra integral equations. Esmail and et al. in [9] presented the collocation Muntz method for the solution of fractional differential equations. Kazem generalized the Jacobi integral operational matrix to solve a class of nonlinear fractional differential equations [11]. The authors in [12] utilized the Tau method together with the d-dimensional orthogonal functions to solve partial differential equations.

In this paper, we consider fractional–order pantograph differential equations as follows:

$$
\begin{aligned}
D^{\gamma}u(t) &= au(t) + \sum_{i=1}^{l} b_i D^{\gamma_i}u(q_i t) + f(t), \qquad m-1 < \gamma \leqslant m, \ \ t \in [0,1], \\
u^{(j)}(0) &= d_j, \qquad j = 0, 1, \ldots, m-1,
\end{aligned}
\tag{1}
$$

where $a, b_i \in \mathbb{R}$, $0 \leqslant \gamma_i < \gamma \leqslant m$, $i = 1, 2, \ldots, l$, $0 < q_i < 1$, D is the fractional derivative in the Caputo sense, and $u(t)$ is an unknown function.

The pantograph equation is one of the kinds of delay differential equations and has many applications in electro-dynamic and biology; see [14]. Several authors have solved the pantograph differential equations of integer-order such as, Jacobi operational method [3], Chebyshev polynomials [21], Bernoullie polynomials [23], variational iteration method [26], and so on. But there exist few methods applied to numerical solution of pantograph differential equations of fractional-order. For example, we can mention the Hermite wavelet method [20], spectral-collocation method [24], and Legendre multiwavelet collocation method [25]. This motivates us to present a low cost algorithm for solving this kind of fractional-order equations. To this end, we define the shifted fractional-order Jacobi functions based on shifted Jacobi polynomial on $[0, 1]$. Then, the operational matrices of fractional integration, product, and pantograph are constructed. The resultant matrices are utilized to approximate the various terms in equation (1). Finally, the main problem is converted to an algebraic equation, which is collocated at the roots of $P_{N+1}^{(\alpha,\beta)}(t)$. Consequently, an algebraic system is achieved, which solving it leads to determine the unknown coefficients vector and thereupon an approximate solution is obtained. It must be mentioned that the resultant nonlinear algebraic system (corresponding to a nonlinear equation) will be solved by the well-known Newton iteration method.

The current paper is organized as follows: Section 1 contains the introduction; in Section 2, some basic definitions of fractional calculus are introduced; the study of existence and uniqueness of solution of (1) is presented in Section 3; the shifted fractional-order Jacobi functions (SFJFs) and their properties are given in Section 4; in Section 5, the Jacobi operational of fractional integration, product, and pantograph are derived; the proposed method is described to solve the fractional-order pantograph differential equations in Section 6; in Section 7, the convergence of proposed approach is studied and an error bound is obtained in the weighted-Jacobi Sobolev space; In section 8, our numerical results are reported. For this purpose, several examples are presented; finally, a conclusion is presented in Section 9.

2 Basic definitions of fractional calculus

Some basic definitions and properties of fractional calculus theory are presented in this section, which are used in this paper.

Definition 1. *The Riemann–Liouville fractional integral operator of order* $\gamma \geqslant 0$ *is defined as follows:*

$$I^\gamma u(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} u(s)ds, \quad \gamma > 0, \ t > 0,$$
$$I^0 u(t) = u(t).$$

Definition 2. *The Caputo fractional derivative operator of order* $\gamma \geqslant 0$ *is given by*

$$D^\gamma u(t) = I^{m-\gamma} D^m u(t) = \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-s)^{m-\gamma-1} \frac{d^m}{ds^m} u(s)ds, \ t > 0,$$

where $m = \lceil \gamma \rceil$.

For the Riemann–Liouville fractional integral and Caputo fractional derivative operators, we have

1. $I^{\gamma_1} I^{\gamma_2} u(t) = I^{\gamma_1+\gamma_2} u(t),$

2. $I^\gamma(\lambda_1 u_1(t) + \lambda_2 u_2(t)) = \lambda_1 I^\gamma u_1(t) + \lambda_2 I^\gamma u_2(t),$

3. $I^\gamma x^\nu = \dfrac{\Gamma(\nu+1)}{\Gamma(\gamma+\nu+1)} x^{\gamma+\nu}, \quad \nu > -1,$

4. $D^\gamma I^\gamma u(t) = u(t),$

5. $I^\gamma D^\gamma u(t) = u(t) - \displaystyle\sum_{i=0}^{m-1} \frac{u^{(i)}(0^+)}{i!} x^i,$

6. $D^\gamma x^\nu = \begin{cases} 0, & \gamma > \nu, \\ \frac{\Gamma(\nu+1)}{\Gamma(\nu-\gamma+1)} x^{\nu-\gamma}, & otherwise, \end{cases}$

7. $D^\gamma \lambda = 0$,

where λ_1, λ_2, and λ are constant.

3 Existence and uniqueness of solutions of fractional-order pantograph differential equations

In this section, the existence of solutions of (1) is established by using the fixed point theorem.

Let Y be a Banach space and let $C(J, Y)$ be the Banach space of continuous $y(t)$ with $y(t) \in Y$ and $t \in J = [0, 1]$ and $\|y\| = \max_{t \in J} |y(t)|$. Moreover, $B_r(y, Y)$ represents the closed ball with center at y and radius r in Y. It is seen that (1) is equivalent to the following integral equation:

$$u(t) = g(t) + \frac{a}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} u(s) ds$$

$$+ \sum_{i=1}^l \frac{b_i}{\Gamma(\gamma-\gamma_i)} \int_0^t (t-s)^{\gamma-\gamma_i-1} u(q_i s) ds$$

$$+ \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds,$$

where

$$g(t) = \sum_{k=0}^{m-1} \frac{d_k\, t^{k+\gamma}}{\Gamma(k+\gamma+1)} - \sum_{i=1}^n \frac{b_i}{\Gamma(\gamma-\gamma_i)} \sum_{k=0}^{m_i-1} q_i^k\, d_k\, t^{k+\gamma-\gamma_i}, \quad m_i - 1 < \gamma_i \leqslant m_i,$$

$$i = 1, 2, \ldots, l.$$

Theorem 1. *If* $(|a|/\Gamma(\gamma+1) + l\, b_0/Z) < 1/2$, *where* $b_0 = \max_{1 \leqslant i \leqslant l}\{b_i\}$ *and* $Z = \min_{1 \leqslant i \leqslant n}\{\Gamma(\gamma - \gamma_i + 1)\}$, *then, the fractional-order pantograph equation* (1) *has a unique solution.*

Proof. Let $W = C(J, Y)$. A mapping $\Psi u(t) : W \to W$ is defined by

$$\Psi u(t) = g(t) + \frac{a}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} u(s) ds$$

$$+ \sum_{i=1}^l \frac{b_i}{\Gamma(\gamma-\gamma_i)} \int_0^t (t-s)^{\gamma-\gamma_i-1} u(q_i s) ds$$

$$+ \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds.$$

It should be shown that Ψ has a fixed point and that this fixed point is a solution of (1). Set $r \geqslant 2\ (G + F/\Gamma(\gamma + 1))$, where $G = \max_{t \in J} \|g(t)\|$ and $F = \max_{t \in J} \|f(t)\|$. Then, it can be shown that $\Psi B_r \subset B_r$, where $B_r = \{y \in W|\ \| y \| \leqslant r\}$. So, one has

$$\| \Psi u(t) \| \leqslant \| g(t) \| + \frac{|a|}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma - 1} \| u(s) \| ds$$

$$+ \sum_{i=1}^l \frac{|b_i|}{\Gamma(\gamma - \gamma_i)} \int_0^t (t - s)^{\gamma - \gamma_i - 1} \| u(q_i s) \| ds$$

$$+ \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma - 1} \| f(s) \| ds$$

$$\leqslant G + r \frac{|a|}{\gamma\Gamma(\gamma)} + r \sum_{i=1}^l \frac{|b_i|}{(\gamma - \gamma_i)\Gamma(\gamma - \gamma_i)} + \frac{F}{\gamma\Gamma(\gamma)}$$

$$\leqslant G + r \left(\frac{|a|}{\Gamma(\gamma + 1)} + \frac{l\ b_0}{Z} \right) + \frac{F}{\Gamma(\gamma + 1)}$$

$$\leqslant r.$$

Thus, Ψ maps B_r into itself. Now, for $u_1(t), u_2(t) \in W$, one has,

$$\| \Psi u_1(t) - \Psi u_2(t) \| \leqslant \frac{|a|}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma - 1} \| u_1(s) - u_2(s) \| ds$$

$$+ \sum_{i=1}^l \frac{|b_i|}{\Gamma(\gamma - \gamma_i)} \int_0^t (t - s)^{\gamma - \gamma_i - 1} \| u_1(q_i s) - u_2(q_i s) \| ds$$

$$\leqslant \frac{|a|}{\Gamma(\gamma + 1)} \| u_1(s) - u_2(s) \| + \frac{l\ b_0}{Z} \| u_1(s) - u_2(s) \|$$

$$= \left(\frac{|a|}{\Gamma(\gamma + 1)} + \frac{l\ b_0}{Z} \right) \| u_1(s) - u_2(s) \|.$$

As $|a|/\Gamma(\gamma+1)+l\ b_0/Z < 1/2$, the mapping Ψ is a contraction and therefore there exists a unique fixed point $u(t) \in B_r$ such that $\Psi u(t) = u(t)$. $\square$

4 Fractional-order Jacobi functions

In this section, first we recall the definitions and some useful properties of the orthogonal Jacobi polynomials in the interval $[0, 1]$. Then, we define the fractional-order Jacobi functions to obtain the operational matrices of fractional integration, product, and pantograph.

4.1 Shifted Jacobi polynomials

The shifted Jacobi polynomials are defined on the interval $[0, 1]$, with weighted function $w^{(\alpha,\beta)}(x) = (1-x)^\alpha \, x^\beta$, and can be determined with the following recurrence formula [22]:

$$P_{i+1}^{(\alpha,\beta)}(x) = A(\alpha,\beta,i) \, P_i^{(\alpha,\beta)}(x) + (2x-1) \, B(\alpha,\beta,i) \, P_i^{(\alpha,\beta)}(x)$$
$$- E(\alpha,\beta,i) \, P_{i-1}^{(\alpha,\beta)}(x), \qquad i = 1, 2, \ldots, \tag{2}$$

where

$$A(\alpha,\beta,i) = \frac{(2i+\alpha+\beta+1)(\alpha^2-\beta^2)}{2(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)},$$

$$B(\alpha,\beta,i) = \frac{(2i+\alpha+\beta+2)(2i+\alpha+\beta+1)}{2(i+1)(i+\alpha+\beta+1)},$$

$$E(\alpha,\beta,i) = \frac{(i+\alpha)(i+\beta)(2i+\alpha+\beta+2)}{(i+1)(i+\alpha+\beta+1)(2i+\alpha+\beta)},$$

and

$$P_0^{(\alpha,\beta)}(x) = 1, \quad P_1^{(\alpha,\beta)}(x) = \frac{\alpha+\beta+2}{2}(2x-1) + \frac{\alpha-\beta}{2}.$$

The orthogonality relation of Jacobi polynomials is defined as follows:

$$\int_0^1 P_j^{(\alpha,\beta)}(x) \, P_k^{(\alpha,\beta)}(x) \, w^{(\alpha,\beta)}(x) \, dx = h_k \, \delta_{jk},$$

where

$$h_k = \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{(2k+\alpha+\beta+1)k! \, \Gamma(k+\alpha+\beta+1)}.$$

The analytic form of Jacobi polynomials is given by

$$P_i^{(\alpha,\beta)}(x) = \sum_{k=0}^i \frac{(-1)^{i-k}\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)x^k}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1) \, (i-k)! \, k!}, \quad i = 0, 1, \ldots.$$

A continuous function $u(t)$, in the interval $[0, 1]$, can be expressed in terms of shifted Jacobi polynomials as follows:

$$u(x) = \sum_{j=0}^\infty C_j \, P_j^{(\alpha,\beta)}(x), \tag{3}$$

where the coefficients C_j are given by

$$C_j = \frac{1}{h_j} \int_0^1 u(t) \, P_j^{(\alpha,\beta)}(x) \, w^{(\alpha,\beta)}(x) \, dx, \quad j = 0, 1, \ldots.$$

In practice, only the first $(N + 1)$-terms of shifted Jacobi polynomials are considered. Therefore, one has

$$u_N(x) = \sum_{j=0}^{N} C_j \ P_j^{(\alpha,\beta)}(x) = \Phi^T(x) \ C = C^T \ \Phi(x), \qquad (4)$$

where the vectors C and $\Phi(t)$ are given by

$$C = [C_0, C_1, \ldots, C_N]^T, \qquad \Phi(x) = [P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), \ldots, P_N^{(\alpha,\beta)}(x)]^T. \qquad (5)$$

Also, the following auxiliary relation is obtained from (2).

$$\begin{aligned}
2x \ P_i^{(\alpha,\beta)}(x) = & \frac{1}{B(\alpha,\beta,i)} \ P_{i+1}^{(\alpha,\beta)}(x) - \frac{A(\alpha,\beta,i)}{B(\alpha,\beta,i)} \ P_i^{(\alpha,\beta)}(x) \\
& + \frac{E(\alpha,\beta,i)}{B(\alpha,\beta,i)} \ P_{i-1}^{(\alpha,\beta)}(x), \qquad i = 1, 2, \ldots, N.
\end{aligned} \qquad (6)$$

Two other properties of the shifted Jacobi polynomials are defined below:

$$\begin{aligned}
P_i^{(\alpha,\beta)}(0) & = (-1)^i \binom{i + \alpha}{i}, \\
\frac{d^i P_n^{(\alpha,\beta)}(x)}{dx^i} & = \frac{\Gamma(n + i + \alpha + \beta + 1)}{\Gamma(n + \alpha + \beta + 1)} P_{n-i}^{(\alpha+i,\beta+i)}(x).
\end{aligned} \qquad (7)$$

4.2 Shifted fractional-order Jacobi functions

We define the shifted fractional-order Jacobi functions (SFJFs) by transformation x to t^σ ($\sigma > 0$) based on Jacobi polynomials (2) in the interval $[0,1]$. For convenience, the fractional-order Jacobi functions, $P_i^{(\alpha,\beta)}(t^\sigma)$, $i = 0, 1, 2, \ldots$, are denoted by $P_i^{(\alpha,\beta,\sigma)}(t)$. It is clear, the classic Jacobi polynomials are resulted for $\sigma = 1$. The weight function is $w^{(\alpha,\beta,\sigma)}(t) = \sigma t^{\sigma(\beta+1)-1}(1 - t^\sigma)^\alpha$. Moreover, the relations (2)–(7) are converted to:

$$\begin{aligned}
P_{i+1}^{(\alpha,\beta,\sigma)}(t) = & A(\alpha,\beta,i) \ P_i^{(\alpha,\beta,\sigma)}(t) + (2t^\sigma - 1) \ B(\alpha,\beta,i) \ P_i^{(\alpha,\beta,\sigma)}(t) \\
& - E(\alpha,\beta,i) \ P_{i-1}^{(\alpha,\beta,\sigma)}(t), \qquad i = 1, 2, \ldots,
\end{aligned} \qquad (8)$$

$$P_0^{(\alpha,\beta,\sigma)}(t) = 1, \qquad P_1^{(\alpha,\beta,\sigma)}(t) = \frac{\alpha + \beta + 2}{2}(2t^\sigma - 1) + \frac{\alpha - \beta}{2},$$

where coefficients $A(\alpha,\beta,i)$, $B(\alpha,\beta,i)$, and $E(\alpha,\beta,i)$ are same in (2).

$$\int_0^1 P_j^{(\alpha,\beta,\sigma)}(t) \ P_k^{(\alpha,\beta,\sigma)}(t) \ w^{(\alpha,\beta,\sigma)}(t) \ dt = h_k \ \delta_{jk},$$

$$P_i^{(\alpha,\beta,\sigma)}(t) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)t^{\sigma k}}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)\ (i-k)!\ k!}, \qquad (9)$$

$$u(t) = \sum_{j=0}^{\infty} C_j\ P_j^{(\alpha,\beta,\sigma)}(t),$$

where

$$C_j = \frac{1}{h_j} \int_0^1 u(t)\ P_j^{(\alpha,\beta,\sigma)}(t)\ w^{(\alpha,\beta,\sigma)}(t)\ dt, \quad j = 0, 1, \ldots,$$

$$u_N(t) = \sum_{j=0}^{N} C_j\ P_j^{(\alpha,\beta,\sigma)}(t) = \Phi^{(\sigma)T}(t)\ C = C^T\ \Phi^{(\sigma)}(t),$$

where the vectors C and $\Phi^{(\sigma)}(t)$ are given by

$$C = [C_0, C_1, \ldots, C_N]^T, \quad \Phi^{(\sigma)}(t) = [P_0^{(\alpha,\beta,\sigma)}(t), P_1^{(\alpha,\beta,\sigma)}(t), \ldots, P_N^{(\alpha,\beta,\sigma)}(t)]^T.$$
$$(10)$$

$$2t^\sigma\ P_i^{(\alpha,\beta,\sigma)}(t) = \frac{1}{B(\alpha,\beta,i)}\ P_{i+1}^{(\alpha,\beta,\sigma)}(t) - \frac{A(\alpha,\beta,i)}{B(\alpha,\beta,i)}\ P_i^{(\alpha,\beta,\sigma)}(t)$$
$$+ \frac{E(\alpha,\beta,i)}{B(\alpha,\beta,i)}\ P_{i-1}^{(\alpha,\beta,\sigma)}(t), \qquad i = 1, 2, \ldots, N, \qquad (11)$$

$$\frac{dP_n^{(\alpha,\beta,\sigma)}(t)}{dt} = \sigma\Gamma(n+\alpha+\beta+1)t^{\sigma-1}P_{n-1}^{(\alpha+1,\beta+1,\sigma)}(t).$$

5 Fractional-order Jacobi operational matrices of integration, product, and pantograph

The objective of this section is to derive the operational matrices of fractional integration, product, and pantograph related to the SFJFs. The fractional-order operational matrices of product and pantograph are utilized for approximating the product of the vectors $\Phi^{(\sigma)}(t)$ and $\Phi^{(\sigma)T}(t)$ and the terms, which include the delay in equations under study, respectively. First, some properties of the shifted fractional-order Jacobi functions are presented as follows.

Lemma 1 (see [19]). *The ith shifted fractional-order Jacobi function,* $P_i^{(\alpha,\beta,\sigma)}(t)$, $t \in [0, 1]$, *can be obtained in the form*

$$P_n^{(\alpha,\beta,\sigma)}(t) = \sum_{k=0}^{n} \gamma_k^{(n)}\ t^{\sigma k},$$

where coefficients $\gamma_k^{(i)}$ are given as

$$\gamma_k^{(n)} = (-1)^{n-k} \begin{pmatrix} n+k+\alpha+\beta \\ k \end{pmatrix} \begin{pmatrix} n+\alpha \\ n-k \end{pmatrix}.$$

Lemma 2. *If $n \in \mathbb{N}$ and $p \geqslant n$, then we have*

$$\int_0^1 t^p\, P_n^{(\alpha,\beta,\sigma)}(t)\, w^{(\alpha,\beta,\sigma)}(t)\, dt$$
$$= \sum_{k=0}^n \frac{(-1)^{n-k}\, \Gamma(n+\beta+1)\, \Gamma(n+k+\alpha+\beta+1)\, \Gamma(k+\beta+p/\sigma+1)\, \Gamma(\alpha+1)}{\Gamma(k+\beta+1)\, \Gamma(n+\alpha+\beta+1)\, \Gamma(k+\alpha+\beta+p/\sigma+2)\, (n-k)!\, k!}.$$

Proof. The lemma can be easily proved by multiplying relation (9) in t^p and then integrating of the resulting equation. $\square$

Now, the Jacobi operational matrix of fractional integration is generally derived.

Theorem 2. *Suppose that $\Phi^{(\sigma)}(t)$ is the shifted fractional-order Jacobi vector in equation (10). Then, we have*

$$I^{(\nu)}\Phi^{(\sigma)}(t) \simeq \mathbf{P}^{(\nu)}\ \Phi^{(\sigma)}(t),$$

where $I^{(\nu)}$ is the Riemann–Liouville fractional integral operator of order ν and $\mathbf{P}^{(\nu)}$ is the $(N+1) \times (N+1)$ operational matrix of fractional integration which defined by

$$\mathbf{P}^{(\nu)} = \begin{bmatrix} \pi(0,0) & \pi(0,1) & \dots & \pi(0,N) \\ \pi(1,0) & \pi(1,1) & \dots & \pi(1,N) \\ \vdots & \vdots & \ddots & \vdots \\ \pi(N,0) & \pi(N,1) & \dots & \pi(N,N) \end{bmatrix},$$

where

$$\pi(i,j) = \sum_{k=0}^i \omega_{ijk}, \quad i = 0,1,\dots,N,\ \ j = 1,\dots,N, \tag{12}$$

and ω_{ijk} are given by

$$\omega_{ijk} = \frac{(-1)^{i-l}\, \Gamma(j+\beta+1)\, \Gamma(i+\beta+1)\, \Gamma(i+l+\alpha+\beta+1)\, \Gamma(\alpha+1)\Gamma(\sigma l+1)}{h_j\, \Gamma(j+\alpha+\beta+1)\, \Gamma(l+\beta+1)\, \Gamma(i+\alpha+\beta+1)\, l!\, (i-l)!\Gamma(\sigma l+\nu+1)}$$
$$\times \sum_{l=0}^j \frac{(-1)^{j-k}\, \Gamma(j+k+\alpha+\beta+1)\, \Gamma(k+l+\beta+\nu/\sigma+2)}{\Gamma(k+\beta+1)\, \Gamma(k+l+\alpha+\beta+\nu/\sigma+2)\, k!(j-k)!},\ \ i,j = 0,1,\dots,N.$$

Proof. By using the analytical form (9) and the properties of operator I^ν, we have

$$I^\nu P_i^{(\alpha,\beta)}(t)\, dt = \sum_{l=0}^i \frac{(-1)^{i-l}\, \Gamma(i+\beta+1)\, \Gamma(i+l+\alpha+\beta+1)\, \Gamma(\sigma l+1)\, t^{\sigma l+\nu}}{\Gamma(l+\beta+1)\, \Gamma(i+\alpha+\beta+1)\, \Gamma(\sigma l+\nu+1)\, l!\, (i-l)!}. \tag{13}$$

Now, $t^{\sigma l + \nu}$ can be approximated in terms of the SFJFs as follows:

$$t^{\sigma l + \nu} \simeq \sum_{j=0}^{N} a_{l,j} \; P_j^{(\alpha,\beta,\sigma)}(t),$$

where

$$a_{l,j} = \frac{1}{h_j} \; \int_0^1 t^{\sigma l + \nu} \; P_j^{(\alpha,\beta,\sigma)}(t) \; w^{(\alpha,\beta,\sigma)}(t) \; dt.$$

According to Lemma 2, one has

$$\int_0^1 t^{\sigma l + \nu} \; P_j^{(\alpha,\beta,\sigma)}(t) \; w^{(\alpha,\beta,\sigma)}(t) \; dt$$

$$= \sum_{k=0}^{j} \frac{(-1)^{j-k} \; \Gamma(\alpha+1) \; \Gamma(j+\beta+1) \; \Gamma(k+l+\beta+\nu/\sigma+1) \; \Gamma(j+k+\alpha+\beta+1)}{\Gamma(k+\beta+1) \; \Gamma(j+\alpha+\beta+1) \; \Gamma(k+l+\alpha+\beta+\nu/\sigma+2) \; k! \; (j-k)!}.$$

Therefore, equation (13) is written as follows:

$$I^\nu P_i^{(\alpha,\beta,\sigma)}(t) \; dt$$

$$\simeq \sum_{j=0}^{N} \left\{ \sum_{k=0}^{i} \frac{(-1)^{i-l} \; \Gamma(j+\beta+1) \; \Gamma(i+\beta+1) \; \Gamma(i+l+\alpha+\beta+1) \; \Gamma(\alpha+1) \; \Gamma(\sigma l+1)}{h_j \; \Gamma(j+\alpha+\beta+1) \; \Gamma(l+\beta+1) \; \Gamma(i+\alpha+\beta+1) \; \Gamma \sigma l + \nu + 1) \; l! \; (i-l)!} \right.$$

$$\left. \times \sum_{k=0}^{j} \frac{(-1)^{j-k} \; \Gamma(j+k+\alpha+\beta+1) \; \Gamma(l+k+\beta+\nu/\sigma+1)}{\Gamma(k+\beta+1) \; \Gamma(l+k+\alpha+\beta+\nu/\sigma+2) \; (j-k)! \; (k)!} \right\} P_j^{(\alpha,\beta,\sigma)}(t)$$

$$= \sum_{j=0}^{N} \pi(i,j) \; P_j^{(\alpha,\beta,\sigma)}(t),$$

$$(14)$$

where $\pi(i,j)$ are given by equation (12). Accordingly, equation (14) can be rewritten as the following vector form:

$$I^\nu P_i^{(\alpha,\beta,\sigma)}(t) \simeq [\pi(i,0), \pi(i,1), \ldots, \pi(i,N)] \; \Phi^{(\sigma)}(t), \quad i = 0, 1, \ldots, N.$$

This leads to the desired result. $\qquad \square$

The following lemmas are useful to obtain the fractional operational matrix of product.

Lemma 3 (see [19]). *If $P_j^{(\alpha,\beta,\sigma)}(t)$ and $P_k^{(\alpha,\beta,\sigma)}(t)$ are respectively jth and kth SFJFs, then the product of $P_j^{(\alpha,\beta,\sigma)}(t)$ and $P_k^{(\alpha,\beta,\sigma)}(t)$ can be written as*

$$Q_{j+k}^{(\alpha,\beta,\sigma)}(t) = \sum_{r=0}^{j+k} \lambda_r^{(j,k)} \; t^{\sigma r},$$

where coefficients $\lambda_r^{(j,k)}$ are determined as follows:

If $j \geq k$ **:**

$r = 0, 1, \ldots, j + k,$

$if \ \ r > j, \ \ then$

$\qquad \lambda_r^{(j,k)} = \sum_{l=r-j}^{k} \gamma_{r-l}^{(j)} \ \gamma_l^{(k)},$

$else$

$\qquad r_1 = min\{r, k\},$

$\qquad \lambda_r^{(j,k)} = \sum_{l=0}^{r_1} \gamma_{r-l}^{(j)} \ \gamma_l^{(k)},$

$end.$

If $j < k$ **:**

$r = 0, 1, \ldots, j + k,$

$if \ \ r \leq j, \ \ then$

$\qquad r_1 = min\{r, j\},$

$\qquad \lambda_r^{(j,k)} = \sum_{l=0}^{r_1} \gamma_{r-l}^{(j)} \ \gamma_l^{(k)},$

$else$

$\qquad r_2 = min\{r, k\},$

$\qquad \lambda_r^{(j,k)} = \sum_{l=r-j}^{r_2} \gamma_{r-l}^{(j)} \ \gamma_l^{(k)},$

$end.$

The quantities $\gamma_l^{(k)}$ and $\gamma_{r-l}^{(j)}$ are introduced, respectively, for $P_k^{(\alpha,\beta,\sigma)}(t)$ and $P_j^{(\alpha,\beta,\sigma)}(t)$ based on the Lemma 1.

Lemma 4. If $P_i^{(\alpha,\beta,\sigma)}(t)$, $P_j^{(\alpha,\beta,\sigma)}(t)$, and $P_k^{(\alpha,\beta,\sigma)}(t)$ are respectively i-, j-, and kth SFJFs, then,

$$q_{ijk} = \int_0^1 P_i^{(\alpha,\beta,\sigma)}(t) \ P_j^{(\alpha,\beta,\sigma)}(t) \ P_k^{(\alpha,\beta,\sigma)}(t) \ w^{(\alpha,\beta,\sigma)}(t) \ dt$$

$$= \sum_{n=0}^{j+k} \sum_{l=0}^{i} \frac{(-1)^{i-l}\lambda_n^{(j,k)} \ \Gamma(i+\beta+1) \ \Gamma(i+l+\alpha+\beta+1) \ \Gamma(n+l+\beta+1) \ \Gamma(\alpha+1)}{\Gamma(l+\beta+1) \ \Gamma(i+\alpha+\beta+1) \ \Gamma(n+l+\alpha+\beta+2) \ (i-l)! \ l!},$$

where $\lambda_l^{(j,k)}$ are computed by Lemma 3.

Now, a general formula is presented for finding the $(N+1) \times (N+1)$ operational matrix of fractional product $\tilde{U}$ whenever,

$$\Phi^{(\sigma)}(t) \ \Phi^{(\sigma)T}(t) \ U \simeq \tilde{U} \ \Phi^{(\sigma)}(t), \tag{15}$$

and U is a $(N+1)$ given vector.

Theorem 3 (see [3]). *The entries of the matrix $\tilde{U}$ in* (15) *are computed as follows:*

$$\tilde{U}_{jk} = \frac{1}{h_k} \ \sum_{i=0}^{N} U_i \ q_{ijk}, \quad j, k = 0, 1, \ldots, N,$$

where q_{ijk} are introduced by Lemma 4 and U_i are the components of the vector U in (15).

Corollary 1. *If $u(t) \simeq U^T \ \Phi^{(\sigma)}(t) = \Phi^{(\sigma)T}(t) \ U$ and $\Phi^{(\sigma)}(t) \ \Phi^{(\sigma)T}(t) \ U \simeq \tilde{U} \ \Phi^{(\sigma)}(t)$, where U is an $(N+1)$ vector and $\tilde{U}$ is the $(N+1) \times (N+1)$ operational matrix of fractional product, corresponding to the vector U, it is easily shown that,*

$$u^2(t) \simeq U^T \ \tilde{U} \ \Phi^{(\sigma)}(t).$$

To approximate the terms including the delay, the fractional-order Jacobi operational matrix of pantograph is derived. For this end, we apply the auxiliary (11).

Replacing t with qt in definition (10), leads to

$$\Phi^{(\sigma)}(qt) = [P_0^{(\alpha,\beta,\sigma)}(qt), P_1^{(\alpha,\beta,\sigma)}(qt), \dots, P_N^{(\alpha,\beta,\sigma)}(qt)]^T.$$

The following theorem expresses the relation between vectors $\Phi^{(\sigma)}(qt)$ and $\Phi^{(\sigma)}(t)$.

Theorem 4. *The relation between the delay vector $\Phi^{(\sigma)}(qt)$, $0 < q < 1$, and vector $\Phi^{(\sigma)}(t)$ can be expressed as follows:*

$$\Phi^{(\sigma)}(tq) = \mathbf{L}_{(\mathbf{q},\sigma)} \ \Phi^{(\sigma)}(t),$$

where $\mathbf{L}_{(\mathbf{q},\sigma)}$ is an $(N+1) \times (N+1)$ lower triangular matrix called fractional operational matrix of pantograph.

Proof. Replacing t with qt in (8) and rearranging leads to

$$
\begin{aligned}
P_{i+1}^{(\alpha,\beta,\sigma)}(qt) =& A(\alpha,\beta,i) \ P_i^{(\alpha,\beta,\sigma)}(qt) + (2q^\sigma t^\sigma - 1) \ B(\alpha,\beta,i) \ P_i^{(\alpha,\beta,\sigma)}(qt) \\
& - E(\alpha,\beta,i) \ P_{i-1}^{(\alpha,\beta,\sigma)}(qt), \qquad i = 1, 2, \dots .
\end{aligned}
$$
(16)

The ith delay SFJF , $P_i^{(\alpha,\beta,\sigma)}(qt)$, can be expressed in terms of the SFJFs themselves as follows:

$$P_i^{(\alpha,\beta,\sigma)}(qt) = \sum_{k=0}^{i} l_{i,k} \ P_k^{(\alpha,\beta,\sigma)}(qt), \qquad i = 0, 1, \dots . \tag{17}$$

From the first few polynomials in (17), it easily obtains

$$l_{0,0} = 1, \quad l_{0,i} = 0, \ i = 1, 2, \dots, N, \quad l_{1,0} = (\beta+1)(q^\sigma - 1), \quad l_{1,1} = q^\sigma. \tag{18}$$

It is clearly found that,

$$l_{i,i} = q^{i\sigma}, \quad i = 0, 1, \dots, N. \tag{19}$$

Using auxiliary (11) and substituting (17) into equation (16), we have

$$\sum_{k=0}^{i+1} l_{i+1,k}\ P_k^{(\alpha,\beta,\sigma)}(t)$$

$$= \sum_{k=0}^{i} l_{i,k}\ \left\{ \left(A(\alpha,\beta,i) + q^\sigma\ B(\alpha,\beta,i)(1 - \frac{A(\alpha,\beta,k)}{B(\alpha,\beta,k)})\ P_{k+1}^{(\alpha,\beta,\sigma)}(t) - B(\alpha,\beta,i) \right)\ P_k^{(\alpha,\beta,\sigma)}(t) \right.$$

$$\left. + q^\sigma B(\alpha,\beta,i)\ \left(\frac{1}{B(\alpha,\beta,k)} + \frac{E(\alpha,\beta,k)}{B(\alpha,\beta,k)}\ P_{k-1}^{(\alpha,\beta,\sigma)}(t) \right) \right\}$$

$$- E(\alpha,\beta,i)\ \sum_{k=0}^{i-1} l_{i-1,k}\ P_k^{(\alpha,\beta,\sigma)}(t).$$

$$(20)$$

Equating the coefficients of $P_k^{(\alpha,\beta,\sigma)}(t)$ on both sides of equation (20), for $k = 0, 1, \ldots, i$, leads to

$$l_{i+1,0} = \left\{ A(\alpha,\beta,i) \right.$$

$$+ q^\sigma\ B(\alpha,\beta,i)\left(1 - \frac{\alpha - \beta}{\alpha + \beta + 2} + \frac{2(\alpha+1)(\beta+1)}{(\alpha+\beta+2)(\alpha+\beta+3)} \right)$$

$$\left. - B(\alpha,\beta,i) \right\}\ l_{i,0} - E(\alpha,\beta,i)\ l_{i-1,0}, \quad i = 1, 2, \ldots, N-1,$$

$$l_{i+1,k} = \left\{ A(\alpha,\beta,i) + q^\sigma\ B(\alpha,\beta,i)(1 - \frac{A(\alpha,\beta,i)}{B(\alpha,\beta,i)} - B(\alpha+\beta+i)) \right\}\ l_{i,k}$$

$$+ \left\{ \frac{1}{B(\alpha,\beta,k-1)}\ l_{i,k-1} + \frac{E(\alpha,\beta,k+1)}{B(\alpha,\beta,k+1)}\ l_{i,k+1} \right\}\ q^\sigma B(\alpha,\beta,i)$$

$$- C(\alpha,\beta,i)\ l_{i-1,k}, \quad k = 1, 2, \ldots, i-1,$$

$$l_{i+1,i} = \left(A(\alpha,\beta,i) + q^\sigma B(\alpha,\beta,i)(1 - \frac{A(\alpha,\beta,i)}{B(\alpha,\beta,i)}) - B(\alpha,\beta,i) \right) q^{i\sigma}$$

$$+ q^\sigma \frac{B(\alpha,\beta,i)}{B(\alpha,\beta,i-1)}\ l_{i,i-1}, \quad i = 1, 2, \ldots, N-1.$$

The starting values are given by equations, (18)–(19). Finally, for the proportional delay q, we have

$$\Phi^{(\sigma)}(qt) = [P_0^{(\alpha,\beta,\sigma)}(qt), P_1^{(\alpha,\beta,\sigma)}(tq), \ldots, P_N^{(\alpha,\beta,\sigma)}(tq)]^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ l_{1,0} & q^\sigma & 0 & \ldots & 0 & 0 \\ l_{2,0} & l_{2,1} & q^{2\sigma} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ l_{N-1,0} & l_{N-1,1} & l_{N-1,2} & \ldots & q^{(N-1)\sigma} & 0 \\ l_{N,0} & l_{N,1} & l_{N,2} & \ldots & l_{N,N-1} & q^{N\sigma} \end{bmatrix} \begin{bmatrix} P_0^{(\alpha,\beta,\sigma)}(t) \\ P_1^{(\alpha,\beta,\sigma)}(t) \\ P_2^{(\alpha,\beta,\sigma)}(t) \\ \vdots \\ P_{N-1}^{(\alpha,\beta,\sigma)}(t) \\ P_N^{(\alpha,\beta,\sigma)}(t) \end{bmatrix}$$

$$= \mathbf{L}_{(\mathbf{q},\sigma)} \; \Phi^{(\sigma)}(x).$$

$$\square$$

Corollary 2. *The operational matrices of fractional integration and product are converted to the following forms for the delay q.*

$$I^{\nu}\Phi^{(\sigma)}(qt) \; \simeq \mathbf{L}_{(\mathbf{q},\sigma)} \; \mathbf{P}^{(\nu)} \; \Phi^{(\sigma)}(t),$$

$$\Phi^{(\sigma)}(qt) \; \Phi^{(\sigma)T}(qt) \; C \simeq \mathbf{L}_{(\mathbf{q},\sigma)} \; \tilde{F} \; \Phi^{(\sigma)}(t), \quad\quad F = \mathbf{L}_{(\mathbf{q},\sigma)}^{\mathbf{T}} \; C,$$

where the matrices $\mathbf{P}^{(\nu)}$ and $\mathbf{L}_{(\mathbf{q},\sigma)}$ were defined by Theorems 2 and 4 and $\tilde{F}$ is the fractional operational matrix of product corresponding to the vector $F = \mathbf{L}_{(\mathbf{q},\sigma)}^{\mathbf{T}} \; C$.

6 Description of method

Consider the problem given in (1) again. We expand $D^{\gamma}u(t)$ by means of the SFJFs as follows:

$$D^{\gamma}u(t) \simeq \Phi^{(\sigma)T}(t)C. \tag{21}$$

According the properties of Caputo fractional derivative, we have

$$u(t) \simeq I^{\gamma}\Phi^{(\sigma)T}(t)C + \sum_{k=0}^{m-1} \frac{d_k}{k!} t^k \tag{22}$$

$$\simeq \Phi^{(\sigma)T}(t)\mathbf{P}^{(\gamma)\mathbf{T}}C + \Phi^{(\sigma)T}(t)F, \quad\quad m-1 < \gamma \leqslant m,$$

where F is a known vector and $\sum_{k=0}^{m-1}\frac{d_k}{k!}t^k$ is approximated as $\Phi^{(\sigma)T}F$. To approximate the $D^{\gamma_i}u(t)$, $0 \leqslant \gamma_i < \gamma \leqslant m$, $i = 1, 2, \ldots, l$, the following procedure can be pursued.

$$D^{\gamma}u(t) = D^{\gamma-\gamma_i}D^{\gamma_i}u(t) \simeq \Phi^{(\sigma)T}(t)C. \tag{23}$$

Using the Caputo derivative, we get

$$D^{\gamma_i}u(t) \simeq I^{\gamma-\gamma_i}\Phi^{(\sigma)T}(t)C + \sum_{k=\lceil \gamma_i \rceil}^{m-1} \frac{d_k}{\Gamma(k+1-\gamma_i)} t^{k-\gamma_i}$$

$$\simeq \Phi^{(\sigma)T}(t)\mathbf{P}^{(\gamma-\gamma_i)\mathbf{T}}C + \Phi^{(\sigma)T}F_i, \quad\quad 0 \leqslant \gamma_i < \gamma \leqslant m, \quad i = 1, 2, \ldots, l. \tag{24}$$

Now, by applying equations (21)–(24) to the terms containing the delay, we get

$$u(qt) \simeq \Phi^{(\sigma)T}(t)\mathbf{L}^{\mathbf{T}}_{(\mathbf{q},\sigma)}\mathbf{P}^{(\gamma)\mathbf{T}}C + \Phi^{(\sigma)T}(t)\mathbf{L}^{\mathbf{T}}_{(\mathbf{q},\sigma)}F,$$

$$D^{\gamma_i}u(qt) \simeq \Phi^{(\sigma)T}(t)\mathbf{L}^{\mathbf{T}}_{(\mathbf{q},\sigma)}\mathbf{P}^{(\gamma-\gamma_i)\mathbf{T}}C + \Phi^{(\sigma)T}(t)\mathbf{L}^{\mathbf{T}}_{(\mathbf{q},\sigma)}F_i, \quad i = 1,2,\ldots,l,$$

$$u^2(t) \simeq U^T\tilde{U}\Phi^{(\sigma)T}(t), \quad U = \mathbf{P}^{(\gamma)\mathbf{T}}C + F, \quad D^\gamma u(qt) \simeq \Phi^{(\sigma)T}(t)\mathbf{L}^{\mathbf{T}}_{(\mathbf{q},\sigma)}C,$$

$$(25)$$

where $\tilde{U}$ is the fractional operational matrix of product corresponding to the vector U. By substituting the expressions (21)–(25) into (1), the following algebraic equations is resulted.

$$\Phi^{(\sigma)T}(t)C - a\Phi^{(\sigma)T}(t)(\mathbf{P}^{(\gamma)\mathbf{T}}C + F) - \sum_{i=1}^{n} b_i\Phi^{(\sigma)T}(t)(\mathbf{L}^{\mathbf{T}}_{(\mathbf{q_i},\sigma)}\mathbf{P}^{(\gamma-\gamma_i)\mathbf{T}}C$$
$$+ \mathbf{L}^{\mathbf{T}}_{(\mathbf{q_i},\sigma)}F_i) - f(t) \approx 0.$$

In this way, the main equation is converted into an algebraic equation. The resultant algebraic equation is collocated at $(N + 1)$ roots of the $(N + 1)$th shifted Jacobi polynomials on the interval $(0, 1)$. By solving the system of generated algebraic equations, the unknown vector C can be determined. Thus, an approximate solution can be acquired for various values of α and β parameters using equation (22).

Corollary 3. *The nonlinear terms in the problems under study are approximated as a matrix form. For this reason, it is necessary that the initial conditions of the problems to be approximated in terms of fractional-order Jacobi basis.*

7 Convergence analysis and error estimation

In this section, the upper bounds are presented for the errors of the residual function, proposed algorithm, and operational matrix of fractional integration, $\mathbf{P}^{(\nu)}$. It is seen that the upper bounds tend to zero when the number of the SFJFs increases. First the following definitions and theorems are stated for $\sigma = 1$.

The set of all algebraic polynomials of degree at most N is denoted by $\mathbb{P}_N$. The orthogonal projection $\mathcal{P}_{N,\alpha,\beta}L^2_{w^{(\alpha,\beta)}}(J) \to \mathbb{P}_N$, $J = [0, 1]$, is considered for $v(t) \in L^2_{w^{(\alpha,\beta)}}(J)$ and defined by

$$(\mathcal{P}_{N,\alpha,\beta}v - v, \ \phi) = 0, \qquad \text{for all } \phi \in \mathbb{P}_N.$$

The Jacobi-weighted space is introduced as

$$\mathcal{J}^r_{w^{(\alpha,\beta)}}(J) = \{v| \ v \ is \ measurable \ and \ \|v\|_{r,w^{(\alpha,\beta)}} < \infty\}, \qquad r \in \mathbb{N},$$

equipped with the following norm and seminorm,

$$\|v\|_{r,w^{(\alpha,\beta)}} = \left(\sum_{k=0}^{r} \| \partial_x^k v \|_{w^{(\alpha+k,\beta+k)}}^2 \right)^{\frac{1}{2}}, \quad |v|_{r,w^{(\alpha,\beta)}} = \| \partial_x^r v \|_{w^{(\alpha+r,\beta+r)}} .$$

Theorem 5 (see [10]). *For any $v \in \mathcal{J}_{w^{(\alpha,\beta)}}^r(J)$, $r \in \mathbb{N}$, and $0 \leqslant \mu \leqslant r$, we have*

$$\|v - \mathcal{P}_{N,\alpha,\beta} v\|_{\mu,w^{(\alpha,\beta)}} \leqslant c(N(N+\alpha+\beta))^{\frac{\mu-r}{2}} |v|_{r,w^{(\alpha,\beta)}},$$

where c is a positive constant independent of N, α, and β.

Lemma 5. *If $n \in \mathbb{N}$, $v \in \mathcal{J}_{w^{(\alpha,\beta)}}^r(J)$, and $D^n v(t)$ is the derivative of $v(t)$ of order n, then*

$$\|D^n v - D^n \mathcal{P}_{N,\alpha,\beta} v\|_{\mu,w^{(\alpha,\beta)}} \leqslant c^* \left((N+n)(N+n+\alpha+\beta) \right)^{\frac{\mu-r}{2}} |v^{(n)}|_{r,w^{(\alpha,\beta)}}.$$

Proof. By derivation of the shifted Jacobi polynomials and applying Theorem 5, the desired result is acquired. $\square$

Corollary 4. *For $v \in \mathcal{J}_{w^{(\alpha,\beta)}}^r(J)$ and $0 < q < 1$, we have*

$$\|v(qt) - \mathcal{P}_{N,\alpha,\beta} v(qt)\|_{\mu,w^{(\alpha,\beta)}} \leqslant c(qN(N+\alpha+\beta))^{\frac{\mu-r}{2}} |v(qt)|_{r,w^{(\alpha,\beta)}}. \quad (26)$$

Theorem 6. *If $\gamma > 0$, $\gamma \in \mathbb{R}$, and $D^\gamma v(t)$ denotes the Caputo fractional derivative of $v(t) \in \mathcal{J}_{w^{(\alpha,\beta)}}^r(J)$ of order γ, then*

$$\|D^\gamma v - D^\gamma \mathcal{P}_{N,\alpha,\beta} v\|_{\mu,w^{(\alpha,\beta)}} \leqslant \frac{c'}{\Gamma(m-\gamma+1)} \left((N+m)(N+m+\alpha+\beta) \right)^{\frac{\mu-r}{2}}$$
$$\times |v^{(m)}|_{r,w^{(\alpha,\beta)}}, \quad m = \lceil \gamma \rceil.$$

Proof. Using properties of the Riemann–Liouville fractional integral operator leads to

$$D^\gamma v(t) = I^{m-\gamma} D^m v(t) = \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-s)^{m-\gamma-1} D^m v(s) ds$$
$$= \frac{1}{\Gamma(m-\gamma)} t^{m-\gamma-1} * D^m v(t),$$

where the star symbol denotes the convolution of two functions $t^{m-\gamma-1}$ and $D^m v(t)$. So, by means of Lemma 5, we have

$$\|D^\gamma v - D^\gamma \mathcal{P}_{N,\alpha,\beta} v\|_{\mu,w^{(\alpha,\beta)}} = \|I^{m-\gamma}(D^m v - D^m \mathcal{P}_{N,\alpha,\beta} v)\|_{\mu,w^{(\alpha,\beta)}}$$
$$= \left\| \frac{t^{m-\gamma-1}}{\Gamma(m-\gamma)} * (D^m v - D^m \mathcal{P}_{N,\alpha,\beta} v) \right\|_{\mu,w^{(\alpha,\beta)}}$$
$$\leqslant \frac{1}{(m-\gamma)\Gamma(m-\gamma)} \|D^m v - D^m \mathcal{P}_{N,\alpha,\beta} v)\|_{\mu,w^{(\alpha,\beta)}}$$

$$\leqslant \frac{c'}{\Gamma(m-\gamma+1)}\bigg((N+m)(N+m+\alpha+\beta)\bigg)^{\frac{\mu-r}{2}}|v^{(m)}|_{r,w^{(\alpha,\beta)}}.$$

□

Now, we consider equation (1). If $u_N(t)$ is the approximate solution to $u(t)$, then the residual function is as follows:

$$H_N(t) = D^\gamma u_N(t) - au_N(t) - \sum_{i=1}^{l} b_i\ D^{\gamma_i} u_N(q_i t) - f(t). \qquad (27)$$

The next lemma estimates a bound for the residual function (27).

Lemma 6. *Consider the residual function given in* (27). *A bound for it can be estimated as follows:*

$$\|H_N(t)\|_{\mu,w^{(\alpha,\beta)}} \leqslant U^*\bigg(\frac{c'}{\Gamma(m-\gamma+1)}\bigg((N+m)(N+m+\alpha+\beta)\bigg)^{\frac{\mu-r}{2}}$$

$$+ \sum_{i=1}^{l} \frac{|b_i|\ c_i'}{\Gamma(m_i-\gamma_i+1)}\bigg(q_i(N+m_i)(N+m_i+\alpha+\beta)\bigg)^{\frac{\mu-r}{2}}$$

$$+ |a|c(N(N+\alpha+\beta))^{\frac{\mu-r}{2}}\bigg),$$

where $U^* = \max_{t\in J} |u^{(k)}(t)|_{r,w^{(\alpha,\beta)}}, k = 0,1,\ldots,m,\ m_i - 1 < \gamma_i \leqslant m_i,$ *and* $m-1 < \gamma \leqslant m.$

Proof. By using Theorems 5 and 6 and (26), the above inequality can be resulted. □

The following theorem presents an error bound for the proposed method.

Theorem 7. *If* $u_N(t)$ *is the Jacobi approximate solution to* $u(t)$ *and* $e_N(t) = u(t) - u_N(t)$ *is the error function, then the error bound can be estimated as follows:*

$$\|e_N(t)\|_{\mu,w^{(\alpha,\beta)}} \leqslant U^*\bigg(\frac{(1/|a|+1)c'}{\Gamma(m-\gamma+1)}\bigg((N+m)(N+m+\alpha+\beta)\bigg)^{\frac{\mu-r}{2}}$$

$$+ \sum_{i=1}^{l} \frac{(1/|a|+1)|b_i|\ c_i'}{\Gamma(m_i-\gamma_i+1)}\bigg(q_i(N+m_i)(N+m_i+\alpha+\beta)\bigg)^{\frac{\mu-r}{2}}$$

$$+ c(N(N+\alpha+\beta))^{\frac{\mu-r}{2}}\bigg).$$

Proof. By subtracting equation (26) from equation (1), we get

$$D^\gamma e_N(t) = ae_N(t) + \sum_{i=1}^{l} b_i \ D^{\gamma_i} e_N(q_i t) - H_N(t).$$

Now, applying Theorems 5 and 6, equation (26), and Lemma 6 leads to the desired result. □

Now, we consider the operational matrix of fractional integration $\mathbf{P}^{(\nu)}$ in Theorem 2. We define the error vector E_ν as follows:

$$E_\nu = I^\nu \Phi^{(\sigma)}(t) - \mathbf{P}^{(\nu)}\Phi^{(\sigma)}(t) = [E_{0,\nu}, E_{1,\nu}, \dots, E_{N,\nu}]^T,$$

where

$$E_{k,\nu} = I^\nu P_k^{(\alpha,\beta,\sigma)}(t) - \sum_{j=0}^{N} \mathbf{P}^{(\nu)}{}_{kj} P_j^{(\alpha,\beta,\sigma)}(t), \quad k = 0, 1, \dots, N.$$

The following theorem presents an upper bound of Riemann–Liouville fractional integral operator.

Theorem 8. *If $E_{k,\nu} = I^\nu P_k^{(\alpha,\beta,\sigma)}(t) - \sum_{j=0}^{N} \mathbf{P}^{(\nu)}{}_{kj} P_j^{(\alpha,\beta,\sigma)}(t)$, then an error bound of Riemann–Liouville fractional integral operator of order ν can be expressed by*

$$\|E_{k,\nu}\|_{\mu,w^{(\alpha,\beta)}}^2 \leqslant c^2 (N(N+\alpha+\beta))^{\mu-r}$$

$$\times \sum_{j=0}^{k} \sum_{i=0}^{k} \left\{ \frac{\gamma_j^{(k)} \gamma_i^{(k)} \Gamma(\sigma j+1)\Gamma(\sigma i+1)\Gamma(\sigma j+\nu+2)\Gamma(\sigma i+\nu+2)}{\Gamma(\sigma j+\nu+1)\Gamma(\sigma i+\nu+1)\Gamma(\sigma j+\nu-r+2)\Gamma(\sigma i+\nu-r+2)} \right.$$

$$\times \left. B(i+j+r+\beta+2(\nu-r)/\sigma+1, \alpha+r+1) \right\},$$

where $\gamma_j^{(k)}$ were introduced by Lemma 1 and $B(u,v)$ is the well-known Beta function.

Proof. By using Lemma 1 and the property of Riemann–Liouville fractional integral, we have

$$I^\nu P_k^{(\alpha,\beta,\sigma)}(t) = \sum_{j=0}^{k} \gamma_j^{(k)} \frac{\Gamma(\sigma j+1)}{\Gamma(\sigma j+\nu+1)} t^{\sigma j+\nu}.$$

By setting $Z = I^\nu P_k^{(\alpha,\beta,\sigma)}(t)$, we get

$$|Z|_{\mu,w^{(\alpha,\beta)}}^2 = \|\partial_t^r Z\|_{\mu,w^{(\alpha+r,\beta+r)}}^2$$

$$= \left\| \frac{\Gamma(\sigma j+1)\Gamma(\sigma j+\nu+2)}{\Gamma(\sigma j+\nu+1)\Gamma(\sigma j+\nu-r+2)} t^{\sigma j+\nu-r} \right\|_{\mu,w^{(\alpha+r,\beta+r)}}^2$$

$$= \sum_{j=0}^{k} \sum_{i=0}^{k} \left\{ \frac{\sigma \gamma_j^{(k)} \gamma_i^{(k)} \Gamma(\sigma j+1)\Gamma(\sigma i+1)\Gamma(\sigma j+\nu+2)\Gamma(\sigma i+\nu+2)}{\Gamma(\sigma j+\nu+1)\Gamma(\sigma i+\nu+1)\Gamma(\sigma j+\nu-r+2)\Gamma(\sigma i+\nu-r+2)} \right.$$

$$\times \int_0^1 t^{\sigma(i+j+r+\beta+1)+2(\nu-r)-1}(1-t^\sigma)^{\alpha+r}dt \Big\}$$

$$= \sum_{j=0}^k \sum_{i=0}^k \Big\{ \frac{\gamma_j^{(k)}\gamma_i^{(k)}\Gamma(\sigma j+1)\Gamma(\sigma i+1)\Gamma(\sigma j+\nu+2)\Gamma(\sigma i+\nu+2)}{\Gamma(\sigma j+\nu+1)\Gamma(\sigma i+\nu+1)\Gamma(\sigma j+\nu-r+2)\Gamma(\sigma i+\nu-r+2)}$$

$$\times \int_0^1 x^{i+j+r+\beta+2(\nu-r)/\sigma}(1-x)^{\alpha+r}dx \Big\}.$$

Therefore, the theorem is proved. $\qquad\square$

8 Numerical examples

In this section, the method presented in this paper is applied to solve some examples taken from [18] to demonstrate the efficiency and reliability of the proposed algorithm. As a result, a comparison is presented.

Example 1. First, consider the following fractional pantograph differential equation from [10],

$$D^\gamma u = \frac{3}{4}u(t) + u(\frac{t}{2}) - t^2 + 2, \quad 1 < \gamma \leqslant 2, \qquad (28)$$

with conditions,

$$u(0) = 0, \qquad u'(0) = 0.$$

The exact solution is $u(t) = t^2$ in the case $\gamma = 2$. We first approximate the unknown function and its derivative of fractional order γ as follows:

$$u^\gamma(t) \simeq \Phi^{(\sigma)T}(t)\, C, \ \ u(t) \simeq \Phi^{(\sigma)T}(t)\, \mathbf{P}^{(\gamma)T}\, C, \ \ u(\frac{t}{2}) \simeq \Phi^{(\sigma)T}(t)\, \mathbf{L}^{\mathbf{T}}_{(\frac{1}{2},\sigma)}\, \mathbf{P}^{(\gamma)\mathbf{T}}\, C,$$

Substituting the approximations into (28) leads to the following algebraic equation:

$$\Phi^{(\sigma)T}(t)\, C - \frac{3}{4}\Phi^{(\sigma)T}(t)\, \mathbf{P}^{(\gamma)T}\, C - \Phi^{(\sigma)T}(t)\, \mathbf{L}^{\mathbf{T}}_{(\frac{1}{2},\sigma)}\, \mathbf{P}^{(\gamma)\mathbf{T}}\, C + t^2 - 2 \approx 0. \ \ (29)$$

By choosing $N = 4$, equation (29) is collocated at roots of $P_5^{(\alpha,\beta)}(t)$. Then, by solving the resultant algebraic system, the unknown vector C can be determined for some values of α and β parameters. Table 1 displays the absolute errors at some arbitrary points for $\alpha = \beta = 0$, $\gamma = 2$, $\sigma = 0.5, 1, 1.5, 2$, and $N = 4$. Table 1 shows that the results of our algorithm are more accurate than those which reported by [18]. In Figure 1, the approximate solutions are compared to the exact solution for $\alpha = \beta = 0$, $N = 4$, $\gamma = 1.3, 1.5, 1.7, 1.9, 2$, and $\sigma = 1$. From these results, it is seen that the approximate solutions converge to the exact solution when γ approaches 2.

Table 1: Errors for $\alpha = \beta = 0$, $N = 4$, and various values of σ for Example 1

t_i	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 1.5$	$\sigma = 2$	$\sigma = 0.5$ in [18]	$\sigma = 1$ in [18]	$\sigma = 1.5$ in [18]
0.0	5.3967×10^{-2}	1.0616×10^{-5}	1.5319×10^{-3}	6.4935×10^{-5}	5.02×10^{-2}	1.64×10^{-1}	1.33×10^{-1}
0.2	9.5185×10^{-4}	5.6130×10^{-5}	4.2342×10^{-4}	1.1404×10^{-4}	$--$	$--$	$--$
0.4	1.8965×10^{-3}	9.4037×10^{-4}	2.0913×10^{-3}	9.7976×10^{-4}	3.08×10^{-2}	6.51×10^{-2}	1.17×10^{-2}
0.6	4.4029×10^{-3}	4.7621×10^{-3}	3.4555×10^{-3}	4.8110×10^{-3}	$--$	$--$	$--$
0.8	1.4375×10^{-2}	1.5160×10^{-2}	1.5934×10^{-2}	1.5206×10^{-2}	1.59×10^{-1}	2.60×10^{-2}	3.78×10^{-1}
1.0	3.8299×10^{-2}	3.7313×10^{-2}	3.1948×10^{-2}	1.5206×10^{-2}	$--$	$--$	$--$
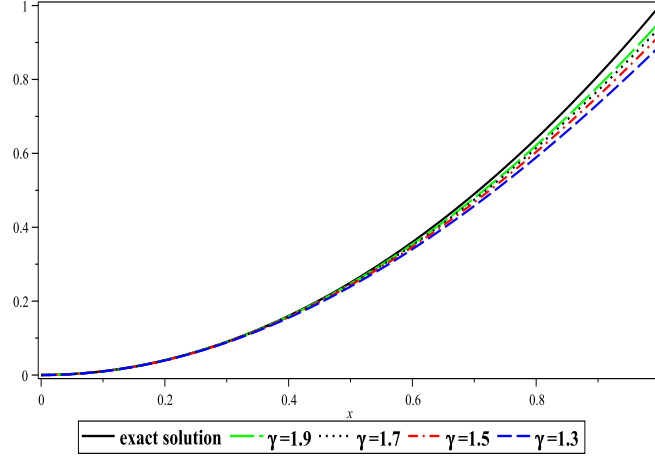
Figure 1: Plots of $u(t)$ for $N = 4$, $\sigma = 1$, $\gamma = 1.3, 1.5, 1.7, 1.9$ and exact solution for Example 1.

Example 2. Consider the following linear fractional-order pantograph differential equation of [18],

$$D^\gamma u(t) = -u(t) + 0.1u(\frac{4}{5}t) + 0.5D^\gamma u(\frac{4}{5}t) + (0.32t - 0.5)\exp(-0.8t) + \exp(-t),$$

where $0 < \gamma \leqslant 1$. With condition $u(0) = 0$ and the exact solution is $u(t) = t\exp(-t)$. By using the fundamental matrices obtained in Section 5, the following algebraic equation is resulted:

$$\Phi^{(\sigma)T}(t)\ C + \Phi^{(\sigma)T}(t)\ \mathbf{P}^{(\gamma)T}\ C - 0.1\Phi^{(\sigma)T}(t)\ \mathbf{L}_{(\frac{4}{5},\sigma)}{}^T\ \mathbf{P}^{(\gamma)\mathbf{T}}\ C$$
$$- 0.5\Phi^{(\sigma)T}(t)\ \mathbf{L}_{(\frac{4}{5},\sigma)}{}^T\ C - (0.32t - 0.5)\exp(-0.8t) - \exp(-t) \approx 0. \tag{30}$$

By choosing $N = 10$, equation (30) is collocated at roots of $P_{11}^{(\alpha,\beta)}(x)$. Then, by solving the resultant algebraic system, the unknown vector C can be determined for some values of α and β parameters. Table 2 displays the maximum absolute errors for various values of α and β parameters, $N = 10$, and $\gamma = \sigma = 1$. The data of Table 2 shows that the approximate and exact solutions are in a good agreement. In Table 3, the values of absolute error are compared to the values reported in [18] at some arbitrary points for $N = 10$, $\alpha = \beta = 1/3$, and $\gamma = \sigma = 1$. It is seen that the errors of our method are lesser than the errors of methods applied in [18] on the interval $[0, 1]$. Also, the absolute errors are listed in Table 4 for $N = 10$, $\alpha = \beta = 1/3$, $\gamma = \sigma$ and various values of γ in some arbitrary points within interval $[0, 1]$. A graphical comparison between the exact and approximate solutions is depicted in part (a) of Figure 2 for values of $\alpha = \beta = 1/3$, $\gamma = \sigma = 1$, and $N = 10$. From part (b), it can be shown that the numerical results are well consistent with the analytical solution. In addition, the plots of $u(t)$ are shown in part (b) of Figure 2 for $\sigma = 1$ and various values of γ. It is seen that the approximate solutions converge to the exact solution as γ approaches 1.

Table 2: Maximum absolute errors for various values of α and β, $\gamma = \sigma = 1$, and $N = 10$ for Example 2

(α, β)	$Error_{Abs}$	(α, β)	$Error_{Abs}$
$(0, 0)$	2.2652×10^{-13}	$(\frac{1}{2}, \frac{1}{2})$	4.5600×10^{-13}
$(1, 1)$	7.6421×10^{-13}	$(-\frac{1}{3}, -\frac{1}{3})$	1.1765×10^{-13}
$(-\frac{2}{3}, -\frac{2}{3})$	8.6704×10^{-14}	$(\frac{2}{3}, \frac{2}{3})$	5.5014×10^{-13}
$(\frac{3}{2}, \frac{3}{2})$	1.1469×10^{-12}	$(2, 2)$	1.5979×10^{-12}

Table 3: Comparison of absolute errors with other methods for Example 2

t_i	present method	Bernoulli method in [18]	Runge–Kutta method in [18]	Variational iteration method in [18]	θ-method in [18]
0.1	6.4471×10^{-14}	1.98×10^{-8}	8.68×10^{-4}	1.30×10^{-3}	4.65×10^{-3}
0.3	4.4701×10^{-14}	7.78×10^{-9}	1.90×10^{-3}	2.63×10^{-3}	2.57×10^{-2}
0.5	4.7819×10^{-16}	6.34×10^{-5}	2.28×10^{-3}	2.83×10^{-3}	4.43×10^{-2}
0.7	4.5170×10^{-14}	4.36×10^{-5}	2.28×10^{-3}	2.39×10^{-3}	5.37×10^{-2}
0.9	6.4785×10^{-14}	2.80×10^{-5}	2.03×10^{-3}	1.64×10^{-3}	5.35×10^{-2}

Example 3. In current example, we consider the following fractional pantograph differential equation of [18] :

$$D^\gamma u(t) = \frac{3}{4} u(t) + u(\frac{t}{2}) + D^{\gamma_1} u(\frac{t}{2}) + \frac{1}{2} D^\gamma u(\frac{t}{2}) - t^2 - t + 1, \qquad (31)$$

Table 4: Absolute errors for various values of $\gamma = \sigma$, $\alpha = \beta = \frac{1}{3}$, and $N = 10$ for Example 2

t_i	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
0.2	2.5022×10^{-5}	1.1429×10^{-4}	2.6347×10^{-4}	3.3616×10^{-4}
0.4	1.0801×10^{-5}	5.3195×10^{-5}	1.3757×10^{-4}	1.9924×10^{-4}
0.6	6.3865×10^{-6}	3.1052×10^{-5}	7.9303×10^{-5}	1.1741×10^{-4}
0.8	4.2784×10^{-6}	1.8443×10^{-5}	4.4219×10^{-5}	6.5747×10^{-5}
1.0	5.0278×10^{-6}	2.0332×10^{-5}	4.6987×10^{-5}	6.8103×10^{-5}
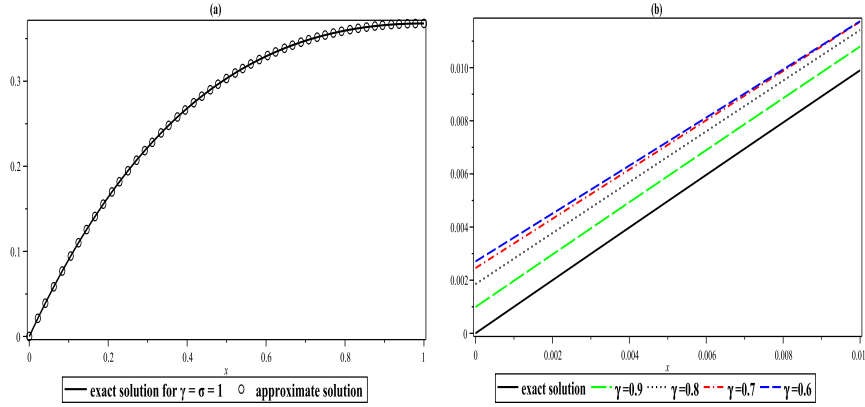
Figure 2: Plots of (a) exact and approximate solutions, (b) $u(t)$ for $\alpha = \beta = \frac{1}{3}$, $\sigma = 1$, $\gamma = 0.6, 0.7, 0.8, 0.9$, and $N = 10$ for Example 2.

where $0 < \gamma_1 < \gamma \leqslant 2$, with conditions $u(0) = u'(0) = 0$. The exact solution is $u(t) = t^2$ for case $\gamma = 2$ and $\gamma_1 = 1$. Substituting the approximations introduced in the previous sections into (31), leads to the following algebraic equation:

$$\Phi^{(\sigma)T}(t)\ C - \frac{3}{4}\Phi^{(\sigma)T}(t)\ \mathbf{P}^{(\gamma)T}\ C - \Phi^{(\sigma)T}(t)\ \mathbf{L}^{\mathbf{T}}_{(\frac{1}{2},\sigma)}\ \mathbf{P}^{(\gamma)\mathbf{T}}\ C$$
$$- \Phi^{(\sigma)T}(t)\ \mathbf{L}^{\mathbf{T}}_{(\frac{1}{2},\sigma)}\ \mathbf{P}^{(\gamma-\gamma_1)\mathbf{T}}\ C - \frac{1}{2}\Phi^{(\sigma)T}(t)\ \mathbf{L}^{\mathbf{T}}_{(\frac{1}{2},\sigma)}\ C + t^2 + t - 1 \approx 0. \tag{32}$$

By choosing $N = 10$, equation (32) is collocated at roots of $P_{11}^{(\alpha,\beta)}(x)$. Then, by solving the resultant algebraic system, the unknown vector C can be determined for some values of α and β parameters. Table 5 displays the maximum absolute errors for various values of α and β parameters, $\gamma = 2$, $\gamma_1 = 1$, $\sigma = 1$, and $N = 10$. The approximate solutions are in good agreement with exact solution. Equation (31) was solved in [18] by Bernoulli collocation method. In Table 6, the absolute errors are compared to the values reported

by [18] at some arbitrary points for $N = 10$, $\alpha = \beta = 2$, $\gamma = 2$, $\gamma_1 = 1$, $\sigma = 1$. Table 7 shows the maximum absolute errors for $\alpha = \beta = 0$, $\gamma = 1.9, 2$, $\gamma_1 = 0.9, 1$, and $\sigma = 0.5, 1, 1.5, 2$. For the case $\gamma = 1.9$, $\gamma_1 = 0.9$, and the case $\gamma = 2$, $\gamma_1 = 1$, the best results obtain for $\sigma = 0.5$ and $\sigma = 1$, respectively. The errors given in [18] are greater than the errors of our method at the selected points on the interval $[0, 0.5]$. A graphical comparison between the exact and approximate solutions for values of $\alpha = \beta = 0$, $\gamma_1 = \sigma = 1$, $\gamma = 1.2, 1.4, 1.6, 1.8$, and $N = 10$ are shown in Figure 3.

Table 5: Maximum absolute errors for various values of α and β, $\gamma = 2$, $\gamma_1 = \sigma = 1$, and $N = 10$ for Example 3

(α, β)	$Error_{Abs}$	(α, β)	$Error_{Abs}$
$(0, 0)$	7.2345×10^{-22}	$(\frac{1}{2}, \frac{1}{2})$	1.9150×10^{-19}
$(1, 1)$	1.8407×10^{-20}	$(-\frac{4}{5}, -\frac{4}{5})$	8.7002×10^{-22}
$(-\frac{1}{3}, -\frac{1}{3})$	2.2935×10^{-20}	$(2, 2)$	1.2297×10^{-20}

Table 6: Comparison of absolute errors with other methods for Example 3

t_i	present method	Bernoulli method in [18]	Runge–Kutta method in [18]	Variational iteration method in [18]	θ-method in [18]
0.1	0.0000	3.47×10^{-16}	1.00×10^{-3}	1.67×10^{-4}	6.10×10^{-3}
0.2	1.0000×10^{-21}	5.55×10^{-16}	2.02×10^{-3}	7.15×10^{-4}	2.58×10^{-2}
0.3	2.0000×10^{-21}	3.05×10^{-16}	3.07×10^{-3}	1.73×10^{-3}	6.47×10^{-2}
0.4	0.0000	1.14×10^{-15}	4.17×10^{-3}	3.30×10^{-3}	1.37×10^{-2}
0.5	1.0000×10^{-20}	1.39×10^{-13}	5.34×10^{-3}	5.50×10^{-3}	2.81×10^{-2}

Table 7: Comparison of maximum absolute errors for different values of γ, γ_1, and σ for Example 3

	$\gamma = 1.9$	$\gamma_1 = 0.9$	
$\sigma = 0.5$	$\sigma = 1$	$\sigma = 1.5$	$\sigma = 2$
2.2058×10^{-3}	1.0822×10^{-2}	1.0081×10^{-2}	1.5358×10^{-1}
	$\gamma = 2$	$\gamma_1 = 1$	
$\sigma = 0.5$	$\sigma = 1$	$\sigma = 1.5$	$\sigma = 2$
3.4229×10^{-18}	7.2345×10^{-22}	6.8343×10^{-3}	1.1564×10^{-1}

Example 4. Consider the following pantograph fractional differential equation [18] :

$$D^\gamma u(t) = -\frac{5}{6}u(t) + 4u(\frac{t}{2}) + 9u(\frac{t}{3}) + t^2 - 1, \quad 0 < \gamma \leqslant 1.$$
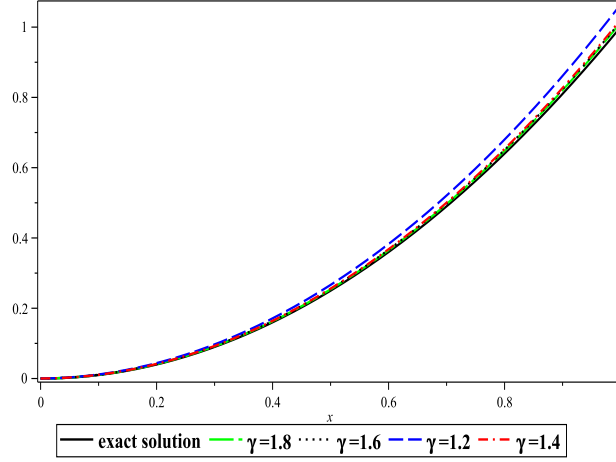
Figure 3: Comparison of exact and approximate solutions for $\gamma_1 = \sigma = 1$, $\gamma = 1.2, 1.4, 1.6, 1.8$, and $N = 10$ for Example 3.

The initial condition is $u(0) = 1$. The exact solution is $u(t) = 1 + \frac{67}{6}x + \frac{1675}{72}x^2 + \frac{12157}{1296}x^3$ in the case $\gamma = 1$. By applying the fundamental matrices presented in the previous sections, we get the following algebraic equation:

$$\Phi^{(\sigma)T}(t) \ C + \frac{5}{6}\{\Phi^{(\sigma)T}(t) \ \mathbf{P}^{(\gamma)T} \ C + \Phi^{(\sigma)T}(t) \ F\} - 4\{\Phi^{(\sigma)T}(t) \ \mathbf{L}^{\mathbf{T}}_{(\frac{1}{2},\sigma)} \ \mathbf{P}^{(\gamma)\mathbf{T}} \ C$$
$$+ \Phi^{(\sigma)T}(t) \ \mathbf{L}^{\mathbf{T}}_{(\frac{1}{2},\sigma)} \ F\} - 9\{\Phi^{(\sigma)T}(t) \ \mathbf{L}^{\mathbf{T}}_{(\frac{1}{3},\sigma)} \ \mathbf{P}^{(\gamma)\mathbf{T}} \ C$$
$$+ \Phi^{(\sigma)T}(t) \ \mathbf{L}^{\mathbf{T}}_{(\frac{1}{3},\sigma)} \ F\} - t^2 + 1 \approx 0.$$
$$\tag{33}$$

where the matrices $\mathbf{L}_{(\frac{1}{2},\sigma)}$ and $\mathbf{L}_{(\frac{1}{2},\sigma)}$ are pantograph operational matrices corresponding to the delays $q = \frac{1}{2}, \frac{1}{3}$ and $u(0)$ is approximated as $\Phi^{(\sigma)T}(t) \ F$. By choosing $N = 10$, equation (33) is collocated at roots of $P_{11}^{(\alpha,\beta)}(x)$. By solving the resultant algebraic systems, the unknown vector C can be determined for some values of α and β parameters. In Table 8, the absolute errors of the approximate solutions are listed at some arbitrary points for $N = 10$, $\alpha = \beta = \gamma = 1$, and various values of σ. As it is seen, the best results are related to the values $\sigma = 0.5, 1$. A graphical comparison between the exact and approximate solutions is found in part (a) of Figure 4 for $\alpha = \beta = \gamma = \sigma = 1$. The plots of $u(t)$ have been compared in part (b) for $\gamma = \sigma = 0.75, 0.85, 0.95$. Also, the error functions of these solutions are depicted in part (c). It is seen that the approximate solutions converge to the exact solution as γ and σ approach 1.

Table 8: Absolute errors for various values of σ, $\alpha = \beta = \gamma = 1$, and $N = 10$ for Example 4

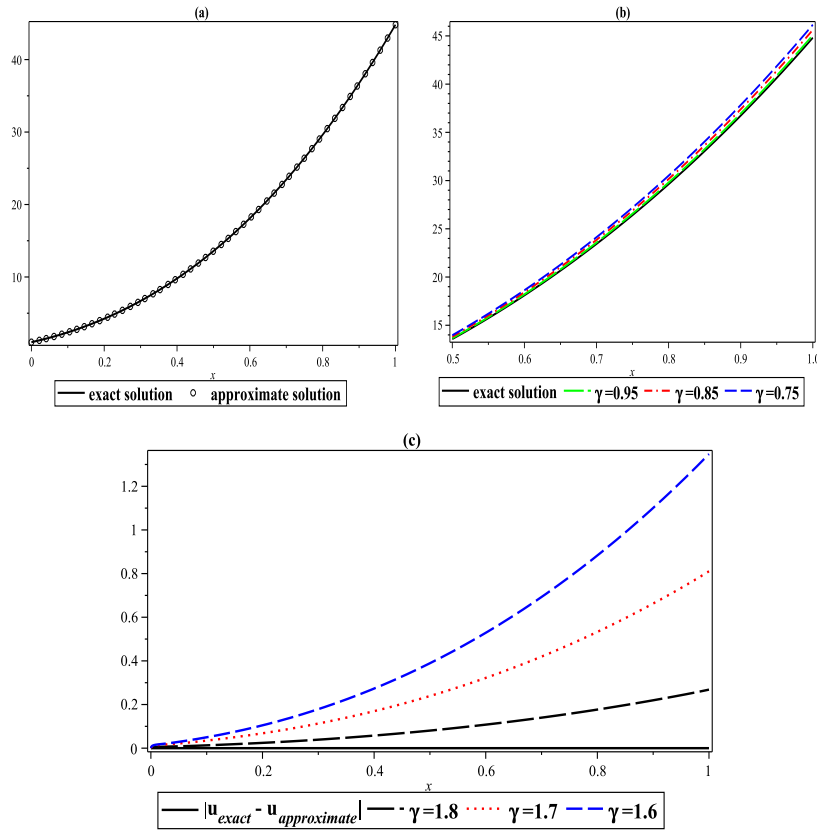| t_i	$\sigma = 0.5$	$\sigma = 0.75$	$\sigma = 1$	$\sigma = 1.5$
0.2	1.5100×10^{-15}	1.1748×10^{-3}	2.0000×10^{-19}	4.5227×10^{-1}
0.4	3.5176×10^{-15}	2.3520×10^{-3}	9.0000×10^{-19}	1.0366×10^{-0}
0.6	6.5190×10^{-15}	4.5162×10^{-3}	0.0000	1.9337×10^{-0}
0.8	1.0666×10^{-14}	7.3187×10^{-3}	5.0000×10^{-18}	3.1543×10^{-0}
1.0	1.6149×10^{-14}	9.6215×10^{-3}	3.0000×10^{-18}	4.7377×10^{-0}

Figure 4: (*a*) Exact and approximate solutions for $\alpha = \beta = \gamma = \sigma = 1$, and $N = 10$, (*b*) comparison of u(t) for $\gamma = \sigma = 0.75, 0.85, 0.95, 1$, (*c*) maximum absolute errors for $\gamma = \sigma = 0.75, 0.85, 0.95$ for Example 4.

Table 9: Absolute errors for various values of σ, and $N = 10$ for Example 5

t_i	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 1.5$	$\sigma = 2$
0.2	2.2499×10^{-18}	0.0000	5.0015×10^{-5}	2.4119×10^{-2}
0.4	1.3489×10^{-17}	1.0000×10^{-21}	6.8587×10^{-5}	7.0770×10^{-2}
0.6	3.9130×10^{-17}	0.0000	5.8770×10^{-6}	1.6689×10^{-1}
0.8	8.4580×10^{-17}	4.0000×10^{-20}	2.1854×10^{-4}	3.1842×10^{-1}
1.0	1.5572×10^{-16}	1.0000×10^{-20}	2.3271×10^{-4}	5.5809×10^{-1}

Example 5. In this example, we consider the following fractional pantograph differential equation of [18] :

$$D^{\gamma}u(t) = u(t) + D^{\gamma_1}u(\frac{t}{2}) + D^{\gamma_2}u(\frac{t}{3}) + \frac{1}{2}D^{\gamma}u(\frac{t}{4}) - t^4 - \frac{1}{2}t^3 - \frac{4}{3}t^2 + 21t,$$

where $0 < \gamma_1 < \gamma_2 \leqslant \gamma \leqslant 3$. The initial conditions are $u(0) = u'(0) = u''(0) = 0$ and the exact solution is $u(t) = t^4$ in case $\gamma_1 = 1$, $\gamma_2 = 2$, and $\gamma = 3$. By using the approximations introduced in the previous sections, we get the following algebraic equation:

$$\Phi^{(\sigma)T}(t) \ C - \Phi^{(\sigma)T}(t) \ \mathbf{P}^{(\gamma)T} \ C - \Phi^{(\sigma)T}(t) \ \mathbf{L}^{\mathbf{T}}_{(\frac{1}{2},\sigma)} \ \mathbf{P}^{(\gamma-\gamma_1)\mathbf{T}} \ C$$

$$- \Phi^{(\sigma)T}(t) \ \mathbf{L}^{\mathbf{T}}_{(\frac{1}{3},\sigma)} \ \mathbf{P}^{(\gamma-\gamma_2)\mathbf{T}} \ C - \frac{1}{2}\Phi^{(\sigma)T}(t) \ \mathbf{L}^{\mathbf{T}}_{(\frac{1}{4},\sigma)} \ C \qquad (34)$$

$$- t^4 + \frac{1}{2}t^3 + \frac{4}{3}t^2 - 21t \approx 0.$$

By choosing $N = 10$, equation (34) is collocated at roots of $P^{(\alpha,\beta)}_{11}(x)$. Then, by solving the resultant algebraic system, the unknown vector C is determined for some values of α and β parameters. Table 9 displays the absolute errors for $\alpha = \beta = 0$, $\gamma = 3$, $\gamma_2 = 2$, $\gamma_1 = 1$, and various values of σ. In Table 10, the values of the approximate solutions are shown at some arbitrary points for $N = 10$, $\alpha = \beta = 0$, $\gamma = 3$, $\gamma_2 = 2$, $\gamma_1 = 1$, and $\sigma = 1$. A graphical comparison between the exact and approximate solutions is found in part (a) of Figure 5 for $\alpha = \beta = 0$, $\sigma = \gamma_1 = 1$, $\gamma_2 = 2$, and $\gamma = 3$. The plots of $u(t)$ have been compared in part (b) for $\gamma = 2.2, 2.4, 2.6, 2.8$, $\sigma = \gamma_1 = 1$, and $\gamma_2 = 2$. It is seen that the approximate solutions converge to the exact solution as γ approaches 3.
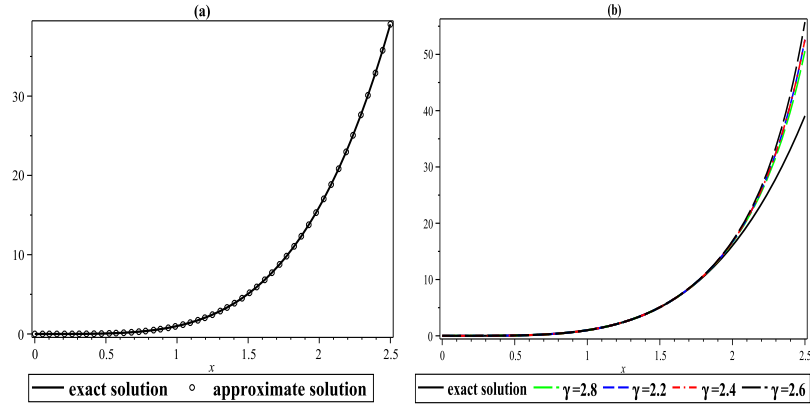
Figure 5: Comparison of (a) exact and approximate solutions, (b) $u(t)$ for $\alpha = \beta = 0$, and $N = 10$, $\gamma_2 = 2$, $\gamma_1 = 1$, and $\sigma = 1$, and $\gamma = 2.2, 2.4, 2.6, 2.8$ for Example 5.

Table 10: Comparison of absolute errors with other methods for Example 5

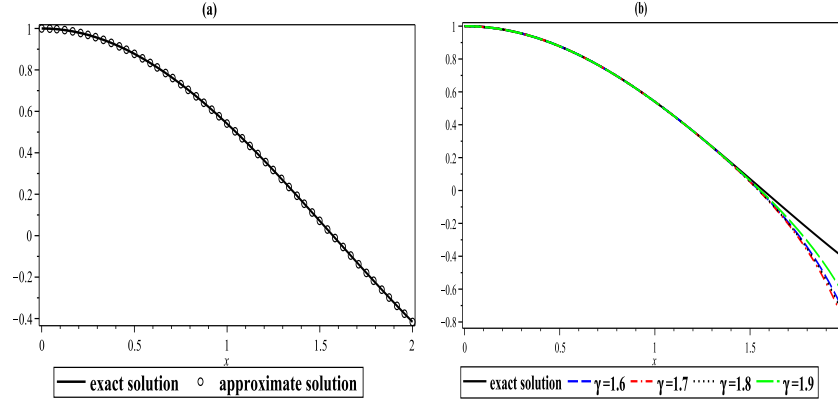| t_i	present method	Bernoulli method in [18]	Runge–Kutta method in [18]	Variational iteration method in [18]
0.2	0.0000	5.31×10^{-13}	4.43×10^{-4}	2.98×10^{-10}
0.4	1.0000×10^{-21}	1.24×10^{-12}	3.85×10^{-3}	1.01×10^{-8}
0.6	0.0000	2.89×10^{-10}	1.39×10^{-2}	8.24×10^{-8}
0.8	4.0000×10^{-20}	6.49×10^{-10}	3.53×10^{-2}	3.76×10^{-7}
1.0	1.0000×10^{-20}	1.23×10^{-9}	7.34×10^{-2}	1.26×10^{-6}

Figure 6: Comparison of (a) exact and approximate solutions, (b) $u(t)$ for $\alpha = \beta = -\frac{1}{4}$, and $N = 7$, $\sigma = \frac{1}{2}$, and $\gamma = 1.6, 1.7, 1.8, 1.9$ for Example 6.

Example 6. Consider the following nonlinear fractional pantograph differential equation of [18] :

$$D^\gamma u(t) = 1 - 2u^2(\frac{t}{2}), \quad 1 < \gamma \leqslant 2.$$

The initial conditions are $u(0) = 1$, $u'(0) = 0$ and the exact solution is $u(t) = \cos(t)$ in case $\gamma = 2$. By using the approximations introduced in the previous sections, we get the following algebraic equation:

$$\Phi^{(\sigma)T}(t)\, C + 2U^T\, \tilde{U}\, \Phi^{(\sigma)}(t) - 1 \approx 0, \quad U = \mathbf{L}^{\mathbf{T}}_{(\frac{1}{2},\sigma)}\, \mathbf{P}^{(\gamma-)\mathbf{T}}\, C + \mathbf{L}^{\mathbf{T}}_{(\frac{1}{2},\sigma)}\, F, \ (35)$$

where $u(0) = 1$ is approximated by $\Phi^{(\sigma)T}(t)\, F$ and $\tilde{U}$ is the operational matrix of product corresponding to the vector U. By choosing $N = 7$, equation (35) is collocated at roots of $P_8^{(\alpha,\beta)}(x)$. Then, by solving the resultant algebraic system by means of Newton's iteration method, the unknown vector C is determined for some values of α and β parameters. Table 11 displays the absolute errors for $\alpha = \beta = 3/2$, $\gamma = 2$, and various values of σ. In Table 12, the values of the approximate solutions are shown at some arbitrary points for $N = 7$, $\alpha = \beta = -1/4$, $\sigma = 1/2$, and various values of γ. A graphical comparison between the exact and approximate solutions is found in part (a) of Figure 6 for $\alpha = \beta = -1/4$, $\sigma = 1$, and $\gamma = 2$. The plots of $u(t)$ have been compared in part (b) for $\gamma = 1.6, 1.7, 1.8, 1.9$, and $\sigma = 1/2$. It is seen that the approximate solutions converge to the exact solution as γ approaches 2.

Table 11: Absolute errors for various values of σ, and $N = 7$ for Example 6

t_i	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 1.5$	$\sigma = 2$
0.0	2.4053×10^{-7}	6.0568×10^{-9}	5.6194×10^{-5}	1.1748×10^{-2}
0.2	4.0376×10^{-8}	7.7581×10^{-11}	2.3175×10^{-3}	1.0518×10^{-2}
0.4	4.1630×10^{-8}	4.1069×10^{-10}	1.7587×10^{-3}	7.6767×10^{-3}
0.6	8.1380×10^{-8}	5.3919×10^{-10}	9.8358×10^{-4}	3.3268×10^{-3}
0.8	2.3864×10^{-7}	2.3463×10^{-10}	3.8368×10^{-4}	2.4282×10^{-3}
1.0	5.8922×10^{-6}	5.4779×10^{-9}	4.2678×10^{-4}	9.5443×10^{-3}

Table 12: Absolute errors for various values of γ, and $N = 7$ for Example 6

t_i	$\gamma = 1.6$	$\gamma = 1.7$	$\gamma = 1.8$	$\gamma = 1.9$
0.0	1.9622×10^{-7}	1.1006×10^{-7}	2.1920×10^{-7}	5.1996×10^{-8}
0.2	1.1452×10^{-5}	2.9183×10^{-5}	4.5715×10^{-5}	2.5122×10^{-5}
0.4	1.4694×10^{-5}	4.2206×10^{-5}	7.3289×10^{-5}	7.9829×10^{-5}
0.6	1.4420×10^{-5}	4.6407×10^{-5}	8.8547×10^{-5}	1.0437×10^{-5}
0.8	1.0683×10^{-5}	4.2124×10^{-5}	9.0686×10^{-5}	1.1652×10^{-4}
1.0	4.2390×10^{-6}	3.0183×10^{-5}	7.9795×10^{-5}	11529×10^{-4}

9 Conclusion

In this paper, the Jacobi operational method was used to numerically solve the linear and nonlinear fractional pantograph differential equations. This approach was based on the shifted fractional-order Jacobi functions on the interval $[0, 1]$, which converted a given linear or nonlinear fractional pantograph differential equation into a set of linear or nonlinear algebraic equations. Another advantage of the proposed method was that the coefficients of the Jacobi series solution were easily found by using the presented algorithms. These algorithms decrease the computational costs compared to other methods such as the Bernoulli collocation method as it applies the lesser numbers of operational matrices. By using the introduced algorithms, the operational matrices of fractional integration, product, and pantograph were simply constructed. The illustrative examples confirmed the validity and simplicity of the proposed approach. It was found that the absolute error of this method is lesser than the errors of other common methods and the errors of our proposed method were almost constant on the studied interval. This demonstrates the stability of the applied Jacobi collocation method. Illustrative examples with satisfactory results were used to demonstrate the application of this method. Suggested approximations make this method very interesting and effective to the good agreement between approximate and exact solutions. Almost the results obtained by the technique developed

in this paper were more accurate than the results obtained by the Bernoulli collocation, Runge–Kutta, variational iteration, and θ- methods.

Acknowledgements

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روش جستجوی همسایگی متغیر برای مسئله مینیم رنگ آمیزی مجموع روی گراف‌های ساده

حمیده ابراهیمی۱ و خدیجه صدری۲

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چکیده : هدف مقاله فعلی، ساختن توابع ژاکوبی کسری انتقال یافته (SFJFs) بر اساس چندجمله‌های ژاکوبی برای حل عددی معادلات دیفرانسیل پانتوگراف مرتبه کسری است. برای دستیابی به این هدف، ابتدا ماتریس‌های عملیاتی انتگرال، حاصل ضرب و پانتوگراف، مربوط به پایه مرتبه کسری، به دست می آیند (ماتریس عملیاتی انتگرال بر حسب تعریف انتگرال کسری ریمان-لیوویل به دست می آید). سپس، از این ماتریس‌ها برای تبدیل مسأله اصلی به مجموعه ای از معادلات جبری استفاده می شود. سرانجام، اعتبار و کارایی روش پیشنهادی به وسیله مثال‌های عددی نشان داده می شود. هم چنین، برخی قضایا در مورد وجود جواب مسأله تحت بررسی و همگرایی روش ارائه می شوند.

کلمات کلیدی : معادله دیفرانسیل پانتوگراف کسری؛ توابع ژاکوبی مرتبه کسری؛ مشتق کاپوتو؛ انتگرال ریمان-لیوویل.