



Singularly perturbed robin type boundary value problems with discontinuous source term in geophysical fluid dynamics

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Abstract

Singularly perturbed robin type boundary value problems with discontinuous source terms applicable in geophysical fluid are considered. Due to the discontinuity, interior layers appear in the solution. To fit the interior and boundary layers, a fitted nonstandard numerical method is constructed. To treat the robin boundary condition, we use a finite difference formula. The stability and parameter uniform convergence of the proposed method is proved. To validate the applicability of the scheme, two model problems are considered for numerical experimentation and solved for different values of the perturbation parameter, ε , and mesh size, h . The numerical result is tabulated, and it is observed that the present method is more accurate and uniformly convergent with order of convergence of $O(h)$.

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Keywords: Singularly perturbed problem; Robin type boundary value problems; Discontinuous source term; Nonstandard fitted method.

1 Introduction

Singular perturbation problems model convection-diffusion processes in applied mathematics that arise in diverse areas, including linearized Navier-Stokes equation at high Reynolds number and the drift-diffusion equation

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of semiconductor device modeling, heat and mass transfer at high Pe'clet number, and so on; see [6, 13, 18, 19]. The novel aspect of the problem under consideration is that we take a source term in the differential equation that has a jump discontinuity at one or more points in the interior of the domain. This gives rise to an interior layer in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point. Problems with discontinuous data were treated theoretically, in the case of the solution of the convection–diffusion with Dirichlet case problem; see [9, 10]. Authors of [2, 8, 14] discussed a self-adjoint Dirichlet type problem with a discontinuous source term. Authors [14, 15, 17] have examined two parameter singularly perturbed boundary value problems for second-order ordinary differential equations with discontinuous source term. Authors of [5, 7] discussed fitted nonstandard finite difference methods for singularly perturbed second-order ordinary differential equations. Singularly perturbed delay differential equation was examined by Mohapatra and Natesan [12] on an adaptively generated grid. Recently, Shandru and shanthi [3] presented a fitted mesh method to solve singularly perturbed robin type boundary value problems with discontinuous source terms. Indeed, still, there is a room to increase the accuracy and show the parameter uniform convergence because the treatment of singular perturbation problem is not trivial distributions and the solution is pended on perturbation parameter, ε and mesh size, h ; see [6]. Due to this, the numerical treatment of singularly perturbed boundary value problems is need improvement. Therefore, it is important to develop a more accurate and convergent numerical method for solving singularly perturbed boundary value problems under consideration.

2 Definition of the problem

Consider the following singularly perturbed problem with Robin boundary condition of the form

$$Ly(x) \equiv \varepsilon y''(x) + a(x)y'(x) - b(x)y(x) = f(x), \quad x \in \Omega^- \cup \Omega^+. \quad (1)$$

Subject to boundary conditions

$$\begin{cases} L_1y(0) = \alpha_1y(0) - \beta_1\varepsilon y'(0) = p, \\ L_2y(1) = \alpha_2y(1) + \beta_2y'(1) = q, \end{cases} \quad (2)$$

where $\alpha_1, \beta_1 \geq 0$, $\alpha_1 + \beta_1 > 0$, $\alpha_2 > 0, \beta_2 \geq 0$, and $\varepsilon > 0$ is a small parameter. The functions $a(x)$ and $b(x)$ are smooth on $\bar{\Omega}$, such that $a(x) \geq a > 0$ and $b(x) \geq b \geq 0$. Furthermore, the notations for the domain are $\Omega = (0, 1)$, $\Omega^- = (0, d)$, and $\Omega^+ = (d, 1)$, where $d \in \Omega$ stands for the jump in the source function. Boundary value problem of the governing problem

under consideration is a model confinement of a plasma column by reaction pressure and geophysical fluid dynamics; see [4].

The solution $y(x)$ of (1)–(2) has a boundary layer near $x = 0$ due to the perturbation parameter, ε and interior layer due to the discontinuous source term.

3 Properties of continuous solution

The differential operator for (1) is given by

$$L_\varepsilon \equiv \varepsilon \frac{d^2}{dx^2} + a \frac{d}{dx} - b,$$

and it satisfies the following minimum principle for boundary value problems. The following lemmas [6] are necessary for the existence and uniqueness of the solution and for the problem to be well-posed.

Lemma 1 (Continuous minimum principle). Suppose that the function $y \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ satisfies $L_1 y(0) \geq 0$, $L_2 y(1) \geq 0$, and $Ly(x) \leq 0$, for all $x \in \Omega^- \cup \Omega^+$ and $[y'](d) \leq 0$. Then, $y(x) \geq 0$ for all $x \in \bar{\Omega}$.

Proof. For the proof, we refer to [3]. □

Lemma 2 (Stability result). Consider the boundary value problem (1)–(2) subject to the conditions $a(x) \geq a > 0$ and $b(x) \geq b \geq 0$. If $y \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$, then

$$\|y\|_{\bar{\Omega}} \leq C \max\{|L_1 y(0)|, |L_2 y(1)|, |Ly|_{\Omega^- \cup \Omega^+}\}.$$

Proof. For the proof, see [3]. □

Lemma 3. For each integer k satisfying $0 \leq k \leq 4$, the solution of (1)–(2) satisfies the bounds $\|y^{(k)}\|_{\bar{\Omega} \setminus \{d\}} \leq C\varepsilon^{-k}$.

Proof. For the proof, see [3]. □

Lemma 4. Let y_ε be the solution of (P_ε) . Then, for $k = 0, 1, 2, 3$,

$$|y_\varepsilon^{(k)}(x)| \leq C(1 + \varepsilon^{-k} \exp(\frac{-a}{\varepsilon}x)) \quad \text{for all } x \in [0, l].$$

Proof. For the proof, see [1]. □

4 Formulation of the method

The theoretical basis of the nonstandard discrete numerical method is based on the development of the exact finite difference method. The author of [11] presented techniques and rules for developing nonstandard finite difference methods for different problem types. In Mickens's rules, to develop a discrete scheme, the denominator function for the discrete derivatives must be expressed in terms of more complicated functions of step sizes than those used in the standard procedures. These complicated functions constitute a general property of the schemes, which is useful while designing reliable schemes for such problems.

For the problem of the form in (1)–(2), in order to construct the exact finite difference scheme, we follow the procedures used in [1]. Let us consider the following singularly perturbed differential equation of the form

$$\varepsilon y''(x) + a(x)y'(x) - b(x)y(x) = f(x). \quad (3)$$

The constant coefficient homogeneous problems corresponding to (3) are

$$\varepsilon y''(x) + ay'(x) - by(x) = 0, \quad (4)$$

$$\varepsilon y''(x) + ay'(x) = 0, \quad (5)$$

where $a(x) \geq a$ and $b(x) \geq b$. Two linear independent solutions of (4) are $\exp(\lambda_1 x)$ and $\exp(\lambda_2 x)$, where

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 + 4\varepsilon b}}{2\varepsilon}. \quad (6)$$

We discretize the domain $[0, 1]$ using the uniform mesh length $\Delta x = h$ such that $\Omega^N = \{x_i = x_0 + ih, 1, 2, \dots, N, x_0 = 0, x_N = 1, h = \frac{1}{N}\}$, where N denotes the number of mesh points. We denote the approximate solution to $y(x)$ at the grid point x_i by Y_i . Now our main objective is to calculate the difference equation, which has the same general solution as the differential equation (4) has at the grid point x_i given by $Y_i = A_1 \exp(\lambda_1 x_i) + A_2 \exp(\lambda_2 x_i)$. Using the theory of difference equations and the procedures used in [1], we have

$$\det \begin{bmatrix} Y_{i-1} \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ Y_i \exp(\lambda_1 x_i) & \exp(\lambda_2 x_i) \\ Y_{i+1} \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{bmatrix} = 0. \quad (7)$$

Simplifying (7), we obtain

$$-\exp\left(\frac{ah}{2\varepsilon}\right)Y_{i-1} + 2 \cosh\left(\frac{h\sqrt{a^2 + 4\varepsilon b}}{2\varepsilon}\right)Y_i - \exp\left(-\frac{ah}{2\varepsilon}\right)Y_{i+1} = 0, \quad (8)$$

which is an exact difference scheme for (4).

After doing the arithmetic manipulation and rearrangement on (8), for the constant coefficient problem (5), we get

$$\varepsilon \frac{Y_{i-1} - 2Y_i + Y_{i+1}}{\frac{h\varepsilon}{a}(\exp(\frac{ah}{\varepsilon}) - 1)} + a \frac{Y_{i+1} - Y_i}{h} = 0. \tag{9}$$

The denominator function becomes $\Psi^2 = \frac{h\varepsilon}{a} \left(\exp\left(\frac{ha}{\varepsilon}\right) - 1 \right)$. Adopting this denominator function for the variable coefficient problem, we write it as

$$\Psi_i^2 = \frac{h\varepsilon}{a_i} \left(\exp\left(\frac{ha_i}{\varepsilon}\right) - 1 \right), \tag{10}$$

where Ψ_i^2 is the function of ε , a_i , and h .

By using the denominator function Ψ_i^2 in to the main scheme, we obtain the difference scheme as

$$L_\varepsilon^N Y_i \equiv \varepsilon \frac{Y_{i+1} - 2Y_i + Y_{i-1}}{\Psi_i^2} + a_i \frac{Y_{i+1} - Y_i}{h} - b_i Y_i = f_i. \tag{11}$$

This can be written as three term recurrence relation as

$$E_i Y_{i-1} + F_i Y_i + G_i Y_{i+1} = H_i, \quad i = 1, 2, \dots, N - 1, \tag{12}$$

where $E_i = \frac{\varepsilon}{\Psi_i^2}$, $F_i = \frac{-2\varepsilon}{\Psi_i^2} - \frac{a_i}{h} - b_i$, $G_i = \frac{\varepsilon}{\Psi_i^2} + \frac{a_i}{h}$, and $H_i = f_i$.

To treat the boundary condition, we use the forward finite difference formula for $i = 0$ and the backward difference formula for $i = N$, respectively, for the first derivative term.

That is, for $i = 0$, from (2), we have $\alpha_1 y(0) - \beta_1 \varepsilon y'_0 = p$ implies $\alpha_1 y_0 - \beta_1 \varepsilon y'_0 = p$, which yields

$$\left(\alpha_1 + \frac{\beta_1 \varepsilon}{h}\right) y_0 - \frac{\beta_1 \varepsilon}{h} y_1 = p, \tag{13}$$

Similarly, for $i = N$, from (2), we have $\alpha_2 y(N) + \beta_2 \varepsilon y'_N = q$ implies $\alpha_2 y_N + \beta_2 \varepsilon y'_N = q$, which yields

$$\left(\alpha_2 + \frac{\beta_2}{h}\right) y_N - \frac{\beta_2}{h} y_{N-1} = q. \tag{14}$$

Therefore, Equation (1) with the given boundary conditions (2), can be solved using the schemes in (12), (13), and (14) which gives the $N \times N$ system of algebraic equations.

5 Uniform convergence analysis

In this section, we need to show that the discrete scheme in (12) satisfies the discrete minimum principle and uniform convergence. Let us define the forward, backward, and second-order central finite difference operators as

$$D^+Y_j = \frac{Y_{j+1} - Y_j}{h}, \quad D^-Y_j = \frac{Y_j - Y_{j-1}}{h}, \quad \delta^2Y_j = D^+D^-Y_j = \frac{D^+Y_j - D^-Y_j}{h}.$$

Lemma 5 (Discrete Minimum principle). Let V_i be any mush function that satisfies $v_0 \geq 0$, $v_N \geq 0$, and $L^h v_i \leq 0$, $i = 1, 2, \dots, N - 1$. Then $v_i \geq 0$, $i = 1, 2, \dots, N$.

Proof. The proof is obtained by contradiction. Let j be such that $V_j = \min V_i$, and suppose that $V_j < 0$. Clearly, $j \notin \{0, N\}$, $V_{j+1} - V_j \geq 0$, and $V_j - V_{j-1} \leq 0$. Therefore,

$$\begin{aligned} L^h V_j &= \frac{\varepsilon}{\Psi_i^2} (V_{j+1} - 2V_j + V_{j-1}) + \frac{a_j}{h} (V_{j+1} - V_j) - bV_j \\ &= \frac{\varepsilon}{\Psi_j^2} [(V_{j+1} - V_j) - (V_j - V_{j-1})] + \frac{a_i}{h} (V_{j+1} - V_j) - bV_j \\ &\geq 0, \end{aligned}$$

where the strict inequality holds if $V_{j+1} - V_j > 0$. This is a contradiction and therefore $V_j \geq 0$. Since j is arbitrary, we have $V_i \geq 0$, $i = 1, 2, \dots, N$. \square

We proved above that the discrete operator L^h satisfies the minimum principle. Next, we analyze the uniform convergence analysis.

Using the Taylor series expansion, the bound for $y(x_{i-1})$ and $y(x_{i+1})$ at x_i are as

$$\begin{cases} y(x_{i-1}) = y(x_i) - hy'(x_i) + \frac{h^2}{2!}y''(x_i) - \frac{h^3}{3!}y^{(3)}(x_i) + \frac{h^4}{4!}y^{(4)}(x_i) + O(h^5), \\ y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y^{(3)}(x_i) + \frac{h^4}{4!}y^{(4)}(x_i) + O(h^5). \end{cases}$$

We obtain the bound for

$$\begin{cases} |D^+D^-y(x_i)| \leq C|y''(x_i)|, \\ |y''(x_i) - D^+D^-y(x_i)| \leq Ch^2|y^{(4)}(x_i)|. \end{cases} \tag{15}$$

Similarly, for the first derivative term, we have

$$|y'(x_i) - D^+y(x_i)| \leq Ch|y^{(2)}(x_i)|, \tag{16}$$

where $|y^{(k)}(x_i)| = \sup_{x_i \in (x_0, x_N)} |y^{(k)}(x_i)|$, $k = 2, 3, 4$.

Theorem 1. Let the coefficients functions $a(x)$ and the source function $f(x)$ in (1)–(2) of the domain Ω be sufficiently smooth, so that $y(x) \in C^4[0, 1]$. Then, the discrete solution Y_i satisfies

$$|L^N(y_i - Y_i)| \leq Ch \left(1 + \sup_{x \in (0,1)} \left(\frac{\exp(\frac{-ax_i}{\varepsilon})}{\varepsilon^3} \right) \right).$$

Proof. We consider the truncation error discretization as

$$\begin{aligned} |L^N(y_i - Y_i)| &= |L^N y_i - L^N Y_i|, \\ &\leq C|\varepsilon y_i'' + a_i y_i' - \{\varepsilon \frac{D^+ D^- h^2}{\Psi_i^2} y_i + a_i D^+ y_i\}|, \\ &\leq C|\varepsilon(y_i'' - \frac{D^+ D^- h^2}{\Psi_i^2} y_i) + a_i(y_i' - D^+ y_i)|, \\ &\leq C\varepsilon|y_i'' - D^+ D^- y_i| + C\varepsilon|(\frac{h^2}{\Psi_i^2} - 1)D^+ D^- y_i| + Ch|y_i''|, \\ &\leq C\varepsilon h^2 |y_i^{(4)}| + Ch|y_i''| + Ch|y_i''|, \\ &\leq C\varepsilon h^2 |y_i^{(4)}| + Ch|y_i''|. \end{aligned}$$

We use the estimate $\varepsilon|\frac{h^2}{\Psi^2} - 1| \leq Ch$, which can be derived from (10).

Indeed, define $\rho = \frac{a_i h}{\varepsilon}, \rho \in (0, \infty)$. Then,

$$\varepsilon|\frac{h^2}{\Psi^2} - 1| = a_i h \left| \frac{1}{\exp(\rho) - 1} - \frac{1}{\rho} \right| =: a_i h Q(\rho).$$

By simplifying and writing the above equation explicitly, we obtain

$$Q(\rho) = \frac{\exp(\rho) - \rho - 1}{\rho(\exp(\rho) - 1)},$$

and we obtain that the limit is bounded as

$$\lim_{\rho \rightarrow 0} Q(\rho) = \frac{1}{2}, \quad \lim_{\rho \rightarrow \infty} Q(\rho) = 0.$$

Hence, for all $\rho \in (0, \infty)$, we have $Q(\rho) \leq C$. So, the error estimate in the discretization is bounded as

$$|L^N(y_i - Y_i)| \leq C\varepsilon h^2 |y_i^{(4)}| + Ch|y_i''|. \tag{17}$$

From (17) and boundedness of derivatives of solution in Lemma 4, we obtain

$$\begin{aligned} |L^N(y(x_i) - Y_i)| &\leq C\varepsilon h^2 \left| \left(1 + \varepsilon^{-4} \exp\left(\frac{-ax_i}{\varepsilon}\right) \right) \right| \\ &\quad + Ch \left| \left(1 + \varepsilon^{-2} \exp\left(\frac{-ax_i}{\varepsilon}\right) \right) \right| \end{aligned}$$

$$\begin{aligned} &\leq Ch^2 \left| \left(\varepsilon + \varepsilon^{-3} \exp\left(\frac{-ax_i}{\varepsilon}\right) \right) \right| \\ &\quad + Ch \left| \left(1 + \varepsilon^{-2} \exp\left(\frac{-ax_i}{\varepsilon}\right) \right) \right| \\ &\leq Ch \left(1 + \sup_{x \in (0,1)} \left(\frac{\exp\left(\frac{-ax_i}{\varepsilon}\right)}{\varepsilon^3} \right) \right), \end{aligned}$$

since $\varepsilon^{-3} > \varepsilon^{-2}$. □

Most of the time during analysis, one encounters with exponential terms involving divided by the power function in ε , which are always the main cause of worry. For their careful consideration while proving the ε -uniform convergence, we prove the following lemma.

Lemma 6. For a fixed mesh and for $\varepsilon \rightarrow 0$, it holds

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \max_{1 \leq i \leq N-1} \left(\frac{\exp\left(\frac{-ax_i}{\varepsilon}\right)}{\varepsilon^m} \right) &= 0, \quad m = 1, 2, 3, \dots, \\ \lim_{\varepsilon \rightarrow 0} \max_{1 \leq i \leq N-1} \left(\frac{\exp\left(\frac{-a(1-x_i)}{\varepsilon}\right)}{\varepsilon^m} \right) &= 0, \quad m = 1, 2, 3, \dots, \end{aligned}$$

where $x_i = ih, h = \frac{1}{N}, i = 1, 2, \dots, N - 1$.

Proof. Consider the partition $[0, 1] := \{0 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1\}$. For the interior grid points, we have

$$\begin{aligned} \max_{1 \leq i \leq N-1} \frac{\exp\left(\frac{-ax_i}{\varepsilon}\right)}{\varepsilon^m} &\leq \frac{\exp\left(\frac{-ax_1}{\varepsilon}\right)}{\varepsilon^m} = \frac{\exp\left(\frac{-ah}{\varepsilon}\right)}{\varepsilon^m}, \\ \max_{1 \leq i \leq N-1} \frac{\exp\left(\frac{-a(1-x_i)}{\varepsilon}\right)}{\varepsilon^m} &\leq \frac{\exp\left(\frac{-a(1-x_{N-1})}{\varepsilon}\right)}{\varepsilon^m} = \frac{\exp\left(\frac{-ah}{\varepsilon}\right)}{\varepsilon^m}, \end{aligned}$$

as $x_1 = 1 - x_{N-1} = h$.

Then, applying L'Hospital's rule m times gives

$$\lim_{\varepsilon \rightarrow 0} \frac{\exp\left(\frac{-ah}{\varepsilon}\right)}{\varepsilon^m} = \lim_{r = \frac{1}{\varepsilon} \rightarrow \infty} \frac{r^m}{\exp(ahr)} = \lim_{r = \frac{1}{\varepsilon} \rightarrow \infty} \frac{m!}{(ah)^m \exp(ahr)} = 0.$$

□

Theorem 2. Under the hypothesis of boundedness of discrete solution (i.e., it satisfies the discrete minimum principle), Lemma 6, and Theorem 1, the discrete solution satisfies the following bound:

$$\sup_{0 \leq \varepsilon \leq 1} \max_i |y_i - Y_i| \leq CN^{-1}. \quad (18)$$

Proof. Results from boundedness of solution, Lemma 6, and Theorem 1 give the required estimates. \square

6 Numerical examples and results

To validate the established theoretical results, we perform numerical experiments using the model problems of the form in (1)–(2).

Example 1. Consider the following problem:

$$\begin{cases} \varepsilon y''(x) + y'(x) = f(x), & x \in \Omega^- \cup \Omega^+, \\ y(0) - \varepsilon y'(0) = 1, & y(1) - y'(1) = -1, \end{cases}$$

where

$$f(x) = \begin{cases} 0.7, & 0 \leq x \leq 0.5, \\ -0.6, & 0.5 < x \leq 1. \end{cases}$$

Example 2. Consider the following problem:

$$\begin{cases} \varepsilon y''(x) + \frac{1}{1+x} y'(x) = f(x), & x \in \Omega^- \cup \Omega^+, \\ y(0) - \varepsilon y'(0) = 1, & y(1) - y'(1) = 1, \end{cases}$$

where

$$f(x) = \begin{cases} 1 + x, & 0 \leq x \leq 0.5, \\ 4, & 0.5 < x \leq 1. \end{cases}$$

Having $y_j \equiv y_j^N$ (the approximated solution is obtained via the fitted operator finite difference method) for different values of h and ε , the maximum errors. Since the exact solution is not available, the maximum errors (denoted by E_ε^N) are evaluated using the double mesh principle [6], for fitted operator finite difference methods using the formula

$$E_\varepsilon^N := \max_{0 \leq j \leq n} |y_j^N - y_{2j}^{2N}|.$$

Furthermore, we will tabulate the ε -uniform error

$$E^N = \max_{0 < \varepsilon \leq 1} E_\varepsilon^N.$$

The numerical rate of convergence is computed using the formula [6]

$$r_\varepsilon^N := \frac{\log(E_\varepsilon^N) - \log(E_\varepsilon^{2N})}{\log(2)}.$$

and the ε -uniform rate of convergence is computed using

$$R^N = \frac{\log(E^N) - \log(E^{2N})}{\log(2)}.$$

Table 1: Maximum absolute errors for different values of ε and number of mesh size, N for Example 1.

ε	N=32	N=64	N=128	N=256	N=512
10^{-4}	2.0313e-02	1.0156 e-02	5.0781e-03	2.5391e-03	1.2695e-03
10^{-8}	2.0313e-02	1.0156 e-02	5.0781e-03	2.5391e-03	1.2695e-03
10^{-12}	2.0313e-02	1.0156 e-02	5.0781e-03	2.5391e-03	1.2695e-03
10^{-16}	2.0313e-02	1.0156 e-02	5.0781e-03	2.5391e-03	1.2695e-03
10^{-20}	2.0313e-02	1.0156 e-02	5.0781e-03	2.5391e-03	1.2695e-03
E^N	2.0313e-02	1.0156 e-02	5.0781e-03	2.5391e-03	1.2695e-03
R^N	1.0001	1.0000	1.0000	1.0001	

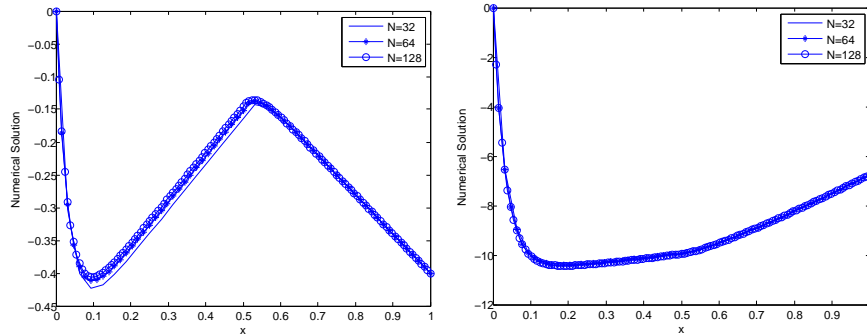


Figure 1: Behavior of numerical solution at $\varepsilon = 2^{-5}$ and different values of N for Examples 1 and 2, respectively.

Table 2: Comparison of maximum absolute errors and order of convergence for Example 1 at number of mesh points N .

ε	N=64	N=128	N=256	N=512	N=1024
Present method					
E^N	1.0156e-02	5.0781e-03	2.5382e-03	1.2467e-03	5.5910e-04
R^N	1.0000	1.0000	1.0257	1.0257	
Method in [3]					
E^N	2.5658e-02	1.4128e-02	7.5359e-03	3.8225e-03	1.7536e-03
R^N	0.8605	0.90671	0.9793	1.1242	

Table 3: Maximum absolute errors for different values of ε and number of mesh size, N for Example 2.

ε	N=32	N=64	N=128	N=256	N=512
10^{-4}	8.3431e-02	4.1854e-02	2.0962e-02	1.0489e-02	5.2329e-03
10^{-8}	8.3431e-02	4.1854e-02	2.0962e-02	1.0489e-02	5.2329e-03
10^{-12}	8.3431e-02	4.1854e-02	2.0962e-02	1.0489e-02	5.2329e-03
10^{-16}	8.3431e-02	4.1854e-02	2.0962e-02	1.0489e-02	5.2329e-03
10^{-20}	8.3431e-02	4.1854e-02	2.0962e-02	1.0489e-02	5.2329e-03
E_N	8.3431e-02	4.1854e-02	2.0962e-02	1.0489e-02	5.2329e-03
R^N	0.9952	0.9976	0.9989	1.0032	

Table 4: Comparison of maximum absolute errors and order of convergence for Example 2 at number of mesh points N .

ε	N=64	N=128	N=256	N=512	N=1024
Present method					
E^N	4.1854e-02	2.0960e-02	1.0408e-02	4.7906e-03	2.0192e-03
R^N	0.9977	1.0099	1.1194	1.2464	
Method in [3]					
E^N	9.6698e-01	5.8056e-01	3.2795e-01	1.7313e-01	8.1501e-02
R^N	0.7364	0.8239	0.0.9216	1.0870	

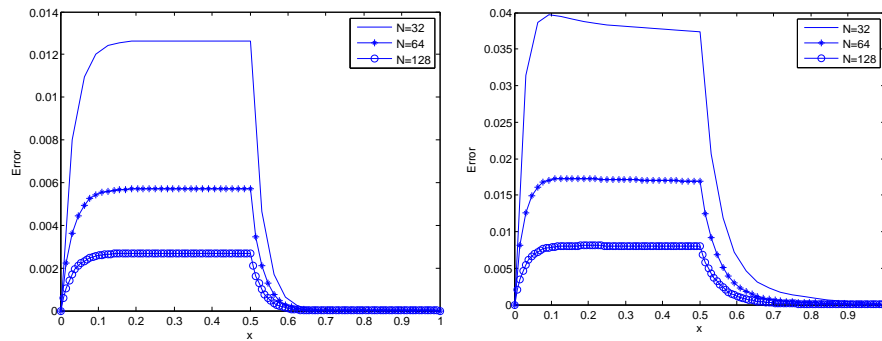


Figure 2: Pointwise absolute error plot at $\varepsilon = 2^{-5}$ and different values of N for Examples 1 and 2, respectively.

7 Discussion and conclusion

This study introduced a uniformly convergent numerical method based on nonstandard finite difference method for solving singularly perturbed second-order ordinary differential equations of Robin type boundary value problems with discontinuous source term. Due to discontinuity in the source term,

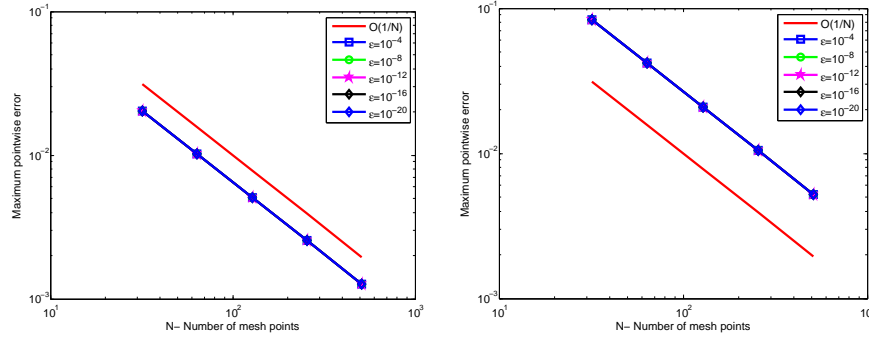


Figure 3: ε -uniform convergence with NSFDM in Log-Log scale for Examples 1 and 2, respectively.

there is an interior layer occurring. To fit the interior and boundary layer, a suitable nonstandard finite difference method on uniform mesh is constructed. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence, and uniform errors (see Tables 1–4) and compared with the results of the previously developed numerical methods existing in the literature (Tables 2 and 4). Furthermore, to see the position of the boundary layer, we plot the behavior of the numerical solution (see Figure 1), as the number of mesh points increases, the maximum pointwise errors decrease (see Figure 2) and the ε -uniform convergence of the method was shown using the log-log plot (see Figure 3). Unlike other fitted operator finite difference methods constructed in standard ways, the method that we presented in this paper is fairly simple to construct.

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