Iranian Journal of Numerical Analysis and Optimization
Vol. 11, No. 2, (2021), pp 385-399
DOI:10.22067/ijnao.2021.69824.1024

Research Article

Application of Newton–Cotes quadrature rule for nonlinear Hammerstein integral equations

A. Shahsavaran

Abstract

A numerical method for solving Fredholm and Volterra integral equations of the second kind is presented. The method is based on the use of the Newton–Cotes quadrature rule and Lagrange interpolation polynomials. By the proposed method, the main problem is reduced to solve some nonlinear algebraic equations that can be solved by Newton's method. Also, we prove some statements about the convergence of the method. It is shown that the approximated solution is uniformly convergent to the exact solution. In addition, to demonstrate the efficiency and applicability of the proposed method, several numerical examples are included, which confirms the convergence results.

AMS subject classifications (2020): 45B05; 45D05; 65R20; 45G10.

Keywords: Fredholm integral equation; Volterra integral equation; Newton–Cotes quadrature rule; Lagrange interpolation; Convergence.

1 Introduction

Integral equations have lots of applications in science and engineering. Fredholm integral equations arising in the theory of signal processing, which is a subfield of mathematics, information, and electrical engineering. In physics, the solution of such integral equations allows for experimental spectra to be related to various underlying distributions, for example, the mass distribution of polymers in a polymeric melt. They also prevalently appear in linear forward modeling and inverse problems. Also, Volterra integral equations arise in many scientific applications such as the population dynamics, the spread of epidemics, and semi-conductor devices. It was also shown that

Received 13 April 2021; revised 24 June 2021; accepted 27 June 2021
Ahmad Shahsavaran
Department of Mathematics, Borujerd Branch, Islamic Azad University, Borujerd, Iran.
e-mail: a.shahsavaran@iaub.ac.ir

Volterra integral equations can be derived from initial value problems; see [21]. Therefore, providing effective methods for solving such equations is consequential. Fredholm and Volterra integral equations have been extensively studied in many papers, and numerical methods have been widely used to solve such equations, for instance, Newton–Kantorovich and Haar wavelets for nonlinear Fredholm and Volterra integral equations [1], hat basis functions for the system of linear and nonlinear integral equations [2], Haar wavelets for nonlinear Fredholm integral equations [3], alternative Legendre polynomials for nonlinear Volterra–Hammerstein integral equations [5], operational matrix approach for nonlinear Volterra-Fredholm integral equations [4], cubic spline method for Fredholm integral equations [6], implicitly linear collocation method for nonlinear Volterra equations [7], Chebyshev approximation for nonlinear Fredholm-Volterra integral equations [9], Chebyshev collocation method for the class of Fredholm integral equations with highly oscillatory kernels [10], collocation-type method for Hammerstein integral equations [12], Sinc-collocation method for nonlinear Fredholm integral equations with weakly singular kernel [15], triangular functions for nonlinear Volterra-Fredholm integral equations [13], wavelets-Galerkin method and wavelets precondition for first kind Fredholm integral equations [14], spectral collocation method for Fredholm integral equations on the half-line [16], hybrid Legendre Block-Pulse functions for the system of nonlinear Fredholm–Hammerstein integral equations [17], single-term Walsh series for nonlinear Volterra-Hammerstein integral equations [18], piecewise constant functions method for nonlinear Fredholm-Volterra integral equations [19], and so on. The existence and uniqueness of such equations were discussed in [8, 11].

In this work, we consider the Fredholm and Volterra integral equations of Hammerstein type

$$u(x) = \nu(x) + \int_a^b \kappa(x,t)\psi(t,u(t))dt, \quad x \in [a,b],$$

$$u(x) = \nu(x) + \int_a^x \kappa(x,t)\psi(t,u(t))dt, \quad x \in [a,b],$$

where $\nu \in L^2[a,b]$ and $\kappa \in L^2[a,b]^2$ are known functions, ψ is a given nonlinear function defined on $[a,b]$, and u is unknown to be determined.

2 Method of solution

In this section, first, we describe the Newton–Cotes quadrature rule.

Let the interval $[a,b]$ be partitioned into n sub-intervals, by n equally spaced points. That is,

$$x_0 = a, \quad x_i = x_0 + ih, \quad for\ i = 0,1,\ldots,n, \tag{1}$$

where the step size h is defined by $h = \frac{x_n - x_0}{n} = \frac{b-a}{n}$.

The Newton–Cotes quadrature rule for a function f defined on $[a, b]$ with known values at equally spaced points x_i, $i = 0, 1, \ldots, n$, is as follows:

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} \omega_i f(x_i), \tag{2}$$

for the set of weights $\{\omega_i\}_{i=0}^{n}$. Now consider the Lagrange basis polynomials $l_i(t) = \prod_{j=0, j\neq i}^{n} (\frac{t-t_j}{t_i-t_j})$ and let $\rho_n(t)$ be the interpolation polynomial in the Lagrange form for the given data points $(t_0, f(t_0)), (t_1, f(t_1)), \ldots, (t_n, f(t_n))$. Then

$$\begin{aligned} \int_a^b f(t)dt &\approx \int_a^b \rho_n(t)dt \\ &= \int_a^b \left(\sum_{i=0}^{n} f(t_i) l_i(t) \right) dt \\ &= \sum_{i=0}^{n} f(t_i) \int_a^b l_i(t)dt \\ &= \sum_{i=0}^{n} \omega_i f(t_i), \end{aligned} \tag{3}$$

where $\omega_i = \int_a^b l_i(t)dt$.

Note that we take the Lagrange interpolation points t_i to be the same as the points x_i for the Newton–Cotes quadrature rule. Now consider the following Fredholm and Volterra integral equations of the second kind

$$u(x) = \nu(x) + \int_a^b \kappa(x, t)\psi(t, u(t))dt, \quad x \in [a, b], \tag{4}$$

$$u(x) = \nu(x) + \int_a^x \kappa(x, t)\psi(t, u(t))dt, \quad x \in [a, b]. \tag{5}$$

These can be written as

$$u(x) = \nu(x) + \int_a^b \kappa(x, t)\Psi(t)dt, \tag{6}$$

$$u(x) = \nu(x) + \int_a^x \kappa(x, t)\Psi(t)dt, \tag{7}$$

where $\Psi(t) = \psi(t, u(t))$. By considering $u_n(x)$ as an approximation for $u(x)$, we can turn equations (6) and (7) into the following equations:

$$u_n(x) = \nu(x) + \int_a^b \kappa(x,t)\Psi_n(t)dt, \tag{8}$$

$$u_n(x) = \nu(x) + \int_a^x \kappa(x,t)\Psi_n(t)dt, \tag{9}$$

where $\Psi_n(t) = \psi(t, u_n(t))$, which immediately implies

$$\Psi_n(t) = \psi\left(t, \nu(t) + \int_a^b \kappa(t,x)\Psi_n(x)dx\right), \tag{10}$$

$$\Psi_n(t) = \psi\left(t, \nu(t) + \int_a^t \kappa(t,x)\Psi_n(x)dx\right). \tag{11}$$

Using quadrature formula (2) to evaluate the integral part of (10), we obtain

$$\begin{aligned}\int_a^b \kappa(t,x)\Psi_n(x)dx &= \sum_{i=0}^n \omega_i \kappa(t,x_i)\Psi_n(x_i) \\ &= \sum_{i=0}^n \omega_i \kappa(t,x_i)\Psi_{n,i},\end{aligned} \tag{12}$$

where $\Psi_{n,i} = \Psi_n(x_i)$ and $\omega_i = \int_a^b l_i(t)dt$. Similarly, using the Lagrange interpolation for the integrand of (11) gives

$$\begin{aligned}\int_a^t \kappa(t,x)\Psi_n(x)dx &= \int_a^t \sum_{i=0}^n \kappa(t,x_i)\Psi_n(x_i)l_i(x)dx \\ &= \sum_{i=0}^n \kappa(t,x_i)\Psi_n(x_i)\int_a^t l_i(x)dx \\ &= \sum_{i=0}^n \omega_i(t)\kappa(t,x_i)\Psi_n(x_i) \\ &= \sum_{i=0}^n \omega_i(t)\kappa(t,x_i)\Psi_{n,i},\end{aligned} \tag{13}$$

where $\omega_i(t) = \int_a^t l_i(x)dx$ and $\Psi_{n,i} = \Psi_n(x_i)$. Substituting (12) into (10) and (13) into (11), we obtain

$$\Psi_n(t) = \psi\left(t, \nu(t) + \sum_{i=0}^n \omega_i \kappa(t,x_i)\Psi_{n,i}\right) \tag{14}$$

and

$$\Psi_n(t) = \psi\left(t, \nu(t) + \sum_{i=0}^n \omega_i(t)\kappa(t,x_i)\Psi_{n,i}\right), \tag{15}$$

respectively. Evaluating (14) and (15) at the points $t = x_j$, $j = 0, 1, \ldots, n$ (the points for Newton–Cotes quadrature rule), respectively, gives

$$\Psi_{n,j} = \psi\left(x_j, \nu(x_j) + \sum_{i=0}^{n} \omega_i \kappa(x_j, x_i)\Psi_{n,i}\right), \quad j = 0, 1, \ldots, n, \tag{16}$$

and

$$\Psi_{n,j} = \psi\left(x_j, \nu(x_j) + \sum_{i=0}^{n} \omega_{i,j} \kappa(x_j, x_i)\Psi_{n,i}\right), \quad j = 0, 1, \ldots, n, \tag{17}$$

where $\omega_{i,j} = \omega_i(x_j) = \int_a^{x_j} l_i(x)dx$. Nonlinear systems of algebraic equations (16) and (17) can be solved by numerical methods such as Newton's method. By solving the above systems, the values $\Psi_{n,i}$, $i = 0, 1, \ldots, n$, will be known. Finally, by substituting (12) into (8) and (13) into (9), we find the numerical solutions of the integral equations (4) and (5) by

$$u_n(x) = \nu(x) + \sum_{i=0}^{n} \omega_i \kappa(x, t_i)\Psi_{n,i} \tag{18}$$

and

$$u_n(x) = \nu(x) + \sum_{i=0}^{n} \omega_i(x) \kappa(x, t_i)\Psi_{n,i}, \tag{19}$$

respectively, where $t_i = x_i$, $i = 0, 1, \ldots, n$.

3 Convergence of the method

In this section, we analyze the convergence of the prescribed method in Section 2, which enables us to control the estimated errors. First, we provide an interpolation polynomial error bound, which is given in the following theorem.

Theorem 1. [20] Suppose that $f \in C^{n+1}[a, b]$, and let p_n be a polynomial of degree $\leq n$ that interpolates the function f at $n + 1$ distinct points $x_0, x_1, \ldots, x_n \in [a, b]$. Then for each $x \in [a, b]$, there exists a point $\zeta_x \in [a, b]$ such that

$$f(x) - p_n(x) = \frac{\pi(x)}{(n+1)!} f^{(n+1)}(\zeta_x), \tag{20}$$

where $\pi(x) = \Pi_{i=0}^{n}(x - x_i)$.

Theorem 2. In the case of equally spaced interpolation points $x_0 = a$ and $x_i = x_0 + ih$, for $i = 0, 1, \ldots n$, where $h = \frac{b-a}{n}$, we have

$$|\pi(x)| \leq \frac{n!}{4} h^{n+1}. \tag{21}$$

Proof. Suppose $x \in [x_i, x_{i+1}]$. Then for the first i terms of $\pi(x)$, that is $\Pi_{j=0}^{i-1}(x - x_j) = (x - x_0)(x - x_1)\dots(x - x_{i-1})$, and due to equally spaced points x_i, we have

$$\begin{aligned} |x - x_{i-j}| &\leq x_{i+1} - x_{i-j} \\ &= (j+1)h, \quad j = 1, 2, \dots, i. \end{aligned} \tag{22}$$

Thus

$$|\Pi_{j=0}^{i-1}(x - x_j)| \leq (i+1)!\, h^i. \tag{23}$$

For the next two terms of $\pi(x)$, that is $(x - x_i)(x - x_{i+1})$, and using the simple identity $\alpha\beta \leq \left(\frac{\alpha+\beta}{2}\right)^2$, we can write

$$\begin{aligned} |(x - x_i)(x - x_{i+1})| &= (x - x_i)(x_{i+1} - x) \qquad (x_i \leq x \leq x_{i+1}) \\ &\leq \left(\frac{x_{i+1} - x_i}{2}\right)^2 \\ &= \frac{h^2}{4}. \end{aligned} \tag{24}$$

For the $n - i - 1$ remaining terms of $\pi(x)$, that is $\Pi_{j=i+2}^{n}(x - x_j) = (x - x_{i+2})(x - x_{i+3})\dots(x - x_n)$, we may proceed as follows:

$$\begin{aligned} |x - x_{i+j}| &= x_{i+j} - x \qquad (x_i \leq x \leq x_{i+1}) \\ &\leq x_{i+j} - x_0 \\ &= (i+j)h, \quad j = 2, 3, \dots, n - i. \end{aligned} \tag{25}$$

Thus

$$|\Pi_{j=i+2}^{n}(x - x_j)| \leq \frac{n!}{(i+1)!}\, h^{n-i-1}. \tag{26}$$

Therefore combining (23), (24), and (26) leads to

$$\begin{aligned} |\pi(x)| &= |\Pi_{i=0}^{n}(x - x_i)| \\ &\leq \frac{n!}{4} h^{n+1}. \end{aligned} \tag{27}$$

□

Theorem 3. Suppose that $\kappa \in C^{n+1}[a, b]^2$ and that ψ in (4) is a function in $C^{n+1}[a, b]$ with $n \geq 0$. If $u(x)$, the exact solution, and $u_n(x)$, the approximate solution defined by (18), are both in $C^{n+1}[a, b]$, then $u_n(x)$ is uniformly convergent to $u(x)$.

Proof. From (6) and (18), for every $x \in [a, b]$, we have

$$u(x) - u_n(x) = \int_a^b \kappa(x,t)\Psi(t)dt - \sum_{i=0}^{n} \omega_i \kappa(x,t_i)\Psi_{n,i} \qquad (\omega_i = \int_a^b l_i(t)dt)$$
$$= \int_a^b \left(\kappa(x,t)\Psi(t) - \sum_{i=0}^{n} \kappa(x,t_i)\Psi_{n,i} l_i(t) \right) dt, \tag{28}$$

but $\sum_{i=0}^{n} \kappa(x,t_i)\Psi_{n,i} l_i(t)$ interpolates $\kappa(x,t)\Psi(t)$ at the points t_i, $i = 0,1,\dots,n$, ($t_i = x_i$, the points for Newton–Cotes quadrature rule). If we set $F(x,t) = \kappa(x,t)\Psi(t)$, from (28) and Theorem 1, then

$$u(x) - u_n(x) = \int_a^b \left(\frac{\pi(t)}{(n+1)!} \frac{\partial^{n+1} F}{\partial t^{n+1}}(x,\zeta_t) \right) dt, \ \zeta_t \in [a,b], \tag{29}$$

where $\pi(t) = \Pi_{i=0}^{n}(t - t_i)$. Since $\psi, u \in C^{n+1}[a,b]$, there is some $M_1 > 0$ with $|\Psi^{(n+1)}(t)| \leq M_1$ for all $t \in [a,b]$, where $\Psi(t) = \psi(t,u(t))$. Also $\kappa \in C^{n+1}[a,b]^2$; thus there is some $M_2 > 0$ with $|\frac{\partial^{n+1}\kappa}{\partial t^{n+1}}(x,t)| \leq M_2$ for all $x,t \in [a,b]$. Then $F(x,t) = \kappa(x,t)\Psi(t)$, necessitates that $|\frac{\partial^{n+1}F}{\partial t^{n+1}}(x,t)| \leq M$ for a real number M and for all $x,t \in [a,b]$. Therefore from (27) and (29), we obtain

$$|u(x) - u_n(x)| \leq \frac{1}{(n+1)!} \int_a^b \left(|\pi(t)| |\frac{\partial^{n+1} F}{\partial t^{n+1}}(x,\zeta_t)| \right) dt$$
$$\leq \frac{(b-a)M}{4(n+1)} h^{n+1} \qquad (h = \frac{b-a}{n})$$
$$\leq \frac{M}{4} \left(\frac{b-a}{n} \right)^{n+2}, \tag{30}$$

where we used $(n+1)n^{n+1} \geq nn^{n+1} = n^{n+2}$ to get the last inequality. We note that

$$\left(\frac{b-a}{n} \right)^{n+2} \leq \left(\frac{b-a}{n} \right)^2 \left(1 + \frac{b-a}{n} \right)^n.$$

Thus from (30), we obtain

$$0 \leq |u(x) - u_n(x)| \leq \frac{M}{4} \left(\frac{b-a}{n} \right)^2 \left(1 + \frac{b-a}{n} \right)^n. \tag{31}$$

It is clear to see that

$$\lim_{n\to\infty} \left(\frac{b-a}{n} \right)^2 = 0, \tag{32}$$

$$\lim_{n\to\infty} \left(1 + \frac{b-a}{n} \right)^n = e^{b-a}. \tag{33}$$

Considering the limit of the both sides of (31) as n approaches infinity and using (32)–(33), yield

$$\lim_{n\to\infty} u_n(x) = u(x), \qquad \text{for every } x \in [a,b];$$

that is, $u_n(x)$ is uniformly convergent to $u(x)$. □

As a result of Theorem 3 and from (30), it is also clearly seen that

$$|u(x) - u_n(x)| = O(h^{n+2}), \text{ where } h = \frac{b-a}{n}. \tag{34}$$

4 Illustrative examples

In this section, we apply the method proposed in Section 2 to some test examples. All numerical calculations are performed by Maple 13.

4.1 Fredholm integral equation

Example 1. Consider

$$u(x) = \frac{7}{8}x + \int_0^1 \frac{1}{2}xtu^2(t)dt, \quad 0 \le x \le 1,$$

with exact solution $u(x) = x$.

For this example, by solving the nonlinear system (16) for $n = 4$, we have

$$\Psi_{4,0} = 0,\ \Psi_{4,1} = 0.0625,\ \Psi_{4,2} = 0.25,\ \Psi_{4,3} = 0.5625,\ \Psi_{4,4} = 1.$$

Also the nodes x_i and the weights ω_i of the Newton–Cotes quadrature rule for the same n are

$$t_0 = 0,\ t_1 = \frac{1}{4},\ t_2 = \frac{1}{2},\ t_3 = \frac{3}{4},\ t_4 = 1,$$

$$\omega_0 = \frac{7}{90},\ \omega_1 = \frac{16}{45},\ \omega_2 = \frac{2}{15},\ \omega_3 = \frac{16}{45},\ \omega_4 = \frac{7}{90}.$$

Substituting the values of x_i, ω_i and $\Psi_{4,i}$ for $i = 0, \ldots, 4$ into (18), we have

$$
\begin{aligned}
u_4(x) =&\nu(x) + \sum_{i=0}^{4} \omega_i \kappa(x,t_i)\Psi_{4,i} \\
=&\frac{7}{8}x + \frac{7}{90}(\frac{1}{2}x \times 0) \times 0 \\
&+ \frac{16}{45}(\frac{1}{2}x \times \frac{1}{4}) \times 0.0625 \\
&+ \frac{2}{15}(\frac{1}{2}x \times \frac{1}{2}) \times 0.25 \\
&+ \frac{16}{45}(\frac{1}{2}x \times \frac{3}{4}) \times 0.5625 \\
&+ \frac{7}{90}(\frac{1}{2}x \times 1) \times 1 \\
=&x,
\end{aligned}
$$

which is the exact solution of Example 1

Example 2. [9] Consider

$$u(x) = ex + 1 - \int_0^1 (x+t)e^{u(t)}dt, \quad 0 \le x \le 1,$$

with the exact solution $u(x) = x$.

The absolute error $|u_n(x) - u(x)|$ of Example 2 for $n = 2, 4, 6$ is shown in Table 1.

Table 1: Absolute error of Example 2

x	$n=2$	$n=4$	$n=6$	method of [9] ($N=7$)
0.0	1.7e−3	3.8e−6	6.0e−9	2.5e−6
0.2	1.5e−3	3.2e−6	5.2e−9	7.3e−6
0.4	1.2e−3	2.6e−6	4.4e−9	7.9e−6
0.6	1.0e−3	2.1e−6	3.6e−9	2.5e−6
0.8	7.7e−4	1.5e−6	2.8e−9	3.9e−6
1.0	5.3e−4	9.6e−7	2.0e−9	2.6e−6

Example 3. [1] Consider

$$u(x) = e^{x+1} - \int_0^1 e^{x-2t}(u(t))^3 dt, \quad 0 \le x \le 1,$$

with exact solution $u(x) = e^x$.

The absolute error $|u_n(x) - u(x)|$ of Example 3 for $n = 2, 4, 6$ is shown in Table 2.

Table 2: Absolute error of Example 3

x	$n=2$	$n=4$	$n=6$	method of [1] $(N=16)$
0.1	1.0e−4	1.5e−7	0.0	4.3e−4
0.2	1.1e−4	1.7e−7	1.0e−9	3.3e−7
0.3	1.2e−4	1.8e−7	0.0	7.9e−4
0.4	1.4e−4	2.0e−7	0.0	2.9e−4
0.5	1.5e−4	2.3e−7	1.0e−9	1.2e−3
0.6	1.7e−4	2.5e−7	0.0	7.1e−4
0.7	1.8e−4	2.8e−7	1.0e−9	5.4e−7
0.8	2.0e−4	3.1e−7	1.0e−9	1.3e−3
0.9	2.3e−4	3.4e−7	1.0e−9	4.8e−4

Example 4. [12] In this example, the proposed method in Section 2 is used to solve an integral equation reformulation of the nonlinear two-point boundary value problem

$$u^{''}(t) - \exp(u(t)) = 0, \quad t \in (0,1), \quad u(0) = u(1) = 0.$$

This problem has the unique solution

$$u(t) = -\ln(2) + 2\ln\left(\frac{c}{\cos\left(\frac{c\left(t-\frac{1}{2}\right)}{2}\right)}\right),$$

where c is the only solution of $\frac{c}{\cos\left(\frac{c}{4}\right)} = \sqrt{2}$, and may be reformulated as the integral equation

$$u(x) = \int_0^1 \kappa(x,t)\exp(u(t))dt, \quad x \in [0,1],$$

where

$$\kappa(x,t) = \begin{cases} -t(1-x), & t \leq x, \\ -x(1-t), & t > x. \end{cases}$$

The uniform norm $||u_n - u|| = \sup\{|u_n(t) - u(t)|,\ t \in [0,1]\}$ of Example 4 for $n = 5, 9$ is shown in Table 3.

Table 3: Numerical results of Example 4

n	$\|\|u_n - u\|\|$ (presented method)	$\|\|u_n - u\|\|$ (method of [12])
5	5.94e−3	5.19e−4
9	2.19e−3	1.28e−4

4.2 Volterra integral equation

Example 5. [5] Consider

$$u(x) = \frac{3}{2} - \frac{1}{2}e^{-2x} - \int_0^x (u^2(t) + u(t))dt, \quad 0 \le x \le 1,$$

with the exact solution $u(x) = e^{-x}$.

For this example, we define $e_n(x) = |u_n(x) - u(x)|$. The maximum norm $||e_n||_\infty = \max\{|e_n(x_i)|,\, x_i = .1 * i,\, i = 0, 1, \ldots, 10\}$ for $n = 1, 3, 5, 7, 9$ is presented in Table 4.

Table 4: Numerical results of Example 5

n	$\|\|e_n\|\|_\infty$ (presented method)	$\|\|e_n\|\|_\infty$ (method of [5])
1	1.174e−1	6.282e−2
3	5.235e−4	1.001e−3
5	3.829e−6	8.294e−6
7	2.020e−8	3.913e−8
9	1.000e−10	1.163e−10

Example 6. [18] Consider

$$u(x) = 1 + \sin^2 x - 3\int_0^x \sin(x-t)(u(t))^2 dt, \quad 0 \le x \le 1,$$

with exact solution $u(x) = \cos x$.

The absolute error $|u_n(x) - u(x)|$ of Example 6 for $n = 3, 5, 7, 9$ is shown in Table 5.

Table 5: Absolute error of Example 6

x	$n=3$	$n=5$	$n=7$	$n=9$	method of [18] ($m=60$)
0.2	4.6e−3	8.4e−5	8.4e−7	6.3e−9	0.0
0.4	3.8e−3	2.2e−5	4.2e−7	3.9e−9	1.0e−5
0.6	2.1e−3	1.0e−6	1.1e−7	5.0e−10	1.0e−5
0.8	3.4e−3	6.3e−5	6.1e−7	3.9e−9	2.0e−5
1.0	1.1e−3	2.2e−5	2.9e−7	2.1e−9	1.0e−5

Example 7 (Constructed by author)**.** Consider

$$u(x) = e^{-x} - e^x(x+1) + \int_{-1}^x e^{x+t}u(t)dt, \quad -1 \le x \le 1. \tag{35}$$

To calculate the error in the interval $[-1, 1]$, we define the error function $e_n(x)$ as

$$e_n(x) = u_n(x) - \nu(x) - \int_{-1}^{x} \kappa(x,t)u_n(t)dt.$$

Actually, on the right-hand side of the above equation, we put the approximated solution $u_n(x)$ instead of the exact solution $u(x)$ for (4). Now the absolute error $|e_n(x)|$ of Example 7 for $n = 2, 3, 5, 7, 9$ and some $x \in [-1, 1]$ is shown in Table 6.

Table 6: Absolute error of Example 7

x	$e_2(x)$	$e_3(x)$	$e_5(x)$	$e_7(x)$	$e_9(x)$
-1.0	0.0	0.0	0.0	0.0	0.0
-0.8	2.0e−3	3.4e−4	2.3e−4	5.7e−6	1.0e−9
-0.6	1.0e−2	1.2e−3	4.5e−4	6.9e−6	1.0e−9
-0.4	2.8e−2	2.2e−3	4.4e−4	3.6e−7	1.2e−9
-0.2	5.2e−2	2.5e−3	4.2e−4	1.2e−5	9.0e−10
0.0	7.4e−2	1.6e−3	6.4e−4	1.9e−5	3.0e−9
0.2	7.3e−2	5.9e−5	9.4e−4	3.2e−5	6.1e−9
0.4	2.0e−2	3.7e−4	9.0e−4	1.8e−5	9.2e−9
0.6	1.0e−1	1.5e−3	7.1e−4	6.9e−6	2.1e−8
0.8	2.9e−1	2.2e−4	1.4e−3	3.8e−5	6.9e−8
1.0	4.4e−1	1.3e−3	1.4e−3	5.1e−5	2.2e−7

Example 8. Consider

$$u(x) = -\frac{x^5}{4} - \frac{2x^4}{3} - \frac{5x^3}{6} - x^2 + 1 + \int_0^x (xt+1)(u(t))^2 dt, \quad 0 \le x \le 1.$$

Similar to Example 7, to calculate the error in the interval $[0, 1]$, we define the error function $e_n(x)$ as

$$e_n(x) = u_n(x) - \nu(x) - \int_0^x \kappa(x,t)(u_n(t))^2 dt.$$

Now the absolute error $|e_n(x)|$ of Example 8 for $n = 2, 3, 5, 7$ and some $x \in [0, 1]$ is shown in Table 7.

5 Conclusion

In this work, the Newton–Cotes quadrature rule together with the Lagrange interpolation were used to transform Fredholm and Volterra integral equations to a system of algebraic equations. As shown in Section 4, via some

Table 7: Absolute error of Example 8

x	$e_2(x)$	$e_3(x)$	$e_5(x)$	$e_7(x)$
0.0	0.0	0.0	0.0	0.0
0.1	4.7e−4	9.5e−3	0.0	0.0
0.2	1.1e−3	2.7e−2	1.0e−10	1.0e−10
0.3	1.4e−3	4.0e−2	1.0e−10	3.0e−10
0.4	1.1e−3	4.1e−2	1.0e−10	2.0e−10
0.5	4.4e−4	3.1e−2	1.0e−10	0.0
0.6	9.4e−4	1.9e−2	0.0	2.0e−9
0.7	3.4e−3	1.7e−2	0.0	8.0e−9
0.8	7.4e−3	3.2e−2	1.0e−9	1.0e−8
0.9	1.2e−2	5.1e−2	1.0e−9	4.0e−8
1.0	1.6e−2	5.7e−2	3.0e−9	7.7e−8

test examples, as n increases, the error decreases. The calculated errors in the test examples were compatible with the presented error bound in (30). It was stated that a high accuracy is achieved even by using a small number of n. Also the method can be extended to solve a system of such equations.

References

1. Abdul Sathar, M.H., Rasedee, A.F.N., Ahmedov, A.A. and Bachok, N. *Numerical solution of nonlinear Fredholm and Volterra integral equations by Newton–Kantorovich and Haar wavelets methods,* Symmetry, 12 (2020) 1–13.

2. Babolian, E. and Mordad, M. *A numerical method for solving system of linear and nonlinear integral equations of the second kind by hat basis functions,* Comput. Math. Appl., 62 (2011) 187–198.

3. Babolian, E. and Shahsavaran, A. *Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelet,* J. Comput. Appl. Math. 225 (2009) 87–95.

4. Basirat, B., Maleknejad, K. and Hashemizadeh, E. *Operational matrix approach for the nonlinear Volterra-Fredholm integral equations: Arising in physics and engineering,* Int. J. Phys. Sci. 7 (2012) 226–233.

5. Bazm, S. *Solution of nonlinear Volterra-Hammerstein integral equations using alternative Legendre collocation method,* Sahand Communications in Mathematical Analysis, 4 (2016) 57–77.

6. Bellour, A., Sbibih, D. and Zidna, A. *Two cubic spline methods for solving Fredholm integral equations,* Appl. Math. Comput. 276 (2016) 1–11.

7. Brunner, H. *Implicitly linear collocation method for nonlinear Volterra equations,* J. Appl. Num. Math. 9 (1982) 235–247.

8. Delves, L.M. and Mohamed, J.L. *Computational methods for integral equations*, Cambridge University Press, 1985.

9. Fattahzadeh, F. *Numerical solution of general nonlinear Fredholm-Volterra integral equations using Chebyshev approximation,*Int. J. Ind. Math. 8 (2016) 81–86.

10. He, G., Xiang, S. and Xu, Z. *A Chebyshev collocation method for a class of Fredholm integral equations with highly oscillatory kernels,* J. Comput. Appl. Math. 300 (2016) 354–368.

11. Ibrahim, I.A. *On the existence of solutions of functional integral equations of Urysohn type,* Comput. Math. with Appl. 57 (2009) 1609–1614.

12. Kumar, S. and Sloan, I.H. *A New collocation-type method for Hammerstein integral equations,* Math. Comput. 48 (1987) 585–593.

13. Maleknejad, K., Almasieh, H. and Roodaki, M. *Triangular functions (TF) method for the solution of nonlinear Volterra–Fredholm integral equations,* Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 3293–3298.

14. Maleknejad, K., Lotfi, T. and Mahdiani, K. *Numerical solution of first kind Fredholm integral equations with wavelets-Galerkin method and wavelets precondition,* Appl. Math. Comput. 186 (2007) 794–800.

15. Maleknejad, K., Mollapourasl, R. and Ostadi, A. *Convergence analysis of Sinc-collocation method for nonlinear Fredholm integral equations with a weakly singular kernel,*J. Comput. Appl. Math. 278 (2015) 1–11.

16. Rahmoune, A. *Spectral collocation method for solving Fredholm integral equations on the half-line,* Appl. Math. Comput. 219 (2013) 9254–9260.

17. Sahu, P.K. and Ray, S.S. *Hybrid Legendre Block-Pulse functions for the numerical solutions of system of nonlinear Fredholm–Hammerstein integral equations,* Appl. Math. Comput. 270 (2015) 871–878.

18. Sepehrian, B. and Razzaghi, M.*A new method for solving nonlinear Volterra-Hammerstein integral equations via single-term Walsh series,* Mathematical Analysis and Convex Optimization, 1 (2020) 59–69.

19. Shahsavaran, A. *Numerical solution of nonlinear Fredholm-Volterra integral equations via piecewise constant functions by collocation method,* Am. J. Comput. Math. 1 (2011) 134–138.

20. Stoer, J. and Bulirsch, R. *Introduction to numerical analysis*, Springer-Verlag, 1991.

21. Wazwaz, A.M. *Linear and nonlinear integral equations: Methods and applications*, Higher education, Springer, 2011.