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Research Article

Numerical method for the solution of high order Fredholm integro-differential difference equations using Legendre polynomials

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Abstract

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This research paper deals with the numerical method for the solution of high-order Fredholm integro-differential difference equations using Legendre polynomials. We obtain the integral form of the problem, which is transformed into a system of algebraic equations using the collocation method. We then solve the algebraic equation using Newton's method. We establish the uniqueness and convergence of the solution. Numerical problems are considered to test the efficiency of the method, which shows that the method competes favorably with the existing methods and, in some cases, approximates the exact solution.

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1 Introduction

The theory of integral equations is one of the most important branches of mathematics. Currently, considerable interest in mixed integro-differential difference equations has been stimulated due to their numerous applications in the areas of engineering, science, and medicine. In integro-differential difference equations, the unknown function appears to be under the integration sign, and it may also include the derivatives and functional arguments of the unknown function [28]. integro-differential difference equations can be grouped into Fredholm integro-differential difference equations and Volterra integro-differential difference equations. The upper bound of the integral part of the Volterra type is variable, while it is a fixed number for that of the Fredholm type [16].

Many numerical methods have been presented in open literature for solving integro-differential difference equations and integro-differential equations, include the Adomian decompositions method by [19], the collocation method [4, 2, 13], Hybrid linear multistep method [17, 6, 21], Homotopy analysis method [18], Bernoulli matrix method [10], Differential transform method [15], Shifted Legendre polynomials [23], Bernstein Polynomials Method [22], Differential transformation [12], Chebyshev polynomials[24], Lucas series and polynomials [14], Optimal Auxiliary Function Method (OAFM) [30], Block pulse functions operational matrices [26], and Spectral Homotopy Analysis Method [8]. Ajileye and Aminu [5] presented the standard collocation method to solve first-order Volterra integro-differential equations. Assuming an approximation solution, the class of integro-differential equations was restated in terms of the derived polynomial. After solving for the unknown, we collocated the resultant equation at many points within the range $[0, 1]$, yielding a system of linear algebraic equations. Ajileye et al. [7] introduced a collocation method for the computational solution of the integro-differential equations with Fredholm- Volterra fractional order. They first obtained the problem in linear integral form, which they then converted into a set of linear algebraic equations using standard collocation points.

This research paper considers the integro-differential difference equation of the form

$$\sum_{n=0}^{\alpha} P_n(x)\, y^{(n)} = \sum_{m=0}^{M} Q_m(x)\, y^{(m)}(x-\tau) + g(x) + \lambda \int_a^b K(x,t)\, y^L(t-\tau)\, d\tau, \quad x \in [a,b] = [-1,1], \tag{1}$$

with the initial condition

$$y^{(m-1)}(a) = y_{m-1}, \tag{2}$$

where $g, P, Q \in C([a,b], \mathbb{R})$, $K \in \left([a,b]^2, \mathbb{R}\right)$, λ and y_{m-1} are known constants. $P_\alpha(x) = 1$, $\alpha > M$.

2 Basic definitions

In this section, we define some basic terms that would be encountered in this research.

Definition 1 (Integral equation [9])**.** Given an integral equation

$$y(x) = u(x) + \int_{x_0}^{x_f} k((t,s), y(s))\, ds, \tag{3}$$

then if

(i) $k\left(t,s\right)=k\left(s,t\right)$, then it is symmetry.

(ii) $k\left(t,s\right)=k\left(a+b-t,\ a+b-s\right)$ is linear then the kernel is centrosymmetric.

(iii) If $\mathrm{k}(t,s,y\left(s\right))=k\left(t-s\right)\ g\left(s,y\left(s\right)\right)$ then the equation is called convolution integral equation and if $g\left(s,y\left(s\right)\right)=y\left(s\right)$, it is called linear.

Definition 2 (Normed space [9])**.** Let X be a nonvector space over k. A norm on x is a function $\|\cdot\|:X\to X$ such that for all $x,y\in X$ and $\alpha\in X$

(i) $\|x\|\geq 0$,

(ii) $\|x\|=0$ if and only if $x=0$,

(iii) $\|\alpha x\|=|\alpha|\,\|x\|$,

(iv) $\|x+y\|\leq\|x\|+\|y\|$.

A vector space X on which there is a norm is called a normed space.

Definition 3 (Banach space [9])**.** Banach space is a complete normed space.

Definition 4 (Lipschitzian continuity [9])**.** Let $(X,\|\cdot\|)$ be a normed space. A mapping $T:X\to X$ is L-Lipschitz if there exists $L>0$ such that $\|Tx-Ty\|_{\infty}\leq L\,\|x-y\|_{\infty}$ for all $x,y\in X$.

Definition 5 (q- contraction [9])**.** Let $(X,\|\cdot\|)$ be a normed space. The mapping $T:X\to X$ is a $q-$contraction if $\|Tx_1-Tx_2\|_{\infty}\leq q\,\|x_1-x_2\|_{\infty}, q\in[0,1)$ fixed for all $x_1,x_2\in X$.

Definition 6 (Strict q-contraction [9])**.** Let $(X,\|\cdot\|)$ be a norm space. The mapping $T:X\to X$ is strict q-contraction when

$$\|T^nx_1-T^nx_2\|_{\infty}\leq q^n\,\|x_1-x_2\|_{\infty}\quad\text{for all } x_1,x_2\in X. \tag{4}$$

Definition 7 (nth integration [20])**.** Let $u(x)$ be an integrable function; then

$${}_aI_x^k\left(u(x)\right)=\frac{1}{\Gamma(k)}\int\limits_a^x\left(x-t\right)^{k-1}u(t)dt, \tag{5}$$

$${}_aI_x^k\left(u^{(k)}(x)\right)=u\left(x\right)-\sum_{i=0}^{k-1}\frac{x^i}{i!}u^{(i)}\left(a\right). \tag{6}$$

Definition 8 (Legendre polynomial [1]). Legendre polynomial on the interval $[-1,1]$ can be determined with the aid of the recurrence formulas

$$L_{n+1}(x)=\frac{2n+1}{n+1}xL_n(x)-\frac{n}{n+1}L_{n-1}(x), \qquad n=1,2,\ldots, \tag{7}$$

where $L_0(x)=1$, $L_1(x)=x$. In order to use these polynomials on the interval $x\in[0,1]$, shifted Legendre polynomial is then defined by the recurrence formula

$$p_{n+1}(x)=\frac{(2n+1)(2x-1)}{(n+1)}p_n(x)-\frac{n}{n+1}p_{n-1}(x), \tag{8}$$

where $p_0=1$, $p_1(x)=2x-1$. The analytical form of degree n is defined as

$$p_n(x)=\sum_{k=0}^{n}\frac{(-1)^{n+k}-\Gamma(n+k+1)}{\Gamma(n-k+1)(\Gamma(k+1))^2}x^k. \tag{9}$$

Theorem 1 (Banach's fixed point theorem [25]). Let $(X,\|\cdot\|)$ be a complete norm space; then each contraction mapping $T:X\to X$ has a unique fixed point x of T in X, such that $T(x)=x$.

3 Methodology

This section considers the development of our method, which was achieved by developing the integral form of the modeled (1) and obtaining the algebraic equations using some lemmas.

Lemma 1. Let $y\in C([a,b],\mathbb{R})$ be the solution to (1) and (2), let $K\in C\left([a,b]^2,\mathbb{R}\right)$, and let g and $Q\in C([a,b],\mathbb{R})$. Then (1) and (2) are equivalent to

$$\begin{aligned} y(x)=H(x)&+\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{M}\int_{x_0}^{x}(x-t)^{\alpha-1}Q_m(t)\,y^{(m)}(t)\,dt\mathbf{M}_{-1}\\ &-\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_{x_0}^{x}(x-t)^{\alpha-1}P_n(t)\,y^{(n)}(t)\,dt\\ &+\frac{\lambda}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}\left[\int_a^b K(t,\tau)\,y^L(\tau)\,d\tau\right]dt\mathbf{M}_{-1}^{L}, \end{aligned} \tag{10}$$

where

$$H(x)=\sum_{n=0}^{\alpha-1}\frac{x^i}{i!}y_i+\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}g(t)\,dt.$$

Proof. Equation (10) can be written as

$$y^{(\alpha)}(x)=\sum_{m=0}^{M}Q_m(x)\,y^{(m)}(x-\tau)-\sum_{n=0}^{\alpha-1}P_n(x)\,y^{(n)}(x)$$
$$+g(x)+\lambda\int_a^b K(x,t)\,y^L(t-\tau)\,dt. \tag{11}$$

Using [29] gives

$$y(x-\tau)=y(x)\,\mathbf{M}_{-1}, \tag{12}$$

where

$$\mathbf{M}_{-1}=\begin{bmatrix}\binom{0}{0}(-\tau)^0 & \binom{1}{0}(-\tau)^0 & \cdots & \binom{N}{0}(-\tau)^N\\ 0 & \binom{1}{1}(-\tau)^0 & \cdots & \binom{N}{1}(-\tau)^{N-1}\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & \binom{N}{N}(-\tau)^0\end{bmatrix}. \tag{13}$$

Substituting (12) into (11) gives

$$y^{(\alpha)}(x)=\sum_{m=0}^{M}Q_m(x)\,y^{(m)}(x)\,\mathbf{M}_{-1}-\sum_{n=0}^{\alpha-1}P_n(x)\,y^{(n)}(x)$$
$$+g(x)+\lambda\int_{x_0}^{b}K(x,t)\,y^L(t)\,dt\mathbf{M}_{-1}^L. \tag{14}$$

Using (6) in (14) ,we have

$$y(x)=\sum_{i=0}^{n-1}\frac{x^i}{i!}y^i(a)+\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{M}\int_{x_0}^{x}(x-t)^{\alpha-1}Q_m(t)\,y^{(m)}(t)\,dt\,|\mathbf{M}_{-1}|$$
$$-\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_{x_0}^{x}(x-t)^{\alpha-1}P_n(t)\,y^{(n)}(t)\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}g(x)\,dt$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}\left[\int_a^b K(x,\tau)\,y^L(\tau)\,d\tau\right]dt\left|\mathbf{M}_{-1}^{L}\right|,$$

$$\begin{aligned}y(x)=H(x)&+\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{M}\int_{x_0}^{x}(x-t)^{\alpha-1}Q_m(t)\,y^{(m)}(t)\,dt\left|\mathbf{M}_{-1}\right|\\&-\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_{x_0}^{x}(x-t)^{\alpha-1}P_n(t)\,y^{(n)}(t)\,dt\\&+\frac{\lambda}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}\left[\int_a^b K(t,\tau)\,y^L(\tau)\,d\tau\right]dt\left|\mathbf{M}_{-1}^{L}\right|,\end{aligned}\tag{15}$$

where

$$H(x)=\sum_{n=0}^{\alpha-1}\frac{x^i}{i!}y_i+\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}g(t)\,dt.$$

□

3.1 Method of solution

Let the solution to (10) be approximated by

$$y(x)=\mathbf{P}(x)\,\mathbf{A},\tag{16}$$

where

$$P_0(x)=1,\quad P_1(x)=x,$$
$$\mathbf{P}(x)=[P_0(x)\quad P_1(x)\ldots P_N(x)]$$

is the polynomial defined by

$$P_n(x)=\sum_{m=0}^{M}\frac{(-1)^m\,\Gamma(2n-2m+1)}{2^n\Gamma(m+1)\,\Gamma(n-m+1)\,\Gamma(n-2m+1)}x^{n-2m},\tag{17}$$

$$P_n(x)=\sum_{m=0}^{M}Q(n;m)\,x^{n-2m},$$

where

$$Q\left(n;m\right)=\sum_{m=0}^{M}\frac{\left(-1\right)^{m}\Gamma\left(2n-2m+1\right)}{2^{n}\Gamma\left(m+1\right)\Gamma\left(n-m+1\right)\Gamma\left(n-2m+1\right)},$$

$$M=floor\left(\frac{n}{2}\right)\text{ and }\mathbf{A}=\left[a_0\ \ a_1\ \ \dots\ \ a_N\right]^T$$

are constant to be determined.

Equation (16) can be written in the form

$$y\left(x\right)=\mathbf{X}\left(x\right)\mathbf{DA}, \tag{18}$$

where (i) when N is even, we have

$$\mathbf{X}\left(x\right)=\left[1\ x^2\ \ \dots,\ x^4\ \dots x^{2n}\right],\qquad n=0,1,\dots,$$

$$\mathbf{D}_{even}=\begin{bmatrix} D\left(0;0\right) & 0 & 0 & \cdots & 0\\ D\left(2;1\right) & D\left(2;0\right) & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ D\left(N;\frac{N}{2}\right) & D\left(N;\frac{N}{2}-1\right) & D\left(N;\frac{N}{2}-2\right) & \cdots & D\left(N;0\right)\end{bmatrix}^T. \tag{19}$$

(ii) when N is odd, we have

$$\mathbf{X}\left(x\right)=\left[x\ \ x^3\ \ x^5\ \ \dots\ \ x^{2n+1}\right],\qquad n=0,\left(1\right),\frac{N-1}{2}, \tag{20}$$

$$\mathbf{D}_{odd}=\begin{bmatrix} D\left(1;0\right) & 0 & 0 & \cdots & 0\\ D\left(3;1\right) & D\left(3;0\right) & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ D\left(N;\frac{N-1}{2}\right) & D\left(N;\frac{N-3}{2}\right) & D\left(N;\frac{N-5}{2}\right) & \cdots & D\left(N;0\right)\end{bmatrix}^T. \tag{21}$$

Hence,

$$y^{(n)}\left(x\right)=\mathbf{X}^{(n)}\left(x\right)\mathbf{DA} \tag{22}$$

writing

$$\mathbf{P}\left(x\right)=\mathbf{X}\left(x\right)\mathbf{D}. \tag{23}$$

Lemma 2. Let $y\in C\left([a,b],\mathbb{R}\right)$ be defined by (18); then

$$y^{(m)}\left(x\right)=\frac{\Gamma\left(n+1\right)}{\Gamma\left(n-m+1\right)}x^{n-m}\mathbf{DA}, \tag{24}$$

Proof. Given

$$y\left(x\right)=x\left(x\right)\mathbf{DA},$$

then

$$y^{(m)}(x) = \frac{d^m}{dx^m} x(x)\,\mathbf{DA},$$

$$y^{(m)}(x) = \frac{d^m}{dx^m} x^n \mathbf{DA},\ \ n = 0\,(1)\,N.$$

We prove by induction, when

$$m = 1, \qquad y^{(1)} = nx^{n-1}\mathbf{DA},$$

$$m = 2, \qquad y^{(2)} = n\,(n-1)\,x^{n-2}\mathbf{DA},$$

$$m = 3, \qquad y^{(3)} = n\,(n-1)\,(n-2)\,x^{n-3}.\mathbf{DA}$$

Therefore, at

$$\begin{aligned} m = n, \qquad y^{(n)} &= (n-1)\,(n-2)\cdots(n-m+1)\,x^{n-m}\mathbf{DA} \\ &= \frac{n\,(n-1)\,(n-2)\cdots(n-m+1)\,(n-m)!}{(n-m)!} x^{n-m}\mathbf{DA} \\ &= \frac{\Gamma\,(n+1)}{\Gamma\,(n-m+1)} x^{n-m}\mathbf{DA} \end{aligned} \tag{25}$$

which is the expected result. □

Lemma 3. Let $y \in C\,([a,b],\mathbb{R})$ be defined by (18), let $K \in C\left([a,b]^2,\mathbb{R}\right)$ be defined by $K\,(x,t) = x^i t^j$, and let

$$V_1 = \int_a^b K\,(x,t)\,y^L\,(t-\tau)\,dt; \tag{26}$$

then (26) is equivalent to

$$V\,(x;n) = \int_a^b x^i t^j \underbrace{VUU^T VV^T U \ldots U^T V}_{L \text{ times}} dt,$$

where

$$V\,(x;n) = x^n\,|\mathbf{M}_{-1}|\,, U = \mathbf{DA}.$$

Proof. Let

$$V_1\,(x) = \int_a^b K\,(x,t)\,y^L\,(t-\tau)\,dt.$$

Using (18)

$$y\,(x) = \mathbf{X}\,(x)\,\mathbf{DA}$$

then

$$y\left(x-\tau\right)=\mathbf{X}\left(x-\tau\right)\mathbf{DA}=x^{n}|\mathbf{M_{-1}}|\,\mathbf{DA},\; n=0\left(1\right)N.$$

Hence

$$\begin{aligned} y^{L}\left(x-\tau\right) &= \left(x\left|\mathbf{M}_{-1}\right|\mathbf{DA}\right)^{L}\\ &= \left(x^{n}\left|\mathbf{M}_{-1}\right|\mathbf{DA}\right)^{L}, \quad n=0\left(1\right)N\\ &= \underbrace{\left(x\mathbf{M}_{-1}\mathbf{DA}\right)\left(x^{n}\left|\mathbf{M}_{-1}\right|\mathbf{DA}\right)^{T}\left(x^{n}\left|\mathbf{M}_{-1}\right|\mathbf{DA}\right)\cdots\left(x^{n}\left|\mathbf{M}_{-1}\right|\mathbf{DA}\right)^{T}}_{L\text{ times}},\\ V_{1}\left(x;n\right) &= \int_{a}^{b} x^{i}t^{j}\underbrace{VUU^{T}VV^{T}U\ldots U^{T}V}_{L\text{ times}}dt, \end{aligned}$$

where

$$V\left(x;n\right)=x^{n}\left|\mathbf{M}_{-1}\right|,\; U=\mathbf{DA}.$$

□

Lemma 4. Let y(x) be approximated by (18); then (10) is equivalent to

$$\sum_{n=0}^{\alpha}P_{n}\left(x\right)\frac{\Gamma\left(k+1\right)}{\Gamma\left(k-n+1\right)}x^{k-n}\mathbf{DA}-\sum_{m=0}^{M}Q_{m}\left(x\right)\frac{\Gamma\left(k+1\right)}{\Gamma\left(k-m+1\right)}x^{k-m}\left|\mathbf{M}_{-1}\right|\mathbf{DA}\\ -\lambda\int_{a}^{b}x^{i}t^{j}\underbrace{VUU^{T}VV^{T}U\ldots U^{T}V}_{L\text{ times}}dt=g\left(x\right).$$

Proof. It holds that

$$y\left(x-\tau\right)=\mathbf{X}\left(x\right)\mathbf{M}_{-1}\mathbf{DA},$$

$$y\left(x-\tau\right)=x^{k}\mathbf{M}_{-1}\mathbf{DA},\; k=0\left(1\right)N,$$

and

$$y\left(x\right)=x^{k}\mathbf{DA},\; k=0\left(1\right)N.$$

Substituting $y\left(x-\tau\right)$ and $y\left(x\right)$ in (1), we have

$$\sum_{n=0}^{\alpha}P_{n}\left(x\right)\frac{\Gamma\left(k+1\right)}{\Gamma\left(k-n+1\right)}x^{k-n}\mathbf{DA}-\sum_{m=0}^{M}Q_{m}\left(x\right)\frac{\Gamma\left(k+1\right)}{\Gamma\left(k-m+1\right)}x^{k-m}\left|\mathbf{M}_{-1}\right|\mathbf{DA}\\ -\lambda\int_{a}^{b}x^{i}t^{j}\underbrace{VUU^{T}VV^{T}U\ldots U^{T}V}_{L\text{ times}}dt=g\left(x\right). \tag{27}$$

Collocating (27) gives

$$\sum_{n=0}^{\alpha} P_n(x_i)\frac{\Gamma(k+1)}{\Gamma(k-n+1)}x_i^{k-n}\mathbf{DA}-\sum_{m=0}^{M} Q_m(x_i)\frac{\Gamma(k+1)}{\Gamma(k-m+1)}x_i^{k-m}|\mathbf{M}_{-1}|\mathbf{DA} \\ -\lambda\int_a^b x_i^i t^j \underbrace{VUU^TVV^TU\dots U^TV}_{L\text{ times}}dt = g(x_i),$$

$$\mathbf{F}(x_i)=\mathbf{W}(x_i)-\mathbf{g}(x_i)=0, \tag{28}$$

where

$$\mathbf{W}(x_i)=\sum_{n=0}^{\alpha} P_n(x_i)\frac{\Gamma(k+1)}{\Gamma(k-n+1)}x_i^{k-n}\mathbf{DA} \\ -\sum_{m=0}^{M} Q_m(x_i)\frac{\Gamma(k+1)}{\Gamma(k-m+1)}x_i^{k-m}|\mathbf{M}_{-1}|\mathbf{DA} \\ -\lambda\int_a^b x^i t^j \underbrace{VUU^TVV^TU\dots U^TV}_{L\text{ times}}dt = g(x).$$

□

3.2 Uniqueness of the method

In this section, we assume that the solution to (1) and (2) exists. We then establish the uniqueness of solution and present solutions from the method of solution.

Theorem 2 (Uniqueness theorem)**.** Let $T : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ be a mapping, let $y \in C([a,b],\mathbb{R})$ be the solution to (10), and let $C([a,b],\mathbb{R})$ be a Banach space.

In order to apply the uniqueness of solution, we have to establish the following:

i. Continuity of T,

ii. T is a $q-\textit{contraction}$,

iii. T is strict contraction.

In order to prove the uniqueness theorem, we use the following hypothesis [3]:

$$H_1: \qquad P^* = \sum_{n=0}^{N-1} \sup_{x\in J} |P_n(x)|,$$

$$H_2: \qquad K^* = \sup_{x\in J} \int_a^b |K(t,s)|\, ds,$$

$$H_3: \quad Q^* = \sum_{n=0}^{N-1} \sup_{x\in J} |Q_m(x)|,$$

$$H_4: \quad \left|y_1^{(m)} - y_2^{(m)}\right| \le L_m |y_1 - y_2| \text{ for all } m \ge 0,$$

$$H_5: \quad \left|y_1^L - y_2^L\right| \le H^L |y_1 - y_2|,$$

$$H_6: \quad \sup_{x\in J} \left|y_N^{(m)}\right| = \zeta_m,$$

$$H_7: \quad \sup_{x\in J} \left|y_N^{(n)}\right| = \zeta_n,$$

where $J = [-1,1]$.

Theorem 3 (Continuity)**.** Let $T : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ be a mapping, let $y \in C([a,b],\mathbb{R})$ be a solution to (10) and let $C([a,b],\mathbb{R})$ be a Banach space. If $\lim_{h\to\infty} y_h(x) = y(x)$, then T is continuous on $C([a,b],\mathbb{R})$ if $\|Ty_h - Ty\|_\infty \to 0$ as $h \to \infty$.

Proof. It holds that

$$\begin{aligned}
&|(Ty_h)(x) - (Ty)(x)| \\
&\le \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\alpha-1} \int_a^x (x-t)^{\alpha-1} |P_n(t)| \left|y_h^{(n)}(t) - y^{(n)}(t)\right| dt \\
&\quad + \frac{1}{\Gamma(\alpha)} \left[\sum_{m=0}^{\alpha} \int_a^x (x-t)^{\alpha-1} \left|Q_m^{(m)}(t)\right| \left|y_h(t) - y^{(m)}(t)\right| dt\right] |\mathbf{M}_{-1}| \\
&\quad + \frac{|\lambda|}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left[\int_a^b |K(t,s)| \left|y_h^L(s) - y^L(s)\right| ds\right] dt \left|\mathbf{M}_{-1}^L\right|.
\end{aligned}$$

Using hypothesis H_4, we have

$$\begin{aligned}
&|(Ty_h)(x) - (Ty)(x)| \\
&\le \frac{L_n}{\Gamma(\alpha)} \sum_{n=0}^{\alpha-1} \int_a^x (x-t)^{\alpha-1} |P_n(t)| |y_h - y|\, dt \\
&\quad + \frac{L_m}{\Gamma(\alpha)} \left[\sum_{m=0}^{\alpha} \int_a^x (x-t)^{\alpha-1} |Q_m(t)| |y_h - y|\, dt\right] |\mathbf{M}_{-1}|
\end{aligned}$$

$$+\frac{|\lambda|\,H^L}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left[\int_a^b |K(t,s)|\,|y_h(s)-y(s)|\,ds\right]dt\,\left|\mathbf{M}_{-1}^L\right|.$$

Taking the supremum of both sides gives

$$\begin{aligned}
&\sup_{x\in J}|(Ty_h)(x)-(Ty)(x)|\\
&\leq \frac{L_n}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_a^x (x-t)^{\alpha-1}\sup_{x\in J}|P_n(t)|\sup_{x\in J}|y_h(t)-y(t)|\,dt\\
&\quad+\frac{L_m}{\Gamma(\alpha)}\left[\sum_{m=0}^{\alpha}\int_a^x (x-t)^{\alpha-1}\sup_{x\in J}|Q_m(t)|\sup_{x\in J}|y_h(t)-y(t)|\,dt\right]|\mathbf{M}_{-1}|\\
&\quad+\frac{|\lambda_1|\,H^L}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left[\sup_{x\in J}\int_a^b |K(t,s)|\sup_{x\in J}|y_h(s)-y(s)|\,ds\right]dt\,\left|\mathbf{M}_{-1}^L\right|.
\end{aligned}$$

Using the hypothesis, we have

$$\begin{aligned}
&\|Ty_h-Ty\|_\infty\\
&\leq \frac{L_mP^*}{\Gamma(\alpha)}\|y_h-y\|_\infty\int_a^x (x-t)^{\alpha-1}\,dt\\
&\quad+\frac{L_nQ^*}{\Gamma(\alpha)}\|y_h-y\|_\infty\int_a^x (x-t)^{\alpha-1}\,dt\,|\mathbf{M}_{-1}|\\
&\quad+\frac{H^L\,|\lambda|}{\Gamma(\alpha)}K^*\,|y_h-y|\left|\mathbf{M}_{-1}^L\right|\int_a^x (x-t)^{\alpha-1}\,dt\\
&\leq \frac{(x-a)^\alpha P^*L_m}{\Gamma(\alpha+1)}\|y_h-y\|_\infty+\frac{(x-a)\,QP^*L_n}{\Gamma(\alpha+1)}|\mathbf{M}_{-1}|\,\|y_h-y\|_\infty\\
&\quad+\frac{H^L\,|\lambda|\,K^*\left|\mathbf{M}_{-1}^L\right|}{\Gamma(\alpha+1)}\|y_h-y\|_\infty\\
&\leq \frac{1}{\Gamma(\alpha+1)}\left(P^*L_m+Q^*\,|\mathbf{M}_{-1}|\,L_n+H^L\,|\lambda|\,K^*\left|\mathbf{M}_{-1}^L\right|\right)\|y_h-y\|_\infty.
\end{aligned}$$

Since $\lim_{h\to\infty} y_h \to y$, hence

$$\|Ty_h-Ty\|_\infty \to 0 \text{ as } h\to\infty.$$

Therefore, T is continuous. □

Theorem 4 (q-contraction)**.** Let $T : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ be a mapping and let $C([a,b],\mathbb{R})$ be a Banach space. Then T is q-contraction if

$$q=\frac{1}{\Gamma(\alpha+1)}\left(P^*L_n+Q^*L_m\,|\mathbf{M}_{-1}|+H^L\,|\lambda|\,K^*\left|\mathbf{M}_{-1}^L\right|\right)<1. \tag{29}$$

Proof. Using Theorem 1, we have

$$Ty\left(x\right)=H\left(x\right)+\frac{1}{\Gamma\left(\alpha\right)}\sum_{m=0}^{M}\int_{x_0}^{x}\left(x-t\right)^{\alpha-1}Q_m\left(t\right)y^{(m)}\left(t\right)\left|\mathbf{M}_{-1}\right|$$
$$-\frac{1}{\Gamma\left(\alpha\right)}\sum_{n=0}^{\alpha-1}\int_{x_0}^{x}\left(x-t\right)^{\alpha-1}P_n\left(t\right)y^{(n)}\left(t\right)dt$$
$$+\frac{\lambda}{\Gamma\left(\alpha\right)}\int_{x_0}^{x}\left(x-t\right)^{\alpha-1}\left[\int_a^b K\left(t,\tau\right)y^L\left(\tau\right)d\tau\right]dt\left|\mathbf{M}_{-1}^L\right|.$$

Then

$$\begin{aligned}&\left|\left(Ty_1\right)(x)-\left(Ty_2\right)(x)\right|\\&\leq\frac{1}{\Gamma\left(\alpha\right)}\sum_{m=0}^{M}\int_{x_0}^{x}\left(x-t\right)^{\alpha-1}\left|Q_m\left(t\right)\right|\left|y_1^{(m)}\left(t\right)-y_2^{(m)}\left(t\right)\right|dt\left|\mathbf{M}_{-1}\right|\\&\quad+\frac{1}{\Gamma\left(\alpha\right)}\left[\sum_{n=0}^{\alpha-1}\int_{x_0}^{x}\left(x-t\right)^{\alpha-1}\left|P_n\left(t\right)\right|\left|y_1^{(n)}\left(t\right)-y_2^{(n)}\left(t\right)\right|dt\right]\\&\quad+\frac{\left|\lambda\right|H^L}{\Gamma\left(\alpha\right)}\int_{x_0}^{x}\left(x-t\right)^{\alpha-1}\left[\int_a^b\left|K\left(t,\tau\right)\right|\left|y_1\left(\tau\right)-y_2\left(\tau\right)\right|d\tau\right]dt\left|\mathbf{M}_{-1}^L\right|,\end{aligned}$$

$$\begin{aligned}&\left|\left(Ty_1\right)(x)-\left(Ty_2\right)(x)\right|\\&\leq\frac{L_m}{\Gamma\left(\alpha\right)}\sum_{m=0}^{M}\int_{x_0}^{x}\left(x-t\right)^{\alpha-1}\left|Q_m\left(t\right)\right|\left|y_1-y_2\right|dt\left|\mathbf{M}_{-1}\right|\\&\quad+\frac{L_n}{\Gamma\left(\alpha\right)}\left[\sum_{n=0}^{\alpha-1}\int_{x_0}^{x}\left(x-t\right)^{\alpha-1}\left|P_n\left(t\right)\right|\left|y_1-y_2\right|dt\right]\\&\quad+\frac{\left|\lambda\right|H^L}{\Gamma\left(\alpha\right)}\int_{x_0}^{x}\left(x-t\right)^{\alpha-1}\left[\int_a^b\left|K\left(t,\tau\right)\right|\left|y_1\left(\tau\right)-y_2\left(\tau\right)\right|d\tau\right]dt\left|\mathbf{M}_{-1}^L\right|\end{aligned}$$

Taking the supremum of both sides gives

$$\begin{aligned}&\sup_{x\in J}\left|\left(Ty_1\right)(x)-\left(Ty_2\right)(x)\right|\\&\leq\frac{L_m}{\Gamma\left(\alpha\right)}\sum_{m=0}^{M}\int_{x_0}^{x}\left(x-t\right)^{\alpha-1}\sup_{x\in J}\left|Q_m\left(t\right)\right|\sup_{x\in J}\left|y_1-y_2\right|dt\left|\mathbf{M}_{-1}\right|\\&\quad+\frac{L_n}{\Gamma\left(\alpha\right)}\left[\sum_{n=0}^{\alpha-1}\int_{x_0}^{x}\left(x-t\right)^{\alpha-1}\sup_{x\in J}\left|P_n\left(t\right)\right|\sup_{x\in J}\left|y_1-y_2\right|dt\right]\end{aligned}$$

$$+\frac{|\lambda|\,H^L}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}\left[\sup_{x\in J}\int_a^b|K(t,\tau)|\sup_{x\in J}|y_1(\tau)-y_2(\tau)|\,d\tau\right]dt\left|\mathbf{M}_{-1}^L\right|$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left(\frac{P^*L_n}{\Gamma(\alpha)}+Q^*\frac{L_m}{\Gamma(\alpha)}|\mathbf{M}_{-1}|+H^L|\lambda|K^*\left|\mathbf{M}_{-1}^L\right|\right)\|y_1-y_2\|_\infty$$

Since q is T contraction, then

$$q=\frac{1}{\Gamma(\alpha+1)}\left(P^*L_n+Q^*L_m|\mathbf{M}_{-1}|+H^L|\lambda|K^*\left|\mathbf{M}_{-1}^L\right|\right)<1. \tag{30}$$

□

Theorem 5 (Convergence of solution)**.** Let $(C([a,b],\mathbb{R}),\|\cdot\|)$ be a norm space, let $y(x)$ and $y_N(t)$ be the exact and approximate solution of (10), respectively. Then

$$\|y_N-y\|_\infty\leq\frac{\|H_N-H\|_\infty+\zeta_mP_n^*+\zeta_nQ_m^*|\mathbf{M}_{-1}|}{1-q}, \tag{31}$$

where

$$q=\frac{1}{\Gamma(\alpha+1)}\left(P^*L_n+Q^*L_m|\mathbf{M}_{-1}|+H^L|\lambda|K^*\left|\mathbf{M}_{-1}^L\right|\right).$$

Proof. Let $H(t), Q(t)$, and $\mathrm{P}(t)$ be expanded in Legendre polynomial. Then

$$|y_N(x)-y(x)|$$
$$\leq|H_N(x)-H(x)|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_a^x(x-t)^{\alpha-1}\left|P_n^N(t)\,y_N^{(n)}(t)-P_n(t)\,y^{(n)}(t)\right|dt$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{M}\int_a^x(x-t)^{\alpha-1}\left|Q_m^N(t)\,y_N^{(m)}(t)-Q_m(t)\,y^{(m)}(t)\right|dt\,|\mathbf{M}_{-1}|$$
$$+\frac{|\lambda|}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\left[\int_a^b|K(t,s)|\left|y_N^L(s)-y^L(s)\right|ds\right]dt\left|\mathbf{M}_{-1}^L\right|$$
$$\leq|H_N(x)-H(x)|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_a^x(x-t)^{\alpha-1}\left|y_N^{(m)}(t)\right|\left|P_n^N(t)-P_n(t)\right|dt$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_a^x(x-t)^{\alpha-1}|P_n(t)|\left|y_N^{(m)}(t)-y^{(m)}(t)\right|dt$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{M}\int_a^x (x-t)^{\alpha-1}\left|y_N^{(n)}(t)\right|\left|Q_m^N(t)-Q_m(t)\right|dt\,|\mathbf{M}_{-1}|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{M}\int_a^x (x-t)^{\alpha-1}|Q_m(t)|\left|y_N^{(n)}(t)-y^{(n)}(t)\right|dt\,|\mathbf{M}_{-1}|$$
$$+\frac{|\lambda|}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left[\int_a^b |K(t,s)|\left|y_N^L(s)-y^L(s)\right|ds\right]dt\left|\mathbf{M}_{-1}^L\right|$$
$$\leq \|H_N(x)-H(x)\|+\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_a^x (x-t)^{\alpha-1}\left|y_N^{(m)}(t)\right|\left|P_n^N(t)-P_n(t)\right|dt$$
$$+\frac{L_1}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_a^x (x-t)^{N-1}|P_n(t)|\,|y_N(t)-y(t)|\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{M}\int_a^x (x-t)^{\alpha-1}\left|y_N^{(n)}(t)\right|\left|Q_m^N(t)-Q_m(t)\right|dt\,|\mathbf{M}_{-1}|$$
$$+\frac{L_2}{\Gamma(\alpha)}\sum_{m=0}^{M}\int_a^x (x-t)^{\alpha-1}|Q_m(t)|\,|y_N(t)-y(t)|\,dt\,|\mathbf{M}_{-1}|$$
$$+\frac{|\lambda|\,H^L}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left[\int_a^b |K(t,s)|\,|y_N(s)-y(s)|\,ds\right]dt\left|\mathbf{M}_{-1}^L\right|.$$

Taking supremum of both sides, we have

$$\sup_{x\in J}|y_N(x)-y(x)|$$
$$\leq \sup_{x\in J}|H_N(x)-H(x)|$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_a^x (x-t)^{\alpha-1}\sup_{x\in J}\left|y_N^{(m)}(t)\right|\sup_{x\in J}|P_N^n(t)-P_n(t)|\,dt$$
$$+\frac{L_m}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_a^x (x-t)^{\alpha-1}\sup_{x\in J}|P_N(t)|\sup_{x\in J}|y_N(t)-y(t)|\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\sum_{m=0}^{M}\int_a^x (x-t)^{\alpha-1}\sup_{x\in J}\left|y_N^{(n)}(t)\right|\sup_{x\in J}|Q_n^m(t)-Q_m(t)|\,dt\,|\mathbf{M}_{-1}|$$
$$+\frac{L_n}{\Gamma(\alpha)}\sum_{n=0}^{\alpha-1}\int_a^x (x-t)^{\alpha-1}\sup_{x\in J}|Q_m(t)|\sup_{x\in J}|y_N(t)-y(t)|\,dt\,|\mathbf{M}_{-1}|$$
$$+\frac{|\lambda|\,H^L}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left[\int_a^b |K(t,s)|\sup_{x\in J}|y_N(s)-y(s)|\,ds\right]dt\left|\mathbf{M}_{-1}^L\right|.$$

Therefore,

$$
\begin{aligned}
\|y_N - y\|_\infty \le \|H_N - H\|_\infty &+ \frac{\zeta_m P_n^*}{\Gamma(\alpha)} \int_a^x (x-t)^{N-1}\, dt \\
&+ \frac{L_m P^* \|y_N - y\|_\infty}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\, dt \\
&+ \frac{\zeta_n Q_m^*}{\Gamma(\alpha)} \int_a^x (x-t)^{N-1}\, dt \,|\mathbf{M}_{-1}| \\
&+ \frac{L_n Q^* \|y_N - y\|_\infty}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} |\mathbf{M}_{-1}| \\
&+ \frac{|\lambda| H^L \|y_N - y\|_\infty}{\Gamma(\alpha)} K^* \int_a^x (x-t)^{\alpha-1}\, dt \left|\mathbf{M}_{-1}^L\right|,
\end{aligned}
$$

$$
\|y_N - y\|_\infty \le \frac{\|H_N - H\|_\infty + \frac{\zeta_m P_n^*}{\Gamma(\alpha+1)} + \frac{\zeta_n Q_m^*}{\Gamma(\alpha+1)} |\mathbf{M}_{-1}|}{1 - \frac{L_m P^*}{\Gamma(\alpha+1)} - \frac{L_n Q^*}{\Gamma(\alpha+1)} - \frac{|\lambda| H^L K^*}{\Gamma(\alpha+1)} |\mathbf{M}_{-1}|}. \tag{32}
$$

Hence, simplification of (32) gives the required result as follows:

$$
\|y_N - y\|_\infty \le \frac{\|H_N - H\|_\infty + \zeta_m P_n^* + \zeta_n Q_m^* |\mathbf{M}_{-1}|}{1-q},
$$

where

$$
q = \frac{1}{\Gamma(\alpha+1)} \left(P^* L_n + Q^* L_m |\mathbf{M}_{-1}| + H^L |\lambda| K^* \left|\mathbf{M}_{-1}^L\right| \right).
$$

□

3.3 Numerical examples

In this section, we present numerical examples to test the efficiency of the method. The results are presented in tables as we consider Chebyshev's points.

Problem 1. [27] Consider a third order Fredholm integro-differential difference equations

$$
\begin{aligned}
&y^{(3)}(x) - xy'(x) + y''(x-1) - xy(x-1) \\
&= -(x+1)(\sin(x-1) + \cos x) - \cos 2 + 1 + \int_{-1}^{1} y(t-1)\, dt
\end{aligned}
$$

subject to

$$
y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0
$$

with the exact solution

$$y\left(x\right)=\sin x.$$

To show q-contraction for Problem 1, we have

$$\begin{aligned}Ty\left(x\right)=&\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)^{\alpha-1}ty'\left(t\right)dt+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)^{\alpha-1}y''\left(t\right)dt\left|\mathbf{M}_{-1}\right|\\&+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)^{\alpha-1}ty\left(t\right)dt\left|\mathbf{M}_{-1}\right|+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)f\left(t\right)dt\\&+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)^{\alpha-1}\left[\int_{-1}^{1}y\left(s\right)ds\right]dt\left|\mathbf{M}_{-1}\right|,\end{aligned}\tag{33}$$

$$\begin{aligned}Ty_1\left(x\right)=&\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}ty_1'\left(t\right)dt+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}y_1''\left(t\right)dt\left|\mathbf{M}_{-1}\right|\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}ty_1\left(t\right)dt\left|\mathbf{M}_{-1}\right|+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)f\left(t\right)dt\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{N-1}\left[\int_{-1}^{1}y_1\left(s\right)ds\right]dt\left|\mathbf{M}_{-1}\right|,\end{aligned}\tag{34}$$

$$\begin{aligned}Ty_2\left(x\right)=&\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}ty_2'\left(t\right)dt+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}y_2''\left(t\right)dt\left|\mathbf{M}_{-1}\right|\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}ty_2\left(t\right)dt\left|\mathbf{M}_{-1}\right|+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)f\left(t\right)dt\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\left[\int_{-1}^{1}y_2\left(s\right)ds\right]dt\left|\mathbf{M}_{-1}\right|,\end{aligned}\tag{35}$$

$$\begin{aligned}\left|Ty_1-Ty_2\right|\leq&\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\left|t\right|\left|y_1'\left(t\right)-y_2'\left(t\right)\right|dt\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\left|y_1''\left(t\right)-y_2''\left(t\right)\right|dt\left|\mathbf{M}_{-1}\right|\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\left|t\right|\left|y_1\left(t\right)-y_2\left(t\right)\right|dt\left|\mathbf{M}_{-1}\right|\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\left[\int_{-1}^{1}\left|y_1\left(s\right)-y_2\left(s\right)\right|ds\right]dt\left|\mathbf{M}_{-1}\right|.\end{aligned}\tag{36}$$

Using H_4 $\left|y_1^{(m)}\left(t\right)-y_2^{(m)}\left(t\right)\right|\leq L_m\left|y_1-y_2\right|$ gives

$$\left|Ty_1-Ty_2\right|$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(3)} \int_0^x (x-t)^2 |t| L_m |y_1 - y_2| \, dt \\
&\quad + \frac{1}{\Gamma(3)} \int_0^x (x-t)^2 L_n |y_1 - y_2| \, dt \, |\mathbf{M}_{-1}| \\
&\quad + \frac{1}{\Gamma(3)} \int_0^x (x-t)^2 |t| \, |y_1(t) - y_2(t)| \, dt \, |\mathbf{M}_{-1}| \\
&\quad + \frac{1}{\Gamma(3)} \int_0^x (x-t)^2 \left[\int_{-1}^1 |y_1(s) - y_2(s)| \, ds \right] dt \, |\mathbf{M}_{-1}| . \qquad (37)
\end{aligned}
$$

Taking supremum of (37) gives

$$
\begin{aligned}
&\sup_{x \in J} |Ty_1 - Ty_2| \\
&\leq \frac{1}{\Gamma(3)} \int_0^x (x-t)^2 \sup_{x \in J} |t| \sup_{x \in J} |y_1'(t) - y_2'(t)| \, dt \\
&\quad + \frac{1}{\Gamma(3)} \int_0^x (x-t)^2 \sup_{x \in J} |y_1''(t) - y_2''(t)| \, dt \, |\mathbf{M}_{-1}| \\
&\quad + \frac{1}{\Gamma(3)} \int_0^x (x-t)^2 \sup_{x \in J} |t| \sup_{x \in J} |y_1(t) - y_2(t)| \, dt \, |\mathbf{M}_{-1}| \\
&\quad + \frac{1}{\Gamma(3)} \int_0^x (x-t)^2 \left[\int_{-1}^1 \sup_{x \in J} |y_1(s) - y_2(s)| \, ds \right] dt \, |\mathbf{M}_{-1}| , \qquad (38)
\end{aligned}
$$

$$
\begin{aligned}
\|Ty_1 - Ty_2\|_\infty \leq & \frac{L_n}{\Gamma(4)} \|y_1 - y_2\|_\infty + \frac{L_m}{\Gamma(4)} \|y_1 - y_2\|_\infty |\mathbf{M}_{-1}| \\
& + \frac{1}{\Gamma(4)} \|y_1 - y_2\|_\infty |\mathbf{M}_{-1}| + \frac{K^*}{\Gamma(4)} \|y_1 - y_2\|_\infty |\mathbf{M}_{-1}| , \qquad (39)
\end{aligned}
$$

where

$$
K^* = \int_{-1}^1 |K(s,t) \, ds| = 2.
$$

Since $\mathrm{K}^* = 2$,

$$
\|Ty_1 - Ty_2\|_\infty \leq \left[\frac{1}{\Gamma 4} L_1 + L_2 |\mathbf{M}_{-1}| + 3 |\mathbf{M}_{-1}| \right] \|y_1 - y_2\|_\infty ,
$$

$$
\text{for } q - contraction \quad \frac{1}{\Gamma 4} L_1 + L_2 |\mathbf{M}_{-1}| + 3 |\mathbf{M}_{-1}| < 1. \qquad (40)
$$

To show the convergence of solution for Problem 1, we have

$$
Ty_N(x) = H_N(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t y_N'(t) \, dt
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} y_N''(t)\,dt\,|\mathbf{M}_{-1}| \\
&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} t y_N(t)\,dt\,|\mathbf{M}_{-1}| \\
&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} \left[\int_{-1}^1 y_N(s)\,ds\right] dt\,|\mathbf{M}_{-1}|, \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
Ty(x) = H(x) &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} t y'(t)\,dt \\
&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} y''(t)\,dt\,|\mathbf{M}_{-1}| \\
&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} t y(t)\,dt\,|\mathbf{M}_{-1}| \\
&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} \left[\int_{-1}^1 y(s)\,ds\right] dt\,|\mathbf{M}_{-1}|,
\end{aligned}
$$

$$
\begin{aligned}
|Ty_N(x) - Ty(x)| \leq |H_N(x) - H(x)| &+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |t|\,|y_N'(t) - y'(t)|\,dt \\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |y_N''(t) - y''(t)|\,dt\,|\mathbf{M}_{-1}| \\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |t|\,|y_N(t) - y(t)|\,dt\,|\mathbf{M}_{-1}| \\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 \left[\int_{-1}^1 |y_N(s) - y(s)|\,ds\right] dt\,|\mathbf{M}_{-1}| \qquad (42)
\end{aligned}
$$

Using (H_4) $\left|y_1^{(m)} - y_2^{(m)}\right| \leq L_m\,|y_1 - y_2|$,

$$
\begin{aligned}
|Ty_N(x) - Ty(x)| \leq |H_N(x) - H(x)| + \frac{L_1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha} |t|\,|y_N - y|\,dt& \\
+\frac{L_2}{\Gamma(3)}\int_0^x (x-t)^2 |y_N - y|\,dt\,|\mathbf{M}_{-1}|& \\
+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |t|\,|y_N - y|\,dt\,|\mathbf{M}_{-1}|& \\
+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 \left[\int_{-1}^1 |y_N - y|\,ds\right] dt\,|\mathbf{M}_{-1}|,& \qquad (43)
\end{aligned}
$$

$$\sup_{x\in J} |Ty_N(x) - Ty(x)|$$

$$\leq \sup_{x\in J}\left|H_N(x)-H(x)\right|+\frac{L_1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha}\sup_{x\in J}|t|\sup_{x\in J}|y_N-y|\,dt$$
$$+\frac{L_2}{\Gamma(3)}\int_0^x (x-t)^2\sup_{x\in J}|y_N-y|\,dt\,|\mathbf{M}_{-1}|$$
$$+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2\sup_{x\in J}|t|\sup_{x\in J}|y_N-y|\,dt\,|\mathbf{M}_{-1}|$$
$$+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2\left[\int_{-1}^1\sup_{x\in J}|y_N-y|\,ds\right]dt\,|\mathbf{M}_{-1}|, \tag{44}$$

$$\|y_N-y\|_\infty\leq\|H_N-H\|_\infty+\frac{L_1}{\Gamma(4)}\|y_N-y\|_\infty$$
$$+\frac{L_2}{\Gamma(4)}\|y_N-y\|_\infty|\mathbf{M}_{-1}|+\frac{1}{\Gamma(4)}\|y_N-y\|_\infty|\mathbf{M}_{-1}|$$
$$+\frac{K^*}{\Gamma(4)}\|y_N-y\|_\infty|\mathbf{M}_{-1}|, \tag{45}$$

$$K^*=\int_{-1}^1|K(s,t)\,ds|=2,$$

$$\left[1-\frac{L_1}{\Gamma 4}-\frac{L_2}{\Gamma 4}|\mathbf{M}_{-1}|-\frac{3|\mathbf{M}_{-1}|}{\Gamma 4}\right]\|y_N-y\|_\infty\leq\|H_N(x)-H(x)\|,$$

$$\|y_N-y\|_\infty\leq\frac{\Gamma 4\,\|H_N(x)-H(x)\|_\infty}{\Gamma 4-L_1-L_2|\mathbf{M}_{-1}|-3|\mathbf{M}_{-1}|},$$

$$\|y_N-y\|_\infty\leq\frac{\Gamma(4)\|H_N(x)-H(x)\|_\infty}{\Gamma(4)-\Gamma(4)q}\leq\frac{\Gamma(4)\|H_N(x)-H(x)\|_\infty}{\Gamma(4)(1-q)}$$
$$\leq\frac{\|H_N(x)-H(x)\|_\infty}{(1-q)}, \tag{46}$$

since $q<1$, $\|y_N-y\|_\infty$ exists. Furthermore since H is not affected by the approximate solution, this implies that $H_N-H=0$. Hence,

$$\|y_N-y\|_\infty\leq 0, \text{ which shows that it converges.}$$

Solving Problem 1 numerically gives the following solution.

Solution 1. Comparing with (1), $l=r=0$, $b=1$, $a=-1$, we have

$$P_3(x)=1,\qquad P_2(x)=0,\qquad P_1(x)=-x,$$
$$Q_2(x)=1,\qquad Q_1(x)=0,\qquad Q_0(x)=x,$$

$$g(x) = -(x+1)(\sin(x-1)+\cos(x)) - \cos 2 + 1,$$
$$\lambda = 1, \qquad k(x,t) = 1.$$

Using $n = 3$,

$$y^{(3)}(x) = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} x^{n-m}\mathbf{DA}, \quad n = 0(1)N, \quad m = 3,$$

$$-xy'(x) = -\frac{\Gamma(n+1)}{\Gamma(n-m+1)} x^{n-m+1}\mathbf{DA}, \quad n = 0(1)N, \quad m = 1.$$

Using Lemma 1,

$$y^{(2)}(x-1) = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} x^{n-m} |\mathbf{M}_{-1}| \mathbf{DA}, \quad n = 0(1)N, \quad m = 2,$$

$$-xy(x-1) = \frac{\Gamma(n+1)}{\Gamma(n-m+1)} x^{n-m+1} |\mathbf{M}_{-1}| \mathbf{DA}, \quad n = 0(1)N, \quad m = 1.$$

Using Lemma 4, then

$$\begin{aligned}\int_{-1}^{1} y(\tau-1)\,dt &= \int_{-1}^{1} y(t)\,dt\,|\mathbf{M}_{-1}|\,\mathbf{DA} \\ &= \int_{-1}^{1} (x-t)^0\, t^n dt\,|\mathbf{M}_{-1}|\,\mathbf{DA},\end{aligned}$$

$$\begin{aligned}(x-t)^0 &= \left|\frac{\Gamma(n+1)}{\Gamma(n+2)} x^{n+1}\right|_{-1}^{1} |\mathbf{M}_{-1}|\,\mathbf{DA} \\ &= \frac{\Gamma(n+1)}{\Gamma(n+2)} - \frac{\Gamma(n+1)}{\Gamma(n+2)}(-1)^{n+1}|\mathbf{M}_{-1}|\,\mathbf{DA}.\end{aligned}$$

Taking $N = 3$ for illustration,

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T, \ \mathbf{X}(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix}, \ \mathbf{X}(x_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} \\ 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Using Lemma 2 with $m = 3$

$$y'''(x) = \begin{bmatrix} 0 & 0 & 0 & 6 \end{bmatrix} \mathbf{DA}, \quad y'''(x_i) = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 \end{bmatrix} \mathbf{DA},$$

$$-xy'(x) = \begin{bmatrix} 0 & -x & -2x^2 & -3x^3 \end{bmatrix} \mathbf{DA}, \quad xy'(x_i) = \begin{bmatrix} 0 & 1 & -2 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{9} & \frac{1}{9} \\ 0 & -\frac{1}{3} & -\frac{2}{9} & \frac{-1}{9} \\ 0 & -1 & -2 & -3 \end{bmatrix} \mathbf{DA},$$

$$y''(x-1) = \begin{bmatrix} 0 & 0 & 2 & 6x \end{bmatrix} \mathbf{DA}, \quad y''(x_i-1) = \begin{bmatrix} 0 & 0 & 2 & -12 \\ 0 & 0 & 2 & -8 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 2 & 0 \end{bmatrix} \mathbf{DA},$$

$$-xy(x-1) = \begin{bmatrix} -x & -x^2 & -x^3 & -x^4 \end{bmatrix} \mathbf{DA}, \quad x_i y(x_i-1) = \begin{bmatrix} 1 & -2 & 4 & -8 \\ \frac{1}{3} & \frac{-4}{9} & \frac{16}{27} & \frac{-64}{81} \\ \frac{-1}{3} & \frac{2}{9} & \frac{-4}{27} & \frac{8}{81} \\ -1 & 0 & 0 & 0 \end{bmatrix} \mathbf{DA},$$

$$\int_{-1}^{1} y(t-1)\,dt = \begin{bmatrix} 2 & -2 & \frac{8}{3} & 4 \\ 2 & -2 & \frac{8}{3} & -4 \\ 2 & -2 & \frac{8}{3} & -4 \\ 2 & -2 & \frac{8}{3} & -4 \end{bmatrix} \mathbf{DA}.$$

Then

$$\mathbf{W} = y_1^3(x_i) - x_i y^1(x_i) + y''(x_i-1) - x_i y(x_i-1) - \int_{-1}^{1} y(t-1)\,dt,$$

$$\mathbf{W} = \begin{bmatrix} -1 & 1 & \frac{4}{3} & -7 \\ \frac{-5}{3} & \frac{17}{9} & \frac{-8}{27} & \frac{107}{81} \\ \frac{-7}{3} & \frac{17}{9} & \frac{-28}{27} & \frac{485}{81} \\ -3 & 1 & \frac{-8}{3} & 7 \end{bmatrix}.$$

Applying the initial conditions,

$$\mathbf{WW}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\-3&1&\frac{-8}{3}&7\end{bmatrix},$$

$$\mathbf{GG}=\begin{bmatrix}0 & 1 & 0 & 0.33554\end{bmatrix}^T.$$

For solving

$$\mathbf{F}(A)=\mathbf{WWA}-\mathbf{GG}=\mathbf{0},$$

using Newton Raphson's method gives

$$\mathbf{A}=\begin{bmatrix}0 & 1 & 0 & \frac{-1060}{111167}\end{bmatrix}.$$

Substituting into the approximate solution gives

$$y_3(x)=-0.09423x^3+x.$$

Solving at $N=5,7,10$, and 12, we have

$$y_5=-0.01145575875x^5+0.1569475319x^4-0.2923832453x^3+x,$$

$$\begin{aligned}y_7=&\,0.00007972949456x^7-0.001012399319x^6+0.007758248986x^5\\&+0.01453036551x^4-0.18476635x^3+x,\end{aligned}$$

$$\begin{aligned}y_{10}=&-0.000001697045863x^{10}+0.00001105299936x^9\\&-0.000004369833648x^8-0.0004760258412x^7+0.0009938127921x^6\\&+0.008908454521x^5-0.01420598521x^4-0.1491471867x^3+x,\end{aligned}$$

$$\begin{aligned}y_{12}=&\,0.0000000005662168848x^{12}-0.00000002473722641x^{11}\\&-0.00000003837633285x^{10}+0.000002937637513x^9\\&-0.00000009352857036x^8-0.000204436741x^7+0.00002155603046x^6\\&+0.008345804444x^5-0.0003081052941x^4-0.1662866423x^3+x,\end{aligned}$$

$$y_{15}=-7.406875094e-13x^{15}+1.298432652e-13x^{14}$$

Table 1: Results of Problem 1

x_i	*exact*	$N=7$	$N=10$	$N=12$	$N=15$
-0.2	-0.19866933	-0.19874343	-0.19858323	-0.19867251	-0.1986693
-0.4	-0.38941834	-0.39009317	-0.38863325	-0.38944734	-0.38941803
-0.6	-0.56464247	-0.56719006	-0.56167599	-0.56475205	-0.56464128
-0.8	-0.71735609	-0.72400265	-0.70961228	-0.71764214	-0.71735297
-1.0	-0.84147098	-0.8555408	-0.82507367	-0.84207669	-0.84146437

Table 2: Absolute error for Problem 1

x_i	ERR_7 [27]	ERR_7	ERR_{10}	ERR_{12}	ERR_{15}
-0.2	2.37×10^{-5}	7.41×10^{-5}	8.61×10^{-5}	3.18×10^{-6}	3.0×10^{-8}
-0.4	1.15×10^{-4}	6.75×10^{-4}	7.85×10^{-4}	2.90×10^{-5}	3.10×10^{-7}
-0.6	8.13×10^{-4}	2.55×10^{-3}	2.97×10^{-3}	1.10×10^{-4}	1.19×10^{-6}
-0.8	2.12×10^{-3}	6.67×10^{-3}	7.74×10^{-3}	2.86×10^{-4}	3.12×10^{-6}
-1.0	4.82×10^{-3}	1.41×10^{-2}	1.64×10^{-2}	6.06×10^{-4}	6.61×10^{-6}

$$
\begin{aligned}
&+0.0000000001564193771x^{13}+1.326069211e-11x^{12}\\
&-0.00000002504317927x^{11}-0.0000000009095730742x^{10}\\
&+0.000002760036704x^{9}-0.000000002211661503x^{8}\\
&-0.000198555331x^{7}+0.0000005103898555x^{6}\\
&+0.008333628614x^{5}-0.000007295143911x^{4}\\
&-0.1666576686x^{3}+x.
\end{aligned}
$$

Problem 2. [27] Let us consider the integro-differential difference equation with variable coefficient

$$y''(x)+xy(x)-xy(x-1)+y'(x-1)+y(x-1)=e^{-x}+e+\int_{-1}^{0}ty(t-1)$$

subject to the initial condition

$$y(0)=1,\qquad y'(0)=-1$$

with the exact solution is given by $y(x)=e^{-x}$.

To show q-contraction for Problem 2 gives

$$
\begin{aligned}
Ty\left(x\right) = & \frac{1}{\Gamma\left(\alpha\right)}\int_0^x \left(x-t\right)^{\alpha-1} ty'\left(t\right)dt + \frac{1}{\Gamma\left(\alpha\right)}\int_0^x \left(x-t\right)^{N-1} y\left(t\right)dt \\
& -\frac{1}{\Gamma\left(\alpha\right)}\int_0^x \left(x-t\right)^{\alpha-1} y'\left(t\right)dt\left|\mathbf{M}_{-1}\right| - \frac{1}{\Gamma\left(\alpha\right)}\int_0^x \left(x-t\right) y\left(t\right)dt \\
& +\frac{1}{\Gamma\left(\alpha\right)}\int_0^x \left(x-t\right) f\left(t\right)dt \\
& +\frac{1}{\Gamma\left(\alpha\right)}\int_0^x \left(x-t\right)^{N-1}\left[\int_{-1}^0 ty\left(s\right)ds\right]dt\left|\mathbf{M}_{-1}\right|, \qquad (47)
\end{aligned}
$$

$$
\begin{aligned}
Ty_1\left(x\right) = & \frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right) ty_1'\left(t\right)dt + \frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right) y_1\left(t\right)dt \\
& -\frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right) y_1'\left(t\right)dt\left|\mathbf{M}_{-1}\right| - \frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right) y_1\left(t\right)dt \\
& +\frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right) f\left(t\right)dt \\
& +\frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right)^{N-1}\left[\int_{-1}^0 ty_1\left(s\right)ds\right]dt\left|\mathbf{M}_{-1}\right|, \qquad (48)
\end{aligned}
$$

$$
\begin{aligned}
Ty_2\left(x\right) = & \frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right) ty_2'\left(t\right)dt + \frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right) y_2\left(t\right)dt \\
& -\frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right) y_2'\left(t\right)dt\left|\mathbf{M}_{-1}\right| - \frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right) y_2\left(t\right)dt \\
& +\frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right) f\left(t\right)dt \\
& +\frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right)^{N-1}\left[\int_{-1}^0 ty_2\left(s\right)ds\right]dt\left|\mathbf{M}_{-1}\right|, \qquad (49)
\end{aligned}
$$

$$
\begin{aligned}
\left|Ty_1 - Ty_2\right| \leq & \frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right)\left|t\right|\left|y_1'\left(t\right) - y_2'\left(t\right)\right|dt \\
& +\frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right)\left|y_1\left(t\right) - y_2\left(t\right)\right|dt \\
& +\frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right)\left|y_1'\left(t\right) - y_2'\left(t\right)\right|dt\left|\mathbf{M}_{-1}\right| \\
& +\frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right)\left|y_1\left(t\right) - y_2\left(t\right)\right|dt \\
& +\frac{1}{\Gamma\left(2\right)}\int_0^x \left(x-t\right)\left[\int_{-1}^0 \left|t\right|\left|y_1\left(s\right) - y_2\left(s\right)\right|ds\right]dt\left|\mathbf{M}_{-1}\right| \qquad (50)
\end{aligned}
$$

Using (H_4) $\left|y_1^{(m)}(t)-y_2^{(m)}(t)\right|\le L_m|y_1-y_2|$ gives

$$\begin{aligned}|Ty_1-Ty_2|\le{}&\frac{L_1}{\Gamma(2)}\int_0^x(x-t)|t||y_1-y_2|\,dt+\frac{1}{\Gamma(2)}\int_0^x(x-t)|y_1-y_2|\,dt\\&+\frac{L_2}{\Gamma(2)}\int_0^x(x-t)|y_1-y_2|\,dt\,|\mathbf{M}_{-1}|\\&+\frac{1}{\Gamma(2)}\int_0^x(x-t)|y_1-y_2|\,dt\\&+\frac{1}{\Gamma(2)}\int_0^x(x-t)^{N-1}\left[\int_{-1}^0|t||y_1-y_2|\,ds\right]dt\,|\mathbf{M}_{-1}|.\end{aligned}\tag{51}$$

Taking supremum of (51) gives

$$\begin{aligned}&\sup_{x\in J}|Ty_1-Ty_2|\\&\le\frac{L_1}{\Gamma(2)}\int_0^x(x-t)\sup_{x\in J}|t|\sup_{x\in J}|y_1-y_2|\,dt\\&\quad-\frac{L_2}{\Gamma(2)}\int_0^x(x-t)\sup_{x\in J}|y_1-y_2|\,dt\,|\mathbf{M}_{-1}|\\&\quad+\frac{1}{\Gamma(2)}\int_0^x(x-t)\left[\int_{-1}^0\sup_{x\in J}|t|\sup_{x\in J}|y_1-y_2|\,ds\right]dt\,|\mathbf{M}_{-1}|,\end{aligned}\tag{52}$$

$$\begin{aligned}\|Ty_1-Ty_2\|_\infty\le{}&\frac{L_1}{\Gamma(3)}\|y_1-y_2\|_\infty-\frac{L_2}{\Gamma(3)}\|y_1-y_2\|_\infty|\mathbf{M}_{-1}|\\&+\frac{K^*}{\Gamma(3)}\|y_1-y_2\|_\infty|\mathbf{M}_{-1}|,\ K^*=\int_{-1}^0|K(s,t)|\,ds=1,\end{aligned}\tag{53}$$

$$\begin{aligned}\|Ty_1-Ty_2\|_\infty&\le\frac{1}{\Gamma3}\left[L_1-L_2|\mathbf{M}_{-1}|-K^*|\mathbf{M}_{-1}|\right]\|y_1-y_2\|_\infty,\text{since }\mathrm{K}^*=1\\&\le\frac{1}{\Gamma3}\left[L_1-L_2|\mathbf{M}_{-1}|-|\mathbf{M}_{-1}|\right]\|y_1-y_2\|_\infty\\&\text{for } q-\text{contraction }\frac{1}{\Gamma3}\left[L_1-L_2|\mathbf{M}_{-1}|-|\mathbf{M}_{-1}|\right]<1.\end{aligned}\tag{54}$$

To show the convergence of solution for Problem 2, we have

$$\begin{aligned}Ty_N(x)={}&H_N(x)+\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}ty_N'(t)\,dt\\&+\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}y_N(t)\,dt\\&-\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}y_N'(t)\,dt\,|\mathbf{M}_{-1}|\end{aligned}$$

$$-\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} y_N(t)\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[\int_{-1}^0 t y_N(s)\,ds\right]dt\,|\mathbf{M}_{-1}|, \qquad (55)$$

$$Ty(x) = H(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} t y'(t)\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} y(t)\,dt$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} y'(t)\,dt\,|\mathbf{M}_{-1}|$$
$$-\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} y(t)\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[\int_{-1}^0 t y(s)\,ds\right]dt\,|\mathbf{M}_{-1}|, \qquad (56)$$

$$|Ty_N(x) - Ty(x)| = |H_N(x) - H(x)| + \frac{1}{\Gamma(2)}\int_0^x (x-t)\,|t|\,|y_N'(t) - y'(t)|\,dt$$
$$+\frac{1}{\Gamma(2)}\int_0^x (x-t)\,|y_N(t) - y(t)|\,dt$$
$$+\frac{1}{\Gamma(2)}\int_0^x (x-t)\,|y_N'(t) - y'(t)|\,dt\,|\mathbf{M}_{-1}|$$
$$+\frac{1}{\Gamma(2)}\int_0^x (x-t)\,|y_N(t) - y(t)|\,dt$$
$$+\frac{1}{\Gamma(2)}\int_0^x (x-t)\left[\int_{-1}^0 |t|\,|y_N(s) - y(s)|\,ds\right]dt\,|\mathbf{M}_{-1}| \qquad (57)$$

Using $\mathrm{H}_4 = \left|y_1^{(m)} - y_2^{(m)}\right| \le L_m\,|y_1 - y_2|$, we have

$$|Ty_N(x) - Ty(x)| = |H_N(x) - H(x)| + \frac{L_n}{\Gamma(2)}\int_0^x (x-t)\,|t|\,|y_N - y|\,dt$$
$$+\frac{1}{\Gamma(2)}\int_0^x (x-t)\,|y_N - y|\,dt$$
$$+\frac{L_m}{\Gamma(2)}\int_0^x (x-t)\,|y_N - y|\,dt\,|\mathbf{M}_{-1}|$$
$$+\frac{1}{\Gamma(2)}\int_0^x (x-t)\,|y_N - y|\,dt$$
$$+\frac{1}{\Gamma(2)}\int_0^x (x-t)\left[\int_{-1}^0 |t|\,|y_N - y|\,ds\right]dt\,|\mathbf{M}_{-1}|\,. \qquad (58)$$

Taking supremum of both sides gives

$$\begin{aligned}\sup_{x\in J}\left|Ty_N(x)-Ty(x)\right| \leq & \sup_{x\in J}\left|H_N(x)-H(x)\right| \\ &+\frac{L_n}{\Gamma(2)}\int_0^x (x-t)\sup_{x\in J}|t|\sup_{x\in J}|y_N-y|\,dt \\ &+\frac{1}{\Gamma(2)}\int_0^x (x-t)\sup_{x\in J}|y_N-y|\,dt \\ &+\frac{L_m}{\Gamma(2)}\int_0^x (x-t)\sup_{x\in J}|y_N-y|\,dt\,|\mathbf{M}_{-1}| \\ &+\frac{1}{\Gamma(2)}\int_0^x (x-t)\sup_{x\in J}|y_N-y|\,dt \\ &+\frac{1}{\Gamma(2)}\int_0^x (x-t)\left[\int_{-1}^0 \sup_{x\in J}|t|\sup_{x\in J}|y_N-y|\,ds\right]dt\,|\mathbf{M}_{-1}|,\end{aligned} \tag{59}$$

$$\begin{aligned}\|y_N-y\|_\infty \leq & \|H_N-H\|_\infty+\frac{L}{\Gamma(3)}\|y_N-y\|_\infty+\frac{1}{\Gamma(3)}\|y_N-y\|_\infty \\ &+\frac{L}{\Gamma(3)}\|y_N-y\|_\infty|\mathbf{M}_{-1}|+\frac{1}{\Gamma(3)}\|y_N-y\|_\infty \\ &+\frac{K^*}{\Gamma(2)}\|y_N-y\|_\infty|\mathbf{M}_{-1}|,\end{aligned} \tag{60}$$

$$K^*=\int_{-1}^0 |K(s,t)|\,ds=1,$$

$$(1-\Gamma(3)-L_1-L_2|\mathbf{M}_{-1}|-|\mathbf{M}_{-1}|)\|y_N-y\|_\infty \leq \|H_N-H\|_\infty, \tag{61}$$

$$\frac{\Gamma(3)\|H_N-H\|_\infty}{\Gamma(3)-L_1-L_2|\mathbf{M}_{-1}|-|\mathbf{M}_{-1}|} \leq \frac{\Gamma(3)\|H_N-H\|_\infty}{\Gamma(3)-\Gamma(3)q} \leq \frac{\|H_N-H\|_\infty}{1-q}. \tag{62}$$

Since $q<1$, $\|y_N-y\|_\infty$ exists. Furthermore since H is not affected by the approximate solution, this implies that $H_N-H=0$. Hence,

$$\|y_N-y\|_\infty \leq 0, \text{ which shows that it converges.}$$

Solving Problem 2 numerically gives as follows.

Solution 2. Comparing with (1),

$$P_2(x)=1,\quad P_1(x)=1,\quad P_0(x)=x,\quad Q_1(x)=1,\quad Q_0(x)=x-1,$$
$$g(x)=e^{-x}+e,\quad L=0,\lambda=1,\quad k(x,t)=t.$$

Hence,

$$y''(x)=\frac{\Gamma(n+1)}{\Gamma(n-m+1)}x^{n-m}\mathbf{DA},\quad m=2,\ n=0\,(1)\,N,$$

$$xy(x)=\frac{\Gamma(n+1)}{\Gamma(n-m+1)}x^{n-m+1}\mathbf{DA},\quad n=0\,(1)\,N,$$

$$xy(x-1)=\frac{\Gamma(n+1)}{\Gamma(n-m+1)}x^{n-m}\mathbf{M}_{-1}\mathbf{DA},\quad m=0,\ n=0\,(1)\,N,$$

$$y'(x-1)=\frac{\Gamma(n+1)}{\Gamma(n-m+1)}x^{n-m}\mathbf{M}_{-1}\mathbf{DA},\quad m=1,\ n=0\,(1)\,N,$$

$$y(x-1)=\frac{\Gamma(n+1)}{\Gamma(n-m+1)}x^{n-m}\mathbf{M}_{-1}\mathbf{DA},\quad m=0,\ n=0\,(1)\,N$$

$$\int_{-1}^{0}ty(t-1)\,dt=\int_{-1}^{0}t\,(X(t))\,dt\mathbf{M}_{-1}\mathbf{DA},$$

$$\int_{-1}^{0}t^{n+1}dt\mathbf{M}_{-1}\mathbf{DA}=\left[\frac{t^{n+2}}{n+2}\right]_{-1}^{0}\mathbf{M}_{-1}\mathbf{DA}=\frac{-(-1)^{n+2}}{n+2}\mathbf{M}_{-1}\mathbf{DA}.$$

Substituting into the approximate solution gives

$$y_3(x)=-0.136348347x^3+0.589442798x^2-x+1,$$

$$\begin{aligned}y_5(x)=&-0.0134885103x^5-0.0135768498x^4-0.0370560278x^3\\&+0.556269874x^2-x+1,\end{aligned}$$

$$\begin{aligned}y_7(x)=&-0.000626560072x^7+0.00170800372x^6-0.0066859407x^5\\&+0.0477358013x^4-0.18169637x^3+0.492416409x^2-x+1,\end{aligned}$$

$$\begin{aligned}y_{10}(x)=&-0.000000191511561x^{10}-0.00000417255283x^9+0.0000275111315x^8\\&-0.000173222035x^7+0.00137098295x^6-0.00841936174x^5\\&+0.0413648129x^4-0.165914444x^3+0.500378553x^2-1x+1,\end{aligned}$$

$$\begin{aligned}y_{12}(x)=&\,0.00000000418649794x^{12}-0.0000000216397711x^{11}+0.000000244349171x^{10}\\&-0.0000282966002x^9+0.0000249459268x^8-0.000197246671x^7\\&+0.00138804522x^6-0.00833728371x^5+0.0416528216x^4-0.166632138x^3\end{aligned}$$

Table 3: Results of Problem 2

x_i	*exact*	$N=5$	$N=7$	$N=10$	$N=12$	$N=15$
-0.2	1.2214028	1.2225298	1.2212289	1.2214114	1.2214032	1.2214027
-0.4	1.4918247	1.4911653	1.4917137	1.4918302	1.4918249	1.4918247
-0.6	1.8221188	1.8075506	1.82332	1.8220587	1.822116	1.8221189
-0.8	2.2255409	2.1738442	2.2304976	2.2252932	2.2255296	2.2255412
-1.0	2.7182818	2.5932376	2.7308691	2.7176529	2.718253	2.7182825

Table 4: Absolute error for Problem 2

x_i	ERR_{13}[27]	$N=3$	$N=7$	$N=10$	$N=12$
-0.2	2.27×10^{-5}	3.266×10^{-3}	1.739×10^{-4}	8.600×10^{-6}	4.000×10^{-7}
-0.4	1.43×10^{-5}	1.121×10^{-2}	1.110×10^{-4}	5.500×10^{-6}	2.000×10^{-7}
-0.6	1.57×10^{-4}	1.953×10^{-2}	1.201×10^{-3}	6.010×10^{-5}	2.800×10^{-6}
-0.8	6.49×10^{-4}	2.151×10^{-2}	4.957×10^{-3}	2.477×10^{-4}	1.130×10^{-5}
-1.0	1.65×10^{-3}	7.509×10^{-3}	1.259×10^{-2}	6.289×10^{-4}	2.880×10^{-5}

$$+0.500017372x^2 - x + 1,$$

$$\begin{aligned} y_{15}(x) = {} & -5.30985676e^{-13}x^{15} + 1.49454482e^{-11}x^{14} - 1.60904091e^{-10}x^{13} \\ & +0.00000000202630179x^{12} - 0.0000000251416995x^{11} + 0.000000276291033x^{10} \\ & -0.00000275410796x^{9} + 0.0000247983934x^{8} - 0.000198437964x^{7} \\ & +0.00138890723x^{6} - 0.00833324776x^{5} + 0.0416669666x^{4} - 0.166667415x^{3} \\ & +0.499999624x^{2} - x + 1. \end{aligned}$$

Problem 3. [11] Consider the third-order nonlinear Fredholm integro-differential difference equation

$$u'''(x) + \frac{1}{2}u'' + xu'(x) + 2u'(x-1) + \frac{1}{2}xu(x) + u(x-1) = e + \int_{-1}^{0} tu^2(t-1)\,dt$$

with the following initial condition

$$u(0) = 1, u'(0) = -\frac{1}{2}, u''(0) = \frac{1}{4},$$

the exact solution

$$u\left(x\right)=e^{-\frac{x}{2}}.$$

To show q-contraction for Problem 3, we have

$$\begin{aligned}Tu\left(x\right)=&\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)^{\alpha-1}\frac{1}{2}u''\left(t\right)dt+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)^{\alpha-1}tu'\left(t\right)dt\\&+\frac{2}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)^{\alpha-1}u'\left(t\right)dt\mathbf{M}_{-1}+\frac{1}{2\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)^{\alpha-1}tu\left(t\right)dt\\&+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)^{\alpha-1}u\left(t\right)dt\mathbf{M}_{-1}+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)f\left(t\right)dt\\&+\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{x}\left(x-t\right)^{\alpha-1}\left[\int_{-1}^{0}tu^{2}\left(s\right)ds\right]dt\mathbf{M}_{-1},\end{aligned}\tag{63}$$

$$\begin{aligned}Tu_{1}\left(x\right)=&\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\frac{1}{2}u_{1}''\left(t\right)dt+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}tu_{1}'\left(t\right)dt\\&+\frac{2}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}u_{1}'\left(t\right)dt\mathbf{M}_{-1}+\frac{1}{2\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}tu_{1}\left(t\right)dt\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}u_{1}\left(t\right)dt\mathbf{M}_{-1}+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)f\left(t\right)dt\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\left[\int_{-1}^{0}tu_{1}^{2}\left(s\right)ds\right]dt\mathbf{M}_{-1},\end{aligned}\tag{64}$$

$$\begin{aligned}Tu_{2}\left(x\right)=&\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\frac{1}{2}u_{2}''\left(t\right)dt+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}tu_{2}'\left(t\right)dt\\&+\frac{2}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}u_{2}'\left(t\right)dt\mathbf{M}_{-1}+\frac{1}{2\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}tu_{2}\left(t\right)dt\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}u_{2}\left(t\right)dt\mathbf{M}_{-1}+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)f\left(t\right)dt\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\left[\int_{-1}^{0}tu_{2}^{2}\left(s\right)ds\right]dt\mathbf{M}_{-1},\end{aligned}\tag{65}$$

$$\begin{aligned}\left|Ty_{1}-Ty_{2}\right|\leq&\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\left|u_{1}''\left(t\right)-u_{2}''\left(t\right)\right|dt\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\left|t\right|\left|u_{1}'\left(t\right)-u_{2}'\left(t\right)\right|dt\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\left|u_{1}'\left(t\right)-u_{2}'\left(t\right)\right|dt\mathbf{M}_{-1}\\&+\frac{1}{\Gamma\left(3\right)}\int_{0}^{x}\left(x-t\right)^{2}\left|t\right|\left|u_{1}\left(t\right)-u_{2}\left(t\right)\right|dt\end{aligned}$$

$$+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |u_1(t)-u_2(t)|\, dt\mathbf{M}_{-1}$$
$$+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 \left[\int_{-1}^0 |t|\left|u_1^2(s)-u_2^2(s)\right| ds\right] dt\mathbf{M}_{-1}. \quad (66)$$

Using $H_4 = \left|y_1^{(m)}(t)-y_2^{(m)}(t)\right| \leq L_m |y_1-y_2|$, gives

$$\begin{aligned}|Tu_1-Tu_2| \leq &\frac{L_n}{\Gamma(3)}\int_0^x (x-t)^2 |u_1-u_2|\, dt + \frac{L_m}{\Gamma(3)}\int_0^x (x-t)^2 |t|\,|u_1-u_2|\, dt\\
&+\frac{L_m}{\Gamma(3)}\int_0^x (x-t)^2 |u_1-u_2|\, dt\mathbf{M}_{-1}\\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |t|\,|u_1-u_2|\, dt\\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |u_1-u_2|\, dt\mathbf{M}_{-1}\\
&+\frac{L^2}{\Gamma(3)}\int_0^x (x-t)^2 \left[\int_{-1}^0 |t|\,|u_1-u_2|\, ds\right] dt\mathbf{M}_{-1}. \quad (67)\end{aligned}$$

Taking supremum of (67) gives

$$\begin{aligned}&\sup_{x\in J}|Tu_1-Tu_2|\\
&\leq \frac{L_n}{\Gamma(3)}\int_0^x (x-t)^2 \sup_{x\in J}|u_1-u_2|\, dt\\
&\quad+\frac{L_m}{\Gamma(3)}\int_0^x (x-t)^2 \sup_{x\in J}|t|\sup_{x\in J}|u_1-u_2|\, dt\\
&\quad+\frac{L_m}{\Gamma(3)}\int_0^x (x-t)^2 \sup_{x\in J}|u_1-u_2|\, dt\,|\mathbf{M}_{-1}|\\
&\quad+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 \sup_{x\in J}|t|\sup_{x\in J}|u_1-u_2|\, dt\\
&\quad+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 \sup_{x\in J}|u_1-u_2|\, dt\,|\mathbf{M}_{-1}|\\
&\quad+\frac{L^2}{\Gamma(3)}\int_0^x (x-t)^2 \left[\int_{-1}^0 \sup_{x\in J}|t|\sup_{x\in J}|u_1-u_2|\, ds\right] dt\,|\mathbf{M}_{-1}|, \quad (68)\end{aligned}$$

$$\begin{aligned}&\|Ty_1-Ty_2\|_\infty\\
&\leq \frac{L_n}{\Gamma(3)}\|y_1-y_2\|_\infty\int_0^x (x-t)^2\, dt + \frac{L_m}{\Gamma(3)}\|y_1-y_2\|_\infty\int_0^x (x-t)^2\, dt\\
&\quad+\frac{L_m}{\Gamma(3)}\|y_1-y_2\|_\infty\int_0^x (x-t)^2\, dt\,|\mathbf{M}_{-1}|\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(3)}\|y_1-y_2\|_\infty\int_0^x (x-t)^2\,dt\\
&+\frac{1}{\Gamma(3)}\|y_1-y_2\|_\infty\int_0^x (x-t)^2\,dt\,|\mathbf{M}_{-1}|\\
&+\frac{L^2K^*}{\Gamma(3)}\|y_1-y_2\|_\infty\int_0^x (x-t)^2\,dt\,|\mathbf{M}_{-1}|,
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
\|Ty_1-Ty_2\|_\infty \le{}& \frac{L_1}{\Gamma(4)}\|y_1-y_2\|_\infty+\frac{L_2}{\Gamma(4)}\|y_1-y_2\|_\infty\\
&+\frac{L_2}{\Gamma(4)}\|y_1-y_2\|_\infty|\mathbf{M}_{-1}|+\frac{1}{\Gamma(4)}\|y_1-y_2\|_\infty\\
&+\frac{1}{\Gamma(4)}\|y_1-y_2\|_\infty|\mathbf{M}_{-1}|\\
&+\frac{L^2K^*}{\Gamma(4)}\|y_1-y_2\|_\infty|\mathbf{M}_{-1}|,
\end{aligned}
$$

$$
K^*=\int_{-1}^{0}|K(s,t)|\,ds=1,
$$

$$
\|Ty_1-Ty_2\|_\infty \le \frac{1}{\Gamma(4)}\left[L_1+L_2+L_2|\mathbf{M}_{-1}|+1+|\mathbf{M}_{-1}|+L^2|\mathbf{M}_{-1}|\right]\|y_1-y_2\|_\infty
$$

$$
\text{for q-contraction, } \frac{1}{\Gamma(4)}\left[L_1+L_2+L_2|\mathbf{M}_{-1}|+1+|\mathbf{M}_{-1}|+L^2|\mathbf{M}_{-1}|\right]<1. \tag{70}
$$

To show the convergence of solution for Problem 3, we have

$$
\begin{aligned}
y_N(x)={}&H_N(x)+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\frac{1}{2}u_N''(t)\,dt+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}tu_N'(t)\,dt\\
&+\frac{2}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}u_N'(t)\,dt\,|\mathbf{M}_{-1}|+\frac{1}{2\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}tu_N(t)\,dt\\
&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}u_N(t)\,dt\,|\mathbf{M}_{-1}|\\
&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[\int_{-1}^{0}tu_N^2(s)\,ds\right]dt\,|\mathbf{M}_{-1}|,
\end{aligned} \tag{71}
$$

$$
\begin{aligned}
y(x)={}&H(x)+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\frac{1}{2}u''(t)\,dt+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}tu'(t)\,dt\\
&+\frac{2}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}u'(t)\,dt\,|\mathbf{M}_{-1}|+\frac{1}{2\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}tu(t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} u(t)\,dt\,|\mathbf{M}_{-1}| \\
&+\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left[\int_{-1}^0 tu^2(s)\,ds\right]dt\,|\mathbf{M}_{-1}|, \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
|y_N(x)-y(x)| \le{}& |H_N(x)-H(x)| + \frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |u''_N(t)-u''(t)|\,dt \\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |t|\,|u'_N(t)-u'(t)|\,dt \\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |u'_N(t)-u'(t)|\,dt\,|\mathbf{M}_{-1}| \\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |t|\,|u_N(t)-u(t)|\,dt \\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |u_N(t)-u(t)|\,dt\,|\mathbf{M}_{-1}| \\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 \left[\int_{-1}^0 |t|\left|u_N^2(s)-u^2(s)\right|ds\right]dt\,|\mathbf{M}_{-1}|. \qquad (73)
\end{aligned}
$$

Applying hypothesis $(H_4)\left|y_1^{(m)}-y_2^{(m)}\right| \le L_m|y_1-y_2|$ for all $m \ge 0$, we have

$$
\begin{aligned}
&|y_N(x)-y(x)| \\
&\le |H_N(x)-H(x)| + \frac{L_n}{\Gamma(3)}\int_0^x (x-t)^2 |u_N-u|\,dt \\
&\quad+\frac{L_n}{\Gamma(3)}\int_0^x (x-t)^2 |t|\,|u_N-u|\,dt \\
&\quad+\frac{L_m}{\Gamma(3)}\int_0^x (x-t)^2 |u_N-u|\,dt\,|\mathbf{M}_{-1}| \\
&\quad+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |t|\,|u_N-u|\,dt \\
&\quad+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2 |u_N-u|\,dt\,|\mathbf{M}_{-1}| \\
&\quad+\frac{L^2}{\Gamma(3)}\int_0^x (x-t)^2 \left[\int_{-1}^0 |t|\,|u_N-u|\,ds\right]dt\,|\mathbf{M}_{-1}|. \qquad (74)
\end{aligned}
$$

Taking supremum of both sides gives

$$
\sup_{x\in J}|y_N(x)-y(x)|
$$

$$
\begin{aligned}
&\leq \sup_{x\in J}\left|H_N(x)-H(x)\right|+\frac{L_n}{\Gamma(3)}\int_0^x (x-t)^2\sup_{x\in J}\left|u_N-u\right|dt\\
&+\frac{L_m}{\Gamma(3)}\int_0^x (x-t)^2\sup_{x\in J}|t|\sup_{x\in J}\left|u_N-u\right|dt\\
&+\frac{L_m}{\Gamma(3)}\int_0^x (x-t)^2\sup_{x\in J}\left|u_N-u\right|dt\left|\mathbf{M}_{-1}\right|\\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2\sup_{x\in J}|t|\sup_{x\in J}\left|u_N-u\right|dt\\
&+\frac{1}{\Gamma(3)}\int_0^x (x-t)^2\sup_{x\in J}\left|u_N-u\right|dt\left|\mathbf{M}_{-1}\right|\\
&+\frac{L^2}{\Gamma(3)}\int_0^x (x-t)^2\left[\int_{-1}^0\sup_{x\in J}|t|\sup_{x\in J}\left|u_N-u\right|ds\right]dt\left|\mathbf{M}_{-1}\right|,
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
\left\|u_N-u\right\|_\infty \leq{}& \left\|H_N-H\right\|_\infty+\frac{L_1}{\Gamma(4)}\left\|u_N-u\right\|_\infty\\
&+\frac{L_2}{\Gamma(4)}\left\|u_N-u\right\|_\infty+\frac{L_2}{\Gamma(4)}\left\|u_N-u\right\|_\infty\mathbf{M}_{-1}\\
&+\frac{1}{\Gamma(4)}\left\|u_N-u\right\|_\infty+\frac{1}{\Gamma(4)}\left\|u_N-u\right\|_\infty\mathbf{M}_{-1}\\
&+\frac{L^2K^*}{\Gamma(4)}\left\|u_N-u\right\|_\infty\mathbf{M}_{-1},
\end{aligned}
\tag{76}
$$

$$
K^*=\int_{-1}^0\left|K(s,t)\right|ds=1,
$$

$$
\begin{bmatrix}
1-\frac{L_1}{\Gamma(3)}+\frac{L_2}{\Gamma(3)}+\frac{L_2}{\Gamma(3)}\left|\mathbf{M}_{-1}\right|\\
+\frac{1}{\Gamma(3)}+\frac{1}{\Gamma(3)}\left|\mathbf{M}_{-1}\right|+\frac{L^2}{\Gamma(3)}\left|\mathbf{M}_{-1}\right|
\end{bmatrix}\left\|u_N-u\right\|_\infty \leq \left\|H_N-H\right\|_\infty,
$$

$$
\begin{aligned}
\left\|u_N-u\right\|_\infty &\leq \frac{\Gamma 3\left\|H_N-H\right\|_\infty}{\Gamma 3-L_1-L_2-L_2\left|\mathbf{M}_{-1}\right|-1-\left|\mathbf{M}_{-1}\right|+L^2\left|\mathbf{M}_{-1}\right|}\\
&\leq \frac{\Gamma 3\left\|H_N-H\right\|_\infty}{\Gamma 3-\Gamma 3q}\leq \frac{\Gamma 3\left\|H_N-H\right\|_\infty}{\Gamma 3(1-q)}\leq \frac{\left\|H_N-H\right\|_\infty}{1-q}.
\end{aligned}
\tag{77}
$$

Since $q<1$, $\left\|u_N-u\right\|_\infty$ exists. Furthermore since H is not affected by the approximate solution, this implies that $H_N-H=0$. Hence,

$$
\left\|y_N-y\right\|_\infty \leq 0, \text{ which shows that it converges.}
$$

Solving Problem 3 numerically gives as follows.

Solution 3. Using $N=2$ for illustration

$$\mathbf{A} = [a_0 \quad a_1 \quad a_2]^T, \ \mathbf{X} = [1 \quad x \quad x^2],$$

$$\mathbf{M}_{-1}(\tau) = \begin{bmatrix} 1 & -\tau & \tau^2 \\ 0 & 1 & -2\tau \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{when } \tau = 1, \mathbf{M}_{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$u'''(x_i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \frac{1}{2}u''(x_i) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$x_i u'(x_i) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 2 \end{bmatrix}, x_i u(x_i) = \begin{bmatrix} 0 & 2 & -4 \\ 0 & 2 & -2 \\ 0 & 2 & 0 \end{bmatrix},$$

$$u(x_i - 1) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \ \int_1^0 t u^2(t-1)\, dt = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -\frac{1}{2} & \frac{1}{4} \\ 1 & 0 & 0 \end{bmatrix}.$$

Solving A using Newton's Raphson's method gives

$$A = \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{8} \end{bmatrix}.$$

Substituting into the approximate solution gives

$$y_2(x) = 1 - \frac{1}{2}x + \frac{1}{8}x^2,$$

which converges to the exact solution.

4 Discussion of results

In this section, we discussed the results obtained from the solved problems using our developed method and the advantages of the new method over the existing methods in the literature. We also established the uniqueness and convergence of the solution.

Theorem 2 was used to establish the uniqueness of the method by first establishing that T is continuous, T is q-contraction, and T is strict contraction using some hypothesis. Theorem 3 shows the proof for continuity. Theorem 4 proves the q-contraction. Using the theorem, we proved that the result is a q-contraction, which shows the uniqueness of the method.

Theorem 5 was used to show the convergence of the solution, and it was established that the method converges.

Problem 1: The approximation gives $y_3(x) = -0.09423x^3 + x$, and solving at $N = 5, 7, 10, 12$ and 15, we obtained Table 1, which shows the result obtained from solving Problem 1 at $x_i = -0.2$ to -1.0 at various values of N and the exact solution. Table 2 shows the error of Problem 1, and it indicates that as our N increases, the error result becomes more consistent, particularly when $N = 12$ and $N = 15$. It can be seen that the error is small and more consistent across all values of x_i and the values of N considered. For instance, the least error in [27] at $N = 15$ is 2.37×10^{-5} while the least error in our method is 0.10×10^{-8} at $N = 15$, this clearly shows that our method performed better.

Problem 2: The approximation gives

$$y_3(x) = -0.136348347x^3 + 0.589442798x^2 - x + 1,$$

and solving at $N = 5, 7, 10, 12$ and 15, we obtained Table 3, which shows the result obtained from solving Problem 2 at $x_i = -0.2$ to -1.0 at various values of N and the exact solution. Table 4 shows the error of Problem 2, and it indicates that as our N increases, the error result becomes more consistent, particularly when $N = 12$ and $N = 15$. It can be seen that the error is small and more consistent across all values of x_i and the values of N considered. For instance, comparing the error of [27] at $N = 15$ and that of our method at $N = 12$, this clearly shows that our method performs better.

Problem 3: The approximation gives $y_2(x) = 1 - \frac{1}{2}x + \frac{1}{8}x^2$, which shows that the results converge to the exact solution.

The numerical method is observed to be consistent and converges faster to the exact solution, as shown in Problems 1, 2, and 3. It is also observed that as N increases, the solution gets better. Hence the stability of the method.

Hence, from the results obtained, one may simply conclude that the numerical method derived is more efficient and computationally reliable than the existing methods in the literature.

5 Conclusion

In conclusion, a new numerical method for solving high-order integro-differential difference equations using Legendre polynomials with some conditions solved Fredholm differential difference equations. Our method has proven to be effective and efficient when compared to other methods of solution. In some of the examples, the result gave the exact solution, and for others, as we increase our value of N, the result approaches the exact solution so fast after a few iterations. The comparison of results also shows that our method performed favorably. All of the computations in this paper were performed using MATLAB 15.

References

[1] Abbas, S. and Mehdi, D. *A new operational matrix for solving fractional order differential equations,* Comput. Math. Appl. 59 (2010), 1326–1336.

[2] Adesanya, A.O., Yahaya, Y.A. Ahmed, B. and Onsachi, R.O. *Numerical solution of linear integral and integro-differential equations using Boubakar collocation method*, Inter. J. Math. Anal. Optim. Theory Appl. 2 (2019), 592–598.

[3] Ahmed, A.H., Kirtiwant, P.G. and Shakir, M.A. *The approximate solutions of fractional integro-differential equations by using modified adomian decomposition method*, Khayyam J. Math. 5 (1) (2019), 21-39.

[4] Ajileye, G. and Aminu, F.A. *A numerical method using collocation approach for the solution of Volterra-Fredholm integro-differential equations*, African Scientific Reports 1 (2022), 205–211.

[5] Ajileye, G. and Aminu, F.A. *Approximate solution to first-order integro-differential equations using polynomial collocation approach*, J. Appl. Computat Math. 11 (2022), 486.

[6] Ajileye, G. Amoo, S.A. and Ogwumu, O.D. *Hybrid block method algorithms for solution of first order initial value problems in ordinary differential equations*, J. Appl. Comput. Math. 7(2018) 390.

[7] Ajileye, G., James, A.A., Ayinde, A.M. and Oyedepo, T. *Collocation approach for the computational solution of Fredholm-Volterra fractional order of integro-differential equations*, J. Nig. Soc. Phys. Sci. 4 (2022), 834.

[8] Atabakan, Z.P., Nasab, A.K., Kiliçman, A. and Eshkuvatov, Z.K. *Numerical solution of nonlinear Fredholm integro-differential equations using spectral homotopy analysis method*, Math. Probl. Eng. 9 (7) (2013) 674364.

[9] Berinde, V. *Iterative approximation of fixed points*, Romania. Editura Efemeride, Baia Mare, 2002.

[10] Bhraway, A.H. Tohidi, E. and Soleymani, F. *A new Bernoulli matrix method for solving high order linear and nonlinear Fredholm integro-differential equations with piecewise interval*, Appl. Math. Comput. 219 (2012), 482–497.

[11] Biazar, J. and Gholami, P.M. *Application of variational iteration method for linear and nonlinear integro-differential-difference equations*, Int. Math. Forum5 (2010), 3335–3341.

[12] Darania, P. and Ebadian, A. *A method for the numerical solution of the integro-differential equations*, Appl. Math. Comput. 188 (2007), 657–668.

[13] Elmaci, D. and Baykus Savasaneril, N. *The Lucas Polynomial solution of linear Volterra-Fredholm integral equations*, Matrix Sci. Math. 6(1) (2022), 21–25.

[14] Elmaci, D. and Baykus Savasaneril, N. *Solutions of high-order linear Volterra integro-differential equations via Lucas polynomials*, Montes Taurus J. Pure Appl. Math. 5 (1) (2023), 22–33.

[15] Ercan, C. and Kharerah, T. *Solving a class of Volterra integral system by the differential transform method*, Int. J. Nonlinear Sci. 16 (2013), 87–91.

[16] Gulsu, M. and Ozturk, Y. *On the numerical solution of linear Fredholm-Volterra integro-differential Difference Equations with Piecewise Intervals*, Appl. Appl. Math. Comput. 7(3) (2012), 556–557.

[17] James A.A. and Ajileye, G., Ayinde A.M. and Dunama, W. *Hybrid-block method for the solution of second order non-linear differential equations*, J. Adv. Math. Comput. Sci. 37(12) (2022), 156–169.2456-9968.

[18] Karakoc, S.B.G., Eryilmaz, A. and Basbuk, M. *The approximate solutions of Fredholm integro-differential difference equations with variable coefficients via homotopy analysis method*, Math. Probl. Eng. (2013) Article ID: 261645.

[19] Khan, R.H. and Bakodah, H.O. *Adomian decomposition method and its modification for nonlinear Abel's integral equations*, Comput. Math. Appl. 7 (2013), 2349–2358.

[20] Matar, M.M. *Nonlocal integro-differential equations with arbitrary fractional order*, Konuralp J. Math. 4(1) (2016), 114–121.

[21] Mehdiyeva, G. Ibrahimov, V. and Imanova, M. *On the construction of the multistep methods to solving the initial-value problem for ODE and the Volterra integro-differential equations*, IAPE, Oxford, United Kingdom, 2019.

[22] Oyedepo, T., Ayinde, M.A., Adenipekun, A.E. and Ajileye, G. *Least-squares collocation Bernstein method for solving system of linear fractional integro-differential equations*, Int. J. Comput. Appl. 183(22) (2021), 0975–8887.

[23] Oyedepo, T., Ayoade, A.A. Ajileye, G. and Ikechukwu, N. J. *Legendre computational algorithm for linear integro-differential equations*, Cumhuriyet Science Journal 44(3) (2023), 561-566.

[24] Oyedepo, T., Ishola, C.Y., Ayoade, A.A. and Ajileye, G. *Collocation computational algorithm for Volterra-Fredholm integro-differential equations*, Electron. J. Math. Anal. Appl. 11(2) (2023), 1–9.

[25] Palais, R.S. *A simple proof of the Banach contraction principle*, J. Fixed Point Theory Appl. 2 (2007) 221–223.

[26] Rahmani, L., Rahimi, B. and Mordad, M. *Numerical solution of Volterra-Fredholm integro-differential equation by block pulse functions and operational matrices*, Gen. Math. Notes 4 (2) (2011), 7–48.

[27] Taiwo, O.A., Alimi, A.T. and Akanmu, M.A. *Numerical solutions for linear Fredholm integro-differential difference equations with variable coefficients by collocation methods*, JEPER 1 (2) (2014), 175–185.

[28] Volterra, V. *Theory of functionals and of integral and integro-differential equations,* Dover Publications, 2005.

[29] Yalcinbas, S. and Akkaya, T. *A numerical approach for solving linear integro-differential-difference equations with Boubaker polynomial bases*, Ain Shams Eng. J. 3(2) (2012), 153–161.

[30] Zada, L., Al-Hamami, M., Nawaz, R., Jehanzeb, S., Morsy, A., Abdel-Aty, A. and Nisar, K.S. *A new approach for solving Fredholm integro-differential equations*. Inform. Sci. Lett. 10(3) (2021), 407–415.