Research Article

An updated two-step method for solving interval linear programming: A case study of the air quality management problem

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Abstract

Many real-world problems occur under uncertainty. In this paper, we consider interval linear programming (ILP) which can be used to tackle uncertainties. Several methods have been proposed by researchers, such as the best and worst cases, Two-step method (TSM), improved TSM, ILP, improved ILP, three-step method, and robust two-step method. First, we define feasibility and optimality conditions in ILP models and review some solving methods shortly, and then show that some solutions of the TSM method are not feasible. Therefore, we propose an updated TSM method (namely, UTSM) by considering the feasibility and optimality conditions. In this paper, the UTSM method was applied to identify the reduction of aerosols by using two controllers with a minimized cost to demonstrate its application under uncertainty. Compared with other methods, the solutions obtained through ILP were presented as interval, which can provide intervals for the decision variables, objective function, and decision-makers. Therefore, the decision-makers can make the best decision based on the obtained solutions through ILP, and then identify desired plans for aerosol-emission control under uncertainty.

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1 Introduction

Interval linear programming (ILP) is described as the lower and upper bounds, which is used as a powerful tool to deal with optimization problems under uncertainty. Some researchers have been working on solving methods of ILP. In the best and worst cases (BWC) method proposed by Tong [30], the ILP model was converted into two submodels: the best and worst submodels, which they, respectively, have the largest and the smallest feasible regions. A given point is feasible for the ILP model if it satisfies the constraints of the best problem, and it is optimal for the ILP model if it is optimal for at least one characteristic model. The BWC method was extended by Chinneck and Ramadan for ILP models with equality constraints; see [7]. A novel ILP method was proposed by Huang and Moore [18], and Huang and Cao [17] also proposed their analyzed principals of two-step method (TSM). Some solutions obtained through the BWC, ILP, and TSM may be infeasible. New solving methods named three-step method (ThSM) and robust two-step method (RTSM) have been developed for solving ILP models; see [10, 17]. To guarantee that the given solutions of ILP method are completely feasible, Zhou et al. [33] proposed the modified ILP method (MILP) in which an extra constraint is added to the second submodel. Since the resulting solutions from the MILP may be nonoptimal, two improved ILP and MILP (IILP and IMILP) methods for solving ILP problems have been proposed; see [4]. The solutions of these methods are completely feasible and optimal; see [3]. Also, there are other methods for solving ILP models including IThSM, a new algorithm by Ashayerinasab et al. and a method proposed by Garajov et al.; see [2, 5, 12].

In this paper, we update the TSM (namely, UTSM) by considering the feasibility and optimality conditions. Following that, a numerical example is solved to demonstrate the effectiveness of UTSM. The air pollution caused by polluted resources, which cause pollution emissions in many sources, receptors, and artificial factors, including emissions of dangerous gases from factories and coal-fired power plants which pose serious threats to humans. These can have a greater impact under stable atmospheric conditions and increased pollution emissions. Pollution control and the reduction in the levels of atmospheric pollution are very important. These would approach the levels of atmospheric stability and prevent the destructive effects of pollutants. The proposed models can identify pollution sources and atmospheric weather conditions. These methods suggest pollution control using efficient methods for the study area. In this case, the levels of emissions of pollutants and environmental loading capacity need to be considered. Using optimization models, the costs should be reduced by controlling methods.

Another aim of the study is to check air pollution management under uncertainty as a range in the ILP model. The suggested methods will be analyzed to show the control and the reduction of dust by mulching and green belt controllers. However, the suggested methods are common methods

for dust and aerosol control. Now, to assess the effectiveness of the two controllers (mulching and green belts), the ILP model will be used and solved by the BWC, ITSM, and UTSM methods. The aim of this model is to minimize the costs in order to reduce aerosols by using two controllers.

2 Interval linear programming

In this section, we define the ILP model and some solving methods. An interval number $[X^-, X^+]$ is shown as $X^\pm$ where $X^- \leq X^+$. If $X^- = X^+$, then $X^\pm$ is degenerate. If A^- and A^+ are two matrices in $\mathbb{R}^{m \times n}$ such that $A^- \leq A^+$, then the set of matrices $A^\pm = [A^-, A^+] = \{A \mid A^- \leq A \leq A^+\}$ is called an interval matrix, and the matrices A^- and A^+ are called bounds of A. Center and radius matrices are defined as $\Delta_{A^\pm} = \frac{1}{2}(A^+ - A^-)$ and $A^c = \frac{1}{2}(A^- + A^+)$. A special case of an interval matrix is an interval vector $\mathbf{x}^\pm = \{x \mid x^- \leq x \leq x^+\}$, where $x^-, x^+ \in \mathbb{R}^n$; see [1]. If $\mathbf{A}^\pm$ is a square interval matrix and each $A \in \mathbf{A}^\pm$ is nonsingular, then $\mathbf{A}^\pm$ is called regular. Consider the following ILP model:

$$
\begin{aligned}
\max \quad & f^\pm = \sum_{j=1}^{n} c_j^\pm x_j^\pm \\
s.t. \quad & \sum_{j=1}^{n} a_{ij}^\pm x_j^\pm \leq b_i^\pm, \quad i = 1, 2, \ldots, m, \\
& 0 \leq x_j^- \leq x_j^+, \quad j = 1, 2, \ldots, n.
\end{aligned} \tag{1}
$$

The characteristic model of ILP model (1) is

$$
\begin{aligned}
\max f &= \sum_{j=1}^{n} c_j x_j \\
s.t. \sum_{j=1}^{n} & a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m, \\
& x_j \geq 0, \quad j = 1, 2, \ldots, n,
\end{aligned}
$$

where $a_{ij} \in a_{ij}^\pm, c_j \in c_j^\pm$, and $b_i \in b_i^\pm$. The feasible solution set of the ILP is defined as

$$
\left\{ x \in R^n : \sum_{j=1}^{n} a_{ij}^- x_j \leq b_i^+, \, x_j \geq 0, \, i = 1, \ldots, m, \, j = 1, \ldots, n \right\}.
$$

Also, the optimal solution set of the ILP is defined as the set of all optimal solutions over all the characteristic models. Some methods for solving ILP models are BWC, TSM, ITSM methods (see [16,30,31]), which are presented

in Table 1. Also f_{opt}^- and f_{opt}^+ are the best and worst optimal values of the objective function and the optimal solution is $x_{j\,opt}^{\pm} = \left[x_{j\,opt}^-,\, x_{j\,opt}^+\right]$.

We also have

$$\left|a_{ij}^{\pm}\right|^- = \begin{cases} a_{ij}^-, & a_{ij}^{\pm} \geq 0, \\ -a_{ij}^+, & a_{ij}^{\pm} < 0, \end{cases} \qquad \left|a_{ij}^{\pm}\right|^+ = \begin{cases} a_{ij}^+, & a_{ij}^{\pm} \geq 0, \\ -a_{ij}^-, & a_{ij}^{\pm} < 0, \end{cases}$$

$$\mathrm{Sign}(a_{ij}^{\pm}) = \begin{cases} 1, & a_{ij}^{\pm} \geq 0, \\ -1, & a_{ij}^{\pm} < 0. \end{cases}$$

Note that in all methods, for the objective functions, the notations "+" and "-" are always used for the first submodel and the second submodel, respectively. But for the variables, in the TSM and ITSM methods, the notation "+" for x_j, $j = 1, \ldots, k$, in the first submodel and $j = k+1, \ldots, n$ in the second submodel is used. Also, the notation "-" for x_j, $j = k+1, \ldots, n$ in the first submodel and $j = 1, \ldots, k$ in the second submodel is used. In the BWC method, these notations are not used.

Remark: If a decision-maker wants to minimize the costs, then the BWC method can provide the lower and upper bounds of the cost. Now, an example is solved and then the results are compared. Consider the ILP model as follows:

$$\begin{aligned} \max \quad & f^{\pm} = [3.5,\, 4]\, x_1^{\pm} - [1.5,\, 1.7]\, x_2^{\pm} \\ s.t. \quad & \\ & [1.1,\, 1.2]\, x_1^{\pm} + [1.7,\, 1.9]\, x_2^{\pm} \leq [11.7,\, 12.1]\,, \\ & [4,\, 5]\, x_1^{\pm} - [3,\, 4]\, x_2^{\pm} \leq [5,\, 7]\,, \\ & 0 \leq x_1^- \leq x_1^+, \\ & 0 \leq x_2^- \leq x_2^+. \end{aligned} \qquad (2)$$

The results obtained through the BWC, TSM, and ITSM are given in Tables 2 and 3 and Figure 1.

Some points of the solution space obtained by the BWC and TSM are infeasible. For example, the point $(5.3839, 4.0076)$ in the BWC and the point $(5.1417, 4.3388)$ in the TSM are infeasible because they do not satisfy the inequality $1.1x_1^+ + 1.7x_2^- \leq 12.1$, which is the first constraint of the best problem. The solution space obtained by the ITSM is feasible.

In the next subsection, we will define optimality condition and show that some points of the BWC, TSM, and ITSM are nonoptimal, and then we propose a new method, which we call an updated TSM method.

Table 1: submodels of BWC, TSM, and ITSM

Methods	The first submodel	The second submodel
BWC	max $f^+ = \sum_{j=1}^n c_j^+ x_j$ s.t. $\sum_{j=1}^n a_{ij}^- x_j \leq b_i^+$, $i = 1, 2, \ldots, m,$ $x_j \geq 0$, $j = 1, 2, \ldots, n.$	max $f^- = \sum_{j=1}^n c_j^- x_j$ s.t. $\sum_{j=1}^n a_{ij}^+ x_j \leq b_i^-$, $i = 1, 2, \ldots, m,$ $x_j \geq 0$, $j = 1, 2, \ldots, n.$
TSM	max $f^+ = \sum_{j=1}^k c_j^+ x_j^+ + \sum_{j=k+1}^n c_j^+ x_j^-$ s.t. $\sum_{j=1}^k |a_{ij}|^- \text{Sign}(a_{ij}^\pm) x_j^+ + \sum_{j=k+1}^n |a_{ij}|^+ \text{Sign}(a_{ij}^\pm) x_j^- \leq b_i^+$ $x_j^+ \geq 0$, $j = 1, \ldots, k$ $x_j^- \geq 0$, $j = k+1, \ldots, n$	max $f^- = \sum_{j=1}^k c_j^- x_j^- + \sum_{j=k+1}^n c_j^- x_j^+$ s.t. $\sum_{j=1}^k |a_{ij}|^+ \text{Sign}(a_{ij}^\pm) x_j^- + \sum_{j=k+1}^n |a_{ij}|^- \text{Sign}(a_{ij}^\pm) x_j^+ \leq b_i^-$ $x_j^- \leq x_{j\,opt}^+$, $j = 1, \ldots, k$ $x_j^\pm \geq x_{j\,opt}^-$, $j = k+1, \ldots, n$ $x_j^\pm \geq 0$, $j = 1, \ldots, n$
ITSM	max $f^+ = \sum_{j=1}^k c_j^+ x_j^+ + \sum_{j=k+1}^n c_j^+ x_j^-$ s.t. $\sum_{j=1}^k |a_{ij}|^- \text{Sign}(a_{ij}^\pm) x_j^+ + \sum_{j=k+1}^n |a_{ij}|^+ \text{Sign}(a_{ij}^\pm) x_j^- \leq b_i^+$ $x_j^+ \geq 0$, $j = 1, \ldots, k$ $x_j^- \geq 0$, $j = k+1, \ldots, n$	max $f^- = \sum_{j=1}^k c_j^- x_j^- + \sum_{j=k+1}^n c_j^- x_j^+$ s.t. $\sum_{j=1}^k |a_{ij}|^+ \text{Sign}(a_{ij}^\pm) x_j^- + \sum_{j=k+1}^n |a_{ij}|^- \text{Sign}(a_{ij}^\pm) x_j^+ \leq b_i^-$ $\sum_{j=1}^n a_{ij}^- x'_j \leq b_i^+$, $x'_j = \begin{cases} x_{j\,opt}^+ & if\ \text{Sign}(a_{ij}^\pm) = \text{Sign}(c_j^\pm), j = 1, \ldots, k \\ x_j^- & if\ \text{Sign}(a_{ij}^\pm) \neq \text{Sign}(c_j^\pm), j = 1, \ldots, k \\ x_{j\,opt}^+ & if\ \text{Sign}(a_{ij}^\pm) = \text{Sign}(c_j^\pm), j = k+1, \ldots, n \\ x_j^- & if\ \text{Sign}(a_{ij}^\pm) \neq \text{Sign}(c_j^\pm), j = k+1, \ldots, n \end{cases}$ $0 \leq x_j^- \leq x_{j\,opt}^+$, $j = 1, \ldots, k$ $x_j^+ \geq x_{j\,opt}^-$, $j = k+1, \ldots, n$

Table 2: submodels of BWC, TSM, and ITSM for ILP model (2).

Methods	The first submodel	The second submodel
BWC	max $\quad f^+ = 4x_1 - 1.5x_2$ $s.t.$ $\quad\quad 1.1x_1 + 1.7x_2 \leq 12.1$ $\quad\quad 4x_1 - 4x_2 \leq 7$ $\quad\quad x_1,\, x_2 \geq 0$	max $\quad f^- = 3.5x_1 - 1.7x_2$ $s.t.$ $\quad\quad 1.2x_1 + 1.9x_2 \leq 11.7$ $\quad\quad 5x_1 - 3x_2 \leq 5$ $\quad\quad x_1,\, x_2 \geq 0$
TSM	max $\quad f^+ = 4x_1^+ - 1.5x_2^-$ $s.t.$ $\quad\quad 1.1x_1^+ + 1.9x_2^- \leq 12.1$ $\quad\quad 4x_1^+ - 4x_2^- \leq 7$ $\quad\quad x_1^+,\, x_2^- \geq 0$	max $\quad f^- = 3.5x_1^- - 1.7x_2^+$ $s.t.$ $\quad\quad 1.2x_1^- + 1.7x_2^+ \leq 11.7$ $\quad\quad 5x_1^- - 3x_2^+ \leq 5$ $\quad\quad x_1^- \leq x_{1\,opt}^+ \rightarrow x_1^- \leq 5.1417$ $\quad\quad x_2^+ \geq x_{2\,opt}^- \rightarrow x_2^+ \geq 3.3917$ $\quad\quad x_1^-,\, x_2^+ \geq 0$
ITSM	max $\quad f^+ = 4x_1^+ - 1.5x_2^-$ $s.t.$ $\quad\quad 1.1x_1^+ + 1.9x_2^- \leq 12.1$ $\quad\quad 4x_1^+ - 4x_2^- \leq 7$ $\quad\quad x_1^+,\, x_2^- \geq 0$	max $\quad f^- = 3.5x_1^- - 1.7x_2^+$ $s.t.$ $\quad\quad 1.2x_1^- + 1.7x_2^+ \leq 11.7$ $\quad\quad 5x_1^- - 3x_2^+ \leq 5$ $\quad\quad 1.1x_{1\,opt}^+ + 1.7x_2^+ \leq 12.1$ $\quad\quad 4x_{1\,opt}^+ - 4x_2^+ \leq 7$ $\quad\quad x_1^- \leq x_{1\,opt}^+$ $\quad\quad x_2^+ \geq x_{2\,opt}^-$ $\quad\quad x_1^-,\, x_2^+ \geq 0$

Table 3: Solutions of BWC, TSM, ITSM for ILP model (2).

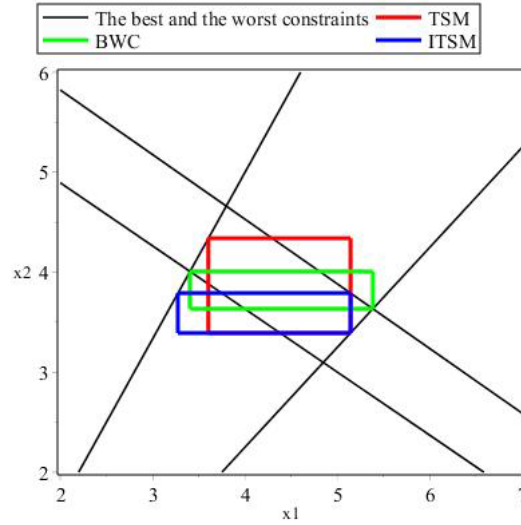| Methods	$x_1^\pm$	$x_2^\pm$	$z^\pm$
BWC	[3.4046,5.3839]	[3.6339,4.0076]	[5.1031,16.0848]
TSM	[3.6033,5.1417]	[3.3917,4.3388]	[5.2355,15.4792]
ITSM	[3.2744,5.1417]	[3.3917,3.7906]	[5.0162,15.4792]

Figure 1: Solutions of ILP model (2)

2.1 A new method: An updated TSM

In this subsection, we suggest an updated TSM by considering both feasibility and optimality conditions. At first we review basis stability, and then obtain two feasibility and optimality conditions.

Definition 1 (see [15]). *The problem* $\max \left\{ c^T x : Ax = b,\, x \geq 0 \right\}$, *where* $c \in C^\pm \subseteq \mathbb{R}^n$, $A \in A^\pm \subseteq \mathbb{R}^{m \times n}$ *and* $b \in b^\pm \subseteq \mathbb{R}^m$ *is called B-stable with basis B, if B is an optimal basis for each characteristic model. The ILP model is called (unique) nondegenerate B-stable if each characteristic model has a (unique) nondegenerate optimal basic solution with the basis B. Let* $B \subseteq \{1, 2, \ldots, n\}$ *be an index set such that (the restriction of A to the columns indexed by B) is nonsingular. Similarly,* $N = \{1, 2, \ldots, n\} \backslash B$ *denotes the index set of nonbasic variables and* A_N *is the restriction of A to the columns indexed by N. Also, B can be computed by solving an arbitrary characteristic model. We now review the conditions for B-stability.*

Regularity: A_B *is regular.*
Feasibility: *The solutions set of the interval system* $A_B x_B = b$ *are nonnegative.*
Optimality: A_B *is optimal, that is,* $c_B^T A_B^{-1} A_N - c_N^T \geq 0^T$.

Theorem 1. *[27] If* $\rho \left(\left| (A^c)_B^{-1} \right| \Delta_{A_B} \right) < 1$, *then* A_B *is regular, where* $\rho(.)$ *denotes the spectral radius.*

Theorem 2. *[27] If* $\max_{1 \leq i \leq n} \left(\left| (A^c)_B^{-1} \right| \Delta_{A_B} \right)_{ii} \geq 1$, *then* A_B *is not regular.*

Some conditions for the regularity of interval matrices have been proposed in [29].

Theorem 3. *[28] The interval vector r is an enclosure to the solution set of a system $A_B x_B = b$, where*

$$r_i^- = \min\left\{-x_i^* + (x_i^c + |x_i^c|)M_{ii}) + \tfrac{1}{2M_{ii}-1}(-x_i^* + (x_i^c + |x_i^c|)M_{ii}\right\},$$
$$r_i^+ = \max\left\{x_i^* + (x_i^c - |x_i^c|)M_{ii}) + \tfrac{1}{2M_{ii}-1}(x_i^* + (x_i^c - |x_i^c|)M_{ii}\right\},$$
$$M = (I - \left|(A^c)_B^{-1}\right|\Delta_{A_B})^{-1}, \; x^c = (A_B^c)^{-1}b^c,$$
$$x^* = M(|x^c| + \left|(A^c)_B^{-1}\right|\Delta_b),$$

with A_B^c is nonsingular and $\rho\left(\left|(A^c)_B^{-1}\right|\Delta_{A_B}\right) < 1$.

Theorem 4. *[15] Let y be an enclosure to the solution set of $A_B y = c_B$. If $\left(\left(A_N^T\right)y\right)^- \geq c_N^+$, then the optimality condition holds.*

Theorem 5. *[15] Let $diag(q)$ denote the diagonal matrix with entries $q_1, \ldots, q_m$. For each $q \in \{\pm 1\}^m$, if the solution set of the system*

$$\begin{cases} ((A_B^c)^T - (\Delta_{A_B})^T \, diag(q)) \, y \leq c_B^+, \\ -((A_B^c)^T + (\Delta_{A_B})^T \, diag(q)) \, y \leq -c_B^-, \\ diag(q)y \geq 0, \end{cases}$$

lies in the solution set of the system

$$\begin{cases} ((A_N^c)^T - (\Delta_{A_N})^T \, diag(q)) \, y \leq c_N^+, \\ diag(q)y \geq 0, \end{cases}$$

then the optimality condition holds.

Theorem 6. *[15] Let the ILP model be unique, nondegenerate B-stable, where $B = (1, \ldots, m)$. Then the optimal solution set of the ILP model is equal to the solution set of the interval system*

$$A_B x_B = b, \; x_B \geq 0, \; x_N = 0, \; or \; A_B^+ x_B \geq b^-, \; A_B^- x_B \leq b^+, \; x_B \geq 0, \; x_N = 0.$$

Now, we introduce two submodels of UTSM. Firstly, we solve the submodel corresponding to f^-, which is the second submodel of the TSM. Secondly, we add two extra constraints in order to ensure that the resulting solution space is completely feasible and optimal in the submodel corresponding to f^+. Submodel 1 is as follows.

Submodel 1

$$\max\ f^- = \sum_{j=1}^{k} c_j^- x_j^- + \sum_{j=k+1}^{n} c_j^- x_j^+$$

s.t.

$$\sum_{j=1}^{k} |a_{ij}|^+ \text{Sign}(a_{ij}^\pm) x_j^- + \sum_{j=k+1}^{n} |a_{ij}|^- \text{Sign}(a_{ij}^\pm) x_j^+ \le b_i^-,\ \ i = 1, \dots m,$$
$$x_j^- \ge 0,\ \ j = 1, \dots, k,$$
$$x_j^+ \ge 0,\ \ j = k+1, \dots, n,$$

$$(3)$$

For Submodel 2, $x_{j\,opt}^-$ for $j = 1, \dots, k$ and $x_{j\,opt}^+$ for $j = k+1, \dots, n$ are the optimal solutions of the first submodel.

$$\max\ f^+ = \sum_{j=1}^{k} c_j^+ x_j^+ + \sum_{j=k+1}^{n} c_j^+ x_j^-,$$

s.t.

$$\sum_{j=1}^{k} |a_{ij}|^- \text{Sign}(a_{ij}^\pm) x_j^+ + \sum_{j=k+1}^{n} |a_{ij}|^+ \text{Sign}(a_{ij}^\pm) x_j^- \le b_i^+,\ \ i = 1, \dots m,$$
$$0 \le x_{j\,opt}^- \le x_j^+,\ \ j = 1, \dots, k,$$
$$0 \le x_j^- \le x_{j\,opt}^+,\ \ j = k+1, \dots, n,$$

Now, we obtain an extra constraint to ensure that the solutions are optimal.

Theorem 6 implies that $\sum_{j=1}^{n} a_{ij}^+ x_j \ge b_i^-,\ i = 1, \dots, m,$ or

$$\sum_{j=1}^{k-p} a_{ij}^+ x_j + \sum_{j=k-p+1}^{k} a_{ij}^+ x_j + \sum_{j=k+1}^{n-q} a_{ij}^+ x_j + \sum_{j=n-q+1}^{n} a_{ij}^+ x_j \ge b_i^-,\ i = 1, \dots, m.$$

Since $a_{ij}^\pm \ge 0\ for\ j = 1, \dots, k-p,\ a_{ij}^\pm \le 0\ for\ j = k-p+1, \dots, k+1, \dots, n-q$ and $a_{ij}^\pm \ge 0\ for\ j = n-q+1, \dots, n,$ therefore

$$\sum_{j=1}^{k-p} a_{ij}^+ x_j + \sum_{j=k-p+1}^{k} a_{ij}^+ x_j + \sum_{j=k+1}^{n-q} a_{ij}^+ x_j + \sum_{j=n-q+1}^{n} a_{ij}^+ x_j$$
$$\ge \sum_{j=1}^{k-p} a_{ij}^+ x_{j\,opt}^- + \sum_{j=k-p+1}^{k} a_{ij}^+ x_j^+ + \sum_{j=k+1}^{n-q} a_{ij}^+ x_{j\,opt}^+ + \sum_{j=n-q+1}^{n} a_{ij}^+ x_j^-.$$

Therefore, for optimality, it is sufficient that

$$\sum_{j=1}^{k-p} a_{ij}^+ x_{j\,opt}^- + \sum_{j=k-p+1}^{k} a_{ij}^+ x_j^+ + \sum_{j=k+1}^{n-q} a_{ij}^+ x_{j\,opt}^+ + \sum_{j=n-q+1}^{n} a_{ij}^+ x_j^- \ge b_i^-,$$

or

$$\sum_{j=1}^{n} a_{ij}^+ x_j'' \ge b_i^-,\ x_j'' = \begin{cases} x_{j\,opt}^- & if\ \text{Sign}(a_{ij}^\pm) = \text{Sign}(c_j^\pm),\ j = 1, \dots, k, \\ x_j^+ & if\ \text{Sign}(a_{ij}^\pm) \ne \text{Sign}(c_j^\pm),\ j = 1, \dots, k, \\ x_{j\,opt}^+ & if\ \text{Sign}(a_{ij}^\pm) = \text{Sign}(c_j^\pm),\ j = k+1, \dots, n, \\ x_j^- & if\ \text{Sign}(a_{ij}^\pm) \ne \text{Sign}(c_j^\pm),\ j = k+1, \dots, n, \end{cases}$$

To ensure that the solutions are feasible, let us obtain an extra constraint:

Theorem 6 implies that $\sum\limits_{j=1}^{n} a_{ij}^- x_j \le b_i^+,\; i = 1, \ldots, m,$ or

$$\sum_{j=1}^{k-p} a_{ij}^- x_j + \sum_{j=k-p+1}^{k} a_{ij}^- x_j + \sum_{j=k+1}^{n-q} a_{ij}^- x_j + \sum_{j=n-q+1}^{n} a_{ij}^- x_j \le b_i^+,\; i = 1, \ldots, m.$$

Since $a_{ij}^\pm \ge 0$ for $j = 1, \ldots, k-p,\; a_{ij}^\pm \le 0$ for $j = k-p+1, \ldots, k+1, \ldots, n-q$ and

$a_{ij}^\pm \ge 0$ for $j = n - q + 1, \ldots, n,$ therefore

$$\sum_{j=1}^{k-p} a_{ij}^- x_j + \sum_{j=k-p+1}^{k} a_{ij}^- x_j + \sum_{j=k+1}^{n-q} a_{ij}^- x_j + \sum_{j=n-q+1}^{n} a_{ij}^- x_j$$

$$\le \sum_{j=1}^{k-p} a_{ij}^- x_j^+ + \sum_{j=k-p+1}^{k} a_{ij}^- x_{j\,opt}^- + \sum_{j=k+1}^{n-q} a_{ij}^- x_j^- + \sum_{j=n-q+1}^{n} a_{ij}^- x_{j\,opt}^+$$

Therefore, for feasibility, it is sufficient that

$$\sum_{j=1}^{k-p} a_{ij}^- x_j^+ + \sum_{j=k-p+1}^{k} a_{ij}^- x_{j\,opt}^- + \sum_{j=k+1}^{n-q} a_{ij}^- x_j^- + \sum_{j=n-q+1}^{n} a_{ij}^- x_{j\,opt}^+ \le b_i^+ \text{ or}$$

$$\sum_{j=1}^{n} a_{ij}^- x_j' \le b_i^+,\; x_j' = \begin{cases} x_j^+ & if\; \text{Sign}(a_{ij}^\pm) = \text{Sign}(c_j^\pm),\; j = 1, \ldots, l, \\ x_{j\,opt}^- & if\; \text{Sign}(a_{ij}^\pm) \ne \text{Sign}(c_j^\pm),\; j = 1, \ldots, l, \\ x_j^- & if\; \text{Sign}(a_{ij}^\pm) = \text{Sign}(c_j^\pm),\; j = l+1, \ldots, n, \\ x_{j\,opt}^+ & if\; \text{Sign}(a_{ij}^\pm) \ne \text{Sign}(c_j^\pm),\; j = l+1, \ldots, n, \end{cases}$$

Finally, we have $f_{opt}^\pm = \left[f_{opt}^-, f_{opt}^+ \right],\; x_{j\,opt}^\pm = \left[x_{j\,opt}^-, x_{j\,opt}^+ \right].$ Therefore, the second submodel is as follows.

Submodel 2

$\max\; f^+ = \sum\limits_{j=1}^{k} c_j^+ x_j^+ + \sum\limits_{j=k}^{n} c_j^+ x_j^-$

$s.t.\; \sum\limits_{j=1}^{k} |a_{ij}|^- \text{Sign}(a_{ij}^\pm) x_j^+ + \sum\limits_{j=k+1}^{n} |a_{ij}|^+ \text{Sign}(a_{ij}^\pm) x_j^- \le b_i^+,\; i = 1, \ldots m,$

$0 \le x_{j\,opt}^- \le x_j^+,\; j = 1, \ldots, k,$

$0 \le x_j^- \le x_{j\,opt}^+,\; j = k+1, \ldots, n,$

$$\sum_{j=1}^{n} a_{ij}^+ x'' \ge b_i^-,\; x''_j = \begin{cases} x_{j\,opt}^- & if\; \text{Sign}(a_{ij}^\pm) = \text{Sign}(c_j^\pm),\; j = 1, \ldots, k, \\ x_j^+ & if\; \text{Sign}(a_{ij}^\pm) \ne \text{Sign}(c_j^\pm),\; j = 1, \ldots, k, \\ x_{j\,opt}^+ & if\; \text{Sign}(a_{ij}^\pm) = \text{Sign}(c_j^\pm),\; j = k+1, \ldots, n, \\ x_j^- & if\; \text{Sign}(a_{ij}^\pm) \ne \text{Sign}(c_j^\pm),\; j = k+1, \ldots, n, \end{cases}$$

$$\sum_{j=1}^{n} a_{ij}^- x' \le b_i^+,\; x'_j = \begin{cases} x_{j\,opt}^+ & if\; \text{Sign}(a_{ij}^\pm) = \text{Sign}(c_j^\pm),\; j = 1, \ldots, l, \\ x_j^- & if\; \text{Sign}(a_{ij}^\pm) \ne \text{Sign}(c_j^\pm),\; j = 1, \ldots, l, \\ x_{j\,opt}^- & if\; \text{Sign}(a_{ij}^\pm) = \text{Sign}(c_j^\pm),\; j = l+1, \ldots, n, \\ x_j^+ & if\; \text{Sign}(a_{ij}^\pm) \ne \text{Sign}(c_j^\pm),\; j = l+1, n. \end{cases}$$

Therefore, the constraints $\sum\limits_{j=1}^{n} a_{ij}^- x_j' \le b_i^+$ and $\sum\limits_{j=1}^{n} a_{ij}^+ x_j'' \ge b_i^-$ are feasibility and optimality conditions, respectively.

Note that, since the solution regions obtained through the UTSM method is completely optimal, the union of these regions will be closer to the exact optimal solution region of the ILP model.

Now, consider ILP model (2), again. Given Figure 1, the points (3.4046, 3.6330), (3.6033, 3.3917), and (3.2744, 3.3917) obtained by the BWC, TSM, and ITSM respectively, are nonoptimal, because they do not satisfy inequality $1.2x_1 + 1.9x_2 \geq 11.7$, which is the first constraint of the worst problem with the reverse sign. Model (2) is equivalent to the following model:

$$\max \quad z^{\pm} = [3.5,\, 4]\, x_1^{\pm} - [1.5,\, 1.7]\, x_2^{\pm}$$
$$s.t.$$
$$[1.1,\, 1.2]\, x_1^{\pm} + [1.7,\, 1.9]\, x_2^{\pm} + x_3^{\pm} = [11.7,\, 12.1]\,,$$
$$[4,\, 5]\, x_1^{\pm} - [3,\, 4]\, x_2^{\pm} + x_4^{\pm} = [5,\, 7]\,,$$
$$0 \leq x_1^{-} \leq x_1^{+}\,,$$
$$0 \leq x_2^{-} \leq x_2^{+}\,,$$
$$0 \leq x_3^{-} \leq x_3^{+}\,,$$
$$0 \leq x_4^{-} \leq x_4^{+}\,.$$

According to Theorems 1, 3, and 4, ILP model 3 is B-stable, where $B = (1, 2)$. Now, we solve the model by the UTSM.

submodel 1
$$\max \quad z^{-} = 3.5x_1^{-} - 1.7x_2^{+}$$
$$s.t.$$
$$1.2x_1^{-} + 1.7x_2^{+} \leq 11.7,$$
$$5x_1^{-} - 3x_2^{+} \leq 5,$$
$$x_1^{-},\, x_2^{+} \geq 0,$$

submodel 2
$$\max \quad z^{+} = 4x_1^{+} - 1.5x_2^{-}$$
$$s.t.$$
$$1.1x_1^{+} + 1.9x_2^{-} \leq 12.1,$$
$$4x_1^{+} - 4x_2^{-} \leq 7,$$
$$x_1^{+} \geq x_{1\,opt}^{-} = 3.6033,$$
$$x_2^{-} \leq x_{2\,opt}^{+} = 4.3388,$$
$$\begin{cases} 1.1x_1^{+} + 1.7x_{2\,opt}^{+} \leq 12.1, \\ 4x_1^{+} - 4x_2^{-} \leq 7, \end{cases}$$
$$\begin{cases} 1.9x_2^{-} \geq 1.7x_{2\,opt}^{+}, \\ 5x_1^{+} - 3x_2^{-} \geq 5x_{1\,opt}^{-} - 3x_{2\,opt}^{+}, \end{cases}$$
$$x_1^{+}, x_2^{-} \geq 0.$$

Now, the results obtained by the UTSM have been given in Table 4 and Figure 2.

Table 4: Solutions of UTSM for ILP model (2).

Methods	$x_1^{\pm}$	$x_2^{\pm}$	$z^{\pm}$
UTSM	[3.6033,4.2945]	[3.8820, 4.3388]	[5.2355,11.3550]

As shown in Figure 2, the UTSM introduces a feasible and optimal space. All

points satisfy the constraints $1.1x_1 + 1.7x_2 \leq 12.1$ and $4x_1 - 4x_2 \leq 7$ (feasibility conditions) and the constraints $1.2x_1 + 1.9x_2 \geq 11.7$ and $5x_1 - 3x_2 \geq 5$ (optimality conditions).
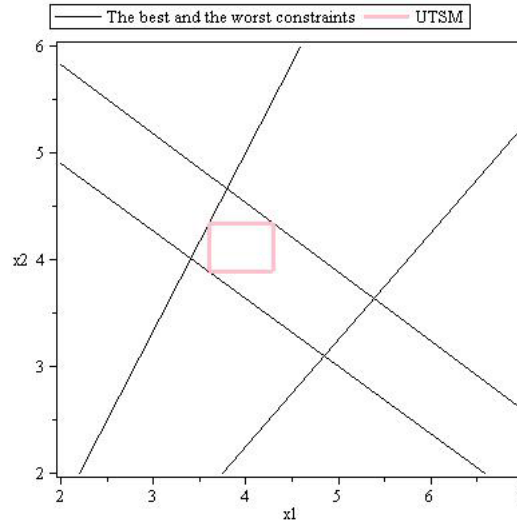

Figure 2: The solution region obtained by the UTSM for ILP model (2)

3 Application to air quality management

Atmospheric conditions can be affected by many factors, including SO2, CO, aerosols, dust, and so on. Aerosols have a very high diversity. As one of the greatest pollutants, this phenomenon occurs in arid, semiarid regions, and deserts, such that their sizes are different according to the concentrations in air. These can have destructive effects on health, nature, and the environment and create numerous economic problems for local people. It can be said that sandstorms and dust-storms are natural events with significant concentrations of particulate matter, which depend on the wind speeds. When the wind speed is almost more than 8 meters per second (and sometimes based on the region, which is stable or not, the wind speed is more than 6 meters/seconds), it can get into the air flow and the earth's atmosphere based on the roughness, particulate size, soil texture, soil moisture, and vegetation; see [9, 32]. The suggested methods are mulching and green belt that should be analyzed to control and reduce dust. These suggested methods are conventional methods for dust control, and many studies have been conducted in this area.

Mulch is a non-life coverage used for sand dune fixation in the desert lands. A variety of organic and inorganic mulches are available that can be used to accelerate production, reduce erosion, and even help control weeds. In Iran, oil mulch is usually used to cover. Most of the causes of oil mulching are the immediate effect, using them is provided in a short time, material source is available in the country, and there is no need for foreign resources.

Vegetation can have a major impact on air pollution and climate quality. It can also directly or indirectly decrease the temperature, increase moisture and dust absorption, and affect urban air pollution control. Studies on this subject have concluded that trees can greatly control air pollution, prevent harmful pollutants like PM10, SO2, NO2 and CO, and particulate matter in the atmosphere. Studies have shown that a lot of PM10 and PM2.5 is destroyed by trees, such that the efficiency and effectiveness of this factor plays a very important role in social and human health, and air quality control. In many countries nowadays, trees are being planted to reduce costs and harmful factors in the atmosphere; see [19, 21, 22].
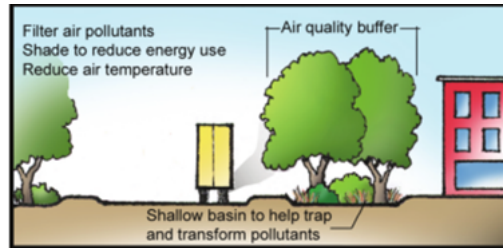

Figure 3: Trees as air quality buffer.

Figure 4: Trees as air quality buffer.

It has been well-documented that the plants destroy air pollutants, including hydrogen fluoride, sulfur dioxide, and some combinations of photochemical reactions in the air. It can be said that the green belt efficiency in the reduction of pollution defined by the following formula:

$$E = \frac{R}{R + A}$$

As E is the relative reduction effect by trees (percent), R is amount removed by trees (kg), and A is the amount of pollution in the atmosphere (kg).

3.1 Methodology and overview of the study system

The Sistan and Baluchestan region in Iran is located in the southeast, with arid and semi-arid weather. The Sistan plain covers a wide range of the western part of the province, and almost a wide catchment area. In addition to Iran, it has expanded into two countries Afghanistan and Pakistan. The border areas between Iran, Pakistan, and Afghanistan are the main dust source region in southwest of Asia; generating some 81 dust storms over Sistan; see [13, 24–26]. The Sistan region is one of the richest regions of Iran in terms of wind, and this sometimes creates difficulties for its people. Dust storms generate great waves, especially in Sistan. Figure 6 shows the geographical location of the Sistan and Baluchestan region.

Figure 5: Geographical location of the Sistan and Baluchestan region

The Sistan's 120-day winds are the most famous local winds of Iran, blowing from mid-May to mid-September, over a large area in the northern part of Sistan-Baluchestan. The strong "Levar" winds in summer favor the uplift of large quantities of dust from the Hamoun basin, which is located in the northern part of Sistan; see [25]. The 120-day winds of Sistan–in terms of time and local distribution and wind speed–have a high variety. In Sistan, the strong winds above eight nodes start in May and continue till November. In Sistan, winds are severe in summer, but can be slowed down in other seasons. Therefore, in this section, we focus on the optimization of an air

quality management model and PM10 emissions control in the Sistan region in a source through controllers, as mentioned in the previous section.

The aim of an air quality management model is to control effects on health by air pollution. These effects are directly related to pollutant concentration. Now, we establish the relationship between air pollution emissions and loading concentrations. The ground concentration at each downwind location (x, y) can be estimated as follows [14]:

$$C\left(x, y\right) = \frac{Q}{\pi u \sigma_y \sigma_z} \exp\left[\left(\frac{-y^2}{2\sigma_z{}^2}\right) - \left(\frac{H^2}{2\sigma_z{}^2}\right)\right],$$

where $C\left(x, y\right)$ is the ground-level concentration of pollutants (mg/m3) at point (x, y); x is the downwind distance from the source; Q denotes pollutant emission rate (mg/sec); u is average wind speed (m/ sec) along x direction; and σ_y and σ_z represent horizontal and vertical dispersion coefficients of plume versus(against) downwind distance x from the pollution source. Generally, σ_y and σ_z can be approximated by the following equations [8, 23]:

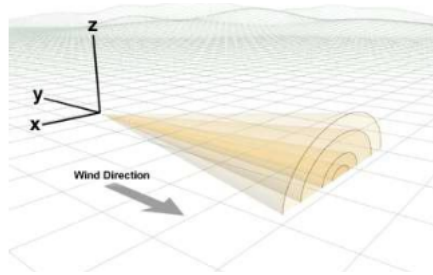$$\sigma_y = \gamma_1 x^{\beta_1} \quad , \quad \sigma_z = \gamma_2 x^{\beta_2}.$$

Figure 6: Dust-plume geometry

where γ_1 and β_1 are constants along the y direction and γ_2 and β_2 are the coefficients along the z direction, respectively. The values of γ_1, β_1, γ_2, and β_2 are related to atmospheric stability classes defined for different meteorological situations through wind speed, solar radiation (during the day), and cloud cover during the night [14, 23].
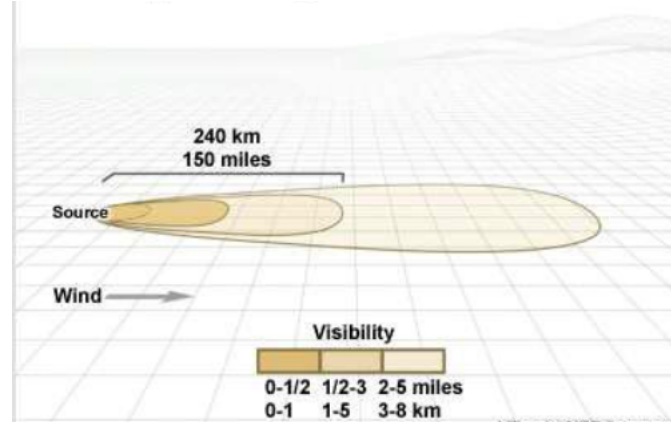
Figure 7: Typical visibility within a dust-storm

A transfer factor t_p can then be determined as follows [14]:

$$t_p = \frac{1}{\pi u \sigma_y \sigma_z} \exp\left[\left(\frac{-y_p^2}{2\sigma_y^2}\right) - \left(\frac{H^2}{2\sigma_z^2}\right)\right],$$

where t_p represents the contributions of pollutant emission rate at the Sistan area to the ground concentration of receptor zone p; the index p indicates different receptors influenced by air pollution emissions; H indicates the effective height of the air pollution plume from the Sistan region [6].

3.2 ILP model for air quality management modeling

The optimization model of air quality management has been presented as fuzzy linear programming [11,20]. Now, we consider the following ILP model:

$$Min \quad \sum_{j=1}^{2} \sum_{k=1}^{5} L_k TC_{jk}^{\pm} X_{jk}^{\pm}$$

$$s.t.$$

$$X_{jk}^{\pm} \leq FC_{jk}^{\pm}, \text{ for all } j, k,$$

$$\sum_{j=1}^{2} X_{jk}^{\pm} \geq S_{k}^{\pm}, \text{ for all } k, \quad (4)$$

$$\sum_{j=1}^{2} \left(1 - \eta_{j}^{\pm}\right) X_{jk}^{\pm} \leq e_{k}^{\pm}, \text{ for all } k,$$

$$\sum_{j=1}^{2} t_{p} \left(1 - \eta_{j}^{\pm}\right) X_{jk}^{\pm} \leq \theta_{kp}^{\pm}, \text{ for all } k, p,$$

$$0 \leq X_{jk}^{-} \leq X_{jk}^{+}, \text{ for all } j, k,$$

where i is the PM10 emission source, j is the type of PM10 control method ($j = 1, 2$, respectively, denotes mulching and green belt), and k is the planning period. The planning horizon is five years. Decision variables are the amount of PM10 allocated to control measure j for the reduction from source i in the period k (t/day). It has been shown by the symbol X_{ijk}. The objective is to minimize the total cost for the abatement of PM10. Now, the model describes the study area according to particulate reduction methods in this article.

Also L_k is length of period k (day), TC_{jk} is operating cost of control measure j during period k (Rials/t), FC_{jk} is maximum mitigation capacity of measure j in period k, ? is PM10 generation amount of emission source in period k (tonne/day); η_j is efficiency of control measure j, e_k is PM10-emission allowance for Sistan during period k (tonne/day), t_p is transfer factor from emission area to receptor zone p (day/m3), and θ_{kp} is environmental loading capacity of receptor zone during period (mg/m3). Note $i = 1$ represents the emission source of PM10 in the Sistan region. All the parameters in model (4) are presented as interval numbers. In this case, the parameters of TC_{jk}, FC_{jk}, S_k, η_j, e_k, and θ_{kp} are assumed to be estimated as interval numbers. Particulate matter in the Sistan region is discussed over four-month periods (the maximum amount of dust and particulate matter) during five years. We estimate-according to two methods, mulching and green belts-the tables are related to the costs of each method, efficiencies, and permitted amounts of dust in the region, and show their reduced values by using the two methods in the respective tables. Here, S_k is the aerosol generation amount of emission sources in period k in Sistan, which has been shown in Table 5.

The parameters related to the model in the Sistan region are now described. Tc_{jk} represents mulching costs per hectare with respect to the inflation rate (million). If we require 10 tonnes mulchs for every hectare and is equal to 20,000 Rials per liter of mulch, then the amount calculated using the inflation rate will be equal to Table 6.

Table 5: S_k-emission rate for ILP model (4)

Months	2012	2013	2014	2015	2016
May	[55,1345]	[76.6,4985]	[62.5,4983.6]	[15.3,460.1]	[45.6,4985]
June	[130.7,1384.8]	[60.1,2570.4]	[140.2,4985]	[17.2,452.4]	[49.1,630.8]
July	[315.6,9985]	[64.6,3044.6]	[4979.4,4985]	[10.5,4985]	[56.2,2489.4]
August	[100,1455.7]	[280.2,3553.5]	[130.2, 4985]	[880.9,4840]	[92.2,4983.6]

Table 6: Costs of different pollution controls for ILP model (4)

Tc_{jk}	2012	2013	2014	2015	2016
Mulching	[8.9,18]	[12.1,24.1]	[15,30]	[16.8,33.8]	[16.6,37.3]
Green belt	[2.5,6]	[3,6]	[3,7]	[4,8]	[5,10]

Figure 8: Dust precipitation

where e_{ik} denotes the aerosol (PM10) emission allowance for the source (Sistan) during period k (t/day) . If FC_{jk} is the maximum mitigation capacity of measure j at area in period k, then the amount of the reduction by the mulching and green belts methods can be seen in Tables 8 and 9.

Table 7: e_k for ILP model (4)

PM10-emission allowance	PM10
Good	0-54
Average (allowance)	55-154
Unhealthy for sensitive groups	155-254
Unhealthy	255-354
Very Unhealthy (crisis)	355-424
Dangerous (emergency)	425-604

Table 8: The value of PM10 emissions controlled by mulching

Months	2012	2013	2014	2015	2016
May	[44,1076]	[61.3,3988]	[50,3986.9]	[12.24,368.08]	[36.48,3988]
June	[104. 6,1108]	[48.08,2056.3]	[112.2,3988]	[13.76,361.9]	[39.28,504.6]
July	[252.5,7988]	[51.68,2435.9]	[4979.4,3989.1]	[8.4,3988]	[44.96,1991.5]
August	[80,1164.6]	[224.16,2842.8]	[104.16, 3988]	[704.72,3872]	[73.76,3986.9]
FC_{jk}	[120,2834.1]	[96.3,2830.7]	[1311.4,3988]	[184.8,2147.5]	[48.62,2617.8]

Table 9: The value of PM10 emissions controlled by green belt

Months	2012	2013	2014	2015	2016
May	[34.4,840.6]	[47.875,3115.6]	[39.06,3114.7]	[9.56,287.6]	[28.5,3115.6]
June	[81.7,865.5]	[37.6,1606.5]	[87.625,3115.6]	[10.7,282.7]	[30.687,394.2]
July	[197.3,6240.6]	[40.375,1903.1]	[3111.9,3115.6]	[6.56,3115.6]	[35.125,1555.9]
August	[62.5,909.8]	[175.1,2220.9]	[81.375,3115.6]	[550.56,3025]	[57.625,3114.7]
FC_{jk}	[193.9,2214.1]	[75.2,2211.5]	[829.98,2336.5]	[144.345,1677.7]	[37.98,2045.1]

Particulate reduction-with an efficiency of 75%-and relative reduction effect impact 0.2 and efficiency 40%, and the relative reduction effect impact 0.375 have been shown by a green belt. $\theta_{kp} = [45, 55]$ denotes environmental loading capacities at area p in period k; transfer factor t_p from source to area p is 1.1587×10^{-26}. The minimum and maximum wind speeds are [6 m/s, 15 m/s] during the 120 days; $e_k = [55, 154]$ shows the PM10 emission allowance during the period k (t/day) in Sistan.

Table 10: PM10 efficiencies of different control measures

Methods efficiency	Interval efficiency
Mulching method η_1	[0.4, 0.6]
Green belt method η_2	[0.40, 0.75]

We now solve the air pollution control model using the methods mentioned in Section 2. The results are given in Table 11.

Table 11: Results obtained from solving ILP model (4)

	BWC	ITSM	UTSM
$Z \times 10^7$	[0.1490,6.0239]	[2.19050165304,5.16560105314]	[2.50191551199,7.7566276346]
x_{1opt}	[0.0,3348.7]	[3634.1,3834.1]	[3834.1 , 6795.9]
x_{2opt}	[0.0,3463.2]	[3730.7, 4830.7]	[4830.7 , 8152.3]
x_{3opt}	[136.0,4963.8]	[4988,5348]	[5071.4 , 8480]
x_{4opt}	[0.0, 2540.1]	[3307.5,4147.5]	[4147.5 , 7222.4]
x_{5opt}	[0.0,3234.3]	[3317.8,4117.8]	[4117.8 , 7181.9]
x_{6opt}	[150.3,194.0]	[193.95,663.7]	[240.6 , 240.6]
x_{7opt}	[75.2,120.4]	[75.2,521.3]	[46.8 , 46.8]
x_{8opt}	[20.9,1192.0]	[16.2,447.4]	[1.1044e-11 , 2.7188e-07]
x_{9opt}	[144.3,231.0]	[144.4,618.9]	[179.6 , 179.6]
x_{10opt}	[379.0,60.8]	[3798,623.2]	[185.4 , 185.4]

Results and analysis

Tables related to the reduction of particulates are given in the previous section, using two controllers as mulching and green belt. It can be said that the Sistan region aerosols were considered over four months in five-year periods, such that the lower bound is the lowest amount of the particulate matter PM10 and upper bound represent the most amount of the particulate matter PM10 in Sistan. Now, using the two above mentioned controller, the reduction of aerosols is calculated according to efficiencies in every method and these estimations are presented in Tables 8–11. Notably, the costs associated with each method can be considered based on the funding in the environmental section and different covering lands. Inflation in the country has been considered over the years according to variable data of costs and inflations as well as other parameters studied in the Sistan region. The greatest reductions in the particulate matter were calculated by the ILP model and the presented formulas.

For greater reliability and accuracy of calculations, the ILP would be solved by three methods: BWC, ITSM, and UTSM. The variations in the reduction of aerosols are studied. The results show that if the aim is to minimize the cost for decision-makers and officials, then the BWC method could be the best method because it can calculate the minimum and maximum of the costs under the two optimistic and pessimistic models (the best and worst) for finding the best answer. For example, the cost per hectare–using mulching–the minimum cost is estimated $\sim 0.1490 \times 10^7$ and the maximum cost is estimated $\sim 6.0239 \times 10^7$; Therefore, these are provided as an interval under the upper and lower bounds.

In the BWC method, as mentioned in Section 2, the resulting solution does not exactly hold into the constraints. For example, in the first constraint, the amount of reduction in particulates in every period may be more than the greatest reduction in particulates in each period. Therefore, UTSM and ITSM methods are used. In this case, maybe because of the reduction area which is created by the solutions, the model cannot be considered good enough in order to estimate the reduction of costs. It can make changes in the costs in some parts to find the best answer and increasing them because the constraints are not violated. Hence, we can say to find the best answer for the reduction of particulate matter, ITSM and UTSM methods are more suitable. In Table 11, 10 optimal solutions are given which are shown with the symbol x_{opt}. The first five numbers represent x_{opt} according to the mulching and the second set of five numbers is x_{opt} according to the green belt over five years. As seen in the ITSM method, the changes are: [3634.1, 3834.1] in 2012; in that year, the minimum value is 55 and the maximum value is 9985, which show the most amount of particulate matter in July. The above models considered all solutions in a feasible area for one another and the maximum value is 9985, which estimated the changes in the reduction of particulates [3634.14, 3834.14].

ITSM method enabled shows a reduction in the intervals. Using the UTSM method, which can be said to be one of the best methods, it seeks to find the best solution in the feasible area. Optimally, the reduction of particulates is [3834.14, 6795.9128], which has obtained good results, and further reductions are anticipated. Also, in 2013, the minimum value is 62.5 for the entire year and the maximum value is 4985, such that the reduction of the particulate matter has been estimated at [3730.7, 4830.7]. This indicates that in 2013, according to the amount of particulates, [3730.7, 4830.7] can undergo a decrease. It can be said that the obtained values from the model show a strong reaction over the years, according to the highest amount of the particulate matter and using the above analysis, we have the amount of particulates [4830.7 , 8152.3416] in the UTSM method similarly. It seeks to solve and attempts to reduce it, but according to the obtained numbers, the reduction is less in some years, it is because of the amount of particulate matter which is very large in all months of the year. However, there are the powerful and useful methods to control particulate matter, those may not be able to greatly reduce particulates, and many environmental factors can cause fewer reductions. The other numbers are explained similarly.

4 Conclusion

In this paper, some methods for solving the ILP models have been reviewed, such as the BWC, TSM, and improved TSM (ITSM). Besides, an updated two-step method (UTSM) has been proposed. This method improves the TSM by considering two feasibility and optimality conditions. The solutions comparison for the BWC, TSM, ITSM, and UTSM methods based on the feasibility and optimality conditions shows that some points of the BWC and TSM may be infeasible. Also, some solutions of the ITSM method may be non-optimal. The advantage of the UTSM is that, two constraints have been added in the second submodel which guarantees the feasibility and optimality conditions, and so the obtained region by the UTSM will be feasible and optimal. Also, a case study related to air quality management has been done.
For solving many uncertain problems in the real-world, different models are used. The ILP model which is one of the models under uncertainty can be used for modeling these problems. In this paper, we consider an ILP model related to minimizing the costs in order to control aerosols and solve it by the BWC, ITSM, and UTSM methods. The results show that the lower and upper bounds of the costs obtain by the BWC method. The reduction of particulates can be clearly seen in two ITSM and UTSM methods.

Acknowledgements

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