

Operational Tau method for nonlinear multi-order FDEs

P. Mokhtary

Abstract

This paper presents an operational formulation of the Tau method based upon orthogonal polynomials by using a reduced set of matrix operations for the numerical solution of nonlinear multi-order fractional differential equations(FDEs). The main characteristic behind the approach using this technique is that it reduces such problems to those of solving a system of nonlinear algebraic equations. Some numerical examples are provided to demonstrate the validity and applicability of the method.

Keywords: Fractional differential equations(FDEs); Caputo derivative; Operational Tau method.

1 Introduction

The mathematical modelling and simulation of systems and processes based upon the description of their properties in terms of fractional derivatives, naturally leads to differential equations of fractional order and to the necessity to solve such equations. However, effective general methods for solving them can not be found even in the most useful works on fractional derivatives and integrals.

There are several approaches to the generalization of the notation of differentiation to fractional orders e.g., Riemann-Liouville, Grunwald-Letnikov and Caputo. We focus on one particular form so-called Caputo derivative.

Recently, linear FDEs based upon the fractional derivatives(such as Riemann-Liouville and Caputo schemes) with general variable coefficients have been solved by adapting various analytical and numerical methods[2, 5, 7, 20]. Nowadays, applications have included some classes of nonlinear FDEs, and this motivates us to consider their effective numerical methods for solution of these type of equations. Among the most recent works concerned with

Received 19 March 2014; revised 29 June 2014; accepted 7 July 2014

P. Mokhtary

Department of Mathematics, Sahand University of Technology, Tabriz, Iran.

Email: mokhtary.payam@gmail.com , mokhtary@sut.ac.ir

nonlinear initial value problems of fractional order, we can consider papers [4, 8, 12, 13, 18, 19, 21, 22, 31, 32].

Spectral methods have been studied intensively in the last two decades because of their good approximation properties. The formulation of spectral methods was first presented in the monograph of Gottlieb and Orszag [11]. The text book of Canuto, et al [3] focuses on practical and theoretical aspects of global spectral methods.

Global spectral methods use a representation of function $u(t)$ throughout the domain via a truncated series expansion with suitable basis functions. This series is then substituted into functional equation and upon the minimization of the residual function the unknown coefficients are computed.

Spectral methods can be broadly classified into three categories, Pseudospectral or Collocation, Galerkin and Tau methods. The Tau method, through which the spectral methods, as shown in [6, 23-29] has found extensive application for the numerical solution of many operator equations in the recent years.

The Tau method, firstly introduced by Lanczos[15-17], involves the projection of the residual function on the span of some appropriate set of basis functions, typically arising as the eigenfunctions of a singular Sturm-Liouville problem. The auxiliary conditions imposed as constraints on the expansion coefficients. It is well known that eigenfunctions of certain singular Sturm-Liouville problems allow the approximation of functions belong to the space $C^\infty[a, b]$ whose truncation error approaches zero faster than any negative power of the number of basis functions used in approximation, as that number (order of truncation N) tends to ∞ . This phenomenon is usually referred to as "Spectral accuracy" (Gottlieb and Orszag [11]). A convergence analysis and error bounds for the Tau method was considered by Ortiz and Pham in the papers [24, 25]. The recursive form of the Tau method, formulated by Ortiz in [26] was extended to the case of systems of ordinary differential equations in [6]. The basic philosophy of the method was extended to the numerical solutions of the linear and nonlinear initial value, boundary value, and mixed problems for ordinary differential equations [23, 25, 27], to the eigenvalue problems [27, 28], to the "Stiff" problems [23], to the partial differential equation[29], among others.

The main objective of the present paper is to provide Ortiz and Samara[27] operational approach to the Tau method for the numerical solution of nonlinear FDEs of the general form

$$L_D(u(t)) = f(t), \quad (1)$$

on $t \in \Lambda = [0, 1]$ with initial conditions

$$u^{(i)}(0) = d_i, \quad i = 0, 1, \dots, \nu - 1, \quad (2)$$

where

$$L_D(u(t)) = \sum_{r=0}^{N_d} p_r(t) \prod_{k=0}^{l_r} (D_C^{\theta_{rk}} u(t))^{\gamma_{rk}}, \quad \theta_{rk} \in \mathbb{Q}^+, \quad N_d, \gamma_{rk}, l_r \in \mathbb{N} \cup \{0\}, \quad (3)$$

where \mathbb{N}, \mathbb{Q}^+ are the collections of the all natural and positive rational numbers, respectively. d_i are constants and $\nu = \max_{0 \leq r \leq N_d} \{l_r\}$. The symbol $\lceil q \rceil$ is the smallest integer greater than or equal to q . $u(t)$ is unknown function, $p_r(t)$ and $f(t)$ are algebraic polynomials or their suitable polynomial approximations. Finally, the fractional derivative is considered in the Caputo sense that is given by

$$D_C^{\theta_{rk}} u(t) = \frac{1}{\Gamma(\lceil \theta_{rk} \rceil - \theta_{rk})} \int_0^t (t - \tau)^{\lceil \theta_{rk} \rceil - \theta_{rk} - 1} u^{(\lceil \theta_{rk} \rceil)}(\tau) d\tau, \quad t \in \Lambda. \quad (4)$$

The properties of Caputo derivative can be found in [30].

In this paper we proceed as follows: In the next section, the spectral Tau method for nonlinear FDEs is described. We reduce the problem to a set of nonlinear algebraic equations using some useful operational matrices. Numerical experiments are carried out in Section 3, to illustrate the efficiency of the proposed method.

2 Numerical approach

Consider the operational Tau solution for nonlinear FDE (2-3) as a polynomial of degree N

$$u_N(t) = \sum_{i=0}^{\infty} u_i J_i^{\alpha, \beta}(t) = \underline{u}_N \underline{\mathbf{J}} = \underline{u}_N \underline{\mathbf{J}} \underline{X}_t, \quad (5)$$

where $\underline{u}_N = [u_0, u_1, \dots, u_N, 0, \dots]$. $\underline{\mathbf{J}}$ is non-singular lower triangular coefficient matrix given by the shifted Jacobi polynomials in Λ , where $\{J_i^{\alpha, \beta}(t)\}_{i=0}^{\infty} = \underline{\mathbf{J}} = \underline{\mathbf{J}} \underline{X}_t$ with a standard basis vector $\underline{X}_t = [1, t, t^2, \dots]^T$ [3]. The effect of $u_N^{(k)}(t)$, $t^s u_N(t)$ and $(u_N(t))^p$ on the coefficients vector of polynomial (5) are

$$u_N^{(k)}(t) = \underline{u}_N \underline{\mathbf{J}} \eta^k \underline{X}_t, \quad t^s u_N(t) = \underline{u}_N \underline{\mathbf{J}} \mu^s \underline{X}_t, \quad (u_N(t))^p = \underline{u}_N \underline{\mathbf{J}} \mathbf{E}^{p-1}(\underline{u}_N, \underline{\mathbf{J}}) \underline{X}_t, \quad (6)$$

where matrices η and μ have the following simple structures ([27])

$$\eta = \begin{bmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \\ 0 & 2 & 0 & \vdots \\ 0 & 0 & 3 & \\ \vdots & \dots & \ddots & \end{bmatrix}, \quad \mu = \begin{bmatrix} 0 & 1 & 0 & 0 \dots \\ 0 & 0 & 1 & 0 \vdots \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \dots & \ddots & \end{bmatrix},$$

and $\mathbf{E}(\underline{u}_N, \mathbf{J})$ is an infinite upper triangular Toeplitz matrix with the following structure

$$\mathbf{E}(\underline{u}_N, \mathbf{J}) = \begin{bmatrix} \underline{u}_N \mathbf{J}_0 & \underline{u}_N \mathbf{J}_1 & \underline{u}_N \mathbf{J}_2 & \dots \\ \mathbf{0} & \underline{u}_N \mathbf{J}_0 & \underline{u}_N \mathbf{J}_1 & \dots \\ \mathbf{0} & \mathbf{0} & \underline{u}_N \mathbf{J}_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where \mathbf{J}_i is the i -th column of the matrix \mathbf{J} . Details for formulation of the matrix $\mathbf{E}(\underline{u}_N, \mathbf{J})$ can be found in [10].

Now, we intend to explain details of the structure of the operational approach to the Tau method with Jacobi polynomial bases for the numerical solution of the nonlinear multi-order FDEs. Firstly, in the Lemma 2.1, we will show that the effect of Caputo fractional derivative $D_C^{\theta_{r^k}}(u_N(t))$, will be represented as the product of a matrix and a vector. Secondly, in the Lemma 2.2, we will prove that the product of polynomials can be written as the product of a matrix and a vector. Finally, in the Theorem 2.3, we will give the matrix representation of $L_D(u_N(t))$ by using the Lemmas 2.1 and 2.2.

Lemma 2.1 *Let $J_j^{\alpha, \beta}(t)$ be the shifted Jacobi polynomials with respect to the weight function $\chi^{\alpha, \beta}(t) = (2-2t)^\alpha(2t)^\beta$ on Λ . Assume that the approximated solution $u_N(t)$ and the fractional derivative $D_C^{\theta_{r^k}}$ are given by the relations (5) and (4) respectively, then*

$$D_C^{\theta_{r^k}}(u_N(t)) = \underline{u}_N \mathbf{J} \Upsilon_{\theta_{r^k}} \mathbf{J},$$

where

$$\Upsilon_{\theta_{r^k}} = \begin{pmatrix} 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \Theta([\theta_{r^k}]\xi_{m_{r^k}, 0}) & \dots & \Theta([\theta_{r^k}]\xi_{[\theta_{r^k}], N}) & \dots \\ \Theta([\theta_{r^k}+1]\xi_{[\theta_{r^k}]+1, 0}) & \dots & \Theta([\theta_{r^k}+1]\xi_{[\theta_{r^k}]+1, N}) & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \Theta(N)\xi_{N, 0} & \dots & \Theta(N)\xi_{N, N} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

with $\Theta(\zeta) = \frac{\zeta!}{\Gamma(\zeta - \theta_{r^k} + 1)}$ and

$$\xi_{k,j} = \frac{1}{\|J_j^{\alpha, \beta}(t)\|_{\chi^{\alpha, \beta}}^2} (t^{k-q}, J_j^{\alpha, \beta}(t))_{\chi^{\alpha, \beta}}, \quad \begin{matrix} k \geq \theta_{r^k}, \\ j = 0, 1, \dots \end{matrix}$$

Proof. See [9]. \square

Lemma 2.2 (a) For two given polynomials $h(t) = \sum_{i=0}^{\infty} h_i v_i(t) = HV\underline{X}_t$ and $s(t) = \sum_{i=0}^{\infty} s_i v_i(t) = SV\underline{X}_t$ with $H = [h_0, h_1, h_2, \dots], S = [s_0, s_1, s_2, \dots]$, we have

$$s(t)h(t) = SVE(H, V)\underline{X}_t.$$

(b) For given polynomials $h_i(t) = \sum_{j=0}^{\infty} a_j T_{ij}(t) = \underline{a}_N T_i \underline{X}_t$, $i = 0, 1, \dots$, where T_i are nonsingular coefficients matrices given by $\{T_{ij}\}_{j=0}^{\infty} = T_i \underline{X}_t$, we have

$$\prod_{i=0}^l h_i(t) = \underline{a}_N T_0 \prod_{i=1}^l E(\underline{a}_N, T_i) \underline{X}_t. \quad (7)$$

Proof. For proof of part(a) see [10]. By using part (a) and mathematical induction we can prove part (b). \square

Theorem 2.3 (Matrix representation for nonlinear part)

Assume that the approximated solution $u_N(t)$ and the nonlinear fractional operator L_D are given by the relations (5) and (2), respectively, then

$$L_D(u_N(t)) = \underline{u}_N \widehat{\Pi} \mathbf{J},$$

where

$$\widehat{\Pi} = \mathbf{J} \left(\sum_{r=0}^{N_d} \Psi_d E^{\gamma_{rd}-1}(\underline{u}_N, \mathbf{J} \Psi_d) \prod_{k=d+1}^{l_r} E(\underline{u}_N, F_{rk}) p_r(\mu) \right) \mathbf{J}^{-1},$$

$$\Psi_k = \begin{cases} \eta^{\theta_{rk}}, & \theta_{rk} \in \mathbb{N}, \\ \Upsilon_{\theta_{rk}} \mathbf{J}, & \theta_{rk} \in \mathbb{Q}^+ - \mathbb{N}, \end{cases} \quad F_{rk} = \mathbf{J} \Psi_k E^{\gamma_{rk}-1}(\underline{u}_N, \mathbf{J} \Psi_k),$$

and d is the smallest index that $\gamma_{rd} \neq 0$.

Proof. From Lemma 2.1 and the third relation in (6) for $k \neq \{p \mid \gamma_{rp} = 0\}$ we have

$$\prod_{k=0}^{l_r} (D_C^{\theta_{rk}}(u_N(t)))^{\gamma_{rk}} = \prod_{k=0}^{l_r} (\underline{u}_N \mathbf{J} \Psi_k \underline{X}_t)^{\gamma_{rk}} = \prod_{k=0}^{l_r} (\underline{u}_N \mathbf{J} \Psi_k E^{\gamma_{rk}-1}(\underline{u}_N, \mathbf{J} \Psi_k) \underline{X}_t).$$

Let d be the smallest index that $\gamma_{rd} \neq 0$, then from (6) we can write

$$\begin{aligned}
\prod_{k=0}^{l_r} (\underline{u}_N \mathbf{J} \Psi_k E^{\gamma_{rk}-1}(\underline{u}_N, \mathbf{J} \Psi_k) \underline{X}_t) &= \underline{u}_N \mathbf{J} \Psi_d E^{\gamma_{rd}-1}(\underline{u}_N, \mathbf{J} \Psi_d) \\
&* \prod_{k=d+1}^{l_r} E(\underline{u}_N, \mathbf{J} \Psi_k E^{\gamma_{rk}-1}(\underline{u}_N, \mathbf{J} \Psi_k)) \underline{X}_t \\
&= \underline{u}_N \Pi_r \mathbf{J},
\end{aligned}$$

where

$$\Pi_r = \mathbf{J} \Psi_d E^{\gamma_{rd}-1}(\underline{u}_N, \mathbf{J} \Psi_d) \prod_{k=d+1}^{l_r} E(\underline{u}_N, F_{rk}) \mathbf{J}^{-1} \text{ and } F_{rk} = \mathbf{J} \Psi_k E^{\gamma_{rk}-1}(\underline{u}_N, \mathbf{J} \Psi_k).$$

By substituting the above relation in (3) and using the second relation in (6) we obtain

$$L_D(u_N(t)) = \underline{u}_N \left(\sum_{r=0}^{N_d} \Pi_r p_r(\mu) \right) \mathbf{J},$$

that is the statement of the Theorem. \square

Also for obtaining the matrix form of the initial conditions (3), we introduce vector $\underline{d} = [d_0, d_1, \dots, d_{\nu-1}, 0, \dots]$ where $\nu = \max_{0 \leq r \leq N_d} \{\theta_{rk}\}_{k=0}^{l_r}$. On the other hand we can write

$$u_N^{(j)}(0) = \underline{u}_N \mathbf{J} \eta^j \underline{X}_t|_{t=0} = \underline{u}_N \mathbf{J} \eta^j e_1 = \underline{u}_N b_j, \quad j = 0, 1, 2, \dots, \nu - 1,$$

where $e_1 = [1, 0, 0, \dots]^T$ and

$$B = (b_j)_{j=0}^{\nu-1} = (\mathbf{J} \eta^j e_1)_{j=0}^{\nu-1}. \quad (8)$$

Now, we are ready to obtain the nonlinear algebraic system of implementing the operational Tau method to the nonlinear multi-order FDE (2-3).

Following Theorem 2.3 and the relation (7) we obtain

$$\begin{cases} \underline{u}_N \widehat{\Pi} \mathbf{J} = \underline{f} \mathbf{J}, \\ \underline{u}_N B = \underline{d}, \end{cases} \quad (9)$$

where $f(t) = \underline{f} \mathbf{J}$ with $\underline{f} = [f_0, f_1, \dots]$. Because of orthogonality of $\{J_i^{\alpha, \beta}(t)\}_{i=0}^{\infty}$, projecting (9) on the $\{J_k^{\alpha, \beta}(t)\}_{k=0}^N$ yields

$$\underline{u}_N \widehat{\Pi}_k = f_k, \quad k = 0, 1, 2, \dots, N.$$

By setting

$$M_N = [b_0, b_1, \dots, b_{\nu-1}, \widehat{\Pi}_0, \widehat{\Pi}_1, \dots, \widehat{\Pi}_N], r_N = [d_0, d_1, \dots, d_{\nu-1}, f_0, f_1, \dots, f_N],$$

we obtain $\underline{u}_N M_N = r_N$. We restrict this system to its first $N + 1$ columns. The square system $\underline{u}_N M_{N-\nu} = r_{N-\nu}$, gives us unknown vector \underline{u}_N .

3 Numerical results

In this section we have considered three test problems. All of these test problems have been solved by the operational Tau method based on the Chebyshev and Legendre bases. In all cases any non-polynomial functions were replaced by a suitable polynomial approximation. All calculations were performed on a PC running Mathematica software. To report some information about the number of operations, we use function `\LeafCount` in the Mathematica software, that gives us total number of indivisible subexpressions. All of achieved nonlinear algebraic systems were solved by the well known iterative Newton method.

Example 1 : [18] Consider the nonlinear FDE with $\alpha = 1.5$

$$D_C^\alpha u(t) + u^2(t) = f(t), \quad u^{(i)}(0) = 0, \quad i = 0, 1,$$

where

$$f(t) = \frac{\Gamma(6)}{\Gamma(6-\alpha)} t^{5-\alpha} - \frac{3\Gamma(5)}{\Gamma(5-\alpha)} t^{4-\alpha} + \frac{2\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} + (t^5 - 3t^4 + 2t^3)^2.$$

the exact solution is $u(t) = t^5 - 3t^4 + 2t^3$.

We apply the proposed operational Tau method to obtain the approximated solution of the problem. The maximal error, with the Chebyshev and Legendre bases have been given in Table 1. A comparison of the Tau method with fractional high order method proposed by R. Lin and F. Liu in [18] shows that our method produces powerful superiority with respect to the proposed method in [18].

Table 1: Numerical results of Example 1, using operational Tau method with different bases

N	Maximal Error	
	Chebyshev Tau	Legendre Tau
5	4.58×10^{-16}	7.28×10^{-17}
7	1.50×10^{-16}	7.67×10^{-17}
10	8.75×10^{-17}	5.52×10^{-16}
Lin and Liu scheme	Max. error is 1.544×10^{-5} in $t = 1$	

Example 2 : Consider the nonlinear FDE

$$D_C^{\frac{1}{4}}u(t)D_C^{\frac{1}{2}}u(t) + u(t) = e^t + e^{2t}\operatorname{erf}(\sqrt{t})\left(1 - \frac{\Gamma(\frac{3}{4}, t)}{\Gamma(\frac{3}{4})}\right), \quad u(0) = 1,$$

where $\operatorname{erf}(z)$ gives the error function and $\Gamma(a, z)$ is the incomplete Gamma function.

the exact solution of this problem is $u(t) = e^t$. We apply the proposed operational Tau method to obtain the approximate solution of this example. We have reported the obtained numerical results in Table 2 and Fig. 1 with the Chebyshev and Legendre bases. In Fig. 1, obtained numerical errors are plotted for several values of approximation degree N in the L_∞ norm. From Table 2 and Fig. 1, we can conclude that desired spectral accuracy is obtained for this nonlinear problem and the approximate solutions are in high agreement with the exact solution. In addition, by using the function `\LeafCount` in the Mathematica software, for $N = 4, 8, 12$ and $N = 16$, we need 213, 812, 1261 and 2153 operations, respectively, to obtain the operational Tau solution with the reported errors in the Table 2 and Fig. 1 based on the Chebyshev polynomial bases.

Table 2: Numerical results of Example 2, using operational Tau method with different bases

N	Maximal Error	
	Chebyshev Tau	Legendre Tau
4	6.79×10^{-5}	5.13×10^{-5}
8	7.19×10^{-11}	5.85×10^{-11}
12	7.31×10^{-16}	7.06×10^{-16}
16	2.85×10^{-16}	2.81×10^{-16}

Example 3 : [14] Consider the following equation of fractional order $\theta = 0.5$:

$$D_C^\theta u(t) = \lambda t^\beta (u(t))^2, \quad (0 < \theta < 1), \quad (10)$$

with $\lambda, \beta \in \mathbb{R} (\lambda \neq 0)$. If $\theta + \beta < 1$, this equation has the exact solution

$$u(t) = \frac{\Gamma(1 - \theta - \beta)}{\lambda \Gamma(1 - 2\theta - \beta)} t^{-(\theta + \beta)}. \quad (11)$$

If $\beta \leq -2\theta$, then the equation (10), has unique solution $u(t) \in C[a, b]$ given by (11) and if $\beta = -(k + \theta), k \in \mathbb{N}$, then the equation (10), has unique solution $u(t) \in C^\infty[a, b]$. (See [14, Chapter 3]).

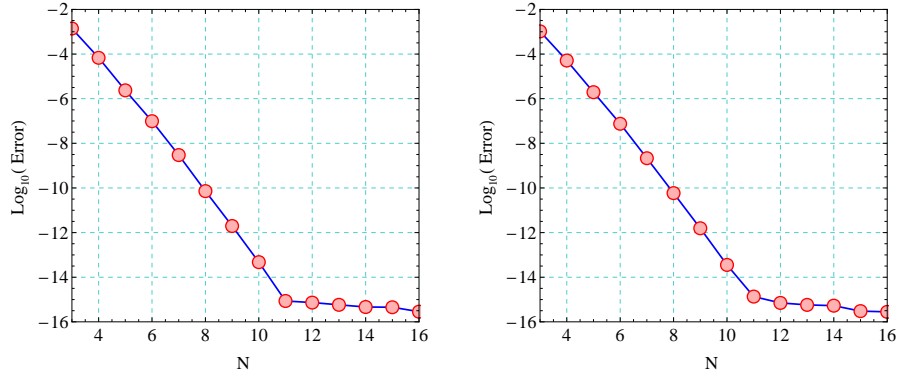


Figure 1: An illustration of the rate of convergence for the Tau method with various N . We observe the errors of Example 2 using Chebyshev bases (left) and Legendre bases (right)

The numerical results for example 3 with the Chebyshev and Legendre bases are presented in Fig. 2 and Table 3. Fig. 2, shows the rate of convergence for various β . Each part of the figure contains numerical errors for several values of N , which are plotted for a special value of β in L_∞ norm. As we can see from Table 3., and Fig. 2, the performance of the spectral Tau method with the Chebyshev and Legendre bases for $\beta \in [-1.5, -2.5]$ almost same, but when β tends to the $\beta = -2.5$ (smooth solution) the rate of convergence increases and we have accurate numerical solutions. For $\beta = -1.5, -2.5$, numerical results have not been presented, since the exact solution is obtained. In addition, by using the function `LeafCount` in the Mathematica software, for $N = 15$ we need 917 operations to obtain the operational Tau solution based on the Legendre polynomial bases.

Table 3: The numerical results of Example 3 with different β , $\lambda = 1$ and $N = 15$

x	Maximal Error for Chebyshev Tau			Maximal Error for Legendre Tau		
	$\beta = -2$	$\beta = -2.2$	$\beta = -2.4$	$\beta = -2$	$\beta = -2.2$	$\beta = -2.4$
0.2	9.92×10^{-6}	3.38×10^{-6}	5.74×10^{-7}	1.29×10^{-5}	4.27×10^{-6}	7.01×10^{-7}
0.4	7.45×10^{-6}	2.57×10^{-6}	4.41×10^{-7}	1.02×10^{-5}	3.46×10^{-6}	5.80×10^{-7}
0.6	4.39×10^{-6}	1.44×10^{-6}	2.38×10^{-7}	7.77×10^{-6}	2.61×10^{-6}	4.31×10^{-7}
0.8	3.47×10^{-6}	1.12×10^{-6}	1.82×10^{-7}	5.41×10^{-6}	1.70×10^{-6}	2.63×10^{-7}
1	2.00×10^{-6}	5.10×10^{-7}	9.23×10^{-8}	5.20×10^{-6}	1.67×10^{-6}	2.64×10^{-7}

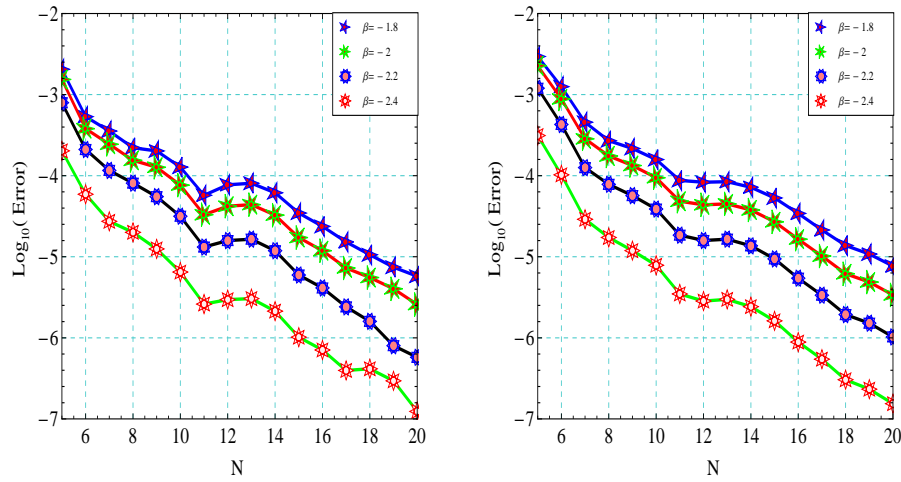


Figure 2: An illustration of the rate of convergence for the Tau method with various β . We observe the errors of Example 3 using Chebyshev bases (left) and Legendre bases (right)

4 Conclusion

In this paper, we presented a numerical scheme for solving nonlinear multi-order fractional differential equations. The operational Tau method was employed. Also, several test problems were used to show the applicability and efficiency of the method. The obtained results indicate that the new approach can solve the problem effectively.

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روش تاو عملیاتی برای معادلات دیفرانسیل کسری چند مرتبه ای

پیام مختاری

دانشگاه صنعتی سهند، گروه ریاضی

چکیده : این مقاله روش تاو عملیاتی مبتنی بر چندجمله ایهای متعامد را برای حل عددی معادلات دیفرانسیل با مشتقات کسری چند مرتبه ای غیر خطی ارائه می دهد. مشخصه اصلی این روش این است که جواب عددی معادله مورد نظر را با استفاده از حل دستگاه غیر خطی جبری بدست می آورد. برخی از مثال های عددی به منظور نمایش کارایی و کاربرد بودن روش ارائه شده است.

کلمات کلیدی : معادلات دیفرانسیل با مشتقات کسری؛ مشتق کاپاتو؛ روش تاو عملیاتی.