Numerical study of sine-Gordon equations using Bessel collocation method

S. Arora and I. Bala*

Abstract

The nonlinear space time dynamics have been discussed in terms of a hyperbolic equation known as a sine-Gordon equation. The proposed equation has been discretized using the Bessel collocation method with Bessel polynomials as base functions. The proposed hyperbolic equation has been transformed into a system of parabolic equations using a continuously differentiable function. The system of equations involves one linear and the other nonlinear diffusion equation. The convergence of the present technique has been discussed through absolute error, L_2-norm, and L_∞-norm. The numerical values obtained from the Bessel collocation method have been compared with the values already given in the literature. The present technique has been applied to different problems to check its applicability. Numerical values obtained from the Bessel collocation method have been presented in tabular as well as in graphical form.

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*Corresponding author

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Shelly Arora
Department of Mathematics, Punjabi University, Patiala, 147002, Punjab, India. e-mail: aroshelly@pbi.ac.in

Indu Bala
Department of Mathematics, Punjabi University, Patiala, 147002, Punjab, India. e-mail: indu13121994@gmail.com

1 Introduction

Nonlinear partial differential equations have wide applications in different branches of science and engineering, which helps to understand the diversity of physical phenomena in a logical manner. The nonlinear wave equations such as Klein–Gordon and sine-Gordon equations have wide applications in optics, plasma physics, quantum mechanics and fluid mechanics, and so on. Finding the analytic solution to these problems is a tedious task due to the complexity of the nonlinear terms. The numerical approximation of the solution in terms of the discrete set of points is often desirable by researchers due to the simplicity of the numerical computations.

The derivation of the Klein–Gordon equation is a generalization of the Schrödinger equation. It was named after the physicists Oskar Klein and Walter Gordon. They together in 1926, proposed that relativistic electrons can be described by the Klein–Gordon equation during the research for the equation describing de Broglie waves. Schrödinger considered the Klein–Gordon equation as a quantum wave equation [18, 19]. Klein–Gordon equation plays a significant role in many scientific applications, such as nonlinear optics and quantum field theory, and solid state physics. Klein–Gordon equation in one dimension can be considered as

$$\frac{\partial^2 y}{\partial t^2} - \beta \frac{\partial^2 y}{\partial \xi^2} + f_1(y) = f_2(\xi, t), \tag{1}$$

where $(\xi, t) \in (\xi_a, \xi_b) \times (0, T)$, $f_1(y)$ is nonlinear force, and β is a constant. The sine-Gordon equation is a special case of (1) for $f_1(y) = \sin(y)$ and $f_2(\xi, t) = 0$. Interestingly, the sine-Gordon equation was discovered separately in 1939 by Frenkel and Kontorova while studying the propagation of slip in an infinite chain of elastically bound atoms lying over a fixed chain. Later it was found that sine-Gordon is a special case of Klein–Gordon equation. The sine-Gordon equation in one dimension can be described as

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial \xi^2} + \sin(y(\xi, t)) = 0, \tag{2}$$

and the initial and boundary conditions can be described as

$$\begin{aligned} y(\xi, 0) = \phi_1(\xi) \quad and \quad y_t(\xi, 0) = \phi_2(\xi), \\ y(\xi_a, t) = \phi_3(t) \quad and \quad y(\xi_b, t) = \phi_4(t). \end{aligned} \tag{3}$$

The present problem of the equation is a continuum model for waves in mechanical systems, coupled-pendulum, the study of the domain wall dynamics in magnetic crystals, magnetic-flux propagation in large Josephson junctions, propagation of crystal dislocations in solids, propagation of ultra-short optical pulses in optical fibers, as a nonlinear effective field theory for strong interactions in particle physics, and so on; see [8, 16, 18, 19, 28, 37].

A variety of numerical techniques have been developed by different researchers to study the behavior of nonlinear sine-Gordon equation, such as finite difference method, inverse scattering method, auxiliary equation method, spectral method, pseudo-spectral method, tanh-sech method, Adomian decomposition method, sine-cosine method, Jacobi elliptic functions, Backlund transformation, Riccati equation expansion method, homotopy perturbation method, and variational iteration method [15, 18, 33, 32, 40].

Method of characteristics and a leapfrog finite difference scheme [3, 24] were the first two methods developed to obtain the numerical solution of sine-Gordon equation. Strauss and Vázquez [36] developed a leapfrog finite difference, an implicit, energy-conserving scheme for the Klein–Gordon equation.

Apart from these, other numerical methods have also been developed for the solution of sine-Gordon equations, which include pseudospectral methods and spectral methods. Pseudospectral methods include the split-step Fourier scheme [1, 2, 41] and spectral methods include energy-conserving, wavelet spectral method, Fourier scheme, Legendre spectral element method, and multiresolution analysis method based on Legendre wavelets [27]. Finite element methods is based on a collocation scheme using Legendre–Gauss–Lobatto points [29], cubic B-splines [31] and Petrov–Galerkin scheme [5].

In the present study, the Bessel collocation method (BCM) has been followed to study the behavior of the nonlinear sine-Gordon equation. The Bessel polynomials of degree n have been taken as base polynomials. To discretize the time direction, the temporal variable has been split by the introduction of a continuously differentiable function. It converts the wave equation into a system of equations involving one linear and the other nonlinear equation.

The present manuscript has been divided into six sections starting from introduction. The BCM has been described in section 2. The explanation of collocation points as well as the implementation of the collocation technique has been described in sections 3 and 4, respectively. Convergence analysis has been discussed in section 5, whereas the numerical application has been given in section 6.

2 Bessel collocation method (BCM)

The collocation method belongs to the general class of approximate methods, known as weighted residual methods. In this method, the residual is set orthogonal to the weight function. In an orthogonal collocation, the trial function $y(\xi, t)$ is represented in a series of known polynomials with unknown coefficients [13, 22, 38]. The residual is set equal to zero at the collocation points.

On the basis of the implementation of the trial function, the collocation technique can be classified into three categories. If the trial function satisfies

the differential equation $\ell \mathbf{V}(\mathbf{y}) = \mathbf{0}$ with volume V, then it is termed as interior collocation. If the trial function satisfies the boundary $\ell \mathbf{B}(\mathbf{y}) = \mathbf{0}$, where B is the boundary adjoining volume V, then it is termed as boundary collocation. If the trail function satisfies neither the equation nor the boundary and is adjusted to both, then it is termed as a mixed collocation.

The choice of base function is the first important step in the technique of collocation. In the present study, Bessel polynomials of order n have been chosen as a trial function, and the technique is called the BCM.
During the study of problems in dynamic astronomy to solve the Kepler's problem, a German astronomer Bessel in 1824, introduced Bessel polynomials, which are the solution to a second order boundary value problem. These polynomials can be written in terms of limit confluent hypergeometric function $_0F_1$. The details of these hypergeometric functions are given in [7, 25, 34, 39]:

$$J_n(\xi) = \frac{\xi^n}{2^n n!} {}_0F_1(-; n+1; -\frac{1}{4}\xi^2). \tag{4}$$

The Bessel coefficients also follow from the power series expansion for small values of ξ

$$\lim_{\xi \to 0} {}_0F_1(-; n+1; -\frac{1}{4}\xi^2) = 1,$$

$$\lim_{\xi \to 0} \xi^{-n} J_n(\xi) = \frac{1}{2^n n!},$$

which shows that as $\xi \to 0$, the Bessel coefficient $J_n(\xi)$ approaches to $\frac{1}{2^n n!}$.

The first order derivative of the Bessel function is defined as

$$\frac{d}{d\xi}(\xi^n J_n(\xi)) = \xi^n J_{n-1}(\xi),$$

$$\frac{d}{d\xi}(\xi^{-n} J_n(\xi)) = -\xi^{-n} J_{n+1}(\xi).$$

3 Collocation points

The next step is the choice of collocation points. It is an important part of the collocation technique. In this study, instead of taking the uniform points, the zeros of orthogonal polynomials, such as Jacobi polynomials, have been taken as collocation points. Legendre and Chebyshev polynomials are special cases of Jacobi polynomials, and the zeros of these orthogonal polynomials are preferably taken as collocation points. Runge's divergence formula also states that nonuniform collocation points give less error as compared to uniform collocation points.

Theorem 1. [26] If $\mathcal{Q}_n(\xi)$ form a simple set of real polynomials and $w(\xi) > 0$ on $a \leq \xi \leq b$, then the necessary and sufficient condition that the set $\mathcal{Q}_n(\xi)$

is orthogonal with respect to the $w(\xi)$ over the interval $a \le \xi \le b$ is that

$$\int_a^b w(\xi)x^k \mathcal{Q}_n(\xi)d\xi = 0, \qquad k = 0, 1, 2, 3, \dots, (n-1).$$

Theorem 2. [26] If the simple set of real polynomials $\mathcal{Q}_n(\xi)$ is orthogonal with respect to the weight function $w(\xi) > 0$ on the interval $a \le \xi \le b$, then the zeros of $\mathcal{Q}_n(\xi)$ are distinct and lie in the interval $a \le \xi \le b$.

Since $\mathcal{Q}_n(\xi)$ is a polynomial of degree n, then it has exactly n roots, multiplicity counted, such that the roots are distinct and all lie in $a \le \xi \le b$.

Usually, the collocation points are selected from the Legendre or Chebyshev polynomials, and these polynomials are also particular cases of Jacobi polynomials. The details of the collocation points are given elsewhere [4, 6, 11, 12, 14, 17, 20, 35]. The zeros of Chebyshev polynomials have been taken as collocation points:

$$\xi_j = \cos\left(\frac{\pi(j-1)}{n}\right), \qquad j = 1, 2, \dots, n+1.$$

The Chebyshev collocation points have been transformed from the interval $[-1, 1]$ to $[\xi_a, \xi_b]$ using one to one correspondence.

4 Implementation of BCM

To discretize the given problem, BCM is applied in space direction. To apply BCM, (2) has been split in time direction. For this purpose, a new continuously differentiable function $z(\xi, t)$, differentiable with respect to t has been introduced:

$$\frac{\partial y}{\partial t} = z(\xi, t), \qquad (\xi, t) \in (\xi_a, \xi_b) \times (0, T),$$

$$\frac{\partial z}{\partial t} = \frac{\partial^2 y}{\partial \xi^2} - \sin(y(\xi, t)), \qquad (\xi, t) \in (\xi_a, \xi_b) \times (0, T). \tag{5}$$

To apply BCM on a system of equations defined by (5), the two functions $y(\xi, t)$ and $z(\xi, t)$ have been approximated in terms of Bessel polynomials as

$$y(\xi, t) = \sum_{i=1}^{n+1} J_i(\xi)c_i(t),$$

$$z(\xi, t) = \sum_{i=1}^{n+1} J_i(\xi)d_i(t), \tag{6}$$

where $J_i(\xi)$ are ith order Bessel polynomials. To simplify (6), the Bessel polynomials can be rewritten as suggested by [10, 42, 43, 44, 45, 46, 47, 48]:

$$y(\xi,t) = \sum_{i=1}^{n+1} \xi^{i-1} R c_i(t),$$

$$z(\xi,t) = \sum_{i=1}^{n+1} \xi^{i-1} R d_i(t),$$ (7)

where R is a square matrix of order $(n+1)\times(n+1)$ and $c_i(t)$ and $d_i(t)$ are the unknown coefficients of t, which are to be determined.

For n being an odd integer, R is defined as

$$R = \begin{bmatrix}
\frac{1}{0!0!2^0} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{0!1!2^1} & 0 & 0 & \cdots & 0 & 0 \\
\frac{-1}{1!1!2^2} & 0 & \frac{1}{0!2!2^2} & 0 & \cdots & 0 & 0 \\
0 & \frac{-1}{1!2!2^3} & 0 & \frac{1}{0!3!2^3} & \cdots & 0 & 0 \\
\frac{1}{2!2!2^4} & 0 & \frac{-1}{1!3!2^4} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{(-1)^{\frac{n-1}{2}}}{(\frac{n-1}{2})!(\frac{n-1}{2})!2^{n-1}} & 0 & \frac{(-1)^{\frac{n-3}{2}}}{(\frac{n-3}{2})!(\frac{n+1}{2})!2^{n-1}} & 0 & \cdots & \frac{1}{0!(n-1)!2^{n-1}} & 0 \\
0 & \frac{(-1)^{\frac{n-1}{2}}}{(\frac{n-1}{2})!(\frac{n+1}{2})!2^n} & 0 & \frac{(-1)^{\frac{n-3}{2}}}{(\frac{n-3}{2})!(\frac{n+3}{2})!2^n} & \cdots & 0 & \frac{1}{0!n!2^n}
\end{bmatrix}$$

However, for n being an even integer, the matrix R can be written as

$$R = \begin{bmatrix}
\frac{1}{0!0!2^0} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{1}{0!1!2^1} & 0 & 0 & \cdots & 0 & 0 \\
\frac{-1}{1!1!2^2} & 0 & \frac{1}{0!2!2^2} & 0 & \cdots & 0 & 0 \\
0 & \frac{-1}{1!2!2^3} & 0 & \frac{1}{0!3!2^3} & \cdots & 0 & 0 \\
\frac{1}{2!2!2^4} & 0 & \frac{-1}{1!3!2^4} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \frac{(-1)^{\frac{n-2}{2}}}{(\frac{n-2}{2})!(\frac{n}{2})!2^{n-1}} & 0 & \frac{(-1)^{\frac{n-4}{2}}}{(\frac{n-4}{2})!(\frac{n+2}{2})!2^{n-1}} & \cdots & \frac{1}{0!(n-1)!2^{n-1}} & 0 \\
\frac{(-1)^{\frac{n}{2}}}{(\frac{n}{2})!(\frac{n}{2})!2^n} & 0 & \frac{(-1)^{\frac{n-2}{2}}}{(\frac{n-2}{2})!(\frac{n+2}{2})!2^n} & 0 & \cdots & 0 & \frac{1}{0!n!2^n}
\end{bmatrix}$$

At jth collocation point, (7) can be written as

$$y(\xi_j,t) = \sum_{i=1}^{n+1} \xi_j^{i-1} R c_i(t), \qquad j=1,2,\ldots,n+1,$$

$$z(\xi_j,t) = \sum_{i=1}^{n+1} \xi_j^{i-1} R d_i(t), \qquad j=1,2,\ldots,n+1.$$ (8)

Now, rewrite (8) in a matrix form at jth collocation point

$$[y_j] = [X][c_i(t)], \qquad [z_j] = [X][d_i(t)],$$ (9)

where $X = [\xi_j^{i-1}]R$ and y_j represents the value of y at jth collocation point. We have

$$[X]^{-1}[y_j] = [\mathbf{c}], \qquad [X]^{-1}[z_j] = [\mathbf{d}]. \tag{10}$$

Substituting collocation coefficients from (10) in (7) results in

$$
\begin{aligned}
y(\xi, t) &= \sum_{i=1}^{n+1} \xi^{i-1} X^{-1} y_i, \\
z(\xi, t) &= \sum_{i=1}^{n+1} \xi^{i-1} X^{-1} z_i.
\end{aligned}
\tag{11}
$$

The first and second order derivatives of $y(\xi, t)$ with respect to ξ can be obtained as

$$
\begin{aligned}
\frac{\partial y}{\partial \xi} &= \sum_{i=1}^{n+1} (i-1)\xi^{i-2} X^{-1} y_i, \\
\frac{\partial^2 y}{\partial \xi^2} &= \sum_{i=1}^{n+1} (i-1)(i-2)\xi^{i-3} X^{-1} y_i.
\end{aligned}
\tag{12}
$$

Using the discretized forms of $y(\xi, t)$ and $z(\xi, t)$ in (5) leads to the following system of equations:

$$
\begin{aligned}
\frac{dy_j}{dt} &= z_j, \\
\frac{dz_j}{dt} &= \beta \sum_{i=1}^{n+1} B_{ji} y_i - \sin(y_j),
\end{aligned}
\tag{13}
$$

where $j = 2, 3, \ldots, n$.

In the above coupled form of equations, A_{ji} and B_{ji} are first and second order discretized forms of derivatives of $y(\xi)$ with respect to ξ at jth collocation point, respectively. Boundary conditions for both $y(\xi, t)$ and $z(\xi, t)$ configurations assumed to be $y(a, t) = y_1 = y_a$, $y(b, t) = y_{n+1} = y_b$, $z(a, t) = z_1 = z_a$, and $z(b, t) = z_{n+1} = z_b$. The matrix representation of (13) of sine-Gordon can be written as:

$$\begin{bmatrix} \frac{dY}{dt} \\ \frac{dZ}{dt} \end{bmatrix} = \begin{bmatrix} I & O \\ O & B \end{bmatrix} \begin{bmatrix} Z \\ Y \end{bmatrix} - F. \tag{14}$$

In the above system of equations, there are $n - 1$ collocation equations and two boundary conditions for each function $y(\xi, t)$ and $z(\xi, t)$, respectively. It results in $2(n-1)$ collocation equations in total and four boundary conditions. There is no effect of boundary conditions in the matrix representation of (14) as they are in scalar form and get merged into F. Moreover, O represents the zero matrix, and I represents the identity matrix in relation (14). Also,

$$O = \begin{bmatrix} 0 \; 0 \; 0 \ldots 0 \\ 0 \; 0 \; 0 \ldots 0 \\ \vdots \; \vdots \; \vdots \; \ddots \; \vdots \\ 0 \; 0 \; 0 \ldots 0 \end{bmatrix}_{(n-1)\times(n-1)}, \qquad I = \begin{bmatrix} 1 \; 1 \; 1 \ldots 1 \\ 1 \; 1 \; 1 \ldots 1 \\ \vdots \; \vdots \; \vdots \; \ddots \; \vdots \\ 1 \; 1 \; 1 \ldots 1 \end{bmatrix}_{(n-1)\times(n-1)},$$

$$Y = \begin{bmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}_{(n-1)\times 1}, \qquad Z = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \end{bmatrix}_{(n-1)\times 1},$$

$$B = \begin{bmatrix} B(2,2) \; B(2,3) \; B(2,4) \ldots B(2,n) \\ B(3,2) \; B(3,3) \; B(3,4) \ldots B(3,n) \\ \vdots \quad\;\; \vdots \quad\;\; \vdots \quad \ddots \quad \vdots \\ B(n,2) \; B(n,3) \; B(3,4) \ldots B(n,n) \end{bmatrix}_{(n-1)\times(n-1)},$$

and

$$F = \begin{bmatrix} O \\ \bar{F} \end{bmatrix}$$

for the Klein–Gordon equation $\bar{F}$ can be represented as

$$\bar{F} = \begin{bmatrix} \sin(y_2) \\ \sin(y_3) \\ \vdots \\ \sin(y_n) \end{bmatrix}_{(n-1)\times 1}$$

The matrix representation corresponds to a nonlinear system of $2(n-1)$ equations form block matrix structure (14). The left-hand side includes the column vector of time derivatives of functions y and z, respectively, and the right-hand side does not include any product with the inverse of a coefficient matrix, which helps to reduce the stiffness of the system of equations. The system of ordinary differential equations is solved using MATLAB with the ode23s module.

5 Convergence analysis

Theorem 3. [26] If $\{\mathcal{Q}_n(\xi)\}$ represents for simple set of polynomials and if $\mathcal{Y}(\xi)$ is a polynomial of degree m, then there exist constants a_k such that

$$\mathcal{Y}(\xi) = \sum_{k=0}^{m} a_k \mathcal{Q}_k(\xi).$$

The a_k's are functions of k and any parameter involved in $\mathcal{Y}(\xi)$.

Theorem 4. [25]There exists a unique polynomial $P_n(\xi)$ of degree n, which assumes prescribed values at $n + 1$ distinct points $\xi_0 < \xi_1 < \cdots < \xi_n$.

Theorem 5. [25] Given any interval $a \leq \xi \leq b$, real number $\varepsilon > 0$ and any real valued continuous function $f(\xi)$ on $a \leq \xi \leq b$, there exists a polynomial $P(\xi)$ such that

$$|| f(\xi) - P(\xi) || < \varepsilon.$$

To study the convergence behavior of orthogonal collocation scheme, the definition given by [21] is quoted here:

Consider a family of mathematical problems parametrized by singular perturbation parameter ε, where ε lies in the semiopen interval $0 < \varepsilon \leq 1$. Assume that each problem in the family has a unique solution denoted by y_ε, and that each y_ε is approximated by a sequence of numerical solutions $\left\{ (Y_\varepsilon, \Omega^N) \right\}_{N=1}^\infty$, where Y_ε is defined on the Ω^N representing the set of points in $\mathbf{R}$, and N is the discretization parameter. Then the numerical solutions Y_ε are said to converge to the exact solution y_ε, if there exists a positive integer N_0 and positive numbers G and p, where N_0, G, and p are all independent of N and ε, such that for all $N \geq N_0$

$$sup_{0 < \varepsilon \leq 1} \left| Y_\varepsilon - y_\varepsilon \right|_{\Omega^N} \leq GN^{-p}.$$

Here p is the rate of convergence and G is the error constant. It shows that the rate of convergence in the case of the collocation technique depends upon the number of collocation points.

Let $y(\xi, t)$ be the exact solution, and let $y_h(\xi, t)$ be the approximate solution. The absolute error between the exact and approximate solution is calculated as

$$E_a = | y(\xi, t) - y_h(\xi, t) | . \tag{15}$$

The error in terms of L_2-norm and L_∞-norm has been calculated with respect to the weight function $w(\xi)$ such that

$$||y - y_h||_2^2 = \sum_{i=1}^{n+1} |w_i(\xi)(y - y_h)_i^2|, \tag{16}$$

where $y(\xi, t)$ represent analytic solutions and $y_h(\xi, t)$ represent approximate solutions [22]. The error between exact and numerical values has been shown by $e = y - y_h$.

L_2-norm is said to converge to the exact solution if $||y - y_h||_2 \to 0$ as $n \to \infty$. Thus

$$||e||_2 = ||y - y_h||_2, \tag{17}$$

Similarly, in L_∞-norm for $||y - y_h||$, it has been taken as

$$||y - y_h||_\infty = \max |(y - y_h)_i|, \qquad i = 1, 2, 3, \ldots, n + 1, \tag{18}$$

$$||e||_\infty = ||y - y_h||_\infty. \tag{19}$$

6 Numerical examples

To verify the applicability of BCM, the scheme has been applied on different nonlinear hyperbolic equations.

Example 1. Consider the sine-Gordon nonlinear hyperbolic equation from the generalized (2) coupled form of two interacting configurations $y(\xi, t)$ and $\frac{\partial y}{\partial t}(\xi, t) = z(\xi, t)$:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial \xi^2} - \sin(y),$$

by defining

$$\frac{\partial y}{\partial t} = z,$$

$$\frac{\partial z}{\partial t} = \frac{\partial^2 y}{\partial \xi^2} - \sin(y),$$

with respect to the initial conditions

$$y(\xi, 0) = 4 \tan^{-1}(\exp(g\xi)),$$

$$z(\xi, 0) = -4cg \frac{\exp(g\xi)}{(1 + \exp(2g\xi))}.$$

Then exact solutions of the above equations have been obtained as

$$y(\xi, t) = 4 \tan^{-1}(\exp(g(\xi - ct))),$$

where $\xi \in [-3, 3]$, $g = \frac{1}{(1-c^2)^{\frac{1}{2}}}$, and $c = 0.5$, and boundary conditions can be extracted from the exact solution [9, 23, 30].

The values of $y(\xi, t)$ are calculated for different numbers of collocation points and compared with the exact solution in Table 1. The numerical values have been presented for a fixed value of $\xi = 0$ and different values of time in Table 1. The numerical results are found to be enough close to the exact solution. It is also observed that no particular change occurs in numerical values after 25 collocation points. A graphical representation of experimental results with respect to the time and collocation points has been presented in Figure 1.

A comparison of error in terms of L_2-norm and L_∞-norm at different numbers of collocation points has also been given in Table 2 for different values of time.

A graphical representation of error in terms of L_2-norm and L_∞-norm at 25 and 31 collocation points with respect to the time values have been presented in Figures 2 and 3, respectively. It can be analyzed from Figures 2 and 3 that error decreases with the increase in collocation points. From these figures, it can also be analyzed that the decrease in error is not so large and almost similar at 25 and 31 collocation points.

Numerical results have also been compared with [9, 23, 30] and have been discussed in Table 3. It is observed that the results obtained by BCM are better and give less error.

Example 2. The sine-Gordon nonlinear hyperbolic equation from the generalized (2) in coupled form of two interacting configurations $y(\xi, t)$ and $z(\xi, t)$ with respect to the initial conditions

$$y(\xi, 0) = 0,$$

$$z(\xi, 0) = 4sech(\xi),$$

the exact solution of equation has been obtained as

$$y(\xi, t) = 4 \tan^{-1}(t.sech(\xi)),$$

$$z(\xi, t) = 4 \frac{sech(\xi)}{1 + t^2 sech^2(\xi)}.$$

The boundary conditions have been taken from the exact solutions [9, 23, 30].

The numerical values calculated at 17 and 19 collocation points have been compared with the exact values and are presented in Table 4 for fixed $\xi = 0.5$ but at different values of the time. It has been observed from Table 4 that the numerical results are close enough to the exact solutions. It is also observed that no particular change in numerical values occurs after 17 collocation points. The graphical representation of numerical values with respect to the time and collocation points has been presented in Figure 4.

A comparison of error in terms of L_2-norm and L_∞-norms at 17 and 19 collocation points has been given in Table 5 at different time levels. The graphical representation of error in terms of L_2-norm and L_∞-norm at 17 and 19 collocation points with respect to the time has been presented in Figures 5 and 6, respectively. It is observed from Figures 5 and 6 that at 17 and 19 collocation points, it does not make much difference in numerical results, but if still count the difference at 19 collocation points, it shows better results than 17 collocation points.

A comparison of numerical results with [9, 23, 30] has been discussed in Table 6 and found to be close enough to be accepted.

7 Conclusion

The given nonlinear sine-Gordon equation has been solved successfully by using the BCM over Chebeshev collocation points. By the above analysis, the proposed method of BCM is proved to have some desired and popular features, such as high order accuracy and preserving energy conservation. Consistency and convergence of the computational technique have been obtained by computing the results of numerical solutions with analytic solutions. Error

analysis in terms of L_2- and L_∞-norms with respect to the weight function employed showed that the Bessel collocation approach is very stable, and the results obtained by this approach are consistent and convergent.

Table 1: Comparison of absolute error (E_a) of Example 1 at $\xi = 0$ for different numbers of collocation points

t	E_a at 8 collocation points	E_a at 16 collocation points	E_a at 25 collocation points	E_a at 31 collocation points
0.1	1.2362e-02	3.6988e-04	1.3449e-07	4.0068e-07
0.2	2.4923e-02	7.4329e-04	9.8035e-07	1.9933e-07
0.3	3.7862e-02	1.1126e-03	2.8106e-06	1.9219e-06
0.4	5.1315e-02	1.4626e-03	5.2349e-06	1.9219e-06
0.5	6.5366e-02	1.7749e-03	7.3092e-06	5.0514e-06
0.6	8.0030e-02	2.0225e-03	7.9431e-06	1.6903e-06
0.7	9.5257e-02	2.1711e-03	6.4173e-06	1.3360e-06
0.8	1.1093e-01	2.1925e-03	2.7694e-06	5.4602e-06
0.9	1.2689e-01	2.0728e-03	2.1323e-06	5.0533e-06
1.0	1.4291e-01	1.8073e-03	6.8670e-06	6.3394e-06

Table 2: Comparison of error for $y(\xi, t)$ of Example 1 at different collocation points

t	At 25 collocation points		At 31 collocation points	
	$||e||_\infty$	$||e||_2$	$||e||_\infty$	$||e||_2$
0.1	6.7963e-07	4.3658e-07	1.2869e-06	0.0000e-00
0.2	2.4566e-06	1.2844e-06	3.5131e-06	1.2717e-07
0.3	4.7272e-06	1.9502e-06	4.9638e-06	1.1174e-07
0.4	6.7848e-06	2.5348e-06	6.4823e-06	1.6430e-07
0.5	7.3091e-06	3.7359e-06	7.1190e-06	2.9715e-07
0.6	7.9430e-06	5.3058e-06	7.7593e-06	2.8624e-07
0.7	6.8875e-06	6.2351e-06	7.9344e-06	1.3433e-07
0.8	7.9192e-06	5.8165e-06	8.1705e-06	0.0000e-00
0.9	9.8062e-06	4.2143e-06	8.1533e-06	2.8093e-07

Table 3: Comparison of $||e||_2$ and $||e||_\infty$ calculated by Bessel collocation for $y(\xi, t)$ with different techniques of Example 1

t	Dehgan & Shokri [9]		Mittal & Bhatia [23]		Shukla & Tamsir [30]		Bessel collocation	
	$||e||_\infty$	$||e||_2$	$||e||_\infty$	$||e||_2$	$||e||_\infty$	$||e||_2$	$||e||_\infty$	$||e||_2$
0.25	4.95e-06	1.76e-05	4.90e-05	3.66e-05	9.61e-06	5.67e-06	4.44e-06	1.46e-07
0.50	8.42e-06	4.31e-05	7.55e-05	9.00e-05	1.10e-05	8.39e-06	7.38e-06	2.97e-07
0.75	1.65e-05	8.25e-05	1.43e-04	1.60e-04	1.26e-05	1.05e-05	7.99e-06	0.00e-00
1.00	2.51e-05	1.27e-04	2.10e-04	2.27e-04	1.44e-05	1.24e-05	1.49e-05	2.81e-07

Table 4: Comparison of absolute error (E_a) of Example 2 at $\xi = 0.5$ for different numbers of collocation points.

t	E_a at 8 collocation points	E_a at 16 collocation points	E_a at 17 collocation points	E_a at 19 collocation points
0.1	4.6603e-06	1.0754e-10	4.4239e-10	5.3072e-11
0.2	2.9466e-05	8.3158e-10	6.1702e-11	2.3970e-10
0.3	6.5426e-05	2.2862e-09	1.1462e-09	6.9454e-10
0.4	7.9905e-05	3.3772e-09	2.5359e-09	1.1188e-09
0.5	5.2043e-05	3.7229e-09	3.3682e-09	1.4447e-09
0.6	4.2174e-05	3.7595e-09	3.1963e-09	1.4496e-09
0.7	3.7627e-05	3.0807e-09	2.0244e-09	1.0735e-09
0.8	3.1964e-05	7.4804e-10	3.7364e-10	4.0962e-10
0.9	2.4734e-05	3.0871e-09	1.3676e-09	4.7658e-10
1.0	3.1170e-05	6.9712e-09	3.3311e-09	1.5792e-09

Table 5: Comparison of error for $y(\xi, t)$ of Example 2 at different collocation points

t	At 17 collocation points		At 19 collocation points	
	$||e||_\infty$	$||e||_2$	$||e||_\infty$	$||e||_2$
0.1	3.6956e-10	2.0235e-10	1.2951e-10	7.3902e-11
0.2	7.5355e-10	4.0045e-10	4.6931e-10	2.9364e-10
0.3	2.5508e-09	1.6149e-09	1.3351e-09	8.3540e-10
0.4	4.2276e-09	2.6935e-09	1.9375e-09	1.2585e-09
0.5	4.5143e-09	3.0129e-09	2.0691e-09	1.4681e-09
0.6	3.0879e-09	2.4971e-09	1.5904e-09	1.3228e-09
0.7	2.2236e-09	1.4627e-09	1.2266e-09	9.1829e-10
0.8	2.2897e-09	1.1368e-09	8.0804e-10	5.1226e-10
0.9	4.4816e-09	2.4093e-09	1.5020e-09	7.6839e-10

Table 6: Comparison of $||e||_2$ and $||e||_\infty$ calculated by Bessel collocation for $y(\xi, t)$ with different techniques of Example 2

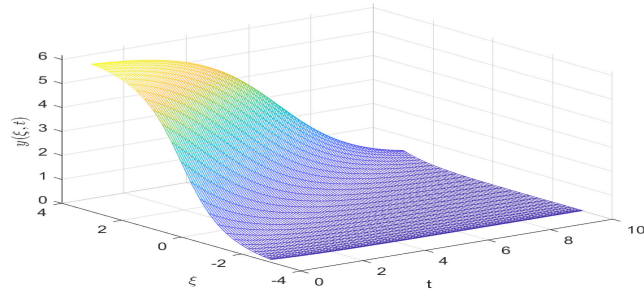| t	Dehgan& Shokri [9]		Mittal& Bhatia [23]		Shukla& Tamsir [30]		Bessel collocation	
	$||e||_\infty$	$||e||_2$	$||e||_\infty$	$||e||_2$	$||e||_\infty$	$||e||_2$	$||e||_\infty$	$||e||_2$
0.25	5.89e-06	3.91e-05	2.32e-05	1.18e-05	5.46e-06	2.43e-06	9.06e-10	5.64e-10
0.50	2.01e-05	1.30e-04	4.11e-05	4.19e-05	7.39e-06	5.54e-06	1.88e-09	1.44e-09
0.75	3.63e-05	2.35e-04	1.02e-04	7.78e-05	7.78e-06	6.45e-06	1.02e-09	6.87e-10
1.00	5.07e-05	3.27e-04	1.64e-04	1.30e-04	8.75e-06	7.84e-06	2.54e-09	1.53e-09

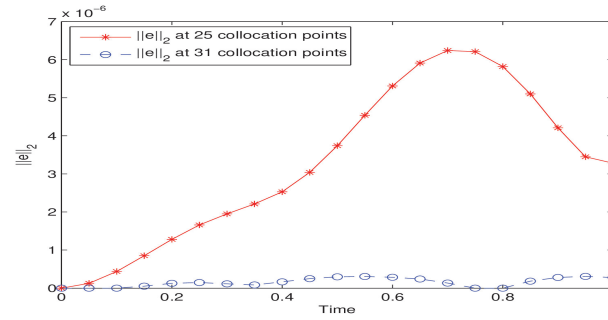Figure 1: Graphical representation of $y(\xi, t)$ of Example 1

Figure 2: Graphical representation of comparison of error in form of L_2-norm with respect to time and number of collocation points of Example 1
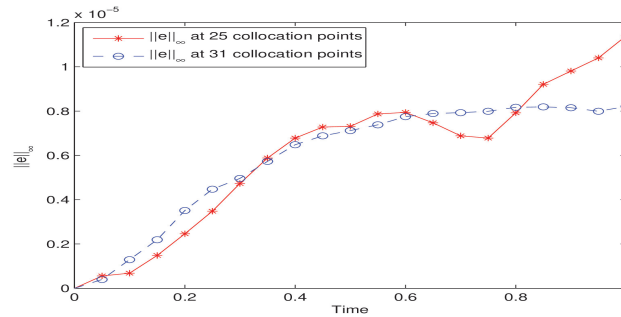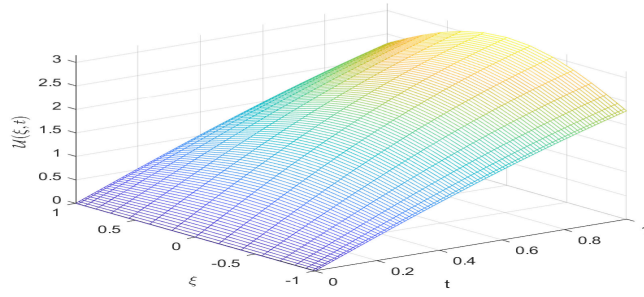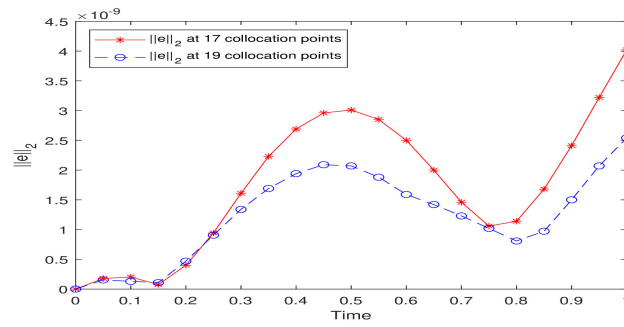

Figure 3: Graphical representation of comparison of error in form of L_∞-norm with respect to time and number of collocation points of Example 1

Figure 4: Graphical representation of $y(\xi, t)$ of Example 2

Figure 5: Graphical representation of comparison of error in form of L_2-norm with respect to time and number of collocation points of Example 2
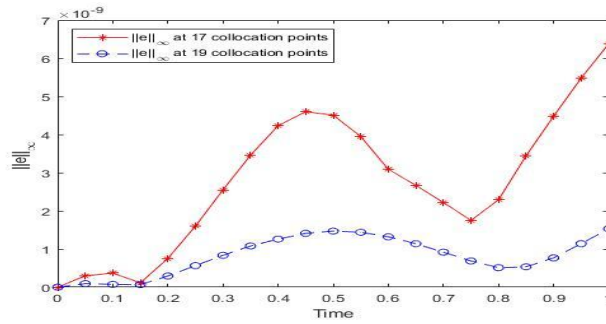

Figure 6: Graphical representation of comparison of error in form of L_∞-norm with respect to time and number of collocation points of Example 2

References

[1] Ablowitz, M.J., Herbst, B.M. and Schober, C.M. *Numerical simulation of quasi-periodic solutions of the sine-Gordon equation*, Phys. D

87 (1995), 37–47.

[2] Ablowitz, M.J., Herbst, B.M. and Schober, C.M. *On the numerical solution of the sine-Gordon equation*, J. Comput. Phys. 131 (1997), 354–367.

[3] Ablowitz, M.J., Kruskal, M.D. and Ladik, J.F. *Solitary wave collisions*, SIAM J. Appl. Math. 36 (1979), 428–443.

[4] Abramowitz, M. and Stegun, I.A. *Handbook of mathematical functions with formulas and mathematical tables*, with corrections Superintendent of Documents. National Bureau of Standards Applied Mathematics Series, No. 55 U. S. Government Printing Office, Washington, D.C., 1965,

[5] Argyris, J. and Haase, M. *An engineer's guide to soliton phenomena: application of the finite element method*, Comput. Meth. Appl. Mech. Eng. 61(1) (1987), 71–122.

[6] Arora, S., Dhaliwal, S.S. and Kukreja, V.K. *Solution of two point boundary value problems using orthogonal collocation on finite elements*, Appl. Math. Comput. 171 (2005), 358–370.

[7] Bailey, W.N. *Generalized Hypergeometric Series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 32 Stechert-Hafner, Inc., New York, 1964,

[8] Barone, A., Esposito, F., Magee, C.J., and Scott, A.C. *Theory and applications of the sine-Gordon equation*, Riv. Nuovo Cimento. 1 (1971), 227–267.

[9] Dehghan, M. and Shokri, A. *A numerical method for one-dimensional nonlinear Sine-Gordon equation using collocation and radial basis functions*, Numer. Methods. Partial. Differ. Eq. 24(2) (2008), 687–698.

[10] Evans, W.D., Everitt, W.N., Kwon, K.H. and Littlejohn, L.L. *Real orthogonalizing weights for Bessel polynomials*, J. Comput. Appl. Math. 49 (1993), 51–57.

[11] Everitt, W.N. and Markett, C. *On a generalization of Bessel functions satisfying higher-order differential equations*, J. Comput. Appl. Math. 54 (1994), 325–349.

[12] Ferguson, N.B. and Finlayson, B.A. *Transient chemical reaction analysis by orthogonal collocation*, Chem. Eng. J. 1(4) (1970), 327–336.

[13] Finlayson, B.A. *Packed bed reactor analysis by orthogonal collocation*, Chem. Eng. Sci. 26 (1971), 1081–1091.

[14] Gautschi, W. *Numerical Analysis*, Second Edition, Springer-Verlag, New York, 2012.

[15] Ilati M. and Dehghan, M. *The use of radial basis functions (RBFs) collocation and RBF-QR methods for solving the coupled nonlinear sine-Gordon equations*, Eng. Anal. Bound. Elem. 52 (2015), 99–109.

[16] Jianga, C., Sun, J., Li, H. and Wang, Y. *A fourth-order AVF method for the numerical integration of sine-Gordon equation*, Appl. Math. Comput. 313 (2017), 144–158.

[17] Koornwinder, T.H. *Orthogonal polynomial with weight function* $(1 - x)^{\alpha}(1 - x)^{\beta} + M\delta(x - 1) + N\delta(x - 1)$, Can. Math. Bull. 27(2) (1984), 205–214.

[18] Kumar, D., Singh, J., Kumar and S., Sushila *Numerical computation of Klein–Gordon equations arising in quantum field theory by using homotopy analysis transform method*, Alex. Eng. J. 53 (2014), 469–474.

[19] Martin-Vergara, F., Rus, F. and Villatoro, F.R. *Padé numerical schemes for the sine-Gordon equation*, Appl. Math. Comput. 358 (2019), 232–243.

[20] McLachlan, N.W. *Bessel functions for engineers*, University of Illinois, Oxford University Press, London, England. 1961.

[21] Miller, J.J.H., O'Riordan, E. and Shishkin, G.I. *Fitted Numerical methods for singular perturbation problems, error estimate in the maximum norm for linear problems in one and two dimensions*, World Scientific, 1996.

[22] Mishra, P., Sharma, K.K., Pani, A.K. and Fairweather, G. *Orthogonal spline collocation for singularly perturbed reaction diffusion problems in one dimension*, Int. J. Numer. Anal. Model. 16(4) (2019), 647–667.

[23] Mittal, R.C., and Bhatia, R. *Numerical solution of nonlinear Sine-Gordon equation by modified cubic B-Spline collocation method*, Int. J. Partial Differ. Equ. 2014 (2014), 1–8.

[24] Perring, J.K. and Skyrme, T.H.R. *A model unified field equation*, Nucl. Phys. 31 (1962), 550–555.

[25] Prenter, P.M. *Splines and variational methods*, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1975.

[26] Rainville, E.D. *Special functions*, Reprint of 1960 first edition. Chelsea Publishing Co., Bronx, N.Y., 1971.

[27] Saray, B.N., Lakestani, M. and Cattani, C. *Evaluation of mixed Crank-Nicolson scheme and tau method for the solution of Klein-Gordon equation*, App. Math. Comput. 331 (2018), 169–181.

[28] Scott, A.C., Chu, F.Y.F. and McLaughlin, D.W. *The soliton: A new concept in applied science*, Proc. IEEE 61 (1973), 1443–1483.

[29] Shan, Y., Liu, W. and Wu, B. *Space-time Legendre-Gauss-Lobatto collocation method for two-dimensional generalized sine-Gordon equation*, Appl. Numer. Math. 122 (2017), 92–107.

[30] Shukla, H.S. and Tamsir M. *Numerical solution of nonlinear Sine–Gordon equation by using the modified cubic B-spline differential quadrature method*, Beni-Seuf Univ. J. Basic Appl. Sci. 7 (2018), 359–366.

[31] Shukla, H.S., Tamsir, M. and Srivastava, V. K. *Numerical simulation of two dimensional sine-Gordon solitons using modified cubic B-spline differential quadrature method*, AIP Adv. 5(1) (2015), 017121.

[32] Sirendaoreji *A new auxiliary equation and exact travelling wave solutions of nonlinear equations*, Phys. Lett. A 356(2) (2006), 124–130.

[33] Sirendaoreji *Auxiliary equation method and new solutions of Klein–Gordon equations*, Chaos Solitons Fractals 31(4) (2007), 943–950.

[34] Sneddon, I. N. *Special functions of mathematical physics and chemistry,* Oliver and Boyd, Edinburgh-London; Interscience Publishers, Inc., New York, 1956.

[35] Stempak, K. *A weighted uniform L^p-estimate of Bessel functions: a note on a paper of K. Guo: "A uniform L^p estimate of Bessel functions and distributions supported on S^{n-1}" [Proc. Amer. Math. Soc. 125 (1997), no. 5, 1329–1340; MR1363462 (97g:46047)],* Proc. Amer. Math. Soc. 128(10) (2000), 2943–2945.

[36] Strauss, W. and Vazquez, L. *Numerical solution of a nonlinear Klein-Gordon equation*, J. Comput. Phys. 28 (1978), 271–278.

[37] Tasbozan, O., Yagmurlu, N.M.,Ucar, Y. and Esen, A. *Numerical solutions of the Sine-Gordon equation by collocation method*, Sohag J. Math. 3(1) (2016), 1–6.

[38] Villadsen, J.V. and Stewart, W.E. *Solution of boundary value problem by orthogonal collocation*, Chem. Eng. Sci. 22 (1967), 1483–1501.

[39] Watson, G.N. *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge, England; The Macmillan Company, New York, 1944.

[40] Wazwaz, A.M. *New travelling wave solutions to the Boussinesq and the Klein–Gordon equations*, Commun. Nonlinear Sci. Numer. Simul. 13(5) (2008), 889–901.

[41] Wingate, C. *Numerical search for a ϕ^4 breather mode*, SIAM J. Appl. Math. 43 (1983), 120–140.

[42] Yüzbaşı, Ş. *A numerical approach for solving a class of the nonlinear Lane–Emden type equations arising in astrophysics*, Math. Methods Appl. Sci. 34(8) (2011), 2218–2230.

[43] Yüzbaşı, Ş. *A numerical approach for solving the high-order linear singular differential-difference equations*, Comput. Math. Appl. 62(5) (2011), 2289–2303.

[44] Yüzbaşı, Ş. *A numerical approximation based on the Bessel functions of first kind for solutions of Riccati type differential-difference equations*, Comput. Math. Appl. 64(6) (2012), 1691–1705.

[45] Yüzbaşı, Ş. *Bessel collocation approach for solving continuous population models for single and interacting species*, Appl. Math. Model. 36 (2012), 3787–3802.

[46] Yüzbaşı, Ş., Şahin, N. and Sezer, M. *A Bessel polynomial approach for solving linear neutral delay differential equations with variable coefficients*, J. Adv. Res. Appl. Math. 3(1) (2011), 81–101.

[47] Yüzbaşı, Ş., Şahin, N. and Sezer, M. *Bessel matrix method for solving high-order linear Fredholm integro-differential equations*, J. Adv. Res. Appl. Math. 3(2) (2011), 23–47.

[48] Yüzbaşı, Ş., Şahin, N. and Sezer, M. *Numerical solutions of systems of linear Fredholm integro-differential equations with Bessel polynomial bases*, Comput. Math. Appl. 61(10) (2011), 3079–3096.

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