Iranian Journal of Numerical Analysis and Optimization
Vol. 13, No. 3, 2023, pp 481–499
https://doi.org/10.22067/ijnao.2022.73126.1189
https://ijnao.um.ac.ir/

How to cite this article
Research Article

Analysis and optimal control of a fractional MSD model

M. Bagherpoorfard and F. Akhavan Ghassabzade*,

Abstract

In this research, we aim to analyze a mathematical model of Maize streak virus disease as a problem of fractional optimal control. For dynamical analysis, the boundedness and uniqueness of solutions have been investigated and proven. Also, the basic reproduction number is obtained, and local stability conditions are given for the equilibrium points of the model. Then, an optimal control strategy is proposed for the purpose of examining the best strategy to fight the maize streak disease. We solve the fractional optimal control problem by a forward-backward sweep iterative algorithm. In this algorithm, the state variable is obtained in a forward and co-state variable by a backward method where an explicit Runge-Kutta method is used to solve differential equations arising from fractional optimal control problems. Some comparative results are presented in order to verify the model and show the efficacy of the fractional optimal control treatments.

AMS subject classifications (2020): Primary 34A08, Secondary 65K10, 92B05.

Keywords: Fractional differential equation; Maize streak virus; Fractional-order optimal control; Sweep method; Numerical simulation.

1 Introduction

Maize is an important annual cereal crop of the world belonging to the family Poaceae. It is considered a staple food in many parts of the world. It

*Corresponding author

Received 8 October 2022; revised 29 February 2023; accepted 1 April 2023

Mina Bagherpoorfard
Department of Mathematics, Fasa Branch, Islamic Azad university, Fasa, Iran.
e-mail: bagherpoorfard@yahoo.com

Fahimeh Akhavan Ghassabzade
Department of Mathematics, Faculty of sciences, University of Gonabad, Gonabad, Iran.
e-mail: akhavan_gh@yahoo.com

is the third leading crop in the world after rice and wheat [20]. Due to its highest yield potential among cereals, it is known globally as the queen of cereals. Maize streak disease (MSD) is the most serious viral crop disease in Sub-Saharan Africa. This disease is caused by the Maize streak virus (MSV), which was first described by the South African entomologist Claude Fuller in 1901 [12]. MSV is mainly transmitted by as many as six leafhopper species in the Genus Cicadulina, but some other leafhopper species are also able to transmit the virus. In addition to maize, this virus can infect over 80 other species in the Family Poaceae. Severe MSD manifests as pronounced, continuous parallel chlorotic streaks on leaves, with severe stunting of the affected plant and, usually, a failure to produce complete cobs or seeds. Erratic epidemics have been occurring every 3-10 years, and the main damage caused is to plants younger than six weeks old [24].

In recent years, mathematical modeling has become a valuable tool to study the mechanisms of plant disease spread, predict the future course of an outbreak, and appraise strategies to control. In most cases, differential equations of the integer order have been used to construct such models; see, for example, [25, 8, 9, 14] and the references therein. The integer-order derivatives and integrals have local properties; that is, the next state is not influenced by the current and previous state. So, the integer-order mathematical models can not describe natural phenomena precisely.

Fractional calculus is an extension of classical calculus that introduces derivatives and integrals of fractional order. Fractional derivatives have nonlocal properties, that is the next state depends on the current state and all previous states. This is the main excellence of fractional derivatives over classical derivatives. Due to this advantage, many applications of fractional calculus can be found in various fields of research, such as biology, economy, physics, control theory, and so on [15, 13, 21, 22, 26, 2]. In [23], a fractional model of tuberculosis disease has presented, and the values of parameters have been evaluated according to the actual clinical cases. In 2020, the dynamics of the fractional HIV infection model were studied by Evirgen Evirgen, Uçar, and Özdemir [11]. Bozkurt et al., in their work [6], have analyzed a fractional model of COVID-19 by considering the fear effects of the media and social networks. In [5], the authors presented a fractional model for the simulation of the Cholera outbreak in Yemen. The authors in [4] proposed a fractional model to study the dynamics of the MSV in the maize plant population by considering the interaction of MSV pathogen with the past invasion.

In light of this significant advantage, we were motivated to develop the model investigated in [3] into a new fractional model involving the Caputo derivatives. The Caputo derivative is of use for modeling phenomena that take account of interactions within the past and also problems with nonlocal properties. In this sense, one can think of the equation as having "memory." After that, we discuss some properties of the fractional version of the model under consideration. Next, fractional optimal control (FOC) is applied as a generalization of the classical optimal control system [3]. The FOC model is

developed with three time-dependent control strategies proposed by Alemneh, Kassa, and Godana [3].

The paper is organized as follows. In section 2, we give a brief review of the Caputo operator and discuss its basic characteristics. In Section 3, the fractional-order model formulation is presented, and the main properties of the fractional model are then given in section 4. Section 5 focuses on the dynamic analysis of the model. The FOC of the model and numerical simulations of the fractional model are presented in section 6. Section 7 also contains concluding remarks.

2 Basic definitions and facts

In this section, we give a brief review of the Caputo operator and discuss its basic characteristics [16].

Definition 1. For a function $f : [0, t_f] \to \mathbb{R}, \nu \in (n-1, n)$, and $n \in \mathbb{N}$, the left- and the right-sided Caputo fractional derivatives of order ν of a function f are defined in the following forms:

$$ {}_0^C\mathfrak{D}_t^\nu f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-u)^{(n-\nu-1)} f^{(n)}(u)du, \quad t > 0, \tag{1} $$

and

$$ {}_{t_f}^C\mathfrak{D}_t^\nu f(t) = \frac{(-1)^n}{\Gamma(n-\nu)} \int_t^{t_f} (t-u)^{(n-\nu-1)} f^{(n)}(u)du, \quad t < t_f. \tag{2} $$

Here, $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2. The integral operators related to (1) and (2), are specified by

$$ {}_0^C\mathfrak{I}_t^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\eta)^{\nu-1} f(\eta)d\eta, \tag{3} $$

$$ {}_t^C\mathfrak{I}_{t_f}^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_t^{t_f} (\eta-t)^{\nu-1} f(\eta)d\eta. \tag{4} $$

Additionally, if $f \in C^n[a, b]$, then

$$ {}_0^C\mathfrak{I}_t^\nu [{}_0^C\mathfrak{D}_t^\nu f(t)] = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k, \tag{5} $$

$$ {}_t^C\mathfrak{I}_{t_f}^\nu [{}_t^C\mathfrak{D}_{t_f}^\nu f(t)] = f(t) - \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(t_f)}{k!} (t_f - t)^k. \tag{6} $$

For any $\alpha_1, \alpha_2 \in \mathbb{R}$ and $f_1, f_2 \in \mathbb{H}^1(0, t_f)$, we have

$$ {}_0^C\mathfrak{D}_t^\nu (\alpha_1 f_1(t) + \alpha_2 f_2(t)) = \alpha_1 \, {}_0^C\mathfrak{D}_t^\nu f_1(t) + \alpha_2 \, {}_0^C\mathfrak{D}_t^\nu f_2(t), \tag{7} $$

$$ {}_t^C\mathfrak{D}_{t_f}^{\nu}(\alpha_1 f_1(t)+\alpha_2 f_2(t)) = \alpha_1\, {}_t^C\mathfrak{D}_{t_f}^{\nu} f_1(t) + \alpha_2\, {}_t^C\mathfrak{D}_{t_f}^{\nu} f_2(t), \tag{8} $$

$$ {}_0^C\mathfrak{I}_{t}^{\nu}(\alpha_1 f_1(t)+\alpha_2 f_2(t)) = \alpha_1\, {}_0^C\mathfrak{I}_{t}^{\nu} f_1(t) + \alpha_2\, {}_0^C\mathfrak{I}_{t}^{\nu} f_2(t), \tag{9} $$

$$ {}_t^C\mathfrak{I}_{t_f}^{\nu}(\alpha_1 f_1(t)+\alpha_2 f_2(t)) = \alpha_1\, {}_t^C\mathfrak{I}_{t_f}^{\nu} f_1(t) + \alpha_2\, {}_t^C\mathfrak{I}_{t_f}^{\nu} f_2(t). \tag{10} $$

Let $f(t)$ be a constant function. Then

$$ {}_0^C\mathfrak{D}_{t}^{\nu} f(t) = {}_t^C\mathfrak{D}_{t_f}^{\nu} f(t) = 0. \tag{11} $$

The Caputo derivatives satisfy the Lipschitz condition.

3 New fractional model of MSV disease in maize plant

In this section, we develop a deterministic eco-epidemiological fractional model for the dynamics of MSV disease in maize plants. The original version of this model is a system of ordinary differential equations that have been before presented in [3]. The effect of previous states in the current states of the disease spread has not been considered in this model. One way to overcome this drawback is to replace the integer-order derivatives in the model with noninteger-order derivatives [19]. Hence, we replace the ordinary derivative with the following Caputo fractional derivative operator

$$ \frac{d}{dt} \longrightarrow \frac{1}{\varrho^{1-\nu}}\, {}_0^C\mathfrak{D}_{t}^{\nu}, \tag{12} $$

where the auxiliary parameter $\varrho > 0$ represents the fractional time components in the system. Thus, the new model is described by the system

$$ \begin{cases} \varrho^{\nu-1}\, {}_0^C\mathfrak{D}_{t}^{\nu} S(t) = rS(1-\dfrac{S+I}{K}) - \dfrac{\beta_1 SY}{A+S}, \\ \varrho^{\nu-1}\, {}_0^C\mathfrak{D}_{t}^{\nu} I(t) = \dfrac{\beta_1 SY}{A+S} - \mu_1 I, \\ \varrho^{\nu-1}\, {}_0^C\mathfrak{D}_{t}^{\nu} H(t) = q - \frac{\beta_2 IH}{C+I} - \mu_2 H, \\ \varrho^{\nu-1}\, {}_0^C\mathfrak{D}_{t}^{\nu} Y(t) = \dfrac{b\beta_2 IH}{C+I} - \mu_3 Y, \end{cases} \tag{13} $$

$$ S(0) = S_0,\ \ I(0) = I_0,\ \ H(0) = H_0,\ \ Y(0) = Y_0, \tag{14} $$

where $0 < \alpha \leq 1, N_1(t) = S(t)+I(t), N_2(t) = H(t)+Y(t)$, and $(S, I, H, Y) \in \mathbb{R}_+^4$. In this model, $S(t)$ denotes the density of the susceptible maize, and $I(t)$ denotes the density of the infected maize. The susceptible and infected leafhopper vector densities are denoted by $H(t)$ and $Y(t)$, respectively. All parameters in the model are nonnegative. Description of the parameters are found in Table 1.

Table 1: Explanation of MSV model parameters

Parameter	Explanation
β_1	Predation and infection rate of infected leafhopper on susceptible maize plant
β_2	Predation and infection rate of susceptible leafhopper on infected maize plant
A	The half-saturation rate of susceptible maize with infected plant
C	The half-saturation rate of susceptible leafhopper with infected maize plant
K	Carrying capacity
q	Recruitment rate of susceptible leafhopper
b	Infected leafhopper conversion rate
r	Maize population intrinsic growth rate
μ_1	Death rate of infected maize
μ_2	Death rate of susceptible leafhopper
μ_3	Infected leafhopper death rate

As can be observed, model (13) involves a system of nonlinear fractional differential equations. The exact solution of this model may not be available in general. However, a mathematical analysis of the existence and uniqueness of the solution ensures that a unique solution exists under some conditions.

4 Properties of the model

In the following, the model's main properties are provided. Our model can be formulated as

$$\begin{aligned} &{}_0^C\mathfrak{D}_t^\nu V(t) = \Phi(t, V(t)), \\ &V(0) = V_0, \end{aligned} \tag{15}$$

where $V(t) = (S(t), I(t), H(t), Y(t))$.

Lemma 1. [17] Let $w(t)$ be a continuous function on $[t_0, \infty)$ and satisfying

$$\begin{cases} {}_0^C\mathfrak{D}_t^\nu w(t) \leq -\lambda w(t) + \mu, \\ w(t_0) = w_0, \end{cases} \tag{16}$$

where $0 < \nu < 1, (\lambda, \mu) \in \mathbb{R}^2$, $\lambda \neq 0$, and $t_0 \geq 0$ is the initial time. Then

$$w(t) \leq (w_0 - \frac{\mu}{\lambda}) E_\nu[-\lambda(t - t_0)^\nu] + \frac{\mu}{\lambda},$$

where E_ν represents Mittag–Leffler function.

Lemma 2. [7] Let $0 < \nu < 1$ and $\lambda < 0$. Then $E_{\nu,\nu}(\lambda t^\nu)$ tends monotonically to zero as $t \to \infty$.

Lemma 3. [10] Let $\Phi : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function and Lipschitz-continuous respecting to the second variable. In addition to, let $\nu \in (0, 1]$ and $V_0 \in \mathbb{R}^n$. Then, the problem

$$\begin{aligned} &{}_0^C\mathfrak{D}_t^\nu V(t) = \Phi(t, V(t)), \quad t > t_0, \\ &V(t_0) = V_0, \end{aligned} \tag{17}$$

has a unique solution in $C([0,\infty);\mathbb{R}^n)$.

Theorem 1. All solutions of system (13) that initiate in $\mathbb{R}^4_+$ are bounded within the region Ω defined by
$\Omega = \{(S,I,H,Y) \in \mathbb{R}^4_+ | S(t) + I(t) + H(t) + \frac{1}{b}Y(t) \leq \frac{L}{\rho} + \varepsilon, \text{ for all } \varepsilon > 0\}$.

Proof. Define a time-dependent function $w(t) = S(t) + I(t) + H(t) + \frac{1}{b}Y(t)$. So, for any positive number ρ, we have

$$\begin{aligned} &{}_0^C\mathfrak{D}_t^\nu w(t) + \rho w(t) \\ &= rS(1 - \frac{S+I}{K}) - \mu_1 I - \mu_2 H - \frac{\mu_3}{b}Y + \rho S + \rho I + \rho H + \frac{\rho}{b}Y + q \\ &\leq rS(1 - \frac{S}{K}) + (\rho - \mu_1)I + (\rho - \mu_2)H + (\rho - \mu_3)\frac{1}{b}Y + q \\ &= (r+\rho)S - \frac{r}{K}S^2 + (\rho - \mu_1)I + (\rho - \mu_2)H + (\rho - \mu_3)\frac{1}{b}Y + q \\ &\leq \frac{K}{4r}(r+\rho)^2 + (\rho - \mu_1)I + (\rho - \mu_2)H + (\rho - \mu_3)\frac{1}{b}Y + q. \end{aligned}$$

Taking $\rho < \min(\mu_1, \mu_2, \mu_3)$, so

$$ {}_0^C\mathfrak{D}_t^\nu w(t) + \rho w(t) \leq L,$$

where $L = \frac{K}{4r}(r+\rho)^2 + q$. Now, we apply Lemma 1 and obtain

$$w(t) \leq (w(0) - \frac{L}{\rho})E_\nu[-\rho t^\nu] + \frac{L}{\rho}.$$

Thus, $w(t) \to \frac{L}{\rho}$ as $t \to \infty$ and $0 < w(t) \leq \frac{L}{\rho}$. Hence all solutions of system (13) that starts from $\mathbb{R}^4_+$ are confined in the region $\Omega = \{(S,I,H,Y) \in \mathbb{R}^4_+ | w(t) \leq \frac{L}{\rho} + \varepsilon, \text{ for all } \varepsilon > 0\}$. □

Now, we study the existence and uniqueness of system (13) in the region $\Lambda \times [0,T]$, where

$$\Lambda = \{(S,I,H,Y)\epsilon\mathbb{R}^4 : \max(|S|,|I|,|H|,|Y|) \leq M\},$$

$T < \infty$ and M is sufficiently large.

Theorem 2. For any nonnegative initial conditions, system (13) has a unique solution.

Proof. Let $X = (S, I, H, Y)$. Consider a mapping

$$Q(X) = (Q_1(X), Q_2(X), Q_3(X), Q_4(X)),$$

where

$$\begin{aligned}
Q_1(X) &= rS(1 - \frac{S+I}{K}) - \frac{\beta_1 SY}{A+S},\\
Q_2(X) &= \frac{\beta_1 SY}{A+S} - \mu_1 I,\\
Q_3(X) &= q - \frac{\beta_2 IH}{C+I} - \mu_2 H,\\
Q_4(X) &= \frac{b\beta_2 IH}{C+I} - \mu_3 Y.
\end{aligned}$$

For any $X, \overline{X} \in \Lambda$, we have

$$\begin{aligned}
&\|Q(X) - Q(\overline{X})\|\\
&\quad = |Q_1(X) - Q_1(\overline{X})| + |Q_2(X) - Q_2(\overline{X})| + |Q_3(X) - Q_3(\overline{X})| + |Q_4(X) - Q_4(\overline{X})|\\
&\quad = |rS(1 - \frac{S+I}{K}) - \frac{\beta_1 SY}{A+S} - r\overline{S}(1 - \frac{\overline{S}+\overline{I}}{K}) + \frac{\beta_1 \overline{S}.\overline{Y}}{A+\overline{S}}|\\
&\qquad + |\frac{\beta_1 SY}{A+S} - \mu_1 I - \frac{\beta_1 \overline{S}.\overline{Y}}{A+\overline{S}} + \mu_1 \overline{I}| + |q - \frac{\beta_2 IH}{C+I} - \mu_2 H - q + \frac{\beta_2 \overline{I}.\overline{H}}{C+\overline{I}} + \mu_2 \overline{H}|\\
&\qquad + |\frac{b\beta_2 IH}{C+I} - \mu_3 Y - \frac{b\beta_2 \overline{I}.\overline{H}}{C+\overline{I}} + \mu_3 \overline{Y}|\\
&\quad = |r(S - \overline{S}) - \frac{r}{K}(S^2 - \overline{S}^2) - \frac{r}{K}(SI - \overline{S}.\overline{I}) - \beta_1(\frac{SY}{A+S} - \frac{\overline{S}.\overline{Y}}{A+\overline{S}})|\\
&\qquad + |\beta_1(\frac{SY}{A+S} - \frac{\overline{S}.\overline{Y}}{A+\overline{S}}) - \mu_1(I - \overline{I})| + |-\beta_2(\frac{IH}{C+I} - \frac{\overline{I}.\overline{H}}{C+\overline{I}}) - \mu_2(H - \overline{H})|\\
&\qquad + |b\beta_2(\frac{IH}{C+I} - \frac{\overline{I}.\overline{H}}{C+\overline{I}}) - \mu_3(Y - \overline{Y})|\\
&\quad \le (r + \frac{3rM}{K})|S - \overline{S}| + 2\beta_1|\frac{SY}{A+S} - \frac{\overline{S}.\overline{Y}}{A+\overline{S}}| + \beta_2(1+b)|\frac{IH}{C+I} - \frac{\overline{I}.\overline{H}}{C+\overline{I}}|\\
&\qquad + \mu_1|I - \overline{I}| + \mu_2|H - \overline{H}| + \mu_3|Y - \overline{Y}|\\
&\quad \le (r + \frac{3rM}{K} + \frac{2\beta_1 M}{A})|S - \overline{S}| + (\frac{2\beta_1(A+M)M}{A^2} + \mu_3)|Y - \overline{Y}|\\
&\qquad + (\frac{\beta_2 M}{C}(1+b) + \mu_1)|I - \overline{I}| + (\frac{\beta_2 M(C+M)}{C^2}(1+b) + \mu_2)|H - \overline{H}|\\
&\quad \le H\|X - \overline{X}\|,
\end{aligned}$$

where

$$\begin{aligned}
H = \max\{&r + \frac{3rM}{K} + \frac{2\beta_1 M}{A}, \frac{2\beta_1(A+M)M}{A^2} + \mu_3,\\
&\frac{\beta_2 M(1+b)}{C} + \mu_1, \frac{\beta_2 M(C+M)(1+b)}{C^2} + \mu_2\}.
\end{aligned}$$

Thus, $Q(X)$ satisfies the Lipschitz condition with respect to X. Hence there exists a unique solution of the system (13) with conditions (14) on $\Lambda \times [0, T]$. □

5 Dynamical behaviors

One of the key concepts in epidemiology is the basic reproduction number (BRN). The aim of this section is to obtain the BRN for model (13) and study the local stability behavior of the model at its disease-free equilibriums.

5.1 Basic reproduction number

Consider the following fractional differential system:

$$\begin{cases} \varrho^{\nu-1}\, {}_0^C\mathfrak{D}_t^\nu X(t) = F(X,Y), \\ \varrho^{\nu-1}\, {}_0^C\mathfrak{D}_t^\nu Y(t) = G(X,Y), \\ G(X,0) = 0, \end{cases} \tag{18}$$

with nonnegative initial conditions $X(0) = X_0 \in \mathbb{R}^2$ and $Y(0) = Y_0 \in \mathbb{R}^2$, where the components of vector $X = (S, H)$ represent the number of susceptible maize and leafhopper, and the components of vector $Y = (I, Y)$ indicate the number of infected maize and leafhopper. Furthermore, we presume that the function G is of class C^1, F is continuous, and the system (18) with the initial conditions $X(0) = X_0$ and $Y(0) = Y_0$ admits a unique solution. Also, suppose that $E = (X^*, 0) \in \mathbb{R}^4$ denotes the disease-free equilibrium point of the system (18). Let $A = \frac{\partial G}{\partial Y}(X^*, 0) = M - D$, where M, D are two square matrices that $D > 0$ is a diagonal matrix and $M \geq 0$. Then the BRN $\mathcal{R}_0$ is obtained as the spectral radius of MD^{-1}.
For system (13), we have

$$A = \begin{bmatrix} -\mu_1 & \frac{\beta_1 S}{A+S} \\ \frac{b\beta_2 HC}{(C+I)^2} & -\mu_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\beta_1 S}{A+S} \\ \frac{b\beta_2 HC}{(C+I)^2} & 0 \end{bmatrix} - \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_3 \end{bmatrix} \tag{19}$$

So,

$$\mathcal{R}_0 := \rho\left(\begin{bmatrix} 0 & \frac{\beta_1 S}{\mu_3(A+S)} \\ \frac{b\beta_2 HC}{\mu_1(C+I)^2} & 0 \end{bmatrix} \right) = \sqrt{\frac{b\beta_1\beta_2 HCS}{\mu_1\mu_3(A+S)(C+I)^2}}. \tag{20}$$

5.2 Local stability analysis

Theorem 3. The disease free-equilibrium point $E_0 = (0, 0, \frac{q}{\mu_2}, 0)$ of system (13) is always unstable while the disease-free equilibrium point $E_1 = (K, 0, \frac{q}{\mu_2}, 0)$, is locally asymptotically stable if $\mathcal{R}_0 < 1$.

Proof. Conforming to Mittag-Leffler function [18], the disease free equilibrium E of system (13) is locally asymptotically stable if all eigenvalues $\lambda_i, i = 1, 2, 3, 4$ of J_E satisfy $|arg(\lambda_i)| > \frac{\nu\pi}{2}, i = 1, 2, 3, 4$. The Jacobian matrix associated to E_0 is given by

$$J_{E_0} = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & -\mu_1 & 0 & 0 \\ 0 & \frac{-\beta_2 q}{C\mu_2} & -\mu_2 & 0 \\ 0 & \frac{b\beta_2 q}{C\mu_2} & 0 & -\mu_3 \end{bmatrix}.$$

The eigenvalues of the matrix J_{E_0} are $\lambda_1 = r > 0, \quad \lambda_2 = -\mu_1 < 0 \quad \lambda_3 = -\mu_2 < 0 \quad \lambda_4 = -\mu_3 < 0$. We observed that $|arg(\lambda_1)| = 0 < \frac{\nu\pi}{2}$. So, the equilibrium point E_0 is unstable.

The Jacobian matrix associated to E_1 is the following one:

$$J_{E_1} = \begin{bmatrix} -r & -r & 0 & \frac{-\beta K}{A+K} \\ 0 & -\mu_1 & 0 & \frac{\beta K}{A+K} \\ 0 & \frac{-\beta_2 q}{C\mu_2} & -\mu_2 & 0 \\ 0 & \frac{b\beta_2 q}{C\mu_2} & 0 & -\mu_3 \end{bmatrix}.$$

The following characteristic equation is obtained from J_{E_1}:

$$\phi(\lambda) = (r + \lambda)(\mu_2 + \lambda)\left(\lambda^2 + (\mu_1 + \mu_3)\lambda + (\mu_1\mu_3 - \frac{Kbq\beta_1\beta_2}{(A + K)\mu_2 C})\right).$$

We observe that two roots of the characteristic equation $\phi(\lambda)$ are

$$\lambda_1 = -r < 0, \quad \lambda_2 = -\mu_2 < 0.$$

It is obvious that $|arg(\lambda_1)| > \frac{\nu\pi}{2}$ and $|arg(\lambda_2)| > \frac{\nu\pi}{2}$. The remaining eigenvalues are given by

$$\lambda^2 + (\mu_1 + \mu_3)\lambda + (\mu_1\mu_3 - \frac{Kbq\beta_1\beta_2}{(A + K)\mu_2 C}) = 0. \tag{21}$$

By the Routh–Hurwitz criteria, all the roots of the polynomial (21) are negative or have negative real part if and only if

$$\mu_1\mu_3 - \frac{Kbq\beta_1\beta_2}{(A + K)\mu_2 C} > 0,$$

or

$$\mathcal{R}_0 = \sqrt{\frac{Kbq\beta_1\beta_2}{(A+K)\mu_1\mu_2\mu_3 C}} < 1. \tag{22}$$

Hence, E_1 is locally asymptotically stable if $\mathcal{R}_0 < 1$. □

6 Optimal control problem

In this section, to attain the minimized number of infected maize and infected leafhoppers, we reconsider the model (13) and formulate an optimal control problem with three control variables $u_1(t), u_2(t)$, and $u_3(t)$. Let

$$\begin{aligned} U = \{(u_1, u_2, u_3) | u_1, u_2, \text{ and } u_3 \text{ are Lebesgue measurable on } [0, t_f], \\ 0 \le u_1, u_2, u_3 \le 1, \text{ for all } t \in [0, t_f]\}, \end{aligned}$$

be the admissible control set. With the existence of control u_1, it is expected to diminish the number of infected maize as compared to those without control cases. The control variable u_2 is used to control the number of infected leafhoppers. Furthermore, u_3 is chemical control that is used as an intervention strategy to optimize the objective functional $\mathcal{F}$. After incorporating the control variables $u_1(t), u_2(t)$, and $u_3(t)$ in the model (13), the optimal control model is as follows:

$$\begin{cases} \varrho^{\nu-1}{}_0^C\mathfrak{D}_t^\nu S(t) = rS(1 - \dfrac{S+I}{K}) - (1-u_1)\dfrac{\beta_1 SY}{A+S}, \\ \varrho^{\nu-1}{}_0^C\mathfrak{D}_t^\nu I(t) = (1-u_1)\dfrac{\beta_1 SY}{A+S} - (\mu_1+u_2)I, \\ \varrho^{\nu-1}{}_0^C\mathfrak{D}_t^\nu H(t) = q - (1-u_2)\dfrac{\beta_2 IH}{C+I} - (u_3+\mu_2)H, \\ \varrho^{\nu-1}{}_0^C\mathfrak{D}_t^\nu Y(t) = (1-u_2)\dfrac{b\beta_2 IH}{C+I} - (u_3+\mu_3)Y, \\ S(0), I(0), H(0), Y(0) \ge 0. \end{cases} \tag{23}$$

Consider the following objective functional:

$$\mathcal{F} = \int_0^{t_f} \left(d_1 I + d_2 Y + \frac{1}{2}(w_1 u_1^2 + w_2 u_2^2 + w_3 u_3^2) \right) dt, \tag{24}$$

where d_1, d_2 are the weights on the state variables and w_1, w_2, and w_3 are relative weights of the treatment related to the control functions u_1, u_2, and u_3.

Our aim is to minimize the cost value $\mathcal{F}$ by the state and control variables I^*, Y^*, u_1^*, u_2^*, and u_3^* satisfying the constraints (23). For this purpose, we use a kind of Pontryagin maximum principle in the fractional order state [1]. We define the Hamiltonian function as below:

$$\begin{aligned}\mathfrak{H}(S,I,H,Y) &= d_1 I + d_2 Y + \frac{1}{2}\big(w_1 u_1^2 + w_2 u_2^2 + w_3 u_3^2\big)\\ &+ \varpi_1\big(rS(1-\frac{S+I}{K}) - (1-u_1)\frac{\beta_1 SY}{A+S}\big)\\ &+ \varpi_2\big((1-u_1)\frac{\beta_1 SY}{A+S} - (\mu_1+u_2)I\big)\\ &+ \varpi_3\big(q-(1-u_2)\frac{\beta_2 IH}{C+I} - (u_3+\mu_2)H\big)\\ &+ \varpi_4\big((1-u_2)\frac{b\beta_2 IH}{C+I} - (u_3+\mu_3)Y\big),\end{aligned}$$

where $\varpi_i, i = 1,2,3,4$ are the co-state variables or adjoint variables. The optimality conditions are obtained from

$$\frac{\partial \mathfrak{H}}{\partial u_1} = \frac{\partial \mathfrak{H}}{\partial u_2} = \frac{\partial \mathfrak{H}}{\partial u_3} = 0.$$

Hence, we have

$$\begin{aligned}u_1 &= \frac{\beta_2(\varpi_2-\varpi_1)IH}{w_1(A+S)},\\ u_2 &= \frac{\varpi_2 I}{w_2} + \frac{\beta_2(b\varpi_4-\varpi_3)IH}{w_2(C+I)},\\ u_3 &= \frac{\varpi_4 Y - \varpi_3 H}{w_3},\end{aligned} \tag{25}$$

where the adjoint variables satisfy

$$\begin{aligned}{}_t^C\mathfrak{D}_{t_f}^{\nu}\varpi_1(t) = \frac{\partial \mathfrak{H}}{\partial S} &= \{r(\frac{K-2S-I}{K}) - (1-u_1)\frac{\beta_1 YA}{(A+S)^2}\}\varpi_1\\ &+ \{\frac{\beta_1(1-u_1)SY}{(A+S)^2}\}\varpi_2,\\ {}_t^C\mathfrak{D}_{t_f}^{\nu}\varpi_2(t) = \frac{\partial \mathfrak{H}}{\partial I} &= d_1 - \frac{rS}{K}\varpi_1 - (\mu_1+u_2)\varpi_2 - (1-u_2)\frac{\beta_2 CH}{(C+I)^2}\varpi_3\\ &+ (1-u_2)\frac{\beta_2 bCH}{(C+I)^2}\varpi_4,\\ {}_t^C\mathfrak{D}_{t_f}^{\nu}\varpi_3(t) = \frac{\partial \mathfrak{H}}{\partial H} &= \{(u_2-1)\frac{\beta_2 I}{C+I} - \mu_2 - u_3\}\varpi_3 + \frac{b(1-u_2)\beta_2 I}{C+I}\varpi_4,\\ {}_t^C\mathfrak{D}_{t_f}^{\nu}\varpi_4(t) = \frac{\partial \mathfrak{H}}{\partial Y} &= d_2 + \frac{\beta_1(u_1-1)S}{A+S}(\varpi_1-\varpi_2) - (u_3+\mu_3)\varpi_4,\\ \varpi_1(t_f) = \varpi_2(t_f) &= \varpi_3(t_f) = \varpi_4(t_f) = 0.\end{aligned} \tag{26}$$

Then, we have the following boundary value problem for optimal treatment:

$$
\begin{cases}
\varrho^{\nu-1}\,{}_0^C\mathfrak{D}_t^\nu S(t) = rS(1-\dfrac{S+I}{K}) - (1-u_1)\dfrac{\beta_1 SY}{A+S},\\
\varrho^{\nu-1}\,{}_0^C\mathfrak{D}_t^\nu I(t) = (1-u_1)\dfrac{\beta_1 SY}{A+S} - (\mu_1+u_2)I,\\
\varrho^{\nu-1}\,{}_0^C\mathfrak{D}_t^\nu H(t) = q - (1-u_2)\frac{\beta_2 IH}{C+I} - (u_3+\mu_2)H,\\
\varrho^{\nu-1}\,{}_0^C\mathfrak{D}_t^\nu Y(t) = (1-u_2)\dfrac{b\beta_2 IH}{C+I} - (u_3+\mu_3)Y,\\
{}_t^C\mathfrak{D}_{t_f}^\nu \varpi_1(t) = \{r(\dfrac{K-2S-I}{K}) - (1-u_1)\dfrac{\beta_1 YA}{(A+S)^2}\}\varpi_1\\
\qquad\qquad +\{\dfrac{\beta_1(1-u_1)SY}{(A+S)^2}\}\varpi_2,\\
{}_t^C\mathfrak{D}_{t_f}^\nu \varpi_2(t) = d_1 - \frac{rS}{K}\varpi_1 - (\mu_1+u_2)\varpi_2 - (1-u_2)\frac{\beta_2 CH}{(C+I)^2}\varpi_3\\
\qquad\qquad +(1-u_2)\frac{\beta_2 bCH}{(C+I)^2}\varpi_4,\\
{}_t^C\mathfrak{D}_{t_f}^\nu \varpi_3(t) = \{(u_2-1)\dfrac{\beta_2 I}{C+I} - \mu_2 - u_3\}\varpi_3 + \dfrac{b(1-u_2)\beta_2 I}{C+I}\varpi_4,\\
{}_t^C\mathfrak{D}_{t_f}^\nu \varpi_4(t) = d_2 + \dfrac{\beta_1(u_1-1)S}{A+S}(\varpi_1-\varpi_2) - (u_3+\mu_3)\varpi_4,\\
\varpi_1(t_f) = \varpi_2(t_f) = \varpi_3(t_f) = \varpi_4(t_f) = 0,\\
S(0) = S_0, I(0) = I_0, H(0) = H_0, Y(0) = Y_0,
\end{cases}
\tag{27}
$$

where $u_1(t), u_2(t)$, and $u_3(t)$ are given by (25). In turn, the optimality conditions of Pontryagin's Minimum Principle establish that the optimal controls $u_1^*(t), u_2^*(t)$, and $u_3^*(t)$ are defined by

$$
\begin{aligned}
u_1^* &= \min\{\max\{0, \frac{\beta_2(\varpi_2-\varpi_1)IH}{w_1(A+S)}\}, 1\},\\
u_2^* &= \min\{\max\{0, \frac{\varpi_2 I}{w_2} + \frac{\beta_2(b\varpi_4-\varpi_3)IH}{w_2(C+I)}\}, 1\},\\
u_3^* &= \min\{\max\{0, \frac{\varpi_4 Y - \varpi_3 H}{w_3}\}, 1\}.
\end{aligned}
$$

Simulation and discussion

In this part, the effects of fractional operators on the behavior of controlled system for the dynamics of MSV disease are investigated. We develop the fractional version of fourth-order Runge-Kutta (RK4) algorithm for the coupled system (27) and apply the iterative process as follows:

We use $S(0) = 1000, I(0) = 20, H(0) = 100$, and $Y(0) = 0$ as initial values. In addition, the parameter values can be seen in [3].

The dynamical behaviors of all variables in the new fractional model without applying any control for different values of the fractional orders and the classic integer-order are plotted in Figure 1. As seen in this figure, infectious

Algorithm 1

Step 1 Set the initial values for the control functions $u_1(t), u_2(t)$, and $u_3(t)$.
Step 2 Use the current values of control functions and apply the forward fractional RK4 method for the control system and obtain the original variables.
Step 3 Apply the backward fractional RK4 method to compute the adjoint variables using the current values of the original variables and control functions.
Step 4 Update the value of control functions.
Step 5 If the updated values of the original variables, adjoint variables, and control functions are not close enough to their previous values, then go to **Step 2**.

maize and leafhopper densities increase with the fractional orders decrease and tend uniformly to the integer-order trajectory. Furthermore, when the fractional orders decrease, the densities of susceptible maize and leafhopper are reduced and go to the $\nu = 1$ state.

To indicate the efficiency of the new optimal control model, the same impact rate has been considered for all three controls, and the numerical results of the new model are compared with the classical integer model, in Figure 2. As can be seen in this figure, the participation of controls leads to a further reduction of infected maize in the new model than in the classical model. Therefore, the effect of controls on the fractional system is more successful than applying controls on the integer system, and the difference between them is significant. Of course, it should be noted that if no control is applied, the fractional model still leads to a significantly lower infection density than the integer model (Figure 2).

In the following, we numerically examine the effect of several optimal control scenarios, where each scenario includes more than one interventionist:

Scenario 1 Applying quarantine (u_2) and chemical control (u_3) along with elimination of prevention (u_1).

Scenario 2 Applying prevention (u_1) and quarantine (u_2) along with elimination of chemical control (u_3).

Scenario 3 Applying prevention (u_1) and chemical control (u_3) along with elimination of quarantine (u_2).

Scenario 4 Applying all three controls u_1, u_2, and u_3.

In the first scenario, the prevention ($u_1(t)$) effect is removed, and two control functions $u_2(t)$ and $u_3(t)$ are used. Figure 3 shows that in this control scenario, the number of infected maize decreases, while if no control is applied, the number of them increases over time. In addition, for fractional derivatives with lower orders, the rate of reduction of infectious cases is more

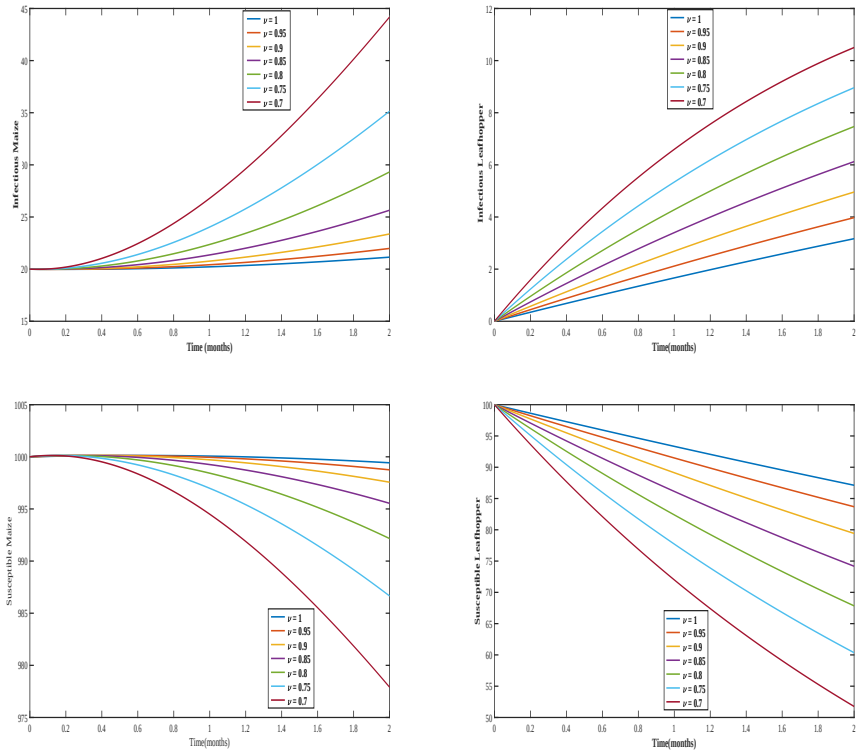

Figure 1: Numerical solutions for classical and fractional order models without controls

significant. Hence, this scenario is effective in the decline of infection in the maize community, especially in the fractional model.

The results of scenario 2 are presented in Figure 4. In this case, the quarantine control $(u_2(t))$ is maintained as in the previous scenario, but the chemical control $(u_3(t))$ is replaced by the prevention $(u_1(t))$. This scenario prevents the spread of infected maize and reduces their number. Therefore, this strategy is also successful in eliminating the disease in the maize community.

Figure 5 indicates the results of using scenario 3. In this scenario, the quarantine control is removed, unlike the previous two scenarios, and the rest of the controls are applied. Based on this figure, the number of infected is reduced compared to the case where there is no control, and this scenario also limits the growth of infectious cases.

Finally, all control interventions are considered together. As you can see in Figure 6, the use of these controls is a successful plan and causes the infected maize to be destroyed as passing the time.

Based on Figures 3–6, the quarantine control is more effective than other controls. Without quarantine control, despite preventive and chemical controls, the number of infected maize will spread, but this increasing process is much slower than the case where there is no control, and also, the difference between their increasing manner is very significant.

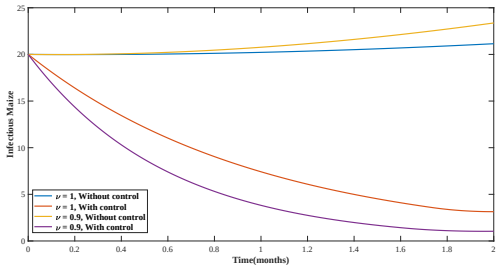

Figure 2: Numerical solutions of $I(t)$, with uncontrolled and controlled conditions for classical and fractional order models

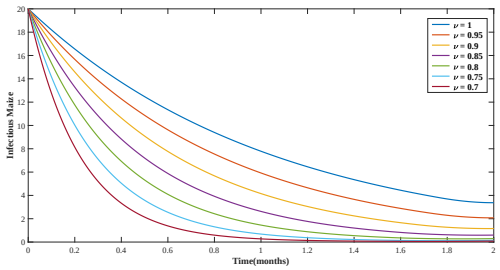

Figure 3: Numerical solutions of $I(t)$ in classic and fractional model, with quarantine and chemical controls ($u_2 \neq 0, u_3 \neq 0, u_1 = 0$)

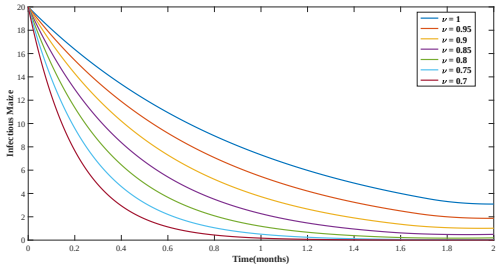

Figure 4: Numerical solutions of $I(t)$ in classic and fractional model, with prevention and quarantine controls ($u_1 \neq 0, u_2 \neq 0, u_3 = 0$)

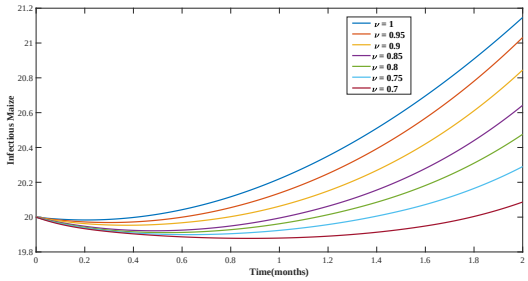

Figure 5: Numerical solutions of $I(t)$ in classic and fractional model, with prevention and chemical controls($u_1 \neq 0, u_3 \neq 0, u_2 = 0$)

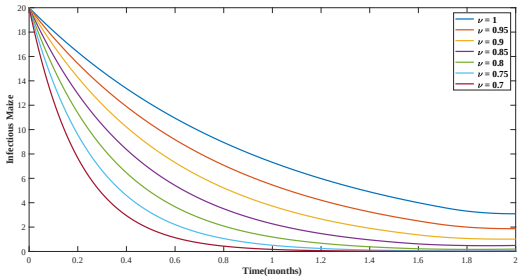

Figure 6: Numerical solutions of $I(t)$ in classic and fractional model, with prevention, quarantine and chemical controls($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0$)

7 Conclusion

In the present study, we developed a new mathematical model involving the Caputo fractional derivative for MSV disease in maize plants. First, we proved that the solution of this model system exists uniquely and that all solutions remain positive and bounded whenever they start with positive initial values, thus justifying the well-posedness of a biological model. We also determined the BRN for the model. Then, we studied the local stability of the disease-free equilibrium points of the model. The study demonstrated that one of the equilibrium points is always unstable, and the other equilibrium point is locally asymptotically stable if the model's BRN is less than unity. Next, an optimization problem is formulated. Our main focus in this work is to investigate the influence of fractional-order derivatives on the optimal control problem. The optimality system was solved numerically by use of a forward and backward RK4 scheme. The effectiveness of preventive, quarantine, and chemical controls on the fractional model is investigated in the figures. Different scenarios for the participation of these controls were evaluated for various fractional-order values. We observed that in all scenarios,

the efficiency of the controls increases by moving away from the integer-order and reducing the fractional orders. Moreover, it was observed that the quarantine control is more effective than other controls. Without quarantine control, despite preventive and chemical controls, the infectious density of maize progresses with an increasing trend. Of course, it should be noted that the increasing trend is much slower than the case where there is no control, and also, the difference between their increasing manner is very significant.

Acknowledgements

Authors are grateful to there anonymous referees and editor for their constructive comments.

References

[1] Agrawal, O.P., Defterli, O. and Baleanu, D. *Fractional optimal control problems with several state and control variables*, J. Vib. Control 16 (2010), 1967–1976.

[2] Akhavan Ghassabzadeh, F., Tohidi, E., Singh, H. and Shateyi, S. *RBF collocation approach to calculate numerically the solution of the nonlinear system of qFDEs*, J. King Saud. Univ. Sci. 33(2) (2021), 101288.

[3] Alemneh, H.T., Kassa, A.S. and Godana, A.A. *An optimal control model with cost effectiveness analysis of Maize streak virus disease in maize plant*, Infect. Dis. Model. 6 (2020), 169–182.

[4] Ameen, I.G., Baleanu, D. and Mohamed Ali, H. *Different strategies to confront maize streak disease based on fractional optimal control formulation*, Chaos, Solitons & Fractals, 164 (2022), 112699.

[5] Baleanu, D., Akhavan Ghassabzade, F., Nieto, J.J. and Jajarmi, A. *On a new and generalized fractional model for a real cholera outbreak*, Alexandria Eng. J. 61(11) (2022), 9175–9186.

[6] Bozkurt, F., Yousef, A., Abdeljawad, T., Kalinli, A. and Al Mdallal, Q. *A fractional-order model of COVID-19 considering the fear effect of the media and social networks on the community*, Chaos Solitons Fract. 152 (2021), 111403.

[7] Choi, S.K., Kang, B. and Koo, N. *Stability for Caputo fractional differential systems*, Abstr. Appl. Anal. (2014), Art. ID 631419, 6 pp.

[8] Collins, O.C. and Duffy, K.J. *Optimal control of maize foliar diseases using the plants population dynamics*, Acta Agriculturae Scandinavica, Section B– Soil and Plant Science 66(1) (2016), 20–26.

[9] Cunniffe, N.J. and Gilligan, C.A. *A theoretical framework for biological control of soil-borne plant pathogens: identifying effective strategies*, J. Theor. Biol. 278 (2011), 32–43.

[10] Diethelm, K., Siegmund, S. and Tuan, H.T. *Asymptotic behavior of solutions of linear multi-order fractional differential systems*, Fract. Calc. Appl. Anal. 20(5) (2017), 1165–1195.

[11] Evirgen, F., Uçar, S. and Özdemir, N. *System analysis of HIV infection model with CD4+T under non-singular kernel derivative*, Appl. Math. Nonlinear Sci. 5(1) (2020), 13–146.

[12] Fuller, C. *Mealie variegation In: 1st Report of the Government Entomologist, Natal, 1899-1900*, Pietermaritzburg, Natal, South Africa: P. Davis & Sons, Government Printers, 1901.

[13] Hristov, J. *Transient heat diffusion with a non-singular fading memory: from the Cattaneo constitutive equation with Jeffrey's kernel to the Caputo-Fabrizio time-fractional derivative*, Thermal Sci. 20(2) (2016), 757–762.

[14] Hugo, A., Lusekelo, E.M. and Kitengeso, R. *Optimal control and cost effectiveness analysis of tomato yellow leaf curl virus disease epidemic model*, Applied Mathematics, 9(3) (2019), 82–88.

[15] Ionescu, C., Lopes, A., Copot, D., Machado, J.A.T. and Bates, J.H.T. *The role of fractional calculus in modeling biological phenomena: A review*, Commun. Nonlinear Sci. Numer. Simul. 51 (2017), 141–159.

[16] Kilbas, A.A., Srivastava, H.H. and Trujillo, J.J. *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[17] Li, H.L., Zhang, L., Hu, C., Jiang, Y.-L. and Teng, Z. *Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge*, J. Appl. Math. Comput. 54 (2017), 435–449.

[18] Matignon, D. *Stability results on fractional differential equations to control processing*, In Proceedings of the Computational Engineering in Systems and Application Multiconference; IMACS, IEEE-SMC: Lille, France, 2 (1996), 963–968.

[19] Saeedian, M., Khalighi, M., Azimi-Tafreshi, N., Jafari, G.R. and Ausloos, M. *Memory effects on epidemic evolution: the susceptible-infected-recovered epidemic model*, Phys. Rev. 95(2) (2017), 0224091–0224099.

[20] Sandhu, K.S., Singh, N. and Malhi, N.S. *Some properties of corn grains and their flours I: Physicochemical, functional and chapati-making properties of flours*, Food Chem. 101(3) (2007), 938–946.

[21] Sene, N. *Integral balance methods for Stokes' first, equation described by the left generalized fractional derivative*, Physics, 1(1) (2019), 154–166.

[22] Sene, N. *Second-grade fluid model with Caputo-Liouville generalized fractional derivative* , Chaos, Solit. Fractals, 133 (2020), 109631.

[23] Shatanawi, W., Abdo, M.S., Abdulwasaa, M.A., Shah, K., Panchal, S.K., Kawale, S.V. and Ghadle K.P. *A fractional dynamics of tuberculosis (TB) model in the frame of generalized Atangana- Baleanu derivative*, Res. Phys. 29 (2021), 104739.

[24] Shepherd, D.N., Martin, D.P., Van Der Walt, E., Dent, K., Varsani, A. and Rybicki, E.P. *Maize streak virus: an old and complex 'emerging' pathogen*, Mol. Plant Pathol. 11(1) (2010), 1–12.

[25] Shi, R., Zhao, H. and Tang, S. *Global dynamic analysis of a vector-borne plant disease model*, Adv. Difference Equ. (59) (2014), 16.

[26] Traore, A. and Sene, N. *Model of economic growth in the context of fractional derivative*, Alex. Eng. J. 59(6) (2020), 4843–4850.

How to cite this article
Bagherpoorfard, M. and Akhavan Ghassabzade, F., Analysis and optimal control of a fractional MSD model. *Iran. j. numer. anal. optim.*, 2023; 13(3): 481-499. https://doi.org/10.22067/ijnao.2022.73126.1189