

# On convergence of He's variational iteration method for nonlinear partial differential equations

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## Abstract

This paper deals with a novel proof of convergence of He's variational iteration method applied to nonlinear partial differential equations by proposing a new formulation for this technique.

**Keywords:** Variational iteration method; Convergence theorem; Partial differential equations; Burger's equation.

## 1 Introduction

Recently He [8] has written a survey article and some new asymptotic techniques with numerous examples. The limitations of traditional perturbation procedures are illustrated. Various modified perturbation techniques are introduced, and some mathematical tools such as variational theory, homotopy technique, and iteration technique are proposed to overcome the shortcomings. For the nonlinear oscillators, all the reviewed schemes produce high approximate periods, but the accuracy of the amplitudes cannot be ameliorated by iteration. The emphasis of this author [8] is on the variational approaches, parameter-expanding methods, parameterized perturbation technique, homotopy perturbation method, iteration perturbation procedure and ancient Chinese methods. Variational approaches to soliton solution, bifurcation, limit cycle, and period solutions of nonlinear equations including the Ritz method, energy technique, variational iteration method are illustrated in his paper [8].

The variational iteration method (VIM) plays an important role in recent researches in this field. This method is proposed by He [4, 6, 5, 7] as a modification of a general Lagrange multiplier method [11]. It has been shown

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that this procedure is a powerful tool for solving various kinds of problems (e.g., see [1, 3, 13, 12]).

In this work, we adapt the technique to nonlinear partial differential equations and we prove the convergence of this method by proposing a new formulation of the method.

## 2 The Variational Iteration Method

The idea of VIM is very simple and straightforward. To explain the basic idea of VIM, we consider an one dimension general first order nonlinear partial differential equation as follows with the assumption that the equation has the unique solution (note that one can consider a nonlinear partial differential equation in higher dimensions and also more general form without loss of generality):

$$\Phi(t, x, u, u_t, u_x, \dots) = 0, \quad (1)$$

with the specified initial condition (i.e.,  $u(x, t_0) = f(x)$ ). We assume that the nonlinear operator  $\Phi$  is continuous with respect to its arguments and  $u(x, t)$  is an unknown. We first consider Eq. (1) as follows:

$$\Lambda[u(x, t)] + N[u(x, t)] = 0, \quad (2)$$

with, for example, the assumption  $\alpha(x, t) \neq 0$

$$\Lambda[u] = \alpha(x, t)u_t + \beta(x, t)u$$

and

$$N[u] = \Phi(t, x, u, u_t, u_x, \dots) - \alpha(x, t)u_t - \beta(x, t)u, \quad (3)$$

where, as shown above,  $\Lambda$  with the property  $\Lambda y \equiv 0$  when  $y \equiv 0$  denotes the linear operator with respect to  $u$  and  $N$  is a nonlinear operator with respect to  $u$ . We then construct a correction functional for Eq. (2) as [7]:

$$u_{n+1}(x, t) = u_n(x, t) + \int_{t_0}^t \lambda_{(x,t,s)} \{ \alpha(x, s)u_{n_s}(x, s) + \beta(x, s)u_n(x, s) + N[\tilde{u}_n(x, s)] \} ds, \quad (4)$$

where  $u_0(x, t)$  is the initial guess and the subscript  $n$  denotes the  $n$ -th iteration, and  $\lambda_{(x,t,s)} \neq 0$  denotes the Lagrange multiplier, which can be identified efficiently via the variational theory, and  $\tilde{u}_n$  is considered as a restricted variation [7], i.e.,  $\delta\tilde{u}_n = 0$ .

Taking variation with respect to the independent variable  $u_n$ , noticing that  $\delta u_n(x, t_0) = 0$  and by making the correction functional stationary, we obtain  $\delta u_{n+1}(x, t) = 0$  and therefore we have the following:

$$\begin{aligned}
\delta u_{n+1}(x, t) &= \delta u_n(x, t) + \delta \int_{t_0}^t \lambda_{(x,t,s)} \{ \alpha(x, s) u_{n_t}(x, s) + \beta(x, s) u_n(x, s) \\
&\quad + N[\tilde{u}_n(x, s)] \} ds \\
&= \delta u_n(x, t) + \alpha(x, s) \lambda_{(x,t,s)} \delta u_n(x, s) \Big|_{s=t} \\
&\quad - \int_{t_0}^t \left\{ \frac{\partial}{\partial s} (\alpha(x, s) \lambda_{(x,t,s)}) - \beta(x, s) \lambda_{(x,t,s)} \right\} \delta u_n(x, s) ds \\
&= [1 + \alpha(x, s) \lambda_{(x,t,s)}] \delta u_n(x, s) \Big|_{s=t} \\
&\quad - \int_{t_0}^t \left\{ \frac{\partial}{\partial s} (\alpha(x, s) \lambda_{(x,t,s)}) - \beta(x, s) \lambda_{(x,t,s)} \right\} \delta u_n(x, s) ds \\
&= 0.
\end{aligned} \tag{5}$$

Therefore, we have the following stationary conditions:

$$\alpha(x, s) \lambda_{(x,t,s)} \Big|_{s=t} = -1, \tag{6}$$

$$\frac{\partial}{\partial s} (\alpha(x, s) \lambda_{(x,t,s)}) - \beta(x, s) \lambda_{(x,t,s)} = 0. \tag{7}$$

Hence, the Lagrange multiplier can be readily identified as

$$\lambda_{(x,t,s)} = \frac{-1}{\alpha(x, t)} \exp \left( \int_s^t \frac{\alpha_\tau(x, \tau) - \beta(x, \tau)}{\alpha(x, \tau)} d\tau \right). \tag{8}$$

As a result, we have the following variational iteration formula:

$$u_{n+1}(x, t) = u_n(x, t) + \int_{t_0}^t \lambda_{(x,t,s)} \Phi(s, x, u_n(x, s), u_{n_s}(x, s), u_{n_x}(x, s), \dots) ds. \tag{9}$$

Accordingly, the successive approximations  $u_n(x, t)$ ,  $n \geq 0$  of VIM will be readily obtained by choosing all the above-mentioned parameters. Consequently, the exact solution may be obtained by using

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t). \tag{10}$$

The initial guess can be freely chosen with possible unknown constants, it can also be solved from its corresponding linear homogeneous equation  $\Lambda[u_0(x, t)] = 0$ . It is important to note that for linear problems, the exact solution can be obtained easily by only one iteration due to the fact that the auxiliary function can be suitably identified [9]. For nonlinear problems, in general, one iteration leads to highly accurate solution by VIM if the initial solution is carefully chosen with some unknown parameters.

### 3 Convergence Theorem

The variational iteration formula makes a recurrence sequence  $\{u_n(x, t)\}$ . Obviously, the limit of the sequence will be the solution of Eq. (1) if the sequence is convergent. In this section, we give a new proof of convergence of VIM in details by introducing a new iterative formulation of this procedure. Here, we suppose that for every positive integer  $n$ ,  $u_n \in C[t_0, T]$ , and  $\{u_{n_t}(x, t)\}$  and  $\{u_{n_x}(x, t)\}$  are uniformly convergent.

**Lemma 3.1.** *The variational iteration formula (9) is equivalent to the following iterative relation*

$$\Lambda[u_{n+1}(x, t) - u_n(x, t)] = -\Phi(t, x, u_n(x, t), u_{n_t}(x, t), u_{n_x}(x, t), \dots), \quad (11)$$

where  $\Lambda$  is as noted in (3).

**Proof.** Suppose  $u_n$  and  $u_{n+1}$  satisfies the variational iteration formula (9). Applying  $\frac{\partial}{\partial t}$  into both sides of (9), results in

$$\frac{\partial}{\partial t}[u_{n+1}(x, t) - u_n(x, t)] = \int_{t_0}^t \frac{\partial \lambda_{(x,t,s)}}{\partial t} \Phi ds + \frac{\partial}{\partial t}[\lambda_{(x,t,s)}|_{s=t} \Phi]. \quad (12)$$

Now, by using conditions (6) and (7), and  $\frac{\partial \lambda_{(x,t,s)}}{\partial t} = -\frac{\beta(x,t)}{\alpha(x,t)}$ , we will have

$$\alpha(x, t) \frac{\partial}{\partial t}[u_{n+1}(x, t) - u_n(x, t)] + \beta(x, t)[u_{n+1}(x, t) - u_n(x, t)] = -\Phi(t, x, u_n(x, t), u_{n_t}(x, t), u_{n_x}(x, t), \dots). \quad (13)$$

From the definition (3) of  $\Lambda$ , we obtain

$$\Lambda[u_{n+1}(x, t) - u_n(x, t)] = -\Phi(t, x, u_n(x, t), u_{n_t}(x, t), u_{n_x}(x, t), \dots). \quad (14)$$

Conversely, suppose  $u_n$  and  $u_{n+1}$  satisfies (11). Multiplying (11) by  $\lambda_{(x,t,s)}$ , in view of the definition of  $\Lambda$  and  $\lambda_{(x,t,s)} \neq 0$ , and next by integrating on both sides of the resulted term from  $t_0$  to  $t$ , yields

$$\begin{aligned} \int_{t_0}^t \lambda_{(x,t,s)} \alpha(x, s) [\frac{\partial}{\partial s} u_{n+1}(x, s) - \frac{\partial}{\partial s} u_n(x, s)] ds + \int_{t_0}^t \lambda_{(x,t,s)} \beta(x, s) [u_{n+1}(x, s) - u_n(x, s)] ds \\ = - \int_{t_0}^t \lambda_{(x,t,s)} \Phi(s) ds. \end{aligned} \quad (15)$$

Using simple integration by parts, the expression (15) becomes

$$\alpha(x, t) \lambda_{(x,t,t)} [u_{n+1}(x, t) - u_n(x, t)] - \int_{t_0}^t (\frac{\partial}{\partial s} (\alpha(x, s) \lambda_{(x,t,s)}) - \beta(x, s) \lambda_{(x,t,s)}) [u_{n+1}(x, s) - u_n(x, s)] ds = - \int_{t_0}^t \lambda_{(x,t,s)} \Phi(s) ds, \quad (16)$$

which exactly results (9) upon imposing conditions (6) and (7), i.e.,

$$u_{n+1}(x, t) = u_n(x, t) + \int_{t_0}^t \lambda_{(x,t,s)} \Phi(s, x, u_n(x, s), u_{n_s}(x, s), u_{n_x}(x, s), \dots) ds. \quad (17)$$

and this ends the proof.  $\square$

**Theorem 3.1.** *If the sequence (10) converges, where  $u_n(x, t)$  is produced by the variational iteration formulation of (9), then it is the exact solution of the equation (1).*

**Proof.** If the sequence  $\{u_n(x, t)\}$  converges, we can write

$$v(x, t) = \lim_{n \rightarrow \infty} u_n(x, t), \quad (18)$$

and it holds

$$v(x, t) = \lim_{n \rightarrow \infty} u_{n+1}(x, t). \quad (19)$$

Using expressions (18) and (19), and by the definition of  $\Lambda$  in (3), we can easily gain

$$\lim_{n \rightarrow \infty} \Lambda [u_{n+1}(x, t) - u_n(x, t)] = \Lambda \lim_{n \rightarrow \infty} [u_{n+1}(x, t) - u_n(x, t)] = 0. \quad (20)$$

From (20) and according to the Lemma 3.1, we obtain

$$\Lambda \lim_{n \rightarrow \infty} [u_{n+1}(x, t) - u_n(x, t)] = - \lim_{n \rightarrow \infty} \Phi(t, x, u_n(x, t), u_{n_t}(x, t), u_{n_x}(x, t), \dots) = 0, \quad (21)$$

which gives us

$$\lim_{n \rightarrow \infty} \Phi(t, x, u_n(x, t), u_{n_t}(x, t), u_{n_x}(x, t), \dots) = 0. \quad (22)$$

From (22) and the continuity of  $\Phi$  operator, it holds

$$\begin{aligned} & \lim_{n \rightarrow \infty} \Phi(t, x, u_n(x, t), u_{n_t}(x, t), u_{n_x}(x, t), \dots) \\ &= \Phi(t, x, \lim_{n \rightarrow \infty} u_n(x, t), \lim_{n \rightarrow \infty} u_{n_t}(x, t), \lim_{n \rightarrow \infty} u_{n_x}(x, t), \dots) \\ &= \Phi\left(t, x, \lim_{n \rightarrow \infty} u_n(x, t), \left(\lim_{n \rightarrow \infty} u_n(x, t)\right)_t, \left(\lim_{n \rightarrow \infty} u_n(x, t)\right)_x, \dots\right) \\ &= \Phi(t, x, v(x, t), v_t(x, t), v_x(x, t), \dots). \end{aligned} \quad (23)$$

Now, from Equations (22) and (23), we have

$$\Phi(t, x, v, v_t, v_x, \dots) = 0, \quad t_0 \leq t \leq T. \quad (24)$$

On the other hand, using the specified initial conditions and the definition of the initial guess, we have

$$v(x, t_0) = \lim_{n \rightarrow \infty} u_n(x, t_0) = f(x), \quad \text{since } u(x, t_0) = u_n(x, t_0) = f(x), \quad n \geq 0, \quad (25)$$

Therefore, according to (24)-(25),  $v(x, t)$  must be the exact solution of the equation (1), this ends the proof.  $\square$

Note that the above theorem is valid for the linear operator  $\Lambda$  defined by (3). This convergence theorem is important. It is because of this theorem we

can focus on ensuring that the approximation sequence converges. It is clear that the convergence of the sequence (10) depends upon the initial guess  $u_0(x, t)$  and the linear operator  $\Lambda$ . Fortunately, VIM provides us with great freedom of choosing them. Thus, as long as  $u_0(x, t)$  and  $\Lambda$  are so properly chosen that the sequence (10) converges in a region  $t_0 \leq t \leq T$ , it must converge to the exact solution in this region. Therefore, the combination of the convergence theorem and the freedom of the choice of the initial guess  $u_0(x, t)$  and the linear operator  $\Lambda$  establishes the cornerstone of the validity and flexibility of VIM.

## 4 An Illustrative Example

In order to illustrate the efficiency of the VIM described in this paper, we present one example.

**Example** A much-considered example is the Burger's equation [12,13]

$$u_t + uu_x - u_{xx} = 0, \quad (26)$$

This equation was only intended as an approach to the study of turbulence because it exhibited some essential characteristics of the more realistic (and difficult) equations. This equation involves nonlinearity, dissipation, and is relatively simple. The VIM solves much more difficult systems. We consider it now to show the simplicity of a proper solution.

According to the VIM procedure, (9), one can obtain the following variational iteration relation ( $\lambda_{(x,t,s)} = -1$ ):

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \{u_{n_t}(x, s) + u_n(x, s)u_{n_x}(x, s) - u_{n_{xx}}(x, s)\} ds. \quad (27)$$

The problem is completely defined when the initial condition is specified. If we specify  $u = x$  when  $t = 0$ , we have

$$\begin{aligned} u_0(x, t) &= x, \\ u_1(x, t) &= x - xt + O(t^2), \\ u_2(x, t) &= x - xt + xt^2 + O(t^3), \\ u_3(x, t) &= x - xt + xt^2 - xt^3 + O(t^4), \\ u_4(x, t) &= x - xt + xt^2 - xt^3 + xt^4 + O(t^5), \\ &\dots \end{aligned} \quad (28)$$

Thus,  $u(x, t) = x/(1 + t)$  which is the exact solution of (26).

## 5 Conclusion

In this paper, we have given a proof of convergence of He's variational iteration method by presenting a new formulation of He's method for nonlinear partial differential equations. The main property of this method is in its flexibility and ability to solve nonlinear equations accurately and conveniently without decomposing the nonlinear terms, which makes the procedure very complex. This technique is a very powerful tool for solving nonlinear problems. Furthermore, it gives an accurate and easily computable solution by means of a truncated series whose convergence is fast.

## References

1. Abdou, M.A. and Soliman, A.A., *Variational iteration method for solving Burger's and coupled Burger's equations*, J. Comput. Appl. Math. **81** (2005) 245-251.
2. Burgers, J.M., *A mathematical model illustrating the theory of turbulence*, Adv. App. Mech. **1** (1948) 171-199.
3. Dehghan, M. and Tatari, M., *The use of He's variational iteration method for solving the Fokker-Planck equation*, Phys. Scripta **74** (2006) 310-316.
4. He, J.H., *Variational iteration method for delay differential equations*, Comm. Non-Linear. Sci. Numer. Simulation **2** (1997) 235-236.
5. He, J.H., *Approximate analytical solution for seepage flow with fractional derivatives in porous media*, Comput. Methods. Appl. Mech. Engng. **167** (1998) 57-68.
6. He, J.H., *Approximate solution of nonlinear differential equations with convolution product non-linearities*, Comput. Methods. Appl. Mech. Engng. **167** (1998) 69-73.
7. He, J.H., *Variational iteration method- a kind of non-linear analytical technique: some examples*, Internat. J. Non-Linear Mech. **34** (1999) 699-708.
8. He, J.H., *Some asymptotic methods for strongly nonlinear equations*, Internat. J. Modern Phys. B **20** (2006) 1141-1199.
9. He, J.H., *Variational iteration method- some recent results and new interpretations*, J. Comput. Appl. Math., **207** (2007) 3-17.
10. Hood, S., *Next exact solutions of Burgers' equation- An extension to the direct method of Clarkson and Kruskal*, J. Math. Phys. **36** (1995) 1971-1990.
11. Inokuti, M., Sekine, H. and Mura, T., *General use of the Lagrange multiplier in non-linear mathematical physics, in: Variational Methods in the Mechanics of Solids*, Pergamon Press, NewYork, (1978), pp. 156-162.
12. Momani, S. and Abusad, S., *Application of He's variational iteration method to Helmholtz equation*, Chaos Solitons Fractals **27** (2006) 1119-1123.
13. Tatari, M. and Dehghan, M., *On the convergence of He's variational iteration method*, J. Comput. Appl. Math. **207** (2007) 121-128.