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Research Article

Nonpolynomial B-spline collocation method for solving singularly perturbed quasilinear Sobolev equation

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Abstract

In this paper, a singularly perturbed one-dimensional initial boundary value problem of a quasilinear Sobolev-type equation is presented. The nonlinear term of the problem is linearized by Newton's linearization method. Time derivatives are discretized by implicit Euler's method on nonuniform step size. A uniform trigonometric B-spline collocation method is used to treat the spatial variable. The convergence analysis of the scheme is proved, and the accuracy of the method is of order two in space and order one in time direction, respectively. To test the efficiency of the method, a model example is demonstrated. Results of the scheme are presented in

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tabular, and the figure indicates the scheme is uniformly convergent and has an initial layer at $t = 0$.

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1 Introduction

A singularly perturbed differential equation is a differential equation in which the highest order derivative is multiplied by a small positive parameter ε that is recognized as a perturbation parameter. While solving these types of problems, the use of classical numerical methods on a uniform mesh may cause large oscillations as the perturbation parameter approaches zero in the entire domain of interest due to the boundary layer behavior. Therefore, to ignore this oscillation, several researchers constructed suitable numerical methods for these problems, whose accuracy does not depend on the perturbation parameter [7, 13, 12, 14, 15, 16].

This study deals also with the singularly perturbed initial boundary value problem of quasilinear Sobolev equation in the domain $\bar{Q} = \bar{\Omega} \times [0, T], \bar{\Omega} = [0, l], Q = (0, l) \times (0, T], \Omega = (0, l)$ of the form:

$$\begin{cases} Lu + f(x,t,u) = \varepsilon\left[\frac{\partial u}{\partial t}\right] - \alpha\frac{\partial^2 u}{\partial x^2} + \beta u\frac{\partial u}{\partial x} + f(x,t,u) = 0, & (x,t) \in Q, \\ u(x,0) = \varphi(x), & x \in \bar{\Omega}, \\ u(0,t) = u(l,t) = 0, & t \in (0,T], \end{cases} \tag{1}$$

where $\left[\frac{\partial u}{\partial t}\right] = -\frac{\partial^3 u}{\partial t \partial x^2} + \frac{\partial u}{\partial t}$, ε is a small perturbation parameter $0 < \varepsilon < 1$, and $\alpha > 0$ and β are given constants. Moreover, $\varphi(x)$ and $f(x, t, u)$ are assumed to be sufficiently continuously differentiable functions in $\bar{\Omega}$ and $\bar{Q} \times \mathbb{R}$, respectively.

Sobolev types equation arises in several mathematical problems, such as homogeneous fluid flow in fissured rocks [4], thermodynamics and propagation of long waves of small amplitude [20], quasi-stationary processes in semiconductors [5], shear in second-order fluid [11], application of control theory [21], and other physical models. The analysis, development, and implementation of numerical methods for the solution of singularly perturbed pseudo-parabolic/Sobolev types of problems have received wide attention and developed in [3, 2, 1, 6, 8, 9, 10, 17].

The numerical method of (1) has been studied in the difference schemes for the singularly perturbed one-dimensional initial boundary value problem of Sobolev equations with initial jump [1]. Finite elements with piece-wise linear functions in space and exponential functions in time variables are applied.

Trigonometric B-spline is a nonpolynomial B-spline with a sine function, which was introduced by Schoenberg in 1964 [19]. Even though the trigonometric B-spline function is used to approximate several types of differential equations, it is not applied to quasilinear Sobolev types of equations. Motivated by this, we present the cubic trigonometric B-spline collocation method for solving the one-dimensional initial boundary value problem of singularly perturbed quasilinear Sobolev types of the equation. Implicit Euler and cubic trigonometric B-spline collocation methods are used to control the time and space variables, respectively.

The outline of this study is the following sequences. A linearization of the numerical scheme is presented in Section 2. In Section 3, the properties of the continuous solution are discussed. Numerical formulation of the problem is presented in Section 4. Convergence analysis and numerical results are considered in Sections 5 and 6, receptively. Finally, the conclusion of the study is given in Section 7.

2 Linearization of the problem

The one-dimensional singularly perturbed initial boundary value problem of Sobolev equation (1) can be rewritten as

$$-\varepsilon\frac{\partial^3 u}{\partial t\partial x^2}+\varepsilon\frac{\partial u}{\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}+F(x,t,u,\frac{\partial u}{\partial x})=0, \quad (x,t)\in Q, \tag{2}$$

where $F(x,t,u,\frac{\partial u}{\partial x})=\beta u\frac{\partial u}{\partial x}+f(x,t,u)$.

To linearize (2), we consider an initial guess for the function $u(x,t)$ by denoting $u^{(0)}(x,t)$ that satisfies both initial and boundary conditions:

$$u^{(0)}(x,t) = \frac{1}{2}\varphi(x)\left(1 + e^{\frac{-2t}{\varepsilon}}\right). \tag{3}$$

Applying Newton's linearization method on $F(x,t,u,\frac{\partial u}{\partial x}))$ for the function $u^{(0)}(x,t)$, we obtain an $(n+1)$th iteration:

$$\begin{aligned} F(x,t,u^{(n+1)},\frac{\partial u^{(n+1)}}{\partial x}) = {} & F(x,t,u^{(n)},\frac{\partial u^{(n)}}{\partial x}) \\ & + \left(u^{(n+1)} - u^{(n)}\right)\frac{\partial F}{\partial u}|_{\left(u^{(n)},\frac{\partial u^{(n)}}{\partial x}\right)} \\ & + \left(\frac{\partial u^{(n+1)}}{\partial x} - \frac{\partial u^{(n)}}{\partial x}\right)\frac{\partial F}{\partial(\frac{\partial u}{\partial x})}|_{\left(u^{(n)},\frac{\partial u^{(n)}}{\partial x}\right)}. \end{aligned} \tag{4}$$

Substituting (4) into (2) and after some rearrangements we obtain

$$\begin{cases} -\varepsilon\frac{\partial^3 u^{(n+1)}}{\partial t\partial x^2} + \varepsilon\frac{\partial u^{(n+1)}}{\partial t} - \alpha\frac{\partial^2 u^{(n+1)}}{\partial x^2} & \\ +a(x,t)\frac{\partial u^{(n+1)}}{\partial x} + b(x,t)u^{(n+1)} = g(x,t), & \\ u(x,0) = \varphi(x), & x \in \bar{\Omega}, \\ u(0,t) = u(l,t) = 0, & t \in (0,T], \end{cases} \tag{5}$$

where

$$\begin{aligned} a(x,t) &= \frac{\partial F}{\partial(\frac{\partial u}{\partial x})}|_{\left(u^{(n)},\frac{\partial u^{(n)}}{\partial x}\right)}, \qquad b(x,t) = \frac{\partial F}{\partial u}|_{\left(u^{(n)},\frac{\partial u^{(n)}}{\partial x}\right)}, \\ g(x,t) &= u^{(n)}\frac{\partial F}{\partial u}|_{\left(u^{(n)},\frac{\partial u^{(n)}}{\partial x}\right)} + \left(\frac{\partial u^{(n)}}{\partial x}\right)\frac{\partial F}{\partial(\frac{\partial u}{\partial x})}|_{\left(u^{(n)},\frac{\partial u^{(n)}}{\partial x}\right)} \\ &\quad - F(x,t,u^{(n)},\frac{\partial u^{(n)}}{\partial x}). \end{aligned}$$

3 Properties of continuous solution

Lemma 1. Let $\varphi(x) \in C^2[0,l]$ and the derivatives $\frac{\partial^s f}{\partial x^s}, \frac{\partial^s f}{\partial u^s} (s = 1,2), \frac{\partial f}{\partial t} \in C(\bar{Q})$. Then, for the solution $u(x,t)$ of (1), the following estimate holds:

$$\left|\frac{\partial^{r+s}u(x,t)}{\partial t^r \partial x^s}\right| \le C\varepsilon^{-r}, \quad \text{for all } (x,t) \in \left(\overline{Q}\right), r = 0,1, \ s = 0,1,2 \tag{6}$$

for any fixed l and T, and provided that

$$M_0\alpha^{-1}\frac{|\beta|\,l}{2\pi}<1,$$

where

$$\alpha_0=\left(\alpha-M_0\alpha^{-1}\frac{|\beta|\,l}{2\pi}\frac{\pi^2}{e^2+\pi^2}\right), \tag{7}$$

$$M_0=\frac{\sqrt{l}}{2}\left(\|\varphi\|_1+\frac{l^2+\pi^2}{\alpha\pi^2}\max_{0\le t\le T}\|f(\cdot,t,0)\|_0\right), \tag{8}$$

and C is a generic positive constant,which is independent of ε and mesh parameters.

Proof. Consider the integral identity

$$(Lu,u)_0+(f,u)_0=0,$$

and taking into account that $\left(u\frac{\partial u}{\partial x},u\right)_0=0$. Then,

$$\left(\varepsilon\frac{\partial u}{\partial t},u\right)_0-\left(\varepsilon\frac{\partial^3 u}{\partial t\partial x^2},u\right)_0-\left(\alpha\frac{\partial^2 u}{\partial x^2},u\right)_0+\left(\beta u\frac{\partial u}{\partial x},u\right)_0+(f(x,t,u),u)_0=0.$$

Estimating these inner products on an interval 0 to l, we obtain

$$\varepsilon\int_0^l\frac{\partial u}{\partial t}udx-\varepsilon\int_0^l\frac{\partial^3 u}{\partial t\partial x^2}udx-\alpha\int_0^l\frac{\partial^2 u}{\partial x^2}udx+\int_0^l f(x,t,u)udx=0.$$

From the linearization by assuming $f(x,t,u)\approx f(x,t,u^{(0)})+\frac{\partial f}{\partial u}u$ and $F=f(x,t,u^{(0)})$, we have

$$\frac{\varepsilon}{2}\frac{d}{dt}(u,u)_0-\frac{\varepsilon}{2}\frac{d}{dt}\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial x}\right)_0-\alpha\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial x}\right)_0+(F,u)_0+\left(\frac{\partial f}{\partial u}u,u\right)_0=0.$$

This become

$$\frac{\varepsilon}{2}\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial x}\right\|_0^2+\|u\|_0^2\right)+\alpha\left\|\frac{\partial u}{\partial x}\right\|_0^2+(F,u)_0+\left(\frac{\partial f}{\partial u}u,u\right)_0=0. \tag{9}$$

Applying an inequality $(F,u)_0\ge-\|F\|_0\|u\|_0$, $\gamma=\frac{l^2}{l^2+\pi^2}$ for $0<\gamma<1$ into (9), and after rearrangement, we get

$$\varepsilon\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial x}\right\|_0^2+\|u\|_0^2\right)+2\alpha\left(\frac{\pi^2}{l^2+\pi^2}\right)\left(\left\|\frac{\partial u}{\partial x}\right\|_0^2+\|u\|_0^2\right)\leq 2\|F\|_0\|u\|_0. \tag{10}$$

Choosing $C_1=\alpha\frac{\pi^2}{l^2+\pi^2}$ and $\delta=\left\|\frac{\partial u}{\partial x}\right\|_0^2+\|u\|_0^2$, the inequality (10) is written as

$$\varepsilon\delta'(t)+2C_1\delta(t)\leq 2\|F\|_0\|u\|_0. \tag{11}$$

Solving the differential inequality (11), we obtain

$$\delta(t)\leq\delta_0 e^{\frac{-C_1 t}{\varepsilon}}+\left(\frac{1}{C_1^2}\max_{0\leq\tau\leq t}\|f(\cdot,t,0)\|_0^2\left(1-e^{\frac{-C_1 t}{\varepsilon}}\right)\right) \tag{12}$$

with $\delta_0=\|\varphi\|_1^2=\|\varphi\|_0^2+\|\varphi'\|_0^2$.

Using by the virtue of embedding inequality $\frac{l}{4}\left\|\frac{\partial u}{\partial x}\right\|_0^2\geq\|u\|_{\infty,\Omega}^2$ into (12) and after some mathematical manipulation, we obtain

$$|u(x,t)|\leq\frac{\sqrt{l}}{2}\left(\delta_0^{\frac{1}{2}}e^{\frac{-C_1 t}{2\varepsilon}}+\frac{1}{C_1}\max_{0\leq\tau\leq t}\|f(\cdot,t,0)\|_0\right). \tag{13}$$

Using an identity

$$\left(Lu,\frac{\partial^2 u}{\partial x^2}\right)_0=\left(f,\frac{\partial^2 u}{\partial x^2}\right)_0, \tag{14}$$

$$\frac{\varepsilon}{2}\frac{d}{dt}\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial x}\right)_0+\frac{\varepsilon}{2}\frac{d}{dt}\left(\frac{\partial^2 u}{\partial x^2},\frac{\partial u}{\partial x}\right)_0+\alpha\left(\frac{\partial^2 u}{\partial x^2},\frac{\partial u}{\partial x}\right)_0$$
$$+\left(F,\frac{\partial^2 u}{\partial x^2}\right)_0+\left(\frac{\partial f}{\partial u}u,\frac{\partial^2 u}{\partial x^2}\right)_0=0. \tag{15}$$

Using of Friedrich's inequality (15) then, after some rearrangement, we obtain

$$\varepsilon\frac{d}{dt}\left(\left\|\frac{\partial^2 u}{\partial x^2}\right\|_0^2+\left\|\frac{\partial u}{\partial x}\right\|_0^2\right)_0+2\alpha\left(\frac{\pi^2}{l^2+\pi^2}\right)\left(\left\|\frac{\partial^2 u}{\partial x^2}\right\|_0^2+\left\|\frac{\partial u}{\partial x}\right\|_0^2\right)$$
$$\leq 2\|F\|_0\left\|\frac{\partial^2 u}{\partial x^2}\right\|_0, \tag{16}$$

which is written as

$$\varepsilon\delta'(t)+2C_1\delta(t)\leq 2\|F\|_0\left\|\frac{\partial^2 u}{\partial x^2}\right\|_0, \tag{17}$$

where $\delta(t)=\left\|\frac{\partial^2 u}{\partial x^2}\right\|_0^2+\left\|\frac{\partial u}{\partial x}\right\|_0^2$ and $C_1=\alpha\frac{\pi^2}{l^2+\pi^2}$.

By solving the differential inequality (17) and applying embedding inequality, we get

$$|\frac{\partial^2 u}{\partial x^2}| \leq C. \tag{18}$$

With the same process from an identity

$$\left(Lu, \frac{\partial^3 u}{\partial t \partial x^2}\right)_0 = \left(f, \frac{\partial^3 u}{\partial t \partial x^2}\right)_0, \tag{19}$$

we obtain

$$\left\|\frac{\partial^3 u}{\partial t \partial x^2}\right\|_0^2 + \left\|\frac{\partial^2 u}{\partial t \partial x}\right\|_0^2 \leq \left(\|u\|_0^2 + \left\|\frac{\partial u}{\partial x}\right\|_0^2 + \left\|\frac{\partial^2 u}{\partial x^2}\right\|_0^2 + \frac{C}{\varepsilon^2}\|F\|_0^2\right), \tag{20}$$

which leads to (6) for $r = 1, \ s = 1, 2$, by using the proved estimate for $\left\|\frac{\partial^s u}{\partial x^2}\right\|_0, s = 0, 1, 2$.

Finally, we can write (1) as a form

$$\varepsilon \frac{\partial^3 u}{\partial t \partial x^2} + \alpha \frac{\partial^2 u}{\partial x^2} = \phi(x, t), \tag{21}$$

where $\phi(x,t) = \varepsilon \frac{\partial u}{\partial t} + \beta u \frac{\partial u}{\partial x} + f(t, x, u)$ and $|\phi(x, t)| \leq C$ with the estimate of (6) immediately for $r = 0, 1, s = 2$. □

Lemma 2. Under the assumption of Lemma 1, the following inequality holds:

$$\left\|\frac{\partial u}{\partial t}\right\|_1 \leq C\{1 + \varepsilon^{-1} e^{-\frac{\varpi_0 t}{\varepsilon}}\}, \qquad t \in [0, T], \tag{22}$$

where $\varpi_0 = \frac{C_1}{2}$ with $C_1 = \alpha \frac{\pi^2}{l^2 + \pi^2}$, which is given in the above.

Proof. Differentiating (1) with respect t and proceeded with $\frac{\partial u}{\partial t}$, we have

$$\frac{d}{dt}\left(\varepsilon \frac{\partial u}{\partial t} - \varepsilon \frac{\partial^3 u}{\partial t \partial x^2} - \alpha \frac{\partial^2 u}{\partial x^2} + \beta u \frac{\partial u}{\partial x} + f(x, t, u) = 0\right). \tag{23}$$

With an assumption $\frac{d}{dt} \beta u \frac{\partial u}{\partial x} = 0$, we can have

$$\left(\varepsilon \frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial t}\right)_0 - \left(\varepsilon \frac{\partial^4 u}{\partial t^2 \partial x^2}, \frac{\partial u}{\partial t}\right)_0 - \left(\alpha \frac{\partial^3 u}{\partial t \partial x^2}, \frac{\partial u}{\partial t}\right)_0 + \left(\frac{\partial f}{\partial t}, \frac{\partial u}{\partial t}\right)_0 = 0.$$

For $\left(\frac{\partial f}{\partial t}, \frac{\partial u}{\partial t}\right)_0 \geq -\left\|\frac{\partial f}{\partial t}\right\|_0 \left\|\frac{\partial u}{\partial t}\right\|_0$, we have

$$\frac{\varepsilon}{2}\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|_0^2+\left\|\frac{\partial^2 u}{\partial t\partial x}\right\|_0^2\right)+\alpha\left\|\frac{\partial^2 u}{\partial t\partial x}\right\|_0^2\leq\left\|\frac{\partial f}{\partial t}\right\|_0\left\|\frac{\partial u}{\partial t}\right\|_0.$$

Applying an inequality relation $\left\|\frac{\partial f}{\partial t}\right\|_0\left\|\frac{\partial u}{\partial t}\right\|_0\leq\frac{1}{C_1}\left\|\frac{\partial f}{\partial t}\right\|_0^2+C_1\left\|\frac{\partial u}{\partial t}\right\|_0^2$, it gives

$$\varepsilon\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|_0^2+\left\|\frac{\partial^2 u}{\partial t\partial x}\right\|_0^2\right)+C_1\left(\left\|\frac{\partial^2 u}{\partial t\partial x}\right\|_0^2+\left\|\frac{\partial u}{\partial t}\right\|_0^2\right)\leq C, \tag{24}$$

where $C=(C_1-2\alpha)\left\|\frac{\partial^2 u}{\partial t\partial x}\right\|_0^2+3C_1\left\|\frac{\partial u}{\partial t}\right\|_0^2+\frac{2}{C_1}\left\|\frac{\partial f}{\partial t}\right\|_0^2$.

Inequality (24) is written as

$$\varepsilon\delta'(t)+C_1\delta(t)\leq C, \tag{25}$$

where $\delta(t)=\left\|\frac{\partial u}{\partial t}\right\|_1^2=\left\|\frac{\partial u}{\partial t}\right\|_0^2+\left\|\frac{\partial^2 u}{\partial t\partial x}\right\|_0^2$.

This is similar to $\delta(t)=\|v\|_0^2+\|v'\|_0^2$ for $v=\left\|\frac{\partial u}{\partial t}\right\|_0^2$ and $C_1=\alpha\frac{\pi^2}{l^2+\pi^2}$.

Solving inequality (25), we get

$$\delta(t)\leq C\left(1+\frac{1}{\varepsilon^2}e^{\frac{-C_1 t}{\varepsilon}}\right), \tag{26}$$

which yields

$$\left\|\frac{\partial u}{\partial t}\right\|_1\leq C\left(1+\frac{1}{\varepsilon}e^{\frac{-C_1 t}{2\varepsilon}}\right).$$

With $\varpi_0=\frac{C_1}{2}$, it gives

$$\left\|\frac{\partial u}{\partial t}\right\|_1\leq C\left(1+\varepsilon^{-1}e^{\frac{-\varpi_0 t}{\varepsilon}}\right).$$

□

4 Numerical scheme formulation

4.1 Temporal discretization

Mesh generation

Based on (22), there is an initial layer in the neighborhood of $t=0$ of order $\varpi_0^{-1}\varepsilon|\ln\varepsilon|$ thickness, where ϖ_0 is given by (7). We divide two nonoverlapping subintervals $[0,\varrho]$ and $[\varrho,T]$, with the transition parameter

$$\varrho = \min\{\frac{T}{2}, \varpi_0^{-1}\varepsilon|\ln\varepsilon|\}.$$

Let $\overline{\Omega}_t^N = \{t_j\}_j^N$ be the set of mesh points. Now, we define piece-wise uniform mesh points as

$$t_j = \begin{cases} -\varpi_0^{-1}\varepsilon\ln N\left[1-(1-\varepsilon)\frac{2j}{N}\right], & j=0,\dots,\frac{N}{2}, \quad if\,\varrho = \frac{T}{2}, \\ -\varpi_0^{-1}\varepsilon\ln N\left[1-\left(1-e^{\frac{-\varpi_0 T}{2\varepsilon}}\right)\frac{2j}{N}\right], & j=0,\dots,\frac{N}{2}, \quad if\,\varrho < \frac{T}{2}, \\ \varrho + \left(1-\frac{N}{2}\right)\tau, & j=\frac{N}{2},\dots,N, \quad \tau = 2\frac{(T-\varrho)}{N}. \end{cases}$$

To discretize time derivative of (5), we use the implicit Euler method on nonuniform step size on the domain: $\Omega_t^N = 0 = t_0 < t_1 < \cdots < t_j < t_{j+1} < \cdots < t_M = T, \quad j = 0.1,\dots,M-1, \quad \tau(j) = t(j+1) - t(j)$ at the point (x, t_j). Then, (5) becomes

$$-\left(\frac{\varepsilon}{\tau(j)} + \frac{\alpha}{2}\right)\frac{\partial^2 u^{j+1}}{\partial x^2}(x) + a\left(x, t_{j+1}\right)\frac{\partial u^{j+1}}{\partial x}(x) + \left(\frac{\varepsilon}{\tau(j)} + b\left(x, t_{j+1}\right)\right)u^{j+1}(x) = \frac{-\varepsilon}{\tau(j)}\frac{\partial^2 u^j}{\partial x^2}(x) + \frac{\varepsilon}{\tau(j)}u^j(x) + g(x, t_{j+1}). \tag{27}$$

Lemma 3 (Semi-discrete maximum principle)**.** For each $j = 1, 2, \dots, N-1$, let Z_{j+1} be a sufficiently smooth function on domain $\bar{\Omega}$. If $Z_{j+1}(0) \geq 0$, $Z_{j+1}(1) \geq 0$, and $L^{\tau(j)}u_{j+1}(x) \geq 0$, $x \in \Omega$, then $Z_{j+1} \geq 0$, for all $x \in \bar{\Omega}$.

Proof. Assume that there is (x^*) such that

$$Z_{j+1}(x^*) = \min_{x\in\overline{\Omega}_x} Z_{j+1}(x) \geq 0.$$

From the assumption it indicates that $x^* \notin \{1, 2\}$, which implies $x^* \in (0, 1)$. Applying the property of extreme values in calculus gives $\frac{d}{dx}Z_{j+1}(x^*) = 0$, and $\frac{d^2}{dx^2}Z_{j+1}(x^*) \geq 0$, given that $L^{\tau(j)}u_{j+1}(x^*) < 0$, which contradicts to $L^{\tau(j)}u_{j+1}(x^*) \geq 0, x \in \Omega$. Therefore, we conclude that $Z_{j+1} \geq 0$, for all $x \in \Omega$. Hence, the operator $L^{\tau(j)}$ satisfies a semi-discrete maximum principle. □

Lemma 4 (Local truncation error)**.** Consider the bound on the derivatives of $u(x,t)$ with respect to $t$given by $\left|\frac{\partial^k u(x,t)}{\partial x^k}\right| \leq C, \quad$ for all $(x,t) \in \left(\overline{\Omega}\right)$. Then the local error estimate in the temporal direction is given by

$$\|e_{j+1}\| \leq C\left(\tau\right)^2,$$

where $e_{j+1} = u^{j+1}(x) - U^{j+1}(x)$ is the local error estimate in the temporal direction at $(j+1)th$ time level.

Proof. From (27), we have

$$\begin{aligned}
&\varepsilon\left(\frac{u^{j+1}(x)-u^{j}(x)}{\tau(j)}-\frac{\tau(j)}{2}\frac{\partial^2 u^j}{\partial x^2}(x)\right)-\varepsilon\frac{\partial^2}{\partial x^2}\left(\frac{u^{j+1}(x)-u^{j}(x)}{\tau(j)}-\frac{\tau(j)}{2}\frac{\partial^2 u^j}{\partial x^2}(x)\right)\\
&\quad-\alpha\frac{\partial^2 u^{j+1}}{\partial x^2}+a(x,t_{j+1})\frac{\partial u^{j+1}}{\partial x}+b(x,t_{j+1})u\\
&\qquad=g(x,t_{j+1})+O(\tau^2(j))^2,\quad (x,t)\in Q.
\end{aligned} \tag{28}$$

Multiplying (28) by $\tau(j)$, it gives

$$\begin{aligned}
&\varepsilon\left(u^{j+1}(x)-u^{j}(x)-\frac{\tau(j)^2}{2}\frac{\partial^2 u^j}{\partial t^2}(x)\right)-\varepsilon\frac{\partial^2}{\partial x^2}\left(u^{j+1}(x)-u^{j}(x)-\frac{\tau(j)^2}{2}\frac{\partial^2 u^j}{\partial t^2}(x)\right)\\
&\quad-\alpha\tau(j)\frac{\partial^2 u^{j+1}}{\partial x^2}+\tau(j)a(x,t_{j+1})\frac{\partial u^{j+1}}{\partial x}+\tau(j)b(x,t_{j+1})u\\
&\qquad=\tau(j)g(x,t_{j+1})+O(\tau(j))^3,\quad (x,t)\in Q.
\end{aligned} \tag{29}$$

By rearranging this, we obtain

$$\begin{aligned}
&\left(\varepsilon-\varepsilon\frac{\partial^2}{\partial x^2}-\frac{\tau(j)}{2}\left(\alpha\frac{\partial^2}{\partial x^2}-a(x,t_{j+1})\frac{\partial}{\partial x}-b(x,t_{j+1})\right)\right)u^{j+1}(x)\\
&\quad=\left(\varepsilon-\varepsilon\frac{\partial^2}{\partial x^2}\right)u^j(x)+\tau(j)\left(g(x,t_{j+1})\right)\\
&\qquad+(\tau(j))^2\left(\frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}-\frac{\varepsilon}{2}\frac{\partial^4}{\partial x^4}\right)u^j(x)+O\left(\Delta t(j)\right)^3,
\end{aligned} \tag{30}$$

which is written as

$$\mathcal{L}_{\varepsilon}^{\tau}u^{j+1}(x)=\mathit{\Gamma}(x,t_{j+1})+O\left(\tau(j)\right)^2, \tag{31}$$

where

$$\begin{aligned}
\mathcal{L}_{\varepsilon}^{\tau}&=\left(\varepsilon-\varepsilon\frac{\partial^2}{\partial x^2}-\frac{\tau(j)}{2}\left(\alpha\frac{\partial^2}{\partial x^2}-a(x,t_{j+1})\frac{\partial}{\partial x}-b(x,t_{j+1})\right)\right)\\
\mathit{\Gamma}(x,t_{j+1})&=\left(\varepsilon-\varepsilon\frac{\partial^2}{\partial x^2}\right)u^j(x,t)+\tau(j)\left(g(x,t_{j+1})\right).
\end{aligned}$$

From the boundedness of the solution, we have

$$\mathcal{L}_{\varepsilon}^{\tau} U^{j+1}(x) = \mathit{\Gamma}(x, t_{j+1}) \text{ for all } x \in \bar{\Omega}. \tag{32}$$

Now, from the desired mesh, we consider two case:

Case 1: For $\tau(j) \in [0, \sigma]$, let us consider $\max\{\tau(j) = \tau_1\}$.
Now, from (31) and (32), it yields

$$\left\|\mathcal{L}_{\varepsilon}^{\tau}\left(u^{j+1}(x) - U^{j+1}(x)\right)\right\| = \|\mathcal{L}_{\varepsilon}^{\tau} e_{j+1}\| \leq C\left(\tau_1\right)^2.$$

Case 2: For $\tau(j) \in [\sigma, 1]$, let us consider $\max\{\tau(j) = \tau_2\}$.

Again from (31) and (32), we have

$$\left\|\mathcal{L}_{\varepsilon}^{\tau}\left(u^{j+1}(x) - U^{j+1}(x)\right)\right\| = \|\mathcal{L}_{\varepsilon}^{\tau} e_{j+1}\| \leq C\left(\tau_2\right)^2$$

with the boundary conditions $u^{j+1}(0) - U^{j+1}(0) = e_{j+1}(0) = 0$ and $u^{j+1}(1) - U^{j+1}(1) = e_{j+1}(1) = 0$. Hence applying the maximum principles and choosing that $\tau = \max\{\tau_1, \tau_2\}$ give

$$\|e_{j+1}\| \leq C\left(\tau\right)^2.$$

□

Lemma 5. [Global error estimate] Under the hypothesis of the Lemma 4, the global error estimate in the temporal direction is given by

$$\|E_{j+1}\|_{\infty} \leq C(\tau)^2, \quad \text{for all } \ j \leq T/\Delta t,$$

where E_{j+1} is the global error estimate in the temporal direction at $(j+1)$th time level.

Proof. Using local error estimates up to $(j+1)$th time step given in Lemma 2, we get the following global error estimates at $(j+1)$th time step

$$\begin{aligned}
\|E_{j+1}\|_{\infty} &= \left\|\sum_{k=1}^{j} e_k\right\|_{\infty}, \quad j+1 \leq T/\tau(j) \\
&\leq \|e_1\|_{\infty} + \|e_2\|_{\infty} + \|e_3\|_{\infty} + \cdots + \|e_+1\|_{\infty} \\
&\leq c_1 j + 1\left(\tau(j)\right)^2 \qquad\qquad \text{(by Lemma 4)} \\
&\leq c_1\left((j+1)\tau(j)\right)\left(\tau(j)\right) \\
&\leq c_1 T\left(\tau(j)\right), \qquad ((j+1)\tau(j) \leq T)
\end{aligned}$$

$$\leq C\left(\tau(j)\right) \quad choosing \qquad \tau = \max\{\tau(j)\}$$
$$\leq C\left(\tau\right).$$

□

4.2 Spatial discretization

Descretizing the interval equally by knots x_i into N subintervals $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, N-1$, such that $0 = x_0 < x_1 < \cdots < x_N = l$ as a uniform partition of the solution domain $0 \leq x \leq l$ with the step length $h = x_{i+1} - x_i = \frac{l}{N},\ i = 0, 1, \ldots, N-1$.

The piece-wise cubic trigonometric B-spline basis function $TB_i(x)$ over the uniform mesh is defined as [22]:

$$TB_i(x) = \frac{1}{\omega(h)}\begin{cases} (\sin)^3\left(\frac{x-x_{i-2}}{2}\right), & x \leq x_{i-1}, \\ \sin\left(\frac{x-x_{i-2}}{2}\right)[\sin\left(\frac{x-x_{i-2}}{2}\right)\sin\left(\frac{x_i-x}{2}\right)+\sin\left(\frac{x_{i+1}-x}{2}\right)\sin\left(\frac{x-x_{i-1}}{2}\right)] & \\ +\sin\left(\frac{x-x_{i-2}}{2}\right)\sin^2\left(\frac{x_i-x}{2}\right), & x_{i-1} \leq x \leq x_i, \\ \sin\left(\frac{x_{i+2}-x}{2}\right)[\sin\left(\frac{x-x_{i-1}}{2}\right)\sin\left(\frac{x_{i+1}-x}{2}\right)+\sin\left(\frac{x_{i+2}-x}{2}\right)\sin\left(\frac{x-x_i}{2}\right)] & \\ +\sin\left(\frac{x-x_{i-2}}{2}\right)\sin^2\left(\frac{x_{i+1}-x}{2}\right), & x_i \leq x \leq x_{i+1}, \\ \sin^3\left(\frac{x_{i+2}-x}{2}\right), & x \leq x_{i+2}, \\ 0, & \text{otherwise}, \end{cases} \tag{33}$$

where $\omega(h = \sin\left(\frac{h}{2}\right)\sin\left(h\right)\sin\left(\frac{3h}{2}\right)$ and $\{TB_{-1}(x), TB_0(x), \ldots, TB_N(x), TB_{N+1}(x)\}$ form a basis over the region $0 \leq x \leq l$. The coefficients of the approximate function $TB_i(x)$ and its derivatives are given in Table 1.

Table 1: Coefficients of cubic B-splines and its derivatives at knots

x	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$TB_i(x)$	0	η_1	η_2	η_1	0
$TB_i^{'}(x)$	0	$-\eta_3$	0	η_3	0
$TB_i^{"}(x)$	0	η_4	η_5	η_4	0

We have

$$\eta_1=\frac{\sin^2\frac{h}{2}}{\sin(h)\sin(\frac{3h}{2})},\qquad \eta_2=\frac{2}{1+2\cos(h)},\quad \eta_3=\frac{3}{4\sin(\frac{3h}{2})},$$
$$\eta_4=\frac{3(1+3\cos(h))}{16\sin^2(\frac{h}{2})\left(2\cos(\frac{h}{2})+\cos(\frac{3h}{2})\right)},\quad \eta_5=-\frac{3\cos^2(\frac{h}{2})}{2\sin^2(\frac{h}{2})\left(1+2\cos(h)\right)}.$$

Let $U(x)$ be the cubic trigonometric B-spline collocation to approximate (1) and given as

$$U(x)\approx\sum_{i=-1}^{M+1}\alpha_i(t)TB_i(x), \tag{34}$$

where $\alpha_i(t)$ is an unknown time-dependent parameter to be determined from the collocation method together with using the boundary and initial conditions. Using (34) and Table 1, an approximate values of $U(x,t)$ and its first and second derivatives at the knots are

$$\begin{cases} U_i=\eta_1\alpha_{i-1}+\eta_2\alpha_i+\eta_1\alpha_{i+1},\\ U_i'=-\eta_3\alpha_{i-1}+\eta_3\alpha_{i+1},\\ U_i''=\eta_4\alpha_{i-1}+\eta_5\alpha_i+\eta_4\alpha_{i+1}.\end{cases} \tag{35}$$

By substituting (35) into (27), we obtain

$$\begin{aligned}&\left(-\eta_4\left(\frac{\varepsilon}{\tau(j)}+\alpha\right)-\eta_3a_i^{j+1}+\eta_1b_i^{j+1}\right)\alpha_{i-1}^{j+1}+\left(-\eta_5\left(\frac{\varepsilon}{\tau(j)}+\alpha\right)+\eta_2b_i^{j+1}\right)\alpha_i^{j+1}\\ &\quad+\left(-\eta_4\left(\frac{\varepsilon}{\tau(j)}+\alpha\right)+\eta_3a_i^{j+1}+\eta_1b_i^{j+1}\right)\alpha_{i+1}^{j+1}\\ &\quad=\left(-\frac{\varepsilon}{\tau(j)}\left(\eta_4-\eta_1\right)\right)\alpha_{i-1}^{j}+\left(-\frac{\varepsilon}{\tau(j)}\left(\eta_5-\eta_2\right)\right)\alpha_i^{j}\\ &\qquad+\left(-\frac{\varepsilon}{\tau(j)}\left(\eta_4-\eta_1\right)\right)\alpha_{i+1}^{j}+g_i^{j+1}.\end{aligned} \tag{36}$$

This can be written as

$$r_i^-\alpha_{i-1}^{j+1}+r_i^c\alpha_i^{j+1}+r_i^+\alpha_{i+1}^{j+1}=q_i^-\alpha_{i-1}^{j}+q_i^c\alpha_i^{j}+q_i^+\alpha_{i+1}^{j}+g_i^{j+1}, \tag{37}$$

where

$$\begin{aligned} r_i^-&=-\eta_4\left(\frac{\varepsilon}{\tau(j)}+\alpha\right)-\eta_3a_i^{j+1}+\eta_1b_i^{j+1},\\ r_i^c&=-\eta_5\left(\frac{\varepsilon}{\tau(j)}+\alpha\right)+\eta_2b_i^{j+1},\end{aligned}$$

$$\begin{aligned}
r_i^+ &= -\eta_4\left(\frac{\varepsilon}{\tau(j)}+\alpha\right)+\eta_3 a_i^{j+1}+\eta_1 b_i^{j+1},\\
q_i^- &= -\frac{\varepsilon}{\tau(j)}\left(\eta_4-\eta_1\right),\\
q_i^c &= -\frac{\varepsilon}{\tau(j)}\left(\eta_5-\eta_2\right),\\
q_i^+ &= -\frac{\varepsilon}{\tau(j)}\left(\eta_4-\eta_1\right).
\end{aligned}$$

Imposing the boundary condition

Using the boundary conditions into (35), we have for $i=0$,

$$\eta_1\alpha_{-1}^j+\eta_2\alpha_0^j+\eta_1\alpha_1^j=\phi_0 \Rightarrow \alpha_{-1}^j=\frac{1}{\eta_1}\phi_0-\frac{\eta_2}{\eta_1}\alpha_0^j-\alpha_1^j, \tag{38}$$

for $i=N$,

$$\eta_1\alpha_{N-1}^j+\eta_2\alpha_N^j+\eta_1\alpha_{N+1}^j=\phi_N \Rightarrow \alpha_{N+1}^j=\frac{1}{\eta_1}\phi_N-\alpha_{N-1}^j-\frac{\eta_2}{\eta_1}\alpha_N^j. \tag{39}$$

Substituting (38) and (39) into (37), we obtain

$$\begin{cases}
\left(r_0^c-\frac{\eta_2}{\eta_1}r_0^-\right)\alpha_0^{j+1}+\left(r_0^+-r_0^-\right)\alpha_0^{j+1} & \\
=\left(q_0^c-\frac{\eta_2}{\eta_1}q_0^-\right)\alpha_0^j+\left(q_0^+-q_0^-\right)\alpha_0^j+\left(\frac{q_0^-}{\eta_1}\phi_0^j-\frac{r_0^-}{\eta_1}\phi_0^{j+1}\right)+g_0^{j+1}, & \\
r_i^-\alpha_{i-1}^{j+1}+r_i^c\alpha_i^{j+1}+r_i^+\alpha_{i+1}^{j+1}=q_i^-\alpha_{i-1}^j+q_i^c\alpha_i^j+q_i^+\alpha_{i+1}^j+g_i^{j+1}, & \\
\left(r_N^--r_N^+\right)\alpha_{N-1}^{j+1}+\left(r_N^c-\frac{\eta_2}{\eta_1}r_N^+\right)\alpha_N^{j+1} & \\
=\left(q_N^+-q_N^+\right)\alpha_{N-1}^j+\left(q_N^c-\frac{\eta_2}{\eta_1}q_N^+\right)\alpha_N^j+\left(\frac{q_N^+}{\eta_1}\phi_N^j-\frac{r_N^+}{\eta_1}\phi_N^{j+1}\right)+g_N^{j+1}, & \\
u(x_i,0)=\varphi(x_i), & x_i\in\bar{\Omega},\\
u(0,t_{j+1})=u(l,t_{j+1})=0, & t_{j+1}\in(0,T].
\end{cases} \tag{40}$$

Equation (40) is an $(N+1)\times(N+1)$ system of linear equations.

Determination of the initial vector α_i^0

An initial vector can be calculated from the initial condition and first space derivative of the initial conditions at the boundaries. At the knots x_i, the following relations are used:

$$\begin{aligned}
U^0(x_0,0)&=\phi_0^0=\eta_1\alpha_{-1}^0+\eta_2\alpha_0^0+\eta_1\alpha_1^0\\
U^0(i,0)&=\phi_0^i=\eta_1\alpha_{i-1}^0+\eta_2\alpha_i^0+\eta_1\alpha_{i+1}^0, \qquad i=1,2,\dots,N-1,\\
U^0(1,0)&=\phi_N^0=\eta_1\alpha_{N-1}^0+\eta_2\alpha_N^0+\eta_1\alpha_{N+1}^0.
\end{aligned} \tag{41}$$

From first derivative of (35), we also have

$$\eta_3\alpha_1^0 - \eta_3\alpha_{-1}^0 = \left(\phi_0^0\right)' \Rightarrow \alpha_{-1}^0 = \alpha_1^0 - \frac{1}{\eta_3}\left(\phi_0^0\right)',$$
$$\eta_3\alpha_{N+1}^0 - \eta_3\alpha_{N-1}^0 = \left(\phi_N^0\right)' \Rightarrow \alpha_{N+1}^0 = \alpha_{N-1}^0 + \frac{1}{\eta_3}\left(\phi_N^0\right)'. \tag{42}$$

Substituting (42) into (41), we obtain an $(N+1)\times(N+1)$ system of linear equations:

$$\begin{aligned}
&\eta_2\alpha_0^0 + 2\eta_1\alpha_1^0 = \phi_0^0 + \frac{\eta_1}{\eta_3}\left(\phi_0^0\right)' \\
&\eta_1\alpha_{i-1}^0 + \eta_2\alpha_i^0 + \eta_1\alpha_{i+1}^0 = U^0(i,0), \qquad i = 1,2,\ldots,N-1, \\
&2\eta_1\alpha_{N-1}^0 + \eta_2\alpha_N^0 = \phi_N^0 - \frac{\eta_1}{\eta_3}\left(\phi_N^0\right)'.
\end{aligned} \tag{43}$$

5 Convergence analysis

Lemma 6. The trigonometric B-spline collocation $\{TB_{-1}(x), TB_0(x), \ldots, TB_N(x), TB_{N+1}(x)\}$ defined in (33) satisfies the inequality

$$\sum_{i=-1}^{N+1} |TB_i(x)| \leq 6, \quad x \in [0,1]. \tag{44}$$

Proof. From the triangular inequality, we have

$$\left|\sum_{i=-1}^{N+1} TB_i(x)\right| \leq \sum_{i=-1}^{N+1} |TB_i(x)|.$$

At any node x_i, we have

$$\begin{aligned}
\sum_{i=-1}^{N+1} |TB_i(x)| &= |TB_{i-1}(x)| + |TB_i(x)| + |TB_{i+1}(x)| \\
&= |\eta_1| + |\eta_2| + |\eta_1| \leq 4.
\end{aligned}$$

At any point in each subinterval $x_{i-1} \leq x \leq x_i$, we also have

$$\sum_{i=-1}^{N+1} |TB_i(x_i)| \leq 2 \quad \textit{and} \quad \sum_{i=-1}^{N+1} |TB_{i-1}(x_{i-1})| \leq 2,$$

and similarly for $x \in [x_{i-1}, x_i]$, we have

$$\sum_{i=-1}^{N+1} |TB_{i+1}(x_i)| \leq 1 \quad and \quad \sum_{i=-1}^{N+1} |TB_{i-2}(x_{i-1})| \leq 1.$$

Therefore

$$\sum_{i=-1}^{N+1} |TB_i(x)| = |TB_{i-2}(x)| + |TB_{i-1}(x)| + |TB_i(x)| + |TB_{i+1}(x)| \leq 6. \quad (45)$$

□

Lemma 7. Let $u(x)$ be the exact solution of the boundary value problem (40) and let $U(x) = \sum_{i=-1}^{N+1} \alpha_i(t) TB_i(x)$ be the trigonometric B-spline collocation approximation of $u(x)$. Then

$$\|u(x) - U(x)\|_\infty \leq C\left(h^2\right) \quad (46)$$

for sufficiently small h, and C is a positive constant.

Proof. Let $\bar{U}(x) = \sum_{i=-1}^{N+1} \bar{\alpha}_i TB_i(x)$ be a unique spline interpolate to be computed B-spline approximation to $u(x)$, where $\bar{\alpha}_i = (\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_N)^T$.

To estimate $\|u(x) - U(x)\|$, we must estimate the errors $\left\|u(x) - \bar{U}(x)\right\|$ and $\left\|\bar{U}(x) - U(x)\right\|$, respectively. Now, (40) is written as

$$A\alpha_i^{j+1} = H, \quad (47)$$

where $H = B\alpha_i^j + D$.

Following (47) for $\bar{U}(x)$, we get

$$A\bar{\alpha}_i^{j+1} = \bar{H}, \quad (48)$$

where $\bar{\alpha}_i^{j+1} = \left(\bar{\alpha}_0^{j+1}, \bar{\alpha}_1^{j+1}, \ldots, \bar{\alpha}_N^{j+1}\right)^T$.

Now, from (47) and (48), we obtain

$$A\left(\alpha_i^{j+1} - \bar{\alpha}_i^{j+1}\right) = \left(H - \bar{H}\right). \quad (49)$$

To proceed this, we consider the following theorem.

Theorem 1. Suppose that $H \in C^2[0,l]$ that $u(x) \in C^4[0,l]$, and that $\bar{\Omega}_x = \{0 = x_0 < x_1 < \cdots < x_N = l\}$ is a uniform partition of $[0,l]$ with the step size h. If $U(x)$ is the unique trigonometric B-spline approximation for

$u(x)$ at the knots $x_0, \ldots, x_N$, then

$$\begin{aligned} &|U(x) - u(x)| \leq O(h^3), \\ &\left|U^{(k)}(x) - u^{(k)}(x)\right| \leq O(h^2), \quad k = 1, 2, \\ &\left|U^{(k)}(x) - u^{(k)}(x)\right| \leq O(h), \quad k = 3, \end{aligned}$$

Proof. See [18]. □

Setting the right-hand side of (27) by H_i and using Theorem 1, we obtain the bound $\left\|H - \breve{H}\right\|$ as

$$\begin{aligned} \left|H_i - \bar{H}_i\right| &= \left|cu_i^{''} + au_i^{'} + bu_i - c\bar{u}_i^{''} + a\bar{u}_i^{'} + b\bar{u}_i\right| \\ &\leq |c| \left|u_i^{''} - \bar{u}_i^{''}\right| + |b| \left|u_i^{'} - \bar{u}_i^{'}\right| + |b| \left|u_i - \bar{u}_i\right|, \end{aligned}$$

where
$c = -\left(\frac{\varepsilon}{\tau(j)} + \frac{\alpha}{2}\right), a = a\left(x, t_{j+1}\right), b = \frac{\varepsilon}{\tau(j)} + b\left(x, t_{j+1}\right),$
which is

$$\left|H_i - \bar{H}_i\right| \leq |c|\, O(h^2) + |b|\, O(h^2) + |b|\, O(h^3) \leq K(h^2), \tag{50}$$

where $K = |c| + |b| + |b|\, O(h)$. Now from (49) and (50), we have

$$\|A\| \left\|\alpha_i^{j+1} - \bar{\alpha}_i^{j+1}\right\| = \left\|H - \bar{H}\right\| \leq Kh^2.$$

This yields

$$\left\|\alpha_i^{j+1} - \bar{\alpha}_i^{j+1}\right\| \leq Kh^2 \left\|A^{-1}\right\|. \tag{51}$$

Moreover, $TB_i(x)$ and its derivative up to the second order have nonvanishing values at the mesh points $[x_{i=2}, x_{i+2}]$ and at other mesh points it is zero. Using these facts, the matrix $\|A\|$ is a tridiagonal and diagonally dominant matrix. Hence, the matrix is nonsingular, and A^{-1} is bounded. Then, we get

$$\left\|\alpha_i^{j+1} - \bar{\alpha}_i^{j+1}\right\| \leq K_1 h^2, \tag{52}$$

where $K_1 = K\left\|A^{-1}\right\|$. Again from $U(x) - \bar{U}(x)$ and Lemma 7, we get

$$U(x) - \bar{U}(x) = \sum_{i=-1}^{N+1} \left(\alpha_i - \bar{\alpha}_i\right) TB_i(x) \leq \bar{k}h^2, \quad \bar{k} = 6k_1. \tag{53}$$

Therefore, from Theorem 1 and (53), we get

$$\begin{aligned}\|u(x)-U(x)\|_\infty &= \left\|u(x)-U(x)+\bar{U}(x)-\bar{U}(x)\right\|_\infty \\ &\leq \left\|u(x)-\bar{U}(x)\right\| + \left\|\bar{U}(x)-U(x)\right\| \\ &\leq O(h^3)+\bar{K}(h^2) \\ &\leq Ch^2, \quad C=\bar{K}+O(h).\end{aligned}$$

□

Theorem 2. Let $u(x,t)$ be the solution of (1) and let $U(x_i,t_{j+1})$ be the solution of the total discretized equation. Under the hypothesis of Lemmas 5 and 7, then the $\varepsilon-$uniform estimate holds

$$\sup_{1\leq i\leq N-1} = \max_{1\leq i\leq N-1, 0<j<M} |u(x_i,t_{j+1})-U(x_i,t_{j+1})| \leq C\left(h^2+\tau\right), \quad (54)$$

where C is the constant independent of ε, h, and τ.

Proof. The proof is obtained by applying the triangle inequality. □

6 Numerical results

To demonstrate the validity of the proposed scheme for the problem, one model example is presented. As the exact solution of this example is not known, the maximum point-wise error for the given example were computed by using the double mesh principle as

$$E_\varepsilon^{N,M} = \max_{1\leq i\leq N-1}\left|U_i^{N,M}-U_i^{2N,2M}\right|,$$

where $U_i^{N,M}$ is the numerical solution obtained on the mesh $D^N=\Omega_x^N\times\Omega_t^M$ with N mesh intervals in the spatial direction and M mesh intervals in the temporal direction. For any value of N and M, the $\varepsilon-$uniform errors are calculated using

$$E^{N,M} = \max_\varepsilon E_\varepsilon^{N,M}.$$

The rate of convergence of the scheme is calculated by the formula

$$r_\varepsilon^{N,M} = \frac{\log(E_\varepsilon^{N,M})-\log(E_\varepsilon^{2N,2M})}{\log(2)},$$

and the $\varepsilon-$uniform convergence is calculated by

$$r^{N,M} = \frac{\log\left(E^{N,M}\right) - \log\left(E^{2N,2M}\right)}{\log(2)}.$$

Example 1. From [1]

$$\begin{cases} \varepsilon\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^3 u}{\partial t\partial x^2} - 2\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}u\frac{\partial u}{\partial x} = \exp(-t)\sin(\pi x), & \\ u(x,0) = \sin(\pi x), & x \in \overline{\Omega}_x, \\ u(0,t) = u(1,t) = 0, & t \in (0,1]. \end{cases}$$

Table 2: $E_\varepsilon^{N,M}$ for $N = M$

ε	32	64	128	256	512
2^0	5.5142e-03	2.8124e-03	1.4209e-03	7.1420e-04	3.5806e-04
2^{-2}	1.9178e-02	1.0254e-02	5.2831e-03	2.6643e-03	1.3257e-03
2^{-4}	5.1476e-02	3.3936e-02	2.1160e-02	1.1892e-02	5.1156e-03
2^{-6}	5.1485e-02	3.3947e-02	2.1168e-02	1.1900e-02	5.1221e-03
2^{-8}	5.1487e-02	3.3950e-02	2.1170e-02	1.1902e-02	5.1239e-03
2^{-10}	5.1488e-02	3.3951e-02	2.1170e-02	1.1903e-02	5.1243e-03
2^{-12}	5.1488e-02	3.3951e-02	2.1170e-02	1.1903e-02	5.1244e-03
2^{-14}	5.1488e-02	3.3951e-02	2.1170e-02	1.1903e-02	5.1245e-03
2^{-16}	5.1488e-02	3.3951e-02	2.1170e-02	1.1903e-02	5.1245e-03
$E^{N,M}$	5.1488e-02	3.3951e-02	2.1170e-02	1.1903e-02	5.1245e-03
$r^{N,M}$	0.60078	0.68143	0.83070	1.2158	

Table 3: $E_{\varepsilon}^{N,M}$ for the proposed method with $N = 60$

ε	N=40	N=80	N=160	N=320	N=640
2^{0}	5.3980e-03	2.8124e-03	1.5094e-03	9.1150e-04	7.7675e-04
2^{-2}	1.9182e-02	1.0254e-02	5.3330e-03	2.7775e-03	1.5158e-03
2^{-4}	5.1414e-02	3.3936e-02	2.1200e-02	1.1953e-02	5.2045e-03
2^{-6}	5.1426e-02	3.3947e-02	2.1207e-02	1.1869e-02	5.2096e-03
2^{-8}	5.1430e-02	3.3950e-02	2.1209e-02	1.1871e-02	5.2110e-03
2^{-10}	5.1431e-02	3.3951e-02	2.1210e-02	1.1871e-02	5.2113e-03
2^{-12}	5.1431e-02	3.3951e-02	2.1210e-02	1.1871e-02	5.2114e-03
2^{-14}	5.1431e-02	3.3951e-02	2.1210e-02	1.1871e-02	5.2114e-03
$E^{N,M}$	5.1431e-02	3.3951e-02	2.1210e-02	1.1871e-02	5.2114e-03
$r^{N,M}$	0.59918	0.67871	8.373	1.1877	

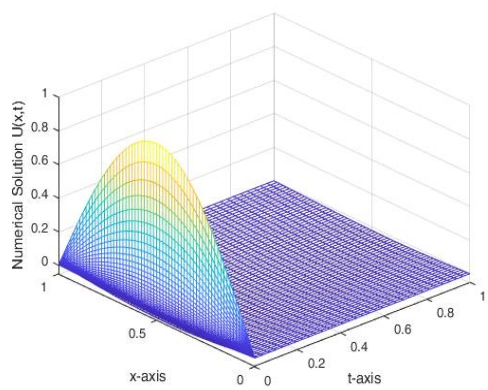

Figure 1: Numerical solution for $N = M = 64$ and $\varepsilon = 2^{-8}$.

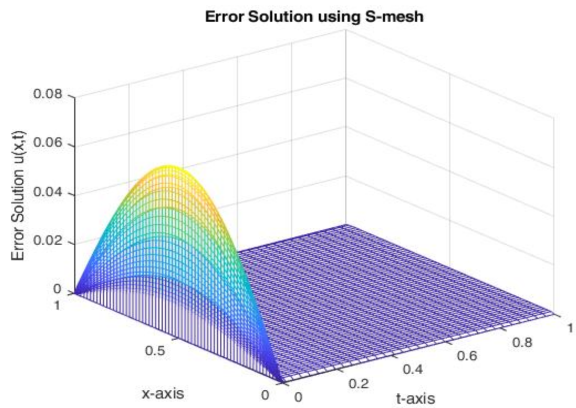

Figure 2: Error solution for $M = N = 64$ and $\varepsilon = 2^{-8}$.

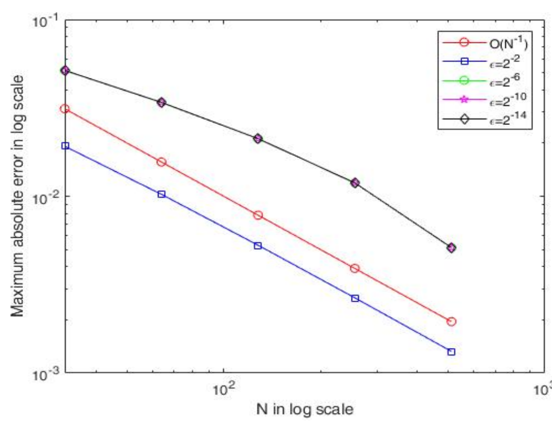

Figure 3: Log-log scale plot for Example 1.

The computed maximum point-wise errors are also presented in Tables 2 and 3. From Table 2, one can observe that as $\varepsilon \to 0$ and time step size decreases with uniform spatial step size, then maximum point-wise also monotonically decreases, and the rate convergence of the method is almost one. Table 3 also yields as temporal step size decreases for fixed spatial step size; then the results of maximum absolute point-wise error also decrease. From Figures 1 and 2, one can also observe that the mesh is dense near the initial, and hence, it indicates that the solution of the model example has an initial layer at $t = 0$. The log-log plot of the scheme is also displayed in Figure 3, which confirms an agreement of the theoretical and numerical results. Finally, the result from the model example confirms that the proposed numerical method is convergent.

7 Conclusions

A nonpolynomial B-spline collocation method was implemented for singularly perturbed quasilinear Sobolev problems with initial boundary value problems. Newton's linearization method was applied to linearize the nonlinear parts. An implicit Euler method in time variable and cubic trigonometric B-spline collocation was used to approximate the space variable and obtain a three-term recurrence relation. Convergence analysis of the scheme was considered, and the scheme was accurate of order $O(h^2 + \tau)$. The results from the model example indicated the method is accurate for different values of ε, M, and N. In general, the effect of the perturbation parameter indicated that the scheme has a layer at initial points $t = 0$.

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