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Research Article

Multiple interpolation with the fast-growing knots in the class of entire functions and its application

I. Sheparovych

Abstract

The conditions for the sequence of complex numbers $(b_{n,k})$ are obtained, such that the interpolation problem $g^{(k-1)}(\lambda_n) = b_{n,k}$, $k \in \overline{1,s}$, $n \in \mathbb{N}$, where $|\lambda_k/\lambda_{k+1}| \leq \Delta < 1$, has a unique solution in some classes of entire functions g for which $M_g(r) \leq c_1 \exp((s-1)N(r) + N(\rho_1 r))$, where $N(r)$ is the counting function of the sequence (λ_n), $\rho_1 \in (\Delta; 1)$, and $c_1 > 0$. Moreover, these results have been applied to the description of the solution of the differential equation $f^{(s)} + A_0(z)f = 0$ for which (λ_n) is zero-sequence and the coefficient A_0 is an entire function from the mentioned class.

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1 Introduction

Let $(b_{n,k})$ be an arbitrary sequence of complex numbers and let (λ_k) be a sequence of distinct complex numbers without finite limit points. Set $N(r) = \int_0^r \frac{n(t)}{t}dt$, where $n(r)$ is equal to the number of points of the sequence (λ_k) in the disk $|z| < r$. Note that $N(r) = \sum_{|\lambda_k| \leq r} \log \frac{r}{|\lambda_k|}$. Let $g(z)$ be an entire function, and let $M_g(r) = \max\{|g(z)| : |z| = r\}$.

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Iryna Sheparovych
Department of Physics, Mathematics, Economy and Innovative Technologies, Ivan Franko State Pedagogical Institute of Drohobych, Drohobych, Ukraine. e-mail: ishep-arovych@ukr.net

Hel'fond [11] and Kaz'min [14] considered an interpolation problem $g(\lambda_k) = b_k$, $k \in \mathbb{N}$, with interpolation knots in $\lambda_k = q^{k-1}$ and $|q| > 1$. From their results, the next theorem follows.

Theorem A (Kaz'min). Let $\lambda_k = q^{k-1}, |q| > 1$. Then for every sequence (b_k) such that

$$\overline{\lim_{k\to\infty}} |q|^{-\frac{(k-1)}{2}} |b_k|^{1/k} \le r_1, \quad r_1 \in (\Delta; 1),$$

the interpolation problem $g(\lambda_k) = b_k$ has a unique solution in the class of entire functions g, that satisfy the condition

$$\ln M_g(r) \le \frac{\ln^2 \rho_1 r}{2 \ln |q|} + \frac{\ln r}{2} + c_2$$

for each $\rho_1 > r_1$ and some $c_2 > 0$ (here and farther c_i are positive constants).

The aim of this paper is to consider the interpolation problem

$$g^{(k-1)}(\lambda_n) = b_{k,n}, \qquad k \in \{1, \dots, s\}, n \in \mathbb{N}, \tag{1}$$

where $s \in \mathbb{N}$ and the sequence (λ_n) satisfies the condition

$$|\lambda_n/\lambda_{n+1}| \le \Delta, \quad n \in \mathbb{N}, \tag{2}$$

for some $\Delta \in (0; 1)$. It should be noted that in the case when $s = 2$, the problem was solved in [21] and the next assertion was proved.

Theorem B. Let (λ_k) be a sequence of complex numbers satisfying condition (2). Then for every sequences $(b_{n,1})$ and $(b_{n,2})$ such that for some $q \in (\Delta; 1)$,

$$|b_{k,1}| \le c_1 \exp\left(2N\left(q|\lambda_k|\right)\right), \ |\lambda_k||b_{k,2}| \le c_2 \exp\left(N\left(|\lambda_k|\right) + N\left(q|\lambda_k|\right)\right), \ k \in \mathbb{N},$$

the interpolation problem $g(\lambda_k) = b_{k,1}, g'(\lambda_k) = b_{k,2}, k \in \mathbb{R}$ has a unique solution in the class of entire functions g that satisfy the condition

$$M_g(r) \le c_3 \exp\left(N(r) + N(\rho_1 r)\right)$$

for each $\rho_1 \in (q; 1)$. The interpolation function has the form

$$g(z) = \sum_{k=1}^{\infty} \left(-\frac{L''(\lambda_k) b_{k,1}}{L'^3(\lambda_k)} \frac{L^2(z)}{z - \lambda_k} + \frac{b_{k,2}}{L'^2(\lambda_k)} \frac{L^2(z)}{z - \lambda_k} + \frac{b_{k,1}}{L'^2(\lambda_k)} \frac{L^2(z)}{(z - \lambda_k)^2} \right),$$

where $L(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{\lambda_j}\right)$.

In this article, this result is generalized. We prove the following theorem.

Theorem 1. Let (λ_k) be a sequence of nonzero numbers satisfying condition (2) for some $\Delta \in (0;1)$, then for every sequences $(b_{k,n})$, $k \in \overline{1,s}$, $n \in \mathbb{N}$, such that

$$|b_{n,k}| \leq c_1 k! |\lambda_n|^{-k} \exp\left(kN\left(|\lambda_n|\right) + (s-k)N\left(q|\lambda_n|\right)\right), \tag{3}$$

where $q \in (\Delta;1)$ and $c_1 > 0$, the interpolation problem (1) has an unique solution in the class of entire functions g, such that

$$M_g(r) \leq c_2 \exp((s-1)N(r) + N(\rho_1 r)) \tag{4}$$

for each $\rho_1 \in (q;1)$ and some $c_2 > 0$.

To prove this, we construct the function g by the known methods used in [5, 15, 16, 21]. The function has the form

$$g(z) = \sum_{n=1}^{\infty} \sum_{k=1}^{s} \frac{L^s(z)}{(s-k)!(z-\lambda_n)^k} \sum_{i=0}^{s-k} C_{s-k}^i b_{n,i} \gamma_{s,s-k-i}(\lambda_n), \tag{5}$$

where $L(z) = \prod_{k=1}^{\infty} (1 - z/\lambda_k)$ and $\gamma_{s,j}(z) = \left(\frac{(z-\lambda_n)^s}{L^s(z)}\right)^{(j)}$.

2 Preliminaries

We need some lemmas.

Lemma 1. [21] Let (λ_k) be a sequence of distinct complex nonzero numbers, satisfying condition (2) for some $\Delta \in (0;1)$. Then there exists a constant $c > 1$ such that $n(\rho_2 r) \leq n(r) + c$ for each $\rho_2 \geq 1$ with $\rho_2 \Delta \leq 1$, and for every $\rho_1, 0 < \rho_1 \leq \rho_2$, the inequality

$$N(\rho_2 r) \leq N(\rho_1 r) + c \log \frac{\rho_2}{\rho_1} + n(r) \log \frac{\rho_2}{\rho_1} \tag{6}$$

is fulfilled.

Lemma 2. If (λ_k) is a sequence of distinct complex nonzero numbers such that the series $\sum_{k=1}^{+\infty} 1/|\lambda_k|$ is convergent and $L(z) = \prod_{k=1}^{\infty} (1 - z/\lambda_k)$, then for $l \geq 2$, it holds that

$$\frac{L^{(l)}(\lambda_k)}{L'(\lambda_k)} = \frac{l}{l-1} \sum_{n=1, n\neq k}^{\infty} \sum_{j=2}^{l} \frac{L^{(j-1)}(\lambda_k)}{L'(\lambda_k)} \frac{(-1)^j (l-j)!}{(\lambda_k - \lambda_n)^{l-j+1}}. \tag{7}$$

Proof. In [2, 13], there is the relation $\frac{L''(\lambda_k)}{L'(\lambda_k)} = 2\sum\limits_{n=1,n\neq k}^{\infty}\frac{1}{\lambda_k-\lambda_n}$, (it was justified in [21]). Now we argue similarly. The next relationships

$$L(z)=\sum_{i=1}^{\infty}\frac{1}{i!}L^{(i)}(\lambda_k)(z-\lambda_k)^i,\ \frac{L(z)}{(z-\lambda_k)}=\sum_{i=1}^{\infty}\frac{1}{i!}L^{(i)}(\lambda_k)(z-\lambda_k)^{i-1},$$

$$\left(\frac{L(z)}{z-\lambda_k}\right)^{(s)}=\sum_{i=s+1}^{\infty}\frac{1}{i(i-s-1)!}L^{(i)}(\lambda_k)(z-\lambda_k)^{i-s-1},$$

$$\left(\frac{L(z)}{z-\lambda_n}\right)^{(s)}=\sum_{j=0}^{s}(-1)^j(s-j)!L^{(j)}(z)(z-\lambda_n)^{-s+j-1},$$

are true. So, by differentiating $l-1$ times the equality $L'(z)=\frac{L(z)}{z-\lambda_k}+\sum\limits_{n=1,n\neq k}^{\infty}\frac{L(z)}{z-\lambda_n}$, we obtain

$$\begin{aligned}
L^{(l)}(z)=&\left(\frac{L(z)}{z-\lambda_k}\right)^{(l-1)}+\sum_{n=1,n\neq k}^{\infty}\left(\frac{L(z)}{z-\lambda_n}\right)^{(l-1)}\\
=&\sum_{i=l}^{\infty}\frac{1}{i(i-l)!}L^{(i)}(\lambda_k)(z-\lambda_k)^{i-l}\\
&+\sum_{n=1,n\neq k}^{\infty}\frac{1}{(z-\lambda_n)^l}\sum_{j=0}^{l-1}(-1)^j(l-j-1)!L^{(j)}(z)(z-\lambda_n)^j\\
=&\frac{1}{l}L^{(l)}(\lambda_k)+o(z-\lambda_k)\\
&+\sum_{n=1,n\neq k}^{\infty}\frac{1}{(z-\lambda_n)^l}\sum_{j=0}^{l-1}(-1)^j(l-j-1)!L^{(j)}(z)(z-\lambda_n)^j\\
=&\frac{1}{l}L^{(l)}(\lambda_k)+o(z-\lambda_k)\\
&+\sum_{n=1,n\neq k}^{\infty}\sum_{j=1}^{l}\frac{(-1)^{j-1}(l-j)!}{(z-\lambda_n)^{l-j+1}}L^{(j-1)}(z),\quad z\to\lambda_k.
\end{aligned}$$

Furthermore, for $z\to\lambda_k$, the next equality

$$\frac{(l-1)L^{(l)}(\lambda_k)}{lL'(\lambda_k)}=\sum_{n=1,n\neq k}^{\infty}\sum_{j=2}^{l}\frac{(-1)^j(l-j)!}{(\lambda_k-\lambda_n)^{l-j+1}}\frac{L^{(j-1)}(\lambda_k)}{L'(\lambda_k)}$$

holds, which completes the proof of Lemma 2. □

Lemma 3. If a sequence (λ_k) satisfies condition (2) for some $\Delta \in (0;1)$, then the next inequality is true

$$\left|\frac{L^{(j)}(\lambda_k)}{L'(\lambda_k)}\right| \le c(j)\left(\frac{k}{|\lambda_k|}\right)^{j-1}, \quad j \in \mathbb{N}\backslash\{1\}.$$

Proof. From (7), we can obtain

$$\left|\frac{L^{(j)}(\lambda_k)}{L'(\lambda_k)}\right| \le c(j)\left(\sum_{n=1,n\neq k}^{\infty}\frac{1}{|\lambda_k-\lambda_n|}\right)^{j-1}, \quad j \ge 2. \tag{8}$$

Indeed, for $j=2$, we have

$$\left|\frac{L''(\lambda_k)}{L'(\lambda_k)}\right| \le 2\left|\sum_{n=1,n\neq k}^{\infty}\frac{1}{\lambda_k-\lambda_n}\right| \le 2\sum_{n=1,n\neq k}^{\infty}\frac{1}{|\lambda_k-\lambda_n|}.$$

Let us use the induction to obtain (8). Assume that (8) is true for $j \le l$ and prove it for $j = l+1$:

$$\begin{aligned}
\left|\frac{L^{(l+1)}(\lambda_k)}{L'(\lambda_k)}\right| &\le \frac{l+1}{l}\left|\sum_{n=1,n\neq k}^{\infty}\sum_{j=2}^{l+1}(-1)^{j+1}\frac{L^{(j-1)}(\lambda_k)}{L'(\lambda_k)}\frac{(l+1-j)!}{(\lambda_k-\lambda_n)^{l-j+2}}\right| \\
&= \frac{l+1}{l}\left|\sum_{n=1,n\neq k}^{\infty}\left((-1)^l\frac{L^{(l)}(\lambda_k)}{L'(\lambda_k)}\frac{1}{(\lambda_k-\lambda_n)} + \sum_{j=2}^{l}(-1)^{j+1}\frac{L^{(j-1)}(\lambda_k)}{L'(\lambda_k)}\frac{(l+1-j)!}{(\lambda_k-\lambda_n)^{l-j+2}}\right)\right| \\
&\le \frac{l+1}{l}\left|\frac{L^{(l)}(\lambda_k)}{L'(\lambda_k)}\right|\sum_{n=1,n\neq k}^{\infty}\frac{1}{|\lambda_k-\lambda_n|} \\
&\quad + \frac{l+1}{l}\sum_{j=2}^{l}(l+1-j)!\left|\frac{L^{(j-1)}(\lambda_k)}{L'(\lambda_k)}\right|\sum_{n=1,n\neq k}^{\infty}\frac{1}{|\lambda_k-\lambda_n|^{l-j+2}} \\
&\le c(l)\left(\sum_{n=1,n\neq k}^{\infty}\frac{1}{|\lambda_k-\lambda_n|}\right)^{l} \\
&\quad + \sum_{j=2}^{l}(l+1-j)!c(j-1)\left(\sum_{n\neq k}\frac{1}{|\lambda_k-\lambda_n|}\right)^{j-2}\sum_{n\neq k}\frac{1}{|\lambda_k-\lambda_n|^{l-j+2}} \\
&\le c(l)\left(\sum_{n=1,n\neq k}^{\infty}\frac{1}{|\lambda_k-\lambda_n|}\right)^{l}\sum_{j=2}^{l+1}(l+1-j)! \le C(l)\left(\sum_{n\neq k}\frac{1}{|\lambda_k-\lambda_n|}\right)^{l}.
\end{aligned}$$

Thus, using a simple mathematical calculation, we have $(c_3 := c(j))$

$$\left|\frac{L^{(j)}(\lambda_k)}{L'(\lambda_k)}\right| \le c_3 \left(\frac{1}{|\lambda_k|}\sum_{n=1}^{k-1}\frac{1}{1-|\lambda_n/\lambda_k|} + \sum_{n=k+1}^{\infty}\frac{1}{|\lambda_n|\,(1-|\lambda_k/\lambda_n|)}\right)^{j-1}$$
$$\le \frac{c_3}{|\lambda_k|^{j-1}}\left(\sum_{n=1}^{k-1}\frac{1}{1-\Delta} + \sum_{n=k+1}^{\infty}\frac{|\lambda_k|}{|\lambda_n|(1-\Delta^{n-k})}\right)^{j-1}$$
$$\le \frac{c_3}{|\lambda_k|^{j-1}}\left(\frac{k-1}{1-\Delta} + \sum_{n=k+1}^{\infty}\Delta^{n-k}\right)^{j-1} \le c_4\left(\frac{k}{|\lambda_k|}\right)^{j-1},$$

where $c_4 := c_4(j, \Delta)$. □

Lemma 4. If a sequence (λ_k) satisfies condition (2), then for $\gamma_{s,j}(z) := \left(\frac{(z-\lambda_n)^s}{L^s(z)}\right)^{(j)}$, $j \in \overline{0, s-1}$, where $L(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)$, the inequalities

$$|\gamma_{s,0}(\lambda_n)| \le c_3|\lambda_n|^s \exp(-sN_\lambda(|\lambda_n|) + cs), \tag{9}$$

$$|\gamma_{s,l}(\lambda_n)| \le c_4 s(s+1)\dots(s+l-1)|\lambda_n|^{s-l} n^l \exp(-sN(|\lambda_n|) + cs) \tag{10}$$

are fulfilled.

Proof. First, we prove (9). By the known equality (see, for example, [20]) $\log|\lambda_k L'(\lambda_k)| = N(|\lambda_k|) + O(1), k \in \mathbb{N}$, we have

$$\left|\lambda_n^{-s}\gamma_{s,0}(\lambda_n)\right| = \frac{1}{|\lambda_n L'(\lambda_n)|^s} = \exp(-sN(\lambda_n) + O(s)), \quad n \in \mathbb{N}.$$

Furthermore, let us prove (10). Since $L(z) = \sum_{i=1}^{\infty}\frac{1}{i!}L^{(i)}(\lambda_n)(z-\lambda_n)^i$ and $L'(z) = \sum_{i=1}^{\infty}\frac{i}{i!}L^{(i)}(\lambda_n)(z-\lambda_n)^{i-1}$, we have

$$L(z) - (z-\lambda_n)L'(z) = -\sum_{i=2}^{\infty}\frac{i-1}{i!}L^{(i)}(\lambda_n)(z-\lambda_n)^i,$$

$$\big(L(z) - (z-\lambda_n)L'(z)\big)^{(l)} = -\sum_{i=l}^{\infty}\frac{i(i-1)\dots(i-l+1)(i-1)}{i!}L^{(i)}(\lambda_n)(z-\lambda_n)^{i-l}.$$

Then

$$\gamma_{s,1}(z) = \left(\frac{(z-\lambda_n)^s}{L^s(z)}\right)' = s\frac{(z-\lambda_n)^{s-1}}{L^{s+1}(z)}(L(z) - (z-\lambda_n)L'(z))$$
$$= -s\frac{(z-\lambda_n)^{s+1}}{L^{s+1}(z)}\sum_{i=2}^{\infty}\frac{i-1}{i!}L^{(i)}(\lambda_n)(z-\lambda_n)^{i-2}$$
$$= -s\frac{(z-\lambda_n)^{s+1}}{L^{s+1}(z)}\left(\frac{1}{2!}L''(\lambda_n) + \frac{2}{3!}L'''(\lambda_n)(z-\lambda_n) + \cdots\right)$$
$$= -s\frac{(z-\lambda_n)^{s+1}}{2L^{s+1}(z)}(L''(\lambda_n) + O(z-\lambda_n)), \quad z \to \lambda_n;$$

$$\gamma_{s,1}(\lambda_n) = -\frac{sL''(\lambda_n)}{2!\,(L'(\lambda_n))^{s+1}} = -\frac{s}{2!\,(L'(\lambda_n))^{s}}\frac{L''(\lambda_n)}{L'(\lambda_n)}.$$

Thus,

$$\begin{aligned}\gamma_{s,2}(z) &= \left(\left(\frac{z-\lambda_n}{L(z)}\right)^s\right)'' = -\left(s\left(\frac{z-\lambda_n}{L(z)}\right)^{s+1}\sum_{i=2}^{\infty}\frac{i-1}{i!}L^{(i)}(\lambda_n)(z-\lambda_n)^{i-2}\right)' \\ &= s\frac{(s+1)(z-\lambda_n)^{s+2}}{L^{s+2}(z)}\left(\sum_{i=2}^{\infty}\frac{i-1}{i!}L^{(i)}(\lambda_n)(z-\lambda_n)^{i-2}\right)^2 \\ &\quad - s\frac{(z-\lambda_n)^{s+1}}{L^{s+1}(z)}\sum_{i=3}^{\infty}\frac{(i-1)(i-2)}{i!}L^{(i)}(\lambda_n)(z-\lambda_n)^{i-3}; \\ \gamma_{s,2}(\lambda_n) &= \frac{s(s+1)}{(2!)^2\,(L'(\lambda_n))^{s+2}}\left(L''(\lambda_n)\right)^2 - \frac{2sL'''(\lambda_n)}{3!\,(L'(\lambda_n))^{s+1}} \\ &= \frac{s}{2\,(L'(\lambda_n))^s}\left(\frac{(s+1)}{2}\left(\frac{L''(\lambda_n)}{L'(\lambda_n)}\right)^2 - \frac{2L'''(\lambda_n)}{3L'(\lambda_n)}\right).\end{aligned}$$

Continuing the process, we can obtain (by mathematical induction) a general formula, for $l \geq 1$,

$$\begin{aligned}\gamma_{s,l}(\lambda_n) =& (-1)^l s(s+1)(s+2)\dots(s+l-1)\frac{(L''(\lambda_n))^l}{2^l\,(L'(\lambda_n))^{s+l}} \\ &+ (-1)^{l-1}C_l^2 s(s+1)(s+2)\dots(s+l-2)\frac{(L''(\lambda_n))^{l-2}\,L'''(\lambda_n)}{(2!)^2 3!\,(L'(\lambda_n))^{s+l-1}} + \cdots \\ &- \frac{s}{(l+1)}\frac{L^{(l+1)}(\lambda_n)}{(L'(\lambda_n))^{s+1}}.\end{aligned}$$

It is not difficult to show that (by Lemma 3)

$$\begin{aligned}\left|\gamma_{s,l}(\lambda_n)\right| \leq& c_4\frac{s(s+1)\dots(s+l-1)n^l}{|\lambda_n|^l|L'(\lambda_n)|^s} \\ \leq& c_4 s(s+1)\dots(s+l-1)|\lambda_n|^{s-l}n^l\exp(-sN(|\lambda_n|)+cs).\end{aligned}$$

□

3 Proof of Theorem 1

Proof of Theorem 1. First, we will prove the *uniqueness*. Assume to the contrary that for some sequences $(b_{n,k})$ with the properties (3) in the class (4), there exist two different entire functions $g = f_1$ and $g = f_2$ that solve problem (1). Then the function $f = f_2 - f_1$ has zeros of order $m_k \geq s$ at all points λ_k and satisfies the condition (2) for some $\rho_1 < 1$. This implies from the Jensen inequality $\ln M_f(r) \geq sN(r) + O(1)$ that $N(r) \leq N(\rho_1 r) + c_0$. This is a contradiction, because $N(r) - N(\rho_1 r) \to +\infty$, if $\rho_1 < 1$ and $r \to +\infty$. Thus uniqueness is proved.

Now, we will prove that the function g of form (5) satisfies condition (4). Since (see [20, 18, 19]) for every $n \in \mathbb{N}$, $r \in [0;+\infty)$, and $\varepsilon > 0$,

$$\max\left\{\left|\frac{L(z)}{z-\lambda_n}\right| : |z| \le r\right\} \le c(\varepsilon)\frac{M_L((1+\varepsilon)r)}{r+|\lambda_n|}, \tag{11}$$

and

$$\exp\left(N(\lambda_n)\right) = \frac{|\lambda_n|^n}{\prod\limits_{k=1}^{n}|\lambda_k|}, \quad \exp\left(N\left(q|\lambda_n|\right)\right) = \frac{(q|\lambda_n|)^{n-1}}{\prod\limits_{k=1}^{n-1}|\lambda_k|} = q^{n-1}\frac{|\lambda_n|^n}{\prod\limits_{k=1}^{n}|\lambda_k|},$$

$q \in (\Delta;1)$, applying Lemmas 2, 3, and 4 and conditions (3), (9), and (10), we obtain ($1 \le k \le s$; $0 \le i \le s-k$)

$$\begin{aligned}\left|C_{s-k}^i b_{n,i}\gamma_{s,s-k-i}(\lambda_n)\right| &\le c_4 i! C_{s-k}^i s(s+1)\dots(s+s-k-i-1)|\lambda_n|^k n^{s-k-i}\\ &\qquad\times\exp\left(cs-(s-i)N\left(|\lambda_n|\right)+(s-i)N\left(q|\lambda_n|\right)\right)\\ &\le c_4\frac{(s-k)!(2s-k-i-1)!}{(s-k-i)!(s-1)!}|\lambda_n|^k n^{s-k-i}q^{(s-i)(n-1)}.\end{aligned}$$

Thus, by (11) and the equality $\ln M_L(r) = N(r)+O(1), r \in [0;+\infty)$, (see, for example, [21]), for each $k \in \overline{1,s}$ and some $s \in \mathbb{N}$, one has (with $\rho_2 := (1+\varepsilon)$)

$$\begin{aligned}I_{k,n} &:= \left|\frac{L^s(z)}{(z-\lambda_n)^k}\sum_{i=0}^{s-k}C_{s-k}^i b_{n,i}\gamma_{s,s-k-i}(\lambda_n)\right|\\ &\le \left|L^{s-k}(z)\right|\left|\frac{L^k(z)}{(z-\lambda_n)^k}\right|\sum_{i=0}^{s-k}C_{s-k}^i|b_{n,i}\gamma_{s,s-k-i}(\lambda_n)|\\ &\le c_4\exp\left((s-k)N_\lambda(r)+kN_\lambda(\rho_2 r)+cs\right)\left(\frac{|\lambda_n|q^n}{r+|\lambda_n|}\right)^k\frac{(s-k)!}{(s-1)!}\\ &\quad\times\sum_{i=0}^{s-k}q^{-s+i}(nq^n)^{s-k-i}\frac{(2s-k-i)!}{(s-k-i)!}.\end{aligned}$$

Furthermore, suppose that $|\lambda_m| \le r < |\lambda_{m+1}|$,that $|z| \le r$, and that $\rho_1 = \rho_2 q$. Then $m = n(r)$ and

$$\left(\frac{|\lambda_n|q^n}{r+|\lambda_n|}\right)^k \le \left(\frac{|\lambda_n|q^n}{r}\right)^k \le \left(\frac{|\lambda_n|q^n}{|\lambda_m|}\right)^k \le \left(q^n\Delta^{m-n}\right)^k = \Delta^{mk}\left(\frac{q}{\Delta}\right)^{nk},$$

So, proceeding from Lemma 1,

$$\begin{aligned}&\exp\left((s-k)N_\lambda(r)+kN_\lambda(\rho_2 r)+cs\right)\left(\tfrac{|\lambda_n|q^n}{r+|\lambda_n|}\right)^k\\&\qquad\le\exp\left((s-k)N_\lambda(r)+kN_\lambda(\rho_1 r)+km\ln\tfrac{1}{q}+cs\right)\Delta^{mk}\left(\tfrac{q}{\Delta}\right)^{nk}\\&\qquad\le\exp\left((s-k)N_\lambda(r)+kN_\lambda(\rho_1 r)+cs\right)\left(\tfrac{\Delta}{q}\right)^{(m-n)k}\end{aligned}$$

for $m>n$. Hence, if $m>n$, then

$$\begin{aligned}I_{k,n}\le&c_4\exp\left((s-k)N_\lambda(r)+kN_\lambda(\rho_1 r)+cs\right)\left(\frac{\Delta}{q}\right)^{(m-n)k}\frac{(s-k)!}{(s-1)!}\\&\times\sum_{i=0}^{s-k}q^{-s+i}(nq^n)^{s-k-i}\frac{(2s-k-i)!}{(s-k-i)!}.\end{aligned}$$

If $m<n$, then

$$\begin{aligned}&\exp\left((s-k)N_\lambda(r)+kN_\lambda(\rho_2 r)+cs\right)\left(\tfrac{|\lambda_n|q^n}{r+|\lambda_n|}\right)^k\\&\quad\le\exp\left((s-k)N_\lambda(r)+kN_\lambda(\rho_1 r)+km\ln\tfrac{1}{q}+cs\right)q^{kn}\\&\quad=\exp\left((s-k)N_\lambda(r)+kN_\lambda(\rho_1 r)+cs\right)q^{k(n-m)}\end{aligned}$$

and

$$\begin{aligned}I_{k,n}\le&\ c_4\exp\left((s-k)N_\lambda(r)+kN_\lambda(\rho_1 r)+cs\right)\\&\times q^{k(n-m)}\tfrac{(s-k)!}{(s-1)!}\sum_{i=0}^{s-k}q^{-s+i}(nq^n)^{s-k-i}\tfrac{(s+s-k-i)!}{(s-k-i)!}.\end{aligned}$$

Therefore, since $nq^{n-1}\le 1$ for $n\ge n_0$ and

$$\sum_{n=1}^{m}\left(\frac{\Delta}{q}\right)^{(m-n)k}\le\frac{1}{1-(\Delta/q)^k}\le\frac{q}{q-\Delta},\quad\sum_{n=m+1}^{\infty}q^{k(n-m)}\le\frac{q^k}{1-q},$$

$$\sum_{k=1}^{s}\frac{1}{(s-1)!}\sum_{i=0}^{s-k}\frac{(2s-k-i)!}{(s-k-i)!}\le\frac{2s^2\Gamma(s-1)}{(s+1)!\Gamma(2s-1)}=:c(s),$$

we have

$$\begin{aligned}|g(z)|\le&\sum_{n=1}^{\infty}\sum_{k=1}^{s}\left|\frac{L^s(z)}{(s-k)!(z-\lambda_n)^k}\right|\sum_{i=0}^{s-k}C^i_{s-k}|b_{n,i}\gamma_{s,s-k-i}(\lambda_n)|\\=&\sum_{n=1}^{m}\sum_{k=1}^{s}\frac{I_{n,k}}{(s-k)!}+\sum_{n=m+1}^{+\infty}\sum_{k=1}^{s}\frac{I_{n,k}}{(s-k)!}\end{aligned}$$

$$
\begin{aligned}
&\le c_5 \sum_{k=1}^{s} \frac{1}{(s-1)!} \exp\left((s-k)N(r) + kN(\rho_1 r) + cs\right) \\
&\quad \times \left(\sum_{n=1}^{m} \left(\frac{\Delta}{q}\right)^{(m-n)k} \sum_{i=0}^{s-k} q^{-s+i} (nq^n)^{s-k-i} \frac{(2s-k-i)!}{(s-k-i)!} \right. \\
&\qquad \left. + \sum_{n=m+1}^{\infty} q^{k(n-m)} \sum_{i=0}^{s-k} q^{-s+i} (nq^n)^{s-k-i} \frac{(2s-k-i)!}{(s-k-i)!} \right) \\
&\le c(s,\Delta) \exp\left((s-1)N(r) + N(\rho_1 r)\right).
\end{aligned}
$$

□

Remark: If (λ_k) satisfies (2), then class (4) can be defined by the condition

$$M_g(r) \le c_2 \exp\left(sN(\rho_3 r)\right) \tag{12}$$

for every $\rho_3 \in (\sqrt[s]{\rho_1}; 1)$.

This statement follows from a similar one in [21, Remark 3].

4 One application to the differential equations

Multiple interpolation can be applied to the problem of oscillation of linear differential equations of higher order, in particular, of equations of the form

$$f^{(s)} + A_{s-1}(z) f^{(s-1)} + \cdots + A_1(z) f' + A_0(z) f = 0,$$

where $A_i(z) (i \in \overline{0, s-1}; s \in \mathbb{N}\backslash\{1\})$ are analytic functions. Many papers are devoted to this one and adjoining problems. See, for example, [3, 17, 7, 12, 9, 8].

Let us consider the equation

$$f^{(s)} + A_0(z) f = 0. \tag{13}$$

Problem. Let (λ_k) be a given sequence of distinct complex numbers having no finite limit points and satisfying (2). Does there exist an entire function $A_0(z)$ such that the differential equation (13) possesses a solution f with the zero-sequence (λ_k)? How does $M_f(r)$ increase?

Note, that in [21] that problem was considered for the case $s = 2$. The following statements were proved.

Corollary 1. Let (λ_k) be a sequence of complex numbers satisfies a condition (2) for some $\Delta < 1$. Then there exists an entire function $A_0(z)$ such that $f'' + A_0(z) f = 0$ possesses a solution f, an entire function with the zero-sequence (λ_k), and

$$\ln M_f(r) \le N(r) + c_3 \exp(2N(\rho_3 r))$$

for each $\rho_3 \in (\rho_1; 1)$.

Corollary 2. If (λ_k) satisfies (2) for some $\Delta < 1$, then there exists an entire function $A_0(z)$ such that $f'' + A_0(z)f = 0$ possesses a solution f with the zero-sequence (λ_k) and

$$|A_0(z)| \le c_2 \exp(4N(R_1 r))$$

for each $R_1 \in (\rho_1; 1)$ and all $r > 0$.

Following [2], we can set the solution of (13) in the form $f(z) = L(z)e^{w(z)}$, where L is an entire function with the simple zeros at points λ_n (for example, $L(z) = \prod_{n=1}^{\infty} (1 - z/\lambda_n)$), w is some entire functions from the class (4) (or (12)). It is not difficult to see (for the case $s = 2$, it was proved in [21, 9]) that $f = Le^w$ is a solution of (13) with zero sequence (λ_n), if w is a solution of the multiple interpolation problem

$$w^{(k+1)}(\lambda_n) = b_{n,k}, \qquad k \in \{0, \dots, s-2\}, n \in \mathbb{N},$$

where

$$\begin{aligned} b_{n,0} =& w'(\lambda_n) = -\frac{L''(\lambda_n)}{2L'(\lambda_n)}, \\ b_{n,1} =& w''(\lambda_n) = -\frac{L'''(\lambda_n)}{3L'(\lambda_n)} + \left(\frac{L''(\lambda_n)}{2L'(\lambda_n)}\right)^2, \\ &\vdots \\ b_{n,k} =& w^{(k+1)}(\lambda_n) \\ =& -\frac{1}{(k+2)L'(\lambda_n)} \times \left(L^{(k+2)}(\lambda_n) + \sum_{j=0}^{k-2} C_{k-2}^j w^{(j+2)} L^{(k-j)}(\lambda_n) \right. \\ &\left. + \sum_{j=0}^{k-1} C_k^j \left(2L^{(k+1-j)} w^{(j+1)} + L^{(k-j)} (w'^2)^{(j)}\right)(\lambda_n) \right). \end{aligned} \tag{14}$$

This problem leads to (1) if we put $g(z) = w'(z)$. From (14), applying Lemmas 2 and 3, we have the next inequality

$$|b_{n,k}| \le c_4 \left(\frac{n}{|\lambda_n|}\right)^{k+1}$$

for every $k \in \overline{0, s-1}$ and $n \in \mathbb{N}$. Thus, $b_{n,k}$, $k \in \overline{0, s-1}$, satisfy the conditions (3) of Theorem 1, which proves the existence of the function w

from the class (4), and consequently, the existence of solution $f = Le^w$ of equation (13).

Therefore, in analogue of Corollaries 1 and 2, the next assertion is true.

Theorem 2. Let (λ_k) be a sequence of complex numbers satisfying condition (2) for some $\Delta < 1$. Then there exists an entire function $A_0(z)$ such that for every $s \geq 2$, equation (13) possesses an entire solution f with the sequence of zeros (λ_k) and for every $\rho_1 \in (q; 1)$ and all $r > 0$, we have

$$\ln M_f(r) \leq c_2 \exp((s-1)N(r) + N(\rho_1 r))$$

or

$$\ln M_f(r) \leq c_2 \exp(sN(\rho_3 r))$$

for every $\rho_3 \in (\rho_1; 1)$ and all $r > 0$.

In addition, $A_0(z) = -\frac{f^{(s)}(z)}{f(z)}$ satisfies the growth estimate

$$M_{A_0}(r) \leq c_3(s) \exp\left(s^2 N(\rho_3 r)\right)$$

for every $\rho_3 \in (\rho_1; 1)$ and all $r > 0$.

5 Conclusion

Problems of multiple interpolation in the classes of entire functions have been investigated in the works of many authors, for example [15, 16, 5, 10, 6, 1, 4].

In this article, there was shown that interpolation problem (1) has the unique solution in class (3) when interpolation knots grow fast (satisfy condition (2)).

Applying counting function $(N(r))$ of the sequence (λ_n) in the definition of the mentioned classes is the peculiarity of this result.

Also, the obtained result was applied to the oscillation problem of the differential equation $f^{(s)} + A_0(z)f = 0$. So, there was shown existing an entire function $A_0(z)$ such that the previous differential equation possesses a solution f with the zero-sequence (λ_k) and there was found growth order of f.

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