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Some efficient Nordsieck integration methods for IVPs

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Abstract

In this paper, in continuation of the construction of efficient numerical methods for stiff IVPs, we construct type two Nordsieck second derivative general linear methods with order $p = s$, where s is the number of internal stages, and stage order $q = p$. Implementation of the constructed methods with fixed and variable stepsize is discussed which verifies their efficiency.

Keywords: Stiff differential equations; Nordsieck second derivative general linear methods; A- and L-stability; Variable stepsize implementation.

1 Introduction

Second derivative general linear methods (SGLMs) for the numerical solution of autonomous ordinary differential equations (ODEs) with initial value problem

$$\begin{aligned} &y'(x) = f(y(x)), \qquad x \in [x_0, \overline{x}], \\ &y(x_0) = y_0, \end{aligned} \tag{1}$$

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where $f : \mathbb{R}^m \to \mathbb{R}^m$ and m is the dimensionality of the system, have been studied in recent years. These methods were introduced by Butcher and Hojjati in [11] and investigated more in [2–6] by Abdi and Hojjati. In the construction of SGLMs which are extension of general linear methods (GLMs) [10, 13–15, 17]), there are a lot of free parameters which allow us to construct high order methods with a small number of internal stages to reduce computational cost [1].

We recall that SGLMs are characterized by four integers, (p, q, r, s) where p and q are, respectively, order and stage order of the method, r is the number of input and output approximations, and s is the number of internal stages. Let $Y^{[n]} = [Y_i^{[n]}]_{i=1}^s$ be an approximation of stage order q to the vector $y(x_{n-1} + ch) = [y(x_{n-1} + c_i h)]_{i=1}^s$, and let the vectors $f(Y^{[n]}) = [f(Y_i^{[n]})]_{i=1}^s$ and $g(Y^{[n]}) = [g(Y_i^{[n]})]_{i=1}^s$ denote the stage first and second derivative values, where $g(\cdot) = f'(\cdot)f(\cdot)$, respectively. If $r = p + 1$, we can assume the input and output vectors at the step number n, $y^{[n-1]}$ and $y^{[n]}$, are approximations of order p to the Nordsieck vectors

$$\begin{bmatrix} y(x_{n-1}) \\ hy'(x_{n-1}) \\ \vdots \\ h^p y^{(p)}(x_{n-1}) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} y(x_n) \\ hy'(x_n) \\ \vdots \\ h^p y^{(p)}(x_n) \end{bmatrix},$$

respectively. In an SGLM used for the numerical solution of (1), these values are related by

$$\begin{aligned} Y^{[n]} &= h(A \otimes I_m) f(Y^{[n]}) + h^2(\overline{A} \otimes I_m) g(Y^{[n]}) + (U \otimes I_m) y^{[n-1]}, \\ y^{[n]} &= h(B \otimes I_m) f(Y^{[n]}) + h^2(\overline{B} \otimes I_m) g(Y^{[n]}) + (V \otimes I_m) y^{[n-1]}, \end{aligned} \tag{2}$$

where $n = 1, 2, \ldots, N$, $Nh = \overline{x} - x_0$, h is the stepsize, and $\otimes$ is the Kronecker product of two matrices. Here $A, \overline{A} \in \mathbb{R}^{s\times s}$, $U \in \mathbb{R}^{s\times r}$, $B, \overline{B} \in \mathbb{R}^{r\times s}$, and $V \in \mathbb{R}^{r\times r}$. The coefficients matrix V in the Nordsieck SGLM (2) has the form

$$V = \left[\begin{array}{c|c} 1 & v^T \\ \hline 0 & \dot{V} \end{array}\right],$$

$v = [v_1 \ \ v_2 \ \ \cdots \ \ v_{r-1}]^T$, $\dot{V} \in \mathbb{R}^{(r-1)\times(r-1)}$. In this paper, the methods will be restricted to the case where $p = q = r - 1 = s$ and the eigenvalues of $\dot{V}$ are zeros. The latter condition ensures zero-stability of the method.

An SGLM in Nordsieck form has order p and stage order $q = p$ if and only if [11]

$$U = C - ACK - \overline{A}CK^2,$$

$$V = E - BCK - \overline{B}CK^2,$$

where the matrices $C \in \mathbb{R}^{s\times(p+1)}$, $K \in \mathbb{R}^{(p+1)\times(p+1)}$, and $E \in \mathbb{R}^{(p+1)\times(p+1)}$ are defined by

$$C := \begin{bmatrix} 1 & \frac{c}{1!} & \frac{c^2}{2!} & \cdots & \frac{c^p}{p!} \end{bmatrix}, \qquad K := [0 \quad e_1 \quad e_2 \quad \cdots \quad e_p\,],$$

and

$$E := \exp(K) = \begin{bmatrix} 1 & \frac{1}{1!} & \frac{1}{2!} & \cdots & \frac{1}{p!} \\ 0 & 1 & \frac{1}{1!} & \cdots & \frac{1}{(p-1)!} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \frac{1}{1!} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

with e_j as the jth vector of canonical basis in $\mathbb{R}^{p+1}$, respectively.

The stability behavior of SGLMs is defined using the standard test problem of Dahlquist $y' = \xi y$, where ξ is a complex number. If method (2) is applied to this problem, then the stability matrix is

$$M(z) = V + \big(zB + z^2\overline{B}\big)\big(I - zA - z^2\overline{A}\big)^{-1}U,$$

where $z = h\xi$ and the stability function of the method is defined as the characteristic polynomial of $M(z)$; that is,

$$p(w, z) = \det\big(wI - M(z)\big).$$

If $p(w, z) = w^{r-1}\big(w - R(z)\big)$, the method is said to possess “Runge–Kutta stability (RKS)”. GLMs and SGLMs with RKS property were studied by Butcher and Jackiewicz in [9, 12–14] and by Abdi and Hojjati in [2–7], respectively.

For economical implementation, it is assumed that the matrices A and $\overline{A}$ have a lower triangular form with the same diagonal entries λ and μ, respectively. SGLMs are divided into four types, depending on the nature of the differential system to be solved and the computer architecture that is used to implement these methods. Types 1 and 2 are those with arbitrary a_{ij} and $\overline{a}_{ij}$, where $\lambda = \mu = 0$ and $\lambda > 0$, $\mu < 0$, respectively. Such methods are appropriate, respectively, for nonstiff and stiff differential systems in a sequential computing environment. Requiring $a_{ij} = \overline{a}_{ij} = 0$, cases $\lambda = \mu = 0$ and $\lambda > 0$, $\mu < 0$ lead, respectively, to types 3 and 4 methods which can be useful, respectively, for nonstiff and stiff systems in a parallel computing environment.

The construction and implementation of type 2 SGLMs with $p = s + 1$ were discussed in [4]. Also, the construction of parallel Nordsieck SGLMs with $p = s$ and their order barriers were studied in [5]. These order barriers were obtained under the assumption of RKS property. In this paper, we are going to construct type 2 Nordsieck SGLMs with $p = s$, which are efficient methods for stiff systems. Also, efficiency of the constructed methods are shown by their implementation in a variable stepsize environment.

Next sections of this paper are organized as follows: In section 2, we construct A- and L-stable SGLMs in the Nordsieck form with RKS property of orders 2, 3, and 4. Considering Nordsieck SGLMs in the variable stepsize mode, implementation issues including local error estimation, and stepsize control are discussed in section 3. Finally, in section 4, some results of numerical experiments on some stiff test systems are presented and compared with those obtained by Nordsieck SGLMs of the same order.

2 Construction of type 2 methods

In this section, the construction of type 2 Nordsieck SGLMs of order p and stage order $q = p$ with some desired stability properties is explained.

2.1 Methods with $p = q = s = r - 1 = 2$

We construct methods with $p = q = s = r - 1 = 2$ and RKS property. We look for methods which their stability function has the form [5]

$$R(z) = \frac{1 + n_1 z + n_2 z^2 + n_3 z^3}{(1 - \lambda z - \mu z^2)^2},$$

where

$$1 + \sum_{k=1}^{3} n_k z^k = \exp(z)(1 - \lambda z - \mu z^2)^2 - \mathcal{C} z^3 + O(z^4)$$

with $\mathcal{C}$ as the error constant of the method. For this method to be A-stable, it is necessary and sufficient that $\lambda > 0$, $\mu < 0$, and so that the $E(y)$ is non-negative for all real y, where the E-polynomial is defined by

$$E(y) = \left|1 - \lambda i y + \mu y^2\right|^4 - \left|1 + n_1 i y - n_2 y^2 - n_3 i y^3\right|^2.$$

By choosing $\mathcal{C} = 10^{-4}$, a detailed calculation shows that

$$E(y) = y^4\big(E_0 + E_1 y^2 + E_2 y^4\big),$$

where

$$\begin{aligned}
E_0 &= \frac{1247}{15000} + 2\mu^2 + \lambda^2 - 2\mu - \frac{4997}{7500}\lambda + 4\lambda\mu,\\
E_1 &= -\frac{24970009}{900000000} - \lambda^4 + 2\lambda^3 - \frac{19997}{15000}\lambda^2 - 4\mu^2 + \frac{4997}{7500}\mu + \frac{4997}{15000}\lambda\\
&\quad +4\mu^3 - 2\lambda^2\mu^2 - 4\lambda^3\mu + 8\lambda\mu^2 + 8\lambda^2\mu - \frac{34997}{7500}\lambda\mu,\\
E_2 &= 7\mu^4.
\end{aligned}$$

Pairs of (λ, μ) with values in domain $[0,2] \times [-2,0]$ giving L-stability are shown in Figure 1.

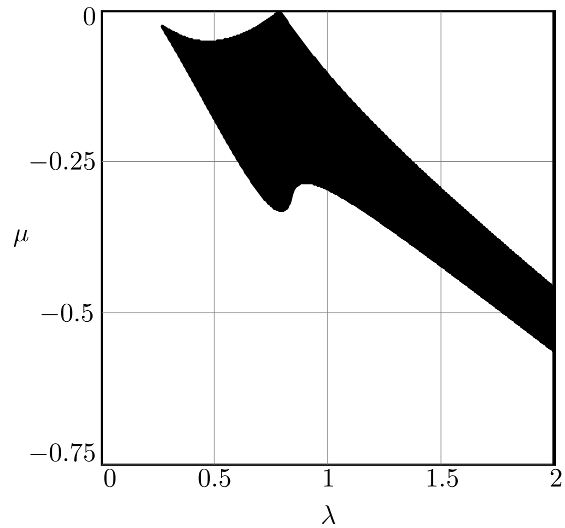


Figure 1: L-stable choices of (λ, μ) for $p = s = 2$ corresponding to $\mathcal{C} = 10^{-4}$.

We select a single example, characterized by $\lambda = \frac{4}{5}$, $\mu = -\frac{1}{5}$ and $c = [\frac{1}{2} \;\; 1]^T$. The coefficients of the method are

$$\left[\begin{array}{c|c|c} A & \overline{A} & U \\ \hline B & \overline{B} & V \end{array}\right] = \left[\begin{array}{cc|cc|ccc}
\frac{4}{5} & 0 & -\frac{1}{5} & 0 & 1 & -\frac{3}{10} & -\frac{3}{40} \\
-\frac{967}{18750} & \frac{4}{5} & \frac{494}{3375} & -\frac{1}{5} & 1 & \frac{4717}{18750} & -\frac{253}{12500} \\ \hline
-\frac{967}{18750} & \frac{4}{5} & \frac{494}{3375} & -\frac{1}{5} & 1 & \frac{4717}{18750} & -\frac{253}{12500} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right].$$

2.2 Methods with $p = q = s = r - 1 = 3$

We construct methods with $p = q = s = r - 1 = 3$ and RKS property. We look for methods which their stability function has the form [5]

$$R(z) = \frac{1 + n_1 z + n_2 z^2 + n_3 z^3 + n_4 z^4 + n_5 z^5}{(1 - \lambda z - \mu z^2)^3},$$

where

$$1 + \sum_{k=1}^{5} n_k z^k = \exp(z)(1 - \lambda z - \mu z^2)^3 - \mathcal{C}_1 z^4 - \mathcal{C}_2 z^5 + O(z^6).$$

Here $\mathcal{C}_1$ is the error constant of the method and $\mathcal{C}_2$ is an arbitrary number. The E-polynomial has the form

$$E(y) = y^6 \big(E_0 + E_1 y^2 + E_2 y^4 + E_3 y^6\big).$$

For these methods to be A-stable, it is necessary and sufficient that $\lambda > 0$, $\mu < 0$, and so that $E_0 + E_1 x + E_2 x^2 + E_3 x^3 + E_4 x^4$ is non-negative for all positive real numbers x, where E_0, E_1, E_2, E_3, and E_4 are complicated expressions in terms of λ and μ. By choosing $\mathcal{C}_1 = \mathcal{C}_2 = 10^{-4}$, pairs of (λ, μ) with values in domain $[0, 2] \times [-2, 0]$ giving L-stability are shown in Figure 2.

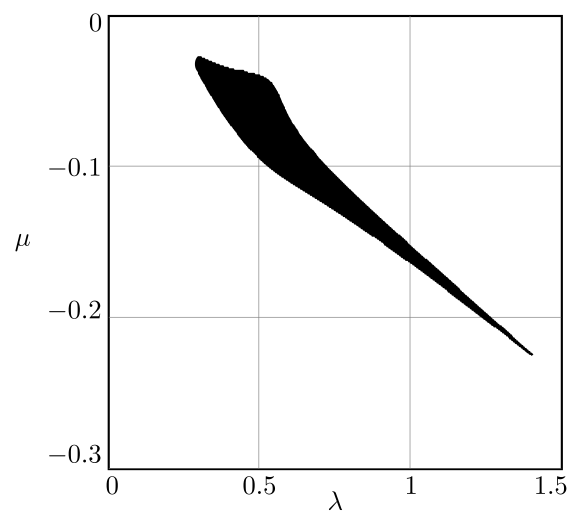


Figure 2: L-stable choices of (λ, μ) for $p = s = 3$ corresponding to $\mathcal{C}_1 = \mathcal{C}_2 = 10^{-4}$.

Here, we represent an example, characterized by

$$\lambda = \frac{1}{2}, \qquad \mu = -\frac{1}{15}, \qquad c = [\frac{1}{3} \quad \frac{2}{3} \quad 1]^T.$$

The coefficients of the method are

$$A = \begin{bmatrix} 0.5000000000000000 & 0 & 0 \\ 1.4279081052775164 & 0.5000000000000000 & 0 \\ 1.0000000000000000 & -0.3168631901664915 & 0.5000000000000000 \end{bmatrix},$$

$$\overline{A} = \begin{bmatrix} -0.0666666666666667 & 0 & 0 \\ -0.3067166674763493 & -0.0666666666666667 & 0 \\ -0.0602082721233515 & 0.0288951398441268 & -0.0666666666666667 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.0000000000000000 & -0.3168631901664915 & 0.5000000000000000 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 84.1340111524194390 & -15.9895442199910120 & -37.9511333307057703 \end{bmatrix},$$

$$\overline{B} = \begin{bmatrix} -0.0602082721233515 & 0.0288951398441268 & -0.0666666666666667 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1.7866934603873189 & 20.0458159571414841 \end{bmatrix},$$

$$U = \begin{bmatrix} 1.0000000000000000 & -0.1666666666666667 & -0.0444444444444444 & 0.0006172839506173 \\ 1.0000000000000000 & -1.2612414386108497 & -0.2136971453939340 & 0.0056267104705261 \\ 1.0000000000000000 & -0.1831368098335086 & -0.0241114076097811 & -0.0010021824846360 \end{bmatrix},$$

$$V = \begin{bmatrix} 1.0000000000000000 & -0.1831368098335086 & -0.0241114076097811 & -0.0010021824846360 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -30.1933336017226565 & 2.3070365964725901 & 0 \end{bmatrix}.$$

2.3 Methods with $p = q = s = r - 1 = 4$

In this part, we construct methods of the Nordsieck SGLMs of type 2 with $p = q = s = r - 1 = 4$ and $c = [0\ \ 0\ \ 0\ \ 1]^T$. Setting some free parameters in order to make calculation easier, RKS conditions make the coefficients matrices of the method to take the following forms

$$\left[\begin{array}{cccc|cccc|ccccc}
\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{12} & 0 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{12} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{4} & -\frac{1}{12} & 0 & 0 & 1 & -1 & \frac{1}{3} & 0 & 0 \\
\frac{1}{2} & 1 & \frac{1}{2} & 0 & -\frac{1}{4} & 1 & -\frac{1}{12} & 0 & 1 & -2 & -\frac{2}{3} & 0 & 0 \\
\frac{1}{2} & -1 & 1 & \frac{1}{2} & -\frac{1}{4} & 1 & -1 & -\frac{1}{12} & 1 & 0 & \frac{1}{3} & 0 & 0 \\
\hline
\frac{1}{2} & -1 & 1 & \frac{1}{2} & -\frac{1}{4} & 1 & -1 & -\frac{1}{12} & 1 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & -6 & 2 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & -12 & 7 & -1 & 0 & 6 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

For this method the only nonzero eigenvalue of $M(z)$ is

$$R(z) = \frac{1 + \frac{z}{2} + \frac{z^2}{12}}{1 - \frac{z}{2} + \frac{z^2}{12}},$$

consequently, by the Ehle conjecture [19], the method is A-stable. The error constant of the method is $\mathcal{C} = \frac{1}{720}$.

3 Implementation of the methods in a variable stepsize mode

In this section, we recall some implementation strategies given in [4], to apply the constructed methods in variable stepsize environment.

A Nordsieck SGLM in the variable stepsize mode takes the form

$$\begin{aligned}
Y^{[n]} &= h_n(A \otimes I_m) f(Y^{[n]}) + h_n^2(\overline{A} \otimes I_m) g(Y^{[n]}) + (UD(\delta_n) \otimes I_m) y^{[n-1]}, \\
y^{[n]} &= h_n(B \otimes I_m) f(Y^{[n]}) + h_n^2(\overline{B} \otimes I_m) g(Y^{[n]}) + (VD(\delta_n) \otimes I_m) y^{[n-1]},
\end{aligned} \tag{3}$$

where

$$D(\delta_n) := \text{diag}(1, \delta_n, \delta_n^2, ..., \delta_n^p), \quad \delta_n = h_n/h_{n-1}.$$

Due to the structure of the matrices V and $D(\delta_n)$, the matrix $VD(\delta_n)$ has one eigenvalue with magnitude one and a zero eigenvalue with multiplicity p, for any value of δ_n.

To achieve a suitable choice of stepsize for the next step, we first need to estimate the leading term in the local truncation error. To do this, we approximate $h^{p+1}y^{(p+1)}(x_n)$; so that $\text{LTE}(x_n) = \mathcal{C}_p h^{p+1} y^{(p+1)}(x_n)$ can be calculated as an approximation to the local truncation error.
For the methods with $p = s = 2$ and abscissas $c = [\frac{1}{2} \;\; 1]^T$, we use the linear combination of the form

$$\text{est}(x_n) := \mathcal{C}_p\Big(\alpha_1 hf(Y_1) + \alpha_2 hf(Y_2) + \beta h^2 g(Y_1)\Big),$$

with

$$\alpha_1 = -8, \quad \alpha_2 = 8, \quad \beta = -4.$$

For the methods with $p = s = 3$ and abscissas $c = [\frac{1}{3}\ \ \frac{2}{3}\ \ 1]^T$, we use the linear combination of the form

$$\text{est}(x_n) := \mathcal{C}_p\Big(\alpha_1 hf(Y_1) + \alpha_2 hf(Y_2) + \alpha_3 hf(Y_3) + \beta h^2 g(Y_1)\Big),$$

with

$$\alpha_1 = \frac{243}{2}, \quad \alpha_2 = -162, \quad \alpha_3 = \frac{81}{2}, \quad \beta = 27.$$

For the methods with $p = s = 4$ and abscissas $c = [0\ \ 0\ \ 0\ \ 1]^T$, we use the linear combination of the form

$$\text{est}(x_n) := \mathcal{C}_p\Big(\alpha_1 hf(Y_3) + \alpha_2 hf(Y_4) + \beta_1 h^2 g(Y_3) + \beta_2 h^2 g(Y_4) + \gamma y_4^{[n-1]}\Big),$$

with

$$\alpha_1 = 72, \quad \alpha_2 = -72, \quad \beta_1 = 48, \quad \beta_2 = 24, \quad \gamma = 12.$$

To control the stepsize, we use the following strategy

$$\text{est}(x_n) \le Rtol \cdot \max\{\|y_n\|, \|y_{n+1}\|\} + Atol, \tag{4}$$

where $Atol$ and $Rtol$ are given absolute and relative tolerances, respectively. If the control (4) is not satisfied, the current step is repeated with a halved stepsize. Otherwise, the current step is accepted and we carry out the next step with the new stepsize defined as

$$h_{n+1} = \delta_{n+1} h_n,$$

where

$$\delta_{n+1} = \min\Big\{facmax, \Big(\frac{fac \cdot tol}{\|\text{est}(x_n)\|_\infty}\Big)^{\frac{1}{p+1}}\Big\}.$$

Here, $facmax$ and fac are safety factors built into a code to prevent the step from increasing too rapidly and to avoid an excessive number of rejected steps. We have chosen $Atol = Rtol = tol$, $facmax = 2$, and $fac = 0.9$.

4 Numerical result

In this section we present the results of numerical experiments to show efficiency of the constructed methods in section 2 for fixed and variable stepsize mode using the provided techniques in section 3. We consider the following test problems:

- Problem 1. The nonlinear stiff system of ODEs

$$\begin{cases} y_1'(x) = -1002y_1(x) + 1000y_2^2(x), \\ y_2'(x) = y_1(x) - y_2(x)(1 + y_2(x)), \end{cases}$$

  whose exact solution is $[y_1(x),\ y_2(x)]^T = [\exp(-2x),\ \exp(-x)]^T$ and $x \in [0, 2]$. This problem is stiff with an approximate stiffness ratio of 10^3 near to $x = 0$.

- Problem 2. A system of differential equation, is called CUSP, resulting from discretization of the diffusion terms by the method of line the periodic boundary-value problem [16], is as below

$$\begin{cases} \frac{\partial y}{\partial t} = -\frac{1}{\varepsilon}(y^3 + ay + b) + \sigma \frac{\partial^2 y}{\partial x^2}, \\ \frac{\partial a}{\partial t} = b + 0.07\nu + \sigma \frac{\partial^2 a}{\partial x^2}, \\ \frac{\partial a}{\partial t} = (1 - a^2)b - a - 0.4y + 0.035\nu + \sigma \frac{\partial^2 b}{\partial x^2}, \end{cases}$$

  where

$$\nu = \frac{u}{0.1 + u}, \qquad u = (y - 0.7)(y - 1.3).$$

  This problem takes the form

$$\begin{cases} \dot{y}_i = -\varepsilon^{-1}(y_i^3 + a_i y_i + b_i) + D(y_{i-1} - 2y_i + y_{i+1}), \\ \dot{a}_i = b_i + 0.07\nu_i + D(a_{i-1} - 2a_i + a_{i+1}), \qquad i = 1, 2, \ldots, N, \\ \dot{b}_i = (1 - a_i^2)b_i - a_i - 0.4y_i + 0.035\nu_i + D(b_{i-1} - 2b_i + b_{i+1}), \end{cases}$$

  where

$$\nu_i = \frac{u_i}{0.1 + u_i}, \qquad u_i = (y_i - 0.7)(y_i - 1.3), \qquad D = \sigma N^2,$$

  with periodic boundary condition

$$\begin{array}{lll} y_0 := y_N, & a_0 := a_N, & b_0 := b_N, \\ y_{N+1} := y_1, & a_{N+1} := a_1, & b_{N+1} := b_1. \end{array}$$

  We take $\sigma = \frac{1}{144}, \epsilon = 10^{-4}, N = 32$, and the initial values as

$$y_i(0) = 0, \quad a_i(0) = -2\cos(\frac{2i\pi}{N}), \quad a_i(0) = 2\sin(\frac{2i\pi}{N}), \quad i = 1, 2, \ldots, N,$$

  with $t_{out} = 1.1$.

- Problem 3. For the ODE case, the Ring modulator problem originates from electrical circuit analysis is of the form [18]

$$\begin{cases} y_1' = C^{-1}(y_8 - 0.5y_{10} + 0.5y_{11} + y_{14} - R^{-1}y_1), \\ y_2' = C^{-1}(y_9 - 0.5y_{12} + 0.5y_{13} + y_{15} - R^{-1}y_2), \\ y_3' = C_s^{-1}\Big(y_{10} - q(U_{D1}) + q(U_{D4})\Big), \\ y_4' = C_s^{-1}\Big(-y_{11} + q(U_{D2}) - q(U_{D3})\Big), \\ y_5' = C_s^{-1}\Big(y_{12} + q(U_{D1}) - q(U_{D3})\Big), \\ y_6' = C_s^{-1}\Big(-y_{13} - q(U_{D2}) + q(U_{D4})\Big), \\ y_7' = C_p^{-1}\Big(R_p^{-1}y_7 + q(U_{D1}) + q(U_{D2}) - q(U_{D3}) - q(U_{D4})\Big), \\ y_8' = -L_h^{-1}y_1, \\ y_9' = -L_h^{-1}y_2, \\ y_{10}' = L_{s2}^{-1}(0.5y_1 - y_3 - R_{g2}y_{10}), \\ y_{11}' = L_{s2}^{-1}(-0.5y_1 + y_4 - R_{g3}y_{11}), \\ y_{12}' = L_{s3}^{-1}(0.5y_2 - y_5 - R_{g2}y_{12}), \\ y_{13}' = L_{s3}^{-1}(-0.5y_2 + y_6 - R_{g3}y_{13}), \\ y_{14}' = L_{s1}^{-1}\Big(-y_1 + U_{in1}(t) - (R_i + R_{g1})y_{14}\Big), \\ y_{15}' = L_{s1}^{-1}\Big(-y_2 - (R_c + R_{g1})y_{15}\Big). \end{cases}$$

The auxiliary functions $U_{D1}, U_{D2}, U_{D3}, U_{D4}, q, U_{in1}$, and U_{in2} are given by

$$\begin{cases} U_{D1} = y_3 - y_5 - y_7 - U_{in2}(t), \\ U_{D2} = -y_4 + y_6 - y_7 - U_{in2}(t), \\ U_{D3} = y_4 + y_5 + y_7 + U_{in2}(t), \\ U_{D4} = -y_3 - y_6 + y_7 + U_{in2}(t), \\ q(U) = \gamma(e^{\delta U} - 1), \\ U_{in1}(t) = 0.5\sin(2000\pi t), \\ U_{in2}(t) = 2\sin(20000\pi t). \end{cases}$$

The values of the parameters are

$$C = 1.6 \times 10^{-8}, \quad C_s = 2 \times 10^{-12}, \quad C_p = 10^{-12}, \quad L_h = 4.45,$$
$$L_{s1} = 0.002, \quad L_{s2} = 5 \times 10^{-4}, \quad L_{s3} = 5 \times 10^{-4},$$
$$R = 25000, \quad R_p = 50, \quad R_{g1} = 36.3, \quad R_{g2} = 17.3,$$
$$R_{g3} = 50, \quad R_i = 50, \quad R_c = 600,$$
$$\delta = 17.74933332, \ \gamma = 40.67286402 \times 10^{-9},$$

with the initial vector $y_0 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ and $t \in [0, 10^{-3}]$.

4.1 Fixed stepsize experiments

We first present fixed stepsize numerical results in order to show accuracy of constructed Nordsieck SGLMs and validate the order of these methods in the integration of stiff differential systems. To do this, we have applied the methods of order 2, 3, and 4 to the Problem 1. We have implemented the methods with a fixed stepsize $h = 1/2^k$, with several integer values of k. The results of numerical experiments for type 2 Nordsieck SGLMs are shown in the Tables 1, 2, and 3. In these tables, we have listed the norm of error $\|e_h(\overline{x})\|$ at the endpoint of integration $\overline{x} = 2$ and numerical estimate to the order of convergence, p, computed by the formula

$$p = \frac{\log(\|e_h(\overline{x})\|/\|e_{h/2}(\overline{x})\|)}{\log(2)},$$

where $e_h(\overline{x})$ and $e_{h/2}(\overline{x})$ are errors corresponding to stepsizes h and $h/2$ for Nordsieck SGLMs. Also, to show that the constructed methods are competitive with the efficient existing methods, we have reported the numerical results of type 2 methods which have been constructed in [2].

Table 1: Numerical results for Nordsieck SGLMs of order $p = q = 2$.

k	Type 2 method of order 2 [2]		Nordsieck SGLM of order 2	
	$e_h(\overline{x})$	p	$e_h(\overline{x})$	p
10	1.78×10^{-11}		1.35×10^{-10}	
11	5.44×10^{-12}	1.71	3.31×10^{-11}	1.88
12	1.43×10^{-12}	1.93	8.18×10^{-12}	1.94
13	2.64×10^{-13}	2.44	2.03×10^{-12}	1.98

Table 2: Numerical results for Nordsieck SGLMs of order $p = q = 3$.

k	Type 2 method of order 3 [2]		Nordsieck SGLM of order 3	
	$e_h(\overline{x})$	p	$e_h(\overline{x})$	p
4	4.07×10^{-6}		6.58×10^{-8}	
5	5.66×10^{-7}	2.85	8.66×10^{-9}	2.93
6	7.43×10^{-8}	2.92	1.11×10^{-9}	2.96
7	9.50×10^{-9}	2.96	1.40×10^{-10}	2.99

Table 3: Numerical results for Nordsieck SGLMs of order $p = q = 4$.

k	Type 2 method of order 4 [2]		Nordsieck SGLM of order 4	
	$e_h(\overline{x})$	p	$e_h(\overline{x})$	p
4	6.22×10^{-8}		5.81×10^{-9}	
5	4.21×10^{-9}	3.88	3.63×10^{-10}	4.00
6	2.74×10^{-10}	3.94	2.27×10^{-11}	4.00
7	1.75×10^{-11}	3.97	1.42×10^{-12}	4.00

4.2 Variable stepsize experiments

We first investigate the potential for efficient implementation in a variable stepsize environment by the reliability of the estimation error for Problem 1. The obtained results for the methods of order 2, 3, and 4 have plotted in Figures 3, 4, and 5, respectively. These figures confirm efficiency of the used estimation for the local truncation error. To compare, we also present the results of numerical experiments of the L-stable Nordsieck SGLM given in [5]. In the implementation of considered SGLMs, we apply the same introduced implementation strategies, including the starting procedures, stage predictors, local error estimation, and the changing stepsize. In our numerical results, we use the following abbreviations:

ns:	the number of steps
nrs:	the number of rejected steps
nfe:	the number of function evaluations
nJe:	the number of Jacobian evaluations
ge:	the global error
tol:	given tolerance
NSGLM2p:	type 2 Nordsieck SGLM of order p
NSGLM4p:	type 4 Nordsieck SGLM of order p

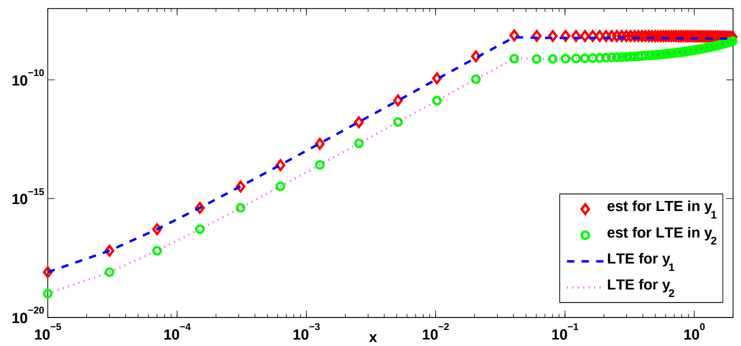


Figure 3: Local errors and local error estimates versus x of the method of order 2 for problem 1 with $h_0 = 10^{-5}$ and $tol = 10^{-8}$.

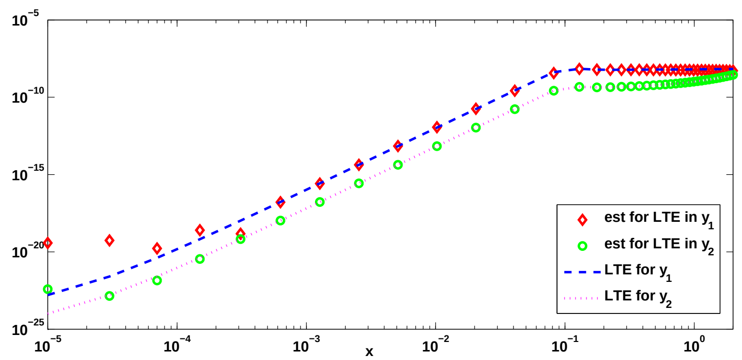


Figure 4: Local errors and local error estimates versus x of the method of order 3 for problem 1 with $h_0 = 10^{-5}$ and $tol = 10^{-8}$.

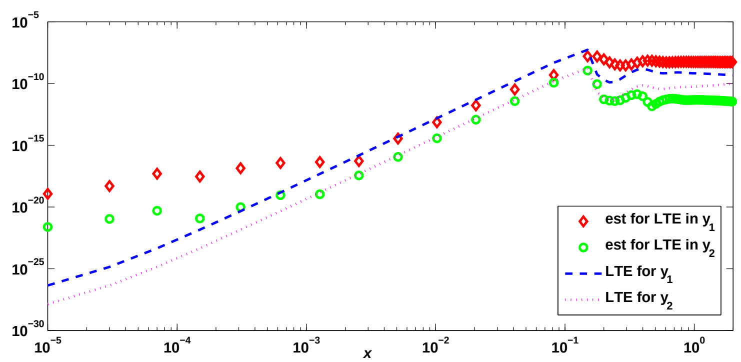


Figure 5: Local errors and local error estimates versus x of the method of order 4 for problem 1 with $h_0 = 10^{-5}$ and $tol = 10^{-8}$.

Table 4: Numerical results for Problem 2 solved by NSGLM2p and NSGLM4p of orders 2, 3, and 4 with $h_0 = 10^{-3}$.

tol	Method	*ge*	*ns*	*nrs*	*nfe*	*nJe*
10^{-6}	NSGLM22	3.61×10^{-5}	169	26	1644	1256
	NSGLM42	2.02×10^{-4}	1313	11	6693	4047
10^{-8}	NSGLM22	1.07×10^{-6}	690	12	3718	2316
	NSGLM42	8.84×10^{-6}	5997	14	24187	12167
10^{-10}	NSGLM22	4.58×10^{-8}	3154	15	12847	6511
	NSGLM42	3.99×10^{-7}	27572	14	110499	55323
10^{-6}	NSGLM23	2.95×10^{-5}	178	36	2339	1700
	NSGLM43	5.80×10^{-5}	426	17	3248	1922
10^{-8}	NSGLM23	6.57×10^{-7}	474	14	4047	2586
	NSGLM43	1.28×10^{-6}	1341	23	8308	4219
10^{-10}	NSGLM23	1.47×10^{-8}	1450	18	8998	4597
	NSGLM43	5.59×10^{-9}	26879	646	165153	82581
10^{-6}	NSGLM24	9.16×10^{-5}	337	25	3308	1864
	NSGLM44	9.66×10^{-5}	313	14	3420	2116
10^{-8}	NSGLM24	2.14×10^{-6}	2040	50	16864	8508
	NSGLM44	2.19×10^{-6}	787	14	6599	2399
10^{-10}	NSGLM24	3.77×10^{-8}	2696	141	22752	11408
	NSGLM44	4.59×10^{-8}	1700	18	13934	7066

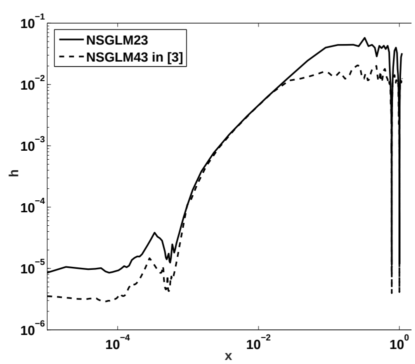
Some numerical results for Problem 2 and 3 demonstrating the computational cost of NSGLM2p are given in Tables 4 and 5 and compared with those in NSGLM4p for $p = 2, 3, 4$. Also, in Figures 6 and 7, we compare the accepted stepsizes of NSGLM23 with those in NSGLM24 and the accepted stepsizes of NSGLM43 with those in NSGLM44 through integration for Problem 2 and 3, recpectively. The numerical results show that the proposed methods are capable in solving stiff problems and competitive with the existing methods.

5 Conclusion

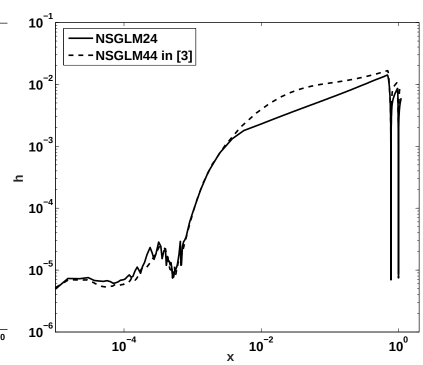
We constructed type 2 Nordsieck SGLMs of orders 2, 3, and 4 with RKS property. Order 2 and 3 methods are L-stable and order 4 method is A-stable. These methods have been equipped to the variable stepsize using Nordsieck technique. The capability of the proposed methods in solving stiff problems with long interval of integration and badly scaled solution have been validated by some numerical experiments and comparisons.

Table 5: Numerical results for Problem 3 solved by NSGLM2p and NSGLM4p of orders $2, 3$, and 4 with $h_0 = 10^{-6}$.

tol	Method	*ge*	*ns*	*nrs*	*nfe*	*nJe*
10^{-6}	NSGLM22	1.18×10^{-5}	3336	492	24233	16579
	NSGLM42	8.95×10^{-5}	27540	475	139942	83914
10^{-8}	NSGLM22	5.94×10^{-7}	14721	486	77469	47057
	NSGLM42	3.94×10^{-6}	126749	427	508703	254353
10^{-10}	NSGLM22	2.05×10^{-8}	67411	469	271519	135761
	NSGLM42	8.61×10^{-8}	272719	396	1092456	546228
10^{-6}	NSGLM23	1.71×10^{-5}	2790	476	30994	21199
	NSGLM43	1.68×10^{-5}	8243	499	61705	35482
10^{-8}	NSGLM23	1.46×10^{-7}	8729	705	76184	74885
	NSGLM43	5.31×10^{-7}	14444	450	155761	77884
10^{-10}	NSGLM23	5.59×10^{-9}	26879	646	165153	82581
	NSGLM43	1.61×10^{-8}	80230	414	483861	241932
10^{-6}	NSGLM24	1.75×10^{-3}	4070	642	42363	23519
	NSGLM44	2.11×10^{-3}	4563	591	51425	30813
10^{-8}	NSGLM24	4.86×10^{-5}	13440	1198	117655	59107
	NSGLM44	5.47×10^{-5}	10954	612	92614	46356
10^{-10}	NSGLM24	8.86×10^{-7}	24087	5512	236786	118394
	NSGLM44	1.35×10^{-6}	26896	602	219980	109992

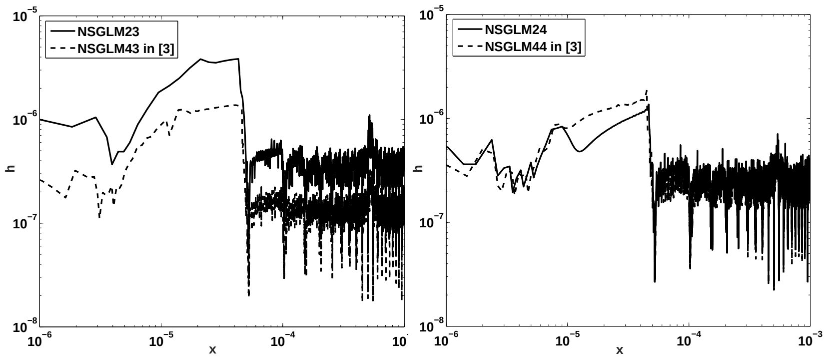


(a)



(b)

Figure 6: Accepted stepsizes versus x for Problem 2 with $h_0 = 10^{-3}$ and $tol = 10^{-6}$: (a) NSGLM23 and NSGLM43, (b) NSGLM24 and NSGLM44.



(a) (b)

Figure 7: Accepted stepsizes versus x for Problem 3 with $h_0 = 10^{-6}$ and $tol = 10^{-6}$: (a) NSGLM23 and NSGLM43, (b) NSGLM24 and NSGLM44.

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برخی روشهای انتگرالگیری نردسیک کارا برای مسایل مقدار اولیه

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دریافت مقاله ۹ مرداد ۱۳۹۵، دریافت مقاله اصلاح شده ۲۰ آبان ۱۳۹۵، پذیرش مقاله ۱ آذر ۱۳۹۶

چکیده : در این مقاله، در ادامه ساخت روش های عددی کارا برای حل مسایل مقدار اولیه سخت، روش های خطی عمومی با مشتق دوم نردسیک نوع دو با مرتبه p=s را می سازیم، که در آن s تعداد مراحل میانی است و مرتبه مرحله ای q=p می باشد. پیاده سازی روش های ساخته شده با طول گام ثابت و متغیر بحث قرار گرفته است که کارایی روشها را تأیید می کند.

کلمات کلیدی : معادلات دیفراتسیل سخت؛ روشهای خطی عمومی با مشتق دوم نردسیک؛ $A-$ و $L-$ پایداری؛ پیاده سازی با طول گام متغیر.