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Research Article

Some applications of Sigmoid functions

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Abstract

In numerical analysis, the process of fitting a function via given data is called interpolation. Interpolation has many applications in engineering and science. There are several formal kinds of interpolation, including linear interpolation, polynomial interpolation, piecewise constant interpolation, trigonometric interpolation, and so on. In this article, by using Sigmoid functions, a new type of interpolation formula is presented. To illustrate the efficiency of the proposed new interpolation formulas, some applications in quadrature formulas (in both open and closed types), numerical integration for double integral, and numerical solution of an ordinary differential equation are included. The advantage of this new approach is shown in the numerical applications section.

AMS subject classifications (2020): 65D05, 65D30, 65D32.

Keywords: Sigmoid function; Interpolation; Numerical integration; Quadrature formula; Numerical solution of ordinary differential equation.

1 Introduction

Interpolation is the process of finding a function via given data. Interpolation has many applications in engineering and science. There are several formal kinds of interpolation, including linear interpolation, polynomial interpolation, piecewise constant interpolation, trigonometric interpolation, and so on. For example, consider the polynomial interpolation. Let $f_0, f_1, \dots, f_n$ be known values for an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ at points $x = x_i$, $i = 0, 1, \dots, n$. Then the Lagrange basis functions are defined as

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$$L_{n,i}(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(x - x_k)}{(x_i - x_k)}, \quad i = 0, 1, \ldots, n. \tag{1}$$

Then the interpolating polynomial of degree n is defined as

$$p(x) = \sum_{i=0}^{n} L_{n,i}(x) f_i. \tag{2}$$

Clearly, $p(x)$ satisfies $p(x_i) = f_i$ for $i = 0, 1, \ldots, n$. Moreover, for any $f \in C^{n+1}[a, b]$, we have

$$f(x) = \sum_{i=0}^{n} L_{n,i}(x) f_i + (x - x_0)(x - x_1) \cdots (x - x_n) \frac{f^{(n+1)}(\eta_x)}{(n+1)!}, \quad \eta_x \in [a, b]. \tag{3}$$

Using relation (2), the classical Newton–Cotes quadrature formulas are obtained [2, 3, 6, 7, 8, 9]. In this article, by using Sigmoid functions [1, 4, 5], new types of interpolation formulas are presented. These new types of interpolation result in new integration formulas and new formulas for solving ordinary differential equations.

Sigmoid functions are mathematical functions with S-shaped curves or sigmoid curves. These functions are bounded and monotone and have the first derivative that is bell shapes. Additionally, these functions are constrained by a pair of horizontal asymptotes as $x \longrightarrow \pm\infty$. Some of them are listed below.

a. **Logistic function** is defined by the formula

$$\phi(x) = \frac{1}{1 + e^{-x}}. \tag{4}$$

b. **Hyperbolic tangent function** is defined by the formula

$$\phi(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \tag{5}$$

c. **Arctangent function** is defined by

$$\phi(x) = \arctan(x). \tag{6}$$

d. **Gudermannian function** is defined by

$$\phi(x) = gd(x) = \int_0^x \frac{1}{\cosh x} dx = 2 \arctan(\tanh(\frac{x}{2})). \tag{7}$$

e. **Error function** is defined as

$$\phi(x) = erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{8}$$

f. **Generalised logistic function** is defined as

$$\phi(x) = (1 + e^{-x})^{-\alpha}, \quad \alpha > 0. \tag{9}$$

g. Some algebraic functions, for example,

$$\phi(x) = \frac{x}{\sqrt{1 + x^2}}. \tag{10}$$

h. **Smoothstep function** is defined for $N \geq 1$ as

$$\phi(x) = \begin{cases} \left(\int_0^1 (1 - u^2)^N du\right)^{-1} \int_0^x (1 - u^2)^N du, & |x| \leq 1, \\ sgn(x), & |x| \geq 1. \end{cases} \tag{11}$$

I. The integral of any continuous, nonnegative, and "bump-shaped" function will be sigmoid function; therefore the cumulative distribution functions for many common probability distributions are sigmoid functions.

This article is organized as follows. In section 2, new methods of interpolation are introduced. In section 3, some applications of the proposed new interpolation formulas are provided. Finally, in section 4, conclusions are presented.

2 Main results

In this section, the new types of interpolation formulas for one- and two-dimensional cases by using Sigmoid functions are obtained.

2.1 One-dimensional case

Theorem 1. Let ϕ be an arbitrary Sigmoid function and let

$$\begin{cases} \lim\limits_{x \to -\infty} \phi(x) = a, \\ \lim\limits_{x \to +\infty} \phi(x) = b. \end{cases} \tag{12}$$

Let $f_0, f_1, \ldots, f_n$ be known values for an arbitrary function at points $x_0 < x_1 < \cdots < x_n$, and let $x_{i+1} - x_i = h_i$ for $i = 0, 1, 2, \ldots, n$. Then

$$S(x) = \sum_{i=0}^{n} \left(\frac{1}{b - a}\right) f_i \left[\phi(a_i x - b_i) - \phi(a_{i+1} x - b_{i+1})\right] \tag{13}$$

is the interpolation formula for the above data points. Moreover, a_i for $i = 0, 1, 2, \ldots, n+1$ are positive and big enough numbers, $\dfrac{b_i}{a_i} = x_i - \dfrac{h_i}{2}$ for $i = 0, 1, 2, \ldots, n$, and $\dfrac{b_{n+1}}{a_{n+1}} = x_n + \dfrac{h_i}{2}$.

Proof. According to the properties of the Sigmoid functions, $0 \leqslant \phi(a_i x - b_i) - \phi(a_{i+1}x - b_{i+1}) \leqslant b - a$. Therefore, for positive and big enough a_i, in a neighborhood of $x = \dfrac{b_i}{a_i}$, the function $[\phi(a_i x - b_i) - \phi(a_{i+1}x - b_{i+1})]$ changes from zero to $b - a$ (for $i = 0, 1, \ldots, n$). Similarly, at $x = \dfrac{a_{i+1}}{b_{i+1}}$, the function $[\phi(a_i x - b_i) - \phi(a_{i+1}x - b_{i+1})]$ changes from $b - a$ to zero (for $i = 0, 1, \ldots, n$). Therefore, for the interior points in intervals $\left[\dfrac{b_i}{a_i}, \dfrac{b_{i+1}}{a_{i+1}}\right]$, for $i = 0, 1, \ldots, n$, $S(x)$ is equal to f_i. In a neighborhood of $x = \dfrac{b_i}{a_i}$ (for $i = 1, 2, \ldots, n$), $S(x)$ is changed from f_{i-1} to f_i. In the same way, in a neighborhood of $x = \dfrac{b_{i+1}}{a_{i+1}}$ (for $i = 0, 1, \ldots, n-1$), $S(x)$ is transformed from f_i to f_{i+1}. In addition, in a neighborhood of $x = \dfrac{b_0}{a_0}$, $S(x)$ is changed from zero to f_0, and in a neighborhood of $x = \dfrac{b_{n+1}}{a_{n+1}}$, $S(x)$ is changed from f_n to zero. Speed of changing can be increased by increasing a_i (for $i = 0, 1, \ldots, n+1$). In other words, a_i and $\dfrac{b_i}{a_i}$, $i = 0, 1, \ldots, n+1$, define the speed and locations of the changes. Therefore, $S(x)$ is the interpolation formula for aforesaid points. □

2.2 Two-dimensional case

By using the above idea for two-dimensional problems, a new type of two-dimensional interpolation formulas can be obtained.

Theorem 2. Let $F(x_i, y_j) = z_{i,j}$ for $i = 0, 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, m$. Let ϕ and ψ be two arbitrary Sigmoid functions and let

$$\begin{cases} \lim\limits_{x \to -\infty} \phi(x) = a, \\ \lim\limits_{x \to +\infty} \phi(x) = b, \end{cases} \qquad \begin{cases} \lim\limits_{x \to -\infty} \psi(x) = c, \\ \lim\limits_{x \to +\infty} \psi(x) = d. \end{cases} \tag{14}$$

Let $x_0 < x_1 < \cdots < x_n$ and $y_0 < y_1 < \cdots < y_m$ (i.e., $x_{i+1} - x_i = h_i$, for $i = 0, 1, \ldots, n-1$, and $y_{j+1} - y_j = k_j$ for $j = 0, 1, \ldots, m-1$). Then

$$S(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m}\left(\frac{1}{b-a}\right)\left(\frac{1}{d-c}\right) z_{i,j} \tag{15}$$
$$[\phi(a_i x - b_i) - \phi(a_{i+1}x - b_{i+1})]\left[\psi(a_j^{'}y - b_j^{'}) - \psi(a_{j+1}^{'}y - b_{j+1}^{'})\right]$$

is the interpolation formula for the above data points. Moreover, a_i for $i = 0, 1, 2, \ldots, n+1$ and $a_j^{'}$ for $j = 0, 1, 2, \ldots, m+1$ are positive and big enough numbers. In addition

$$\begin{cases} \dfrac{b_i}{a_i} = x_i - \dfrac{h_i}{2}, \quad i = 0, 1, 2, \ldots, n \quad \textit{and} \quad \dfrac{b_{n+1}}{a_{n+1}} = x_n + \dfrac{h_n}{2}, \\ \dfrac{b_j^{'}}{a_j^{'}} = y_j - \dfrac{k_j}{2}, \quad j = 0, 1, 2, \ldots, m \quad \textit{and} \quad \dfrac{b_{m+1}^{'}}{a_{m+1}^{'}} = y_m + \dfrac{k_m}{2}. \end{cases} \tag{16}$$

Proof. By using relation (13) two times, relation (15) is obtained. □

In the next section, some numerical applications in interpolation, numerical integration, and numerical solution of ordinary differential equations are included.

3 Numerical applications

In this section, some applications of formula (13) in one-dimensional interpolation, numerical integration, and numerical solving of ordinary differential equations by using the Hyperbolic tangent function are given. Other formulas (i.e., (4)–(11)) can easily be applied. Also, in Example 6, formula (15) by using the Hyperbolic tangent function is applied.

Example 1 (Interpolation problem)**.** Let $f(0) = 1$, $f(1) = 2$, $f(2) = 4$, $f(3) = 7$, $f(4) = 5$, and $f(5) = 6$. Then using formula (13), the interpolation function is obtained as

$$\begin{aligned} S(x) &= \left(\frac{1}{2}\right)\sum_{i=0}^{5} f(x_i)\left[\tanh(a_i x - b_i) - \tanh(a_{i+1}x - b_{i+1})\right] \\ &= \left(\frac{1}{2}\right)(1 \times [\tanh(a_0 x - b_0) - \tanh(a_1 x - b_1)] \\ &\quad +2 \times [\tanh(a_1 x - b_1) - \tanh(a_2 x - b_2)] \\ &\quad +4 \times [\tanh(a_2 x - b_2) - \tanh(a_3 x - b_3)] \\ &\quad +7 \times [\tanh(a_3 x - b_3) - \tanh(a_4 x - b_4)] \\ &\quad +5 \times [\tanh(a_4 x - b_4) - \tanh(a_5 x - b_5)] \\ &\quad + 6 \times [\tanh(a_5 x - b_5) - \tanh(a_6 x - b_6)]) \,. \end{aligned} \tag{17}$$

This interpolation function in two cases (i.e., $a_0 = a_1 = \cdots = a_6 = 5$ and $a_0 = a_1 = \cdots = a_6 = 10$) is plotted in Figures 1 and 2.

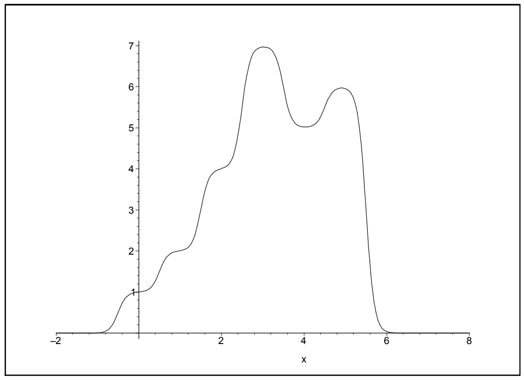

Figure 1: Interpolation function with $a_i = 5$ for $i = 0, 1, \ldots, 6$.

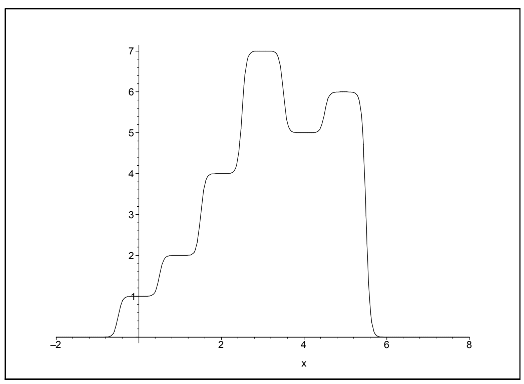

Figure 2: Interpolation function with $a_i = 10$ for $i = 0, 1, \ldots, 6$.

Example 2 (Interpolation problem (Runge's function))**.** Consider the polynomial interpolation of Runge's function defined as (see [9])

$$g(x) = \frac{1}{1 + 25x^2}, \quad x \in [-1, 1]. \tag{18}$$

The polynomial interpolating with $N = 15$ equidistant nodes is plotted in Figure 3. Also, using formula (13), the new interpolation is obtained. This interpolation function is plotted in Figure 4. The error function of polynomial interpolation is plotted in Figure 5. As you can see, the error at nonnodal points does not decrease with the increase of nodes; see [3, 9]. Also, the error function of the new interpolation is plotted in Figure 6.

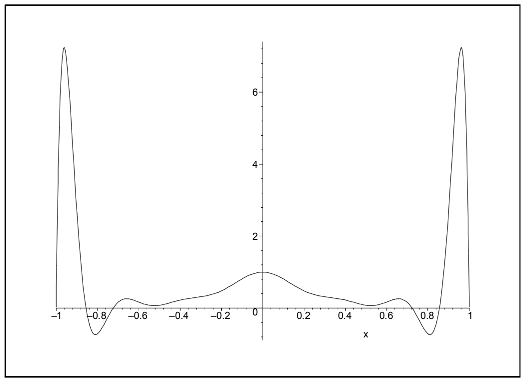

Figure 3: Polynomial interpolation function of Runge's function with $N = 15$ equidistant nodes in $[-1, 1]$.

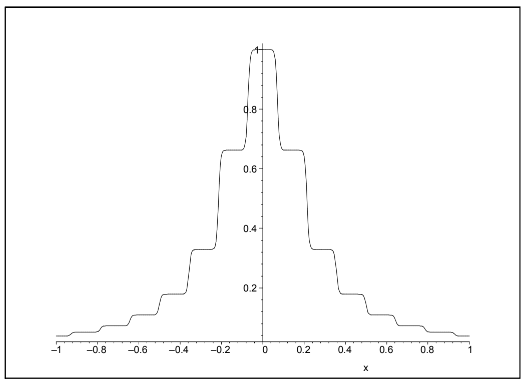

Figure 4: The new interpolation function of Runge's function with $N = 15$ equidistant nodes in $[-1, 1]$ and $a[i] = 100$ for $i = 0, 1, 2, \ldots, 15$.

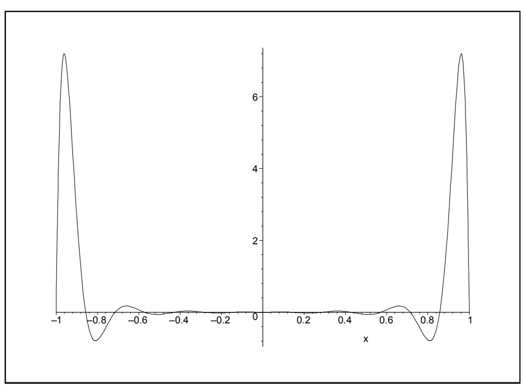

Figure 5: Error function of the polynomial interpolation function for Runge's function with $N = 15$ equidistant nodes in $[-1, 1]$.

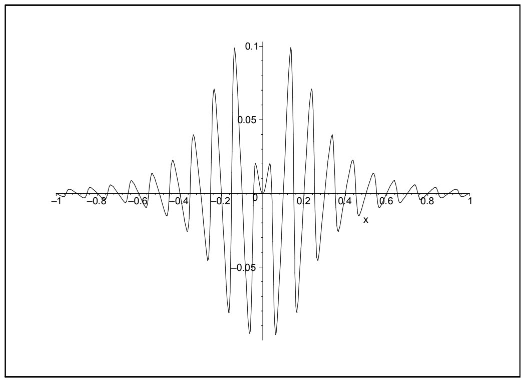

Figure 6: Error function of the new interpolation formula for Runge's function with $N = 15$ equidistant nodes in $[-1, 1]$ and $a[i] = 100$ for $i = 0, 1, 2, \ldots, 15$.

Example 3 (Numerical integration (closed type))**.** Quadrature formula of Newton–Cotes on the finite interval $[a, b]$ is defined as (see [7, 8])

$$\int_a^b f(x)dx = \sum_{i=0}^{n} w_i f(x_i) + R_n(f), \tag{19}$$

where x_i are equidistantly distributed with the step size $\dfrac{b-a}{n}$ and $x_i = a+ih$, $i = 0, 1, \ldots, n$. In addition, $R_n(f) = 0$ whenever $f \in \mathcal{P}_{n-1}$, where $\mathcal{P}_{n-1}$ is the space of all algebraic polynomials of degree at most $n-1$. Now, using relation (13) for $\phi(x) = \tanh(x)$ results in

$$\begin{aligned}\int_a^b f(x)dx &\approx \int_a^b S(x)dx \\ &= \int_a^b \sum_{i=0}^{n} \left(\frac{1}{2}\right) f_i \left[\tanh(a_i x - b_i) - \tanh(a_{i+1}x - b_{i+1})\right] dx \end{aligned} \tag{20}$$

Therefore, relation (20) is simplified as

$$\begin{aligned}\int_a^b f(x)dx \approx \sum_{i=0}^{n} \Bigg[&\left(\frac{1}{2a_i}\right) \left[\ln(\cosh(a_i b - b_i)) - \ln(\cosh(a_i a - b_i)\right] \\ &- \left(\frac{1}{2a_{i+1}}\right) \left[\ln(\cosh(a_{i+1} b - b_{i+1})) - \ln(\cosh(a_{i+1} a - b_{i+1})\right] \Bigg] f_i. \end{aligned} \tag{21}$$

As it is mentioned above, explicit forms of quadrature formulas based on this new interpolation formula can be directly obtained. As an example, consider the following integral:

Table 1: Absolute error of quadrature formula (21) for $n = 5(5)30$ and $f(x) = \exp(x^2)$.

n	Absolute error
5	1.8(-2)
10	4.5(-3)
15	2.0(-3)
20	1.0(-3)
25	4.0(-4)
30	1.2(-4)

$$\int_0^1 \exp(x^2)dx \approx 1.46265174590718. \tag{22}$$

The absolute error of formula (21) to approximate the above integral, for $a_i = 100, i = 0, 1, \dots, n$, is presented in Table 1.

Example 4 (Numerical integration (open type))**.** In relation (19), if $w_0 = w_n = 0$, then the open type of Newton–Cotes formula is obtained. In this case, $R_n(f) = 0$ whenever $f \in \mathcal{P}_{n-3}$. The new quadrature formula in this case is as follows:

$$\begin{aligned}\int_a^b f(x)dx &\approx \int_a^b S(x)dx \qquad (23)\\ &= \int_a^b \sum_{i=1}^{n-1} \left(\frac{1}{2}\right) f_i \left[\tanh(a_i x - b_i) - \tanh(a_{i+1}x - b_{i+1})\right] dx.\end{aligned}$$

Therefore, relation (23) is simplified as

$$\begin{aligned}\int_a^b f(x)dx \approx \sum_{i=1}^{n-1} &\left[\left(\frac{1}{2a_i}\right)[\ln(\cosh(a_i b - b_i)) - \ln(\cosh(a_i a - b_i)]\right.\\ &\left. - \left(\frac{1}{2a_{i+1}}\right)[\ln(\cosh(a_{i+1}b - b_{i+1})) - \ln(\cosh(a_{i+1}a - b_{i+1})]\right] f_i.\end{aligned}$$

As an example, consider the following integral:

$$\int_0^1 x^{100} \sin\left(\frac{1}{x}\right) dx \approx 0.008384044187. \tag{24}$$

The absolute error of formula (24) to approximate the above integral, for $a_i = 100, i = 0, 1, \dots, n$, is presented in Table 2.

Example 5 (Numerical methods for solving ordinary differential equation)**.** Consider the following ODE:

Table 2: Absolute error of quadrature formula (24) for $n = 5(5)30$ and $f(x) = x^{100}\sin(\frac{1}{x})$.

n	Absolute error
5	8.38(-3)
10	8.38(-3)
15	8.32(-3)
20	8.12(-3)
25	7.79(-3)
30	7.39(-3)

$$\begin{aligned} &\frac{dy}{dx} = x + y, \\ &y(0) = 0, \end{aligned} \tag{25}$$

with exact solution $y(x) = \exp(x) - x - 1$. Integrating both sides of relation (25) on interval $[0, x]$ and using initial condition result in

$$y(x) = y(0) + \int_0^x (s + y)ds = \frac{1}{2}x^2 + \int_0^x yds. \tag{26}$$

Now, using the new interpolation formula for y results in

$$\begin{aligned} &\sum_{i=0}^{n} \left(\frac{1}{2}\right) y(x_i)\left[\tanh(a_i x - b_i) - \tanh(a_{i+1}x - b_{i+1})\right] \\ &= \frac{1}{2}x^2 + \int_0^x \left[\sum_{i=0}^{n} \left(\frac{1}{2}\right) y(x_i)\left[\tanh(a_i s - b_i) - \tanh(a_{i+1}s - b_{i+1})\right]\right] ds. \end{aligned} \tag{27}$$

Let $Q_i(x) = \tanh(a_i x - b_i) - \tanh(a_{i+1}x - b_{i+1})$ for $i = 0, 1, 2, \ldots, n$. Then relation (27) is simplified as

$$\begin{aligned} \sum_{i=0}^{n} \left(\frac{1}{2}\right) y(x_i)Q_i(x) &= \frac{1}{2}x^2 + \int_0^x \left[\sum_{i=0}^{n} \left(\frac{1}{2}\right) y(x_i)Q_i(s)\right] ds \\ &= \frac{1}{2}x^2 + \left(\frac{1}{2}\right)\sum_{i=0}^{n} y(x_i)\int_0^x Q_i(s)ds. \end{aligned} \tag{28}$$

Finally, relation (28) is simplified as

$$\sum_{i=0}^{n} y(x_i)\left(Q_i(x) - \int_0^x Q_i(s)ds\right) = x^2. \tag{29}$$

By collocating (29) at x_i, $i = 1, 2, \ldots, n$, the unknown values of $y(x_i)$ are obtained. For example, the error function on $[0, 1]$ for $n = 10$, and $x_i = \dfrac{i}{10}$, $i = 1, 2, \ldots, 10$, is plotted in Figure 7. Also, the error function on $[0, 1]$ for $n = 20$ and $x_i = \dfrac{i}{20}$, $i = 1, 2, \ldots, 20$, is plotted in Figure 8.

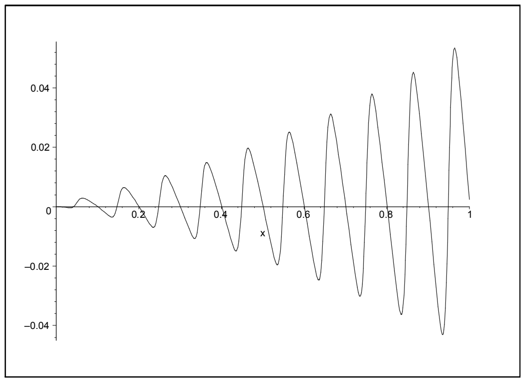

Figure 7: The error function of Example 5, with $n = 10$ in $[0, 1]$ and $a[i] = 100$ for $i = 0, 1, 2, \ldots, 10$.

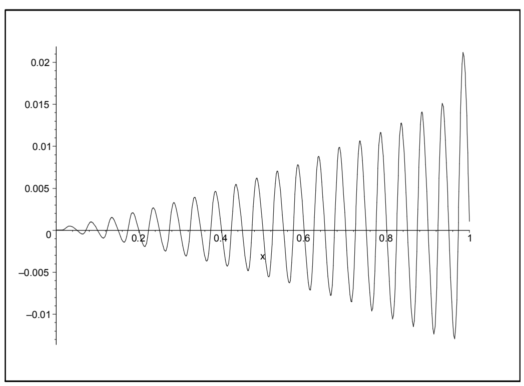

Figure 8: The error function of Example 5, with $n = 20$ in $[0, 1]$ and $a[i] = 100$ for $i = 0, 1, 2, \ldots, 20$.

Example 6 (Numerical integration for double integral)**.** Now using relation (15) results in

Table 3: Absolute error of quadrature formula (30) for $n = 5(5)30$ and $F(x, y) = \exp(x^2 + y^2)$.

n	Absolute error
5	5.30(-2)
10	1.32(-2)
15	5.83(-3)
20	2.97(-3)
25	1.19(-3)
30	3.46(-4)

$$\int_a^b \int_c^d F(x,y)dydx \approx \int_a^b \int_c^d S(x,y)dydx \tag{30}$$

$$= \int_a^b \int_c^d \sum_{i=0}^{n} \sum_{i=0}^{m} \left(\frac{1}{2}\right)^2 z_{i,j} \times [\tanh(a_i x - b_i) - \tanh(a_{i+1}x - b_{i+1})] \times \left[\tanh(a_i^{'} y - b_i^{'}) - \tanh(a_{i+1}^{'} y - b_{i+1}^{'})\right] dxdy. \tag{31}$$

As an example, consider the following integral:

$$\int_0^1 \int_0^1 \exp(x^2 + y^2)dydx \approx 2.139350130. \tag{32}$$

The absolute error of formula (30) to approximate the above integral, for $a_i = 100, i = 0, 1, \ldots, n$, is presented in Table 3.

4 Conclusions

In this article, by using Sigmoid functions, the new types of interpolation formulas were presented. To show the efficiency of the proposed new interpolation formulas, some applications in quadrature formulas (in both open and closed types), numerical integration for double integral, and numerical solution of ordinary differential equation were included. Numerical results were obtained by using the Hyperbolic tangent function, which can be easily changed by other Sigmoid functions.

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